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The Bethe ansatz for superconformal Chern-Simons

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Abstract: We study the anomalous dimensions for scalar operators for a three-dimensional Chern-Simons theory recently proposed in arXiv:0806.1218. We show that the mixing matrix at two-loop order is that for an integrable Hamiltonian of an SU(4) spin chain with sites alternating between the fundamental and the anti-fundamental representations. We find a set of Bethe equations from which the anomalous dimensions can be determined and give a proposal for the Bethe equations to the full superconformal group of $OSp(2,2|6)$.

Keywords: AdS-CFT Correspondence, Chern-Simons Theories, Bethe Ansatz, Lattice Integrable Models.

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Contents

1. Introduction

Integrability has proven to be a powerful tool in analyzing $\mathcal{N}=4$ Super Yang-Mills in the planar limit. An interesting question is whether or not there are other gauge theories with a high degree of supersymmetry that are also integrable at the planar level.

Recently, a proposal was made by Aharony, Bergman, Jafferis and Maldacena (ABJM) [1], following a large body of work on multiple M2-branes [2 − 10], for a three dimensional superconformal $\mathrm{SU}(N)\times\mathrm{SU}(N)$ Chern-Simons theory that seems to be the effective theory for a stack of M2 branes at a Z_k orbifold point. In the large N limit, the gravitational dual becomes M-theory on $AdS_4\times S^7/Z_k$. The integer k is the level of the first $\mathrm{SU}(N)$ and the level of the second $\mathrm{SU}(N)$ is $-k$. The theory has manifest $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry and two sets of scalar fields transforming in bifundamental representations of $\mathrm{SU}(N)\times\mathrm{SU}(N)$. The first set of scalars, A_a are doublets under one SU(2) of the R-symmetry group and transform in the $(N,\overline{N})$ representation and the second set of scalars $B_{\dot{a}}$ are doublets under the second SU(2) and transform under the $(\overline{N},N)$ representation.

The scalars can be conveniently expressed as $N\times N$ matrices, in which case the superpotential takes the form

$$W=\frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\,\mathrm{tr}(A_aB_{\dot{a}}A_bB_{\dot{b}})\,. \tag{1.1}$$

Remarkably, as was argued in [1] and proven in [11], the R-symmetry is enhanced to SO(6) due to contributions from the Chern-Simons terms, and the theory has $\mathcal{N}=6$ supersymmetry if $k > 2$. If $k = 1$ or 2, then there is $\mathcal{N} = 8$ supersymmetry.

The ABJM model has the large-N limit with the 't Hooft coupling $\lambda = N/k$ [1]. For infinite N and finite λ, k is infinite and λ is essentially continuous. In the case of large k, the orbifold effectively compactifies to a cylinder and M-theory approaches type IIA string theory on $AdS_4 \times CP^3$. String theory propagating on this space is classically integrable, so one might expect integrability to appear in the dual gauge theory as well. The classical limit of string theory corresponds to $\lambda \gg 1$. We will analyze the opposite regime of weak coupling.

The scalar fields can be grouped into SU(4) multiplets Y^A as follows

$$Y^A = (A_1, A_2, B_{\dot{1}}^\dagger, B_{\dot{2}}^\dagger) \qquad Y_A^\dagger = (A_1^\dagger, A_2^\dagger, B_{\dot{1}}, B_{\dot{2}})\,, \tag{1.2}$$

and a class of gauge invariant operators can be built out of these scalars in the form[1]

$$\mathcal{O} = \mathrm{tr}(Y^{A_1} Y_{B_1}^\dagger Y^{A_2} Y_{B_2}^\dagger \ldots Y^{A_L} Y_{B_L}^\dagger)\chi_{A_1 \ldots A_L}^{B_1 \ldots B_L}\,. \tag{1.3}$$

The bare dimension of $\mathcal{O}$ is L and $\mathcal{O}$ is a chiral primary if χ is symmetric in all A_i indices, symmetric in all B_i indices and all traces are zero. If $\mathcal{O}$ is not a chiral primary, then it has a nonzero anomalous dimension. The leading order contribution to the anomalous dimension comes at two-loop order since the contributions all come with even powers of k, and in general leads to operator mixing.

In this paper we compute the leading order operator mixing matrix for the scalar operators in (1.3) and show that it is isomorphic to an integrable Hamiltonian of an SU(4) spin-chain with sites alternating between the fundamental and anti-fundamental representations and next to nearest neighbor interactions. The details of the calculation parallel the arguments in [14] for scalar operators in $\mathcal{N} = 4$ SYM$_4$. In that case there was also SU(4) R-symmetry and the scalar operators had a mixing matrix that is isomorphic to a Hamiltonian for an integrable SU(4) spin chain. One can then find a set of Bethe equations whose solutions lead to the eigenvalues of the mixing matrix. For the ABJM model we will also find a set of Bethe equations, but with different weights than the $\mathcal{N} = 4$ SYM$_4$ case. One can then try extending the calculation to the full superconformal group as in [15, 16]. While we do not compute the Hamiltonian explicitly, we propose a natural extension of the SU(4) chain to an $OSp(2, 2|6)$ chain. Extension to higher orders in λ should be also possible as in the $\mathcal{N} = 4$, $D = 4$ super-Yang-Mills [17 − 20], but will not be discussed in this paper.

Previously, Gaiotto and Yin [21] studied a different version of a supersymmetric Chern-Simons theory, with lower supersymmetry and an SU(2) R-symmetry group. In that theory the gauge invariant operators can be mapped to an SU(2) spin chain with both nearest neighbor and next to nearest neighbor interactions, and so the theory cannot be integrable.

[1] Recently considered BMN operators in the ABJM model [12] are particular cases of these more general operators, which as a matter of fact resemble scalar operators in the orbifolds of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions [13].

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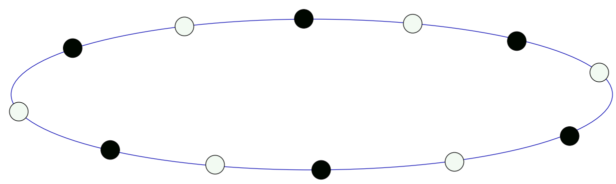

Figure 1: The alternating spin chain.

In the ABJM theory the larger R-symmetry group and cancelation of the nearest neighbor interactions allow integrability to be possible.

In section 2 we construct the Hamiltonian by explicitly computing a two-loop Feynman diagram containing a six-point vertex, a two-loop diagram containing a fermion loop and a two loop diagram containing gauge propagators. The resulting Hamiltonian has next to nearest neighbor interactions of two types. It turns out that the nearest neighbor interactions cancel out between the three types of diagrams. In section 3 we show that the resulting Hamiltonian is integrable and we find the corresponding Bethe equations for this system. In section 4 we propose an extension of the Bethe equations to the full $OSp(2,2|6)$ superconformal group. In section 5 we summarize our results and offer suggestions for further study. In appendices we give some technical details and consider the explicit example of an operator with four sites.

2. Two-loop amplitudes and the hamiltonian

The operators (1.3) need to be renormalized to make their correlation functions finite. Transition to the basis where the renormalization is multiplicative leads to the operator mixing:

$$\mathcal{O}^{\mathbf{A}}_{\text{ren}} = Z^{\mathbf{A}}{}_{\mathbf{B}}(\Lambda)\mathcal{O}^{\mathbf{B}}_{\text{bare}}, \tag{2.1}$$

where $\mathbf{A}$ is a multi-index that enumerates all possible operators, Λ is a UV cutoff, and the Z-factor subtracts all the UV divergences from the correlation functions. The mixing matrix (the quantum part of the dilatation operator) is defined as

$$\varGamma = Z^{-1}\frac{dZ}{d\ln\Lambda}\,. \tag{2.2}$$

Its eigenstates are conformal operators and the eigenvalues are their anomalous dimensions.

It convenient to represent the operators (1.3) as states in a quantum spin chain with $2L$ sites. The spin is alternating between the fundamental representation of $su(4)$ on odd sites and the anti-fundamental representation on the even sites (figure 1). The mixing matrix can then be regarded as the Hamiltonian acting in the Hilbert space $(V\otimes\bar{V})^{\otimes L}$, where V $(\bar{V})$ is the the $\mathbf{4}$ $(\bar{\mathbf{4}})$ of SU(4). We will compute this Hamiltonian to the lowest order in λ and in $1/N$.

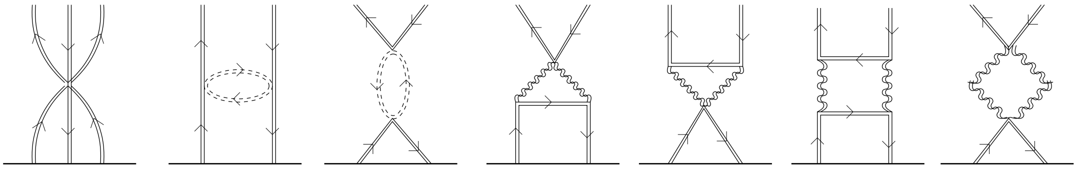

Figure 2: The planar diagrams that contribute to operator mixing at two loops. The horizontal bar denotes the operator. The directions of the arrows refer to the flow of the SU(4) flavor. Since the superpartners of the scalars are in the conjugate representation of SU(4), the fermion arrows in (b) and (c) have the opposite orientation. The gauge propagators in (d), (e), (f) and (g) do not have arrows since they do not carry SU(4) charges. It turns out that only (a), (b) and (d) contribute to the anomalous dimension.

The action of the $\mathcal{N}=6$ Chern-Simons [1] is[2]

$$\begin{aligned} S = \frac{k}{4\pi}\int d^3x \ \mathrm{tr}\ \Big[& \varepsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) + D_\mu Y_A^\dagger D^\mu Y^A \\ & + \frac{1}{12}Y^AY_A^\dagger Y^BY_B^\dagger Y^CY_C^\dagger + \frac{1}{12}Y^AY_B^\dagger Y^BY_C^\dagger Y^CY_A^\dagger - \frac{1}{2}Y^AY_A^\dagger Y^BY_C^\dagger Y^CY_B^\dagger \\ & + \frac{1}{3}Y^AY_B^\dagger Y^CY_A^\dagger Y^BY_C^\dagger + \text{fermions}\Big], \end{aligned} \tag{2.3}$$

where $D_\mu Y = \partial Y + A_\mu Y - Y\hat{A}_\mu$, $D_\mu Y^\dagger = \partial_\mu Y^\dagger + \hat{A}_\mu Y^\dagger - Y^\dagger A_\mu$. Since the interactions are of the Y^6 type (and $Y^2\Psi^2$, if we include fermions), the lowest order contribution to the mixing matrix arises at two loops (figure 2). The scalar diagram (a) connects three sites on the spin chain, so the Hamiltonian will involve interactions of three adjacent spins:

$$\Gamma = \frac{\lambda^2}{4}\sum_{l=1}^{2L} H_{l,l+1,l+2}, \tag{2.4}$$

where $H_{l,l+1,l+2}$ acts on $\bar{V}\otimes V\otimes\bar{V}$ for l even and on $V\otimes\bar{V}\otimes V$ for l odd. The diagrams in (b) and (c) contribute only to the nearest-neighbor interactions. And finally there are also diagrams with the gauge-boson exchange and the self-energy graphs. We will compute the scalar diagram here and the other diagrams in appendix A. The gauge-boson exchange and self-energy contribute only to the diagonal term in the Hamiltonian, and we will reconstruct them by supersymmetry.

The loop integral in the scalar diagram can be easily calculated in the coordinate representation:

$$\int d^3x\left(\frac{1}{4\pi|x|}\right)^3 = \frac{1}{16\pi^2}\ln\Lambda\,.$$

This contains three 3-dimensional propagators in the coordinate representation. The non-trivial part is combinatorics of the SU(4) indices, which can be handled graphically. Omitting unnecessary details, we show the odd-site Hamiltonian in figure 3. The even-site Hamiltonian is obtained by flipping the arrows. The Hamiltonian can be expressed

[2] The fermion terms in the action [11] are listed in the appendix A.

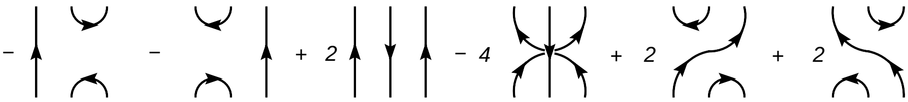

Figure 3: The two-loop Hamiltonian. The arrows denote SU(4) index contractions.

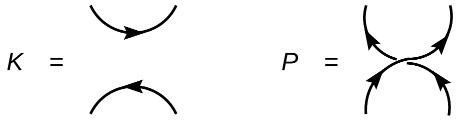

Figure 4: The permutation and trace operators.

in terms of the two basic operators (figure 4): the permutation $P : V \otimes V \to V \otimes V$ (or $P : \bar{V} \otimes \bar{V} \to \bar{V} \otimes \bar{V}$) and the trace $K : V \otimes \bar{V} \to V \otimes \bar{V}$ (or $K : \bar{V} \otimes V \to \bar{V} \otimes V$), defined as

$$\begin{aligned} P^{AB}{}_{A'B'} &= \delta^{A}{}_{B'}\, \delta^{B}{}_{A'} \\ K^{A\ B'}_{B\ A'} &= \delta^{A}{}_{A'}\ \delta_{B}{}^{B'} . \end{aligned} \tag{2.5}$$

The spin-chain operator in figure 3 then reads

$$\Gamma_{\rm sc} = \frac{\lambda^2}{2} \sum_{l=1}^{2L} \left(-K_{l,l+1} + 1 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}\right) \tag{2.6}$$

If we add the fermion loops and gauge contributions from appendix A, a remarkable cancelation happens. These terms contribute the two-site trace operator with the coefficient $+\lambda^2/2$, which exactly cancels the first term in (2.6) and leaves no nearest-neighbor terms in the Hamiltonian. We have not computed the constant piece, but supersymmetry requires that the ground state energy is zero, which happens when the constant term and the permutation combine into a projection $1 - P$ on the symmetric traceless states. From this we can find the missing constant contribution and get the full two-loop dilatation operator:

$$\Gamma = \frac{\lambda^2}{2} \sum_{l=1}^{2L} \left(2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}\right) . \tag{2.7}$$

This is our main result.

The ground states of the Hamiltonian are symmetric traceless chiral primary operators. Because chiral primaries are protected by supersymmetry, it should be possible to directly compare their spectrum with the supergravity harmonics on $AdS_4 \times CP^3$. We have just checked that the chiral primaries are in one-to-one correspondence with the spherical functions on CP^3 (appendix B). We also worked out the complete spectrum of the Hamiltonian for operators of length four ($L = 2$) in appendix C.

Is the Hamiltonian (2.7) integrable? Integrable alternating spin chains have been studied before [22 − 25], and although we were unable to find the Hamiltonian (2.7) in the literature, there is a general formalism [24] that allows one to build an alternating integrable Hamiltonian in any representations starting with appropriate R-matrices. The resulting Hamiltonian indeed involves nearest-neighbor and three-site interactions, but in general breaks charge conjugation symmetry $\mathbf{4} \leftrightarrow \bar{\mathbf{4}}$. It turns out for SL($n$) groups the nearest neighbor interactions always cancel out. If one further makes a special choice of paramters then the conjugation symmetry is preserved and the spin-chain Hamiltonian exactly coincides with the dilatation operator (2.7)! We can then make use of the general formulas [26, 27] that describe the spectrum via the algebraic Bethe ansatz [28].

3. Integrability for an SU(4) spin chain with alternating sites

In this section we show that the Hamiltonian derived in the previous section is that of an integrable SU(4) spin chain with sites alternating between the fundamental and anti-fundamental representation. We will actually generalize the derivation for any SU(n) group, specializing to SU(4) at the end.

In order to establish integrability, one first defines an R-matrix $R_{ab}(u)$ which is a linear map from a tensor product of two vector spaces in the fundamental representation of SU(n)

$$R_{ab}(u): \, V_a \otimes V_b \, \rightarrow \, V_a \otimes V_b \,, \tag{3.1}$$

where the parameter u is the spectral parameter. If we let

$$R_{ab}(u) = u - P_{ab} \,, \tag{3.2}$$

then $R_{ab}(u)$ satisfies the Yang-Baxter equation

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \,. \tag{3.3}$$

The results in (3.1), (3.2) and (3.3) can be generalized to all representations using a universal R matrix, but for our purposes we only need the cases where V_1 and V_2 are the fundamental or anti-fundamental representations. We therefore introduce two other R-matrices

$$\begin{aligned} R_{a\bar{b}}(u) &= u + K_{a\bar{b}} \\ R_{\bar{a}\bar{b}}(u) &= u - P_{\bar{a}\bar{b}} \end{aligned} \tag{3.4}$$

where $P_{\bar{a}\bar{b}}$ and $K_{a\bar{b}}$ were defined in (2.5). We then have the additional Yang-Baxter equations

$$\begin{aligned} R_{\bar{a}\bar{b}}(u-v)R_{\bar{a}c}(u)R_{\bar{b}c}(v) &= R_{\bar{b}c}(v)R_{\bar{a}c}(u)R_{\bar{a}\bar{b}}(u-v) \\ R_{ab}(u-v)R_{a\bar{c}}(u)R_{b\bar{c}}(v) &= R_{b\bar{c}}(v)R_{a\bar{c}}(u)R_{ab}(u-v) \,. \end{aligned} \tag{3.5}$$

To show these formulae, the following identities are useful:

$$P_{ab}P_{ab} = 1 \qquad K_{a\bar{b}}K_{a\bar{b}} = nK_{a\bar{b}} \qquad P_{ab}K_{b\bar{c}} = K_{a\bar{c}}K_{b\bar{c}} \,. \tag{3.6}$$

In addition, there are a set of modified Yang-Baxter equations

$$\begin{aligned} R_{a\bar{b}}(u-v-n)R_{ac}(u)R_{\bar{b}c}(v) &= R_{\bar{b}c}(v)R_{ac}(u)R_{a\bar{b}}(u-v-n) \\ R_{\bar{a}b}(u-v-n)R_{\bar{a}\bar{c}}(u)R_{b\bar{c}}(v) &= R_{b\bar{c}}(v)R_{\bar{a}\bar{c}}(u)R_{\bar{a}b}(u-v-n) \end{aligned} \tag{3.7}$$

Given these R-matrices we can construct the monodromy matrices $T_a(u,\alpha)$ and $T_{\bar{a}}(u,\alpha)$

$$\begin{aligned}
T_a(u,\alpha) &= C\, R_{a1}(u)R_{a\bar{1}}(u+\alpha)R_{a2}(u)R_{a\bar{2}}(u+\alpha)\ldots R_{aL}(u)R_{a\bar{L}}(u+\alpha) \\
T_{\bar{a}}(u,\alpha) &= C\, R_{\bar{a}1}(u+\alpha)R_{\bar{a}\bar{1}}(u)R_{\bar{a}2}(u+\alpha)R_{\bar{a}\bar{2}}(u)\ldots R_{\bar{a}L}(u+\alpha)R_{\bar{a}\bar{L}}(u)
\end{aligned}$$

where a and $\bar{a}$ refer to auxiliary spaces in the fundamental and anti-fundamental representations, α is a constant parameter and C is a normalization constant. It then follows from the Yang-Baxter equations in (3.5) that

$$\begin{aligned}
R_{ab}(u-v)T_a(u,\alpha)T_b(v,\alpha) &= T_b(v,\alpha)T_a(u,\alpha)R_{ab}(u-v) \\
R_{\bar{a}\bar{b}}(u-v)T_{\bar{a}}(u,\alpha)T_{\bar{b}}(v,\alpha) &= T_{\bar{b}}(v,\alpha)T_{\bar{a}}(u,\alpha)R_{\bar{a}\bar{b}}(u-v)
\end{aligned} \tag{3.8}$$

where b ($\bar{b}$) refers to a different fundamental (anti-fundamental) auxiliary space, but otherwise $T_b(v,\alpha)$ and $T_{\bar{b}}(v,\alpha)$ act on the same $(V\otimes\overline{V})^L$ space. If we define the transfer matrices $\tau(u)$ and $\bar{\tau}(u)$ as the trace of $T_a(u,\alpha)$ and $T_{\bar{b}}(u,\alpha)$ over the auxiliary spaces,

$$\tau(u,\alpha) = \mathrm{tr}_a T_a(u,\alpha) \qquad \bar{\tau}(u,\alpha) = \mathrm{tr}_{\bar{a}} T_{\bar{a}}(u,\alpha) \tag{3.9}$$

then (3.8) leads to

$$[\tau(u,\alpha),\tau(v,\alpha)] = 0 \qquad [\tau(u,\alpha),\tau(v,\alpha)] = 0 \tag{3.10}$$

for any u and v. Since $\tau(u,\alpha)$ and $\bar{\tau}(u,\alpha)$ are polynomials of order $2L$, each one gives up to $2L$ independent commuting quantities. Of particular interest are $\tau(0,\alpha)$ and $\bar{\tau}(0,\alpha)$

$$\begin{aligned}
\tau(0,\alpha) &= C\left(\frac{1}{\alpha(n+\alpha)}\right)^L \prod_{i=1}^{L}(\alpha+K_{2i-1,2i})\prod P_{2L-2i+1,2L-2i-1} \\
\bar{\tau}(0,\alpha) &= C\left(\frac{1}{\alpha(n+\alpha)}\right)^L \prod_{i=1}^{L}(\alpha+K_{2i,2i+1})\prod P_{2L-2i+2,2L-2i}
\end{aligned}$$

which are the analogs of the shift operator for a homogeneous chain, and the two Hamiltonians

$$\begin{aligned}
H_{\mathrm{odd}} &= (\tau(0,\alpha))^{-1}\frac{d}{du}\tau(u,\alpha)\Big|_{u=0} \\
&= \sum_{i=1}^{L}\left(\frac{1}{\alpha}-P_{2i-1,2i+1}-\frac{1}{\alpha}K_{2i-1,2i}K_{2i,2i+1}+\frac{1}{n+\alpha}K_{2i,2i+1}K_{2i-1,2i}\right) \\
H_{\mathrm{even}} &= (\tau(0,\alpha))^{-1}\frac{d}{du}\tau(u,\alpha)\Big|_{u=0} \\
&= \sum_{i=1}^{L}\left(\frac{1}{\alpha}-P_{2i,2i+2}-\frac{1}{\alpha}K_{2i,2i+1}K_{2i,2i+2}+\frac{1}{n+\alpha}K_{2i+1,2i+2}K_{2i,2i+1}\right)\,.
\end{aligned}$$

We can see that H_{odd} and H_{even} are proportional to the contribution of the odd and even sites in the gauge theory spin chain if $n=4$ and $\alpha=-n/2$. However, we still have to establish that $[\tau(u,\alpha),\bar\tau(v,\alpha)]=0$ in order that $H_{\rm odd}+H_{\rm even}$ is an integrable Hamiltonian. In order to show this there should be a Yang-Baxter equation of the form

$$R_{a\bar b}(u-v+\beta)T_a(u,\alpha)T_{\bar b}(v,\alpha)=T_{\bar b}(v,\alpha)T_a(u,\alpha)R_{a\bar b}(u-v+\beta)\,, \tag{3.11}$$

where β can be any constant. In order for (3.11) to work, we have to satisfy *both* the equations

$$\begin{aligned}R_{a\bar b}(u-v+\beta)R_{ac}(u)R_{\bar bc}(v+\alpha) &= R_{\bar bc}(v+\alpha)R_{ac}(u)R_{a\bar b}(u-v+\beta)\\ R_{a\bar b}(u-v+\beta)R_{a\bar c}(u+\alpha)R_{\bar b\bar c}(v) &= R_{\bar b\bar c}(v)R_{a\bar c}(u+\alpha)R_{a\bar b}(u-v+\beta)\,.\end{aligned} \tag{3.12}$$

Using (3.5) and (3.7), we see that both equations in (3.12) are true only if $\beta=\alpha=-n/2$. But this is precisely the value of α that matches the Hamiltonian derived from the gauge theory spin chain! Furthermore, if $\alpha=-n/2$ and we now choose $C=(2/n)^L$, then the product of $\tau(0,-n/2)$ and $\bar\tau(0,-n/2)$ is

$$\tau(0,-n/2)\bar\tau(0,-n/2)=\prod_{i=1}^{2L}P_{2L+2-2i,2L-2i}\,, \tag{3.13}$$

which is the operator that shifts every flavor index by two sites. Since the trace is invariant under such a shift, we must have $\tau(0,-n/2)\bar\tau(0,-n/2)\,|\text{phys}\rangle=|\text{phys}\rangle$ for all physical operators.

From now on we let $\alpha=-n/2$ and define $\tau(u)\equiv\tau(u,-n/2)$, $\bar\tau(u)\equiv\bar\tau(u,-n/2)$. One can construct the eigenvalues for $\tau(u)$ and $\bar\tau(u)$ using the algebraic Bethe ansatz. This was originally done in [26] for an inhomogenous spin chain with different representations on the $2L$ sites. For the case considered here, one finds the eigenvalues of $\Lambda(u)$ are

$$\begin{aligned}\Lambda(u) \,=\,& (u-1)^L(u-2)^L\prod_{j=1}^{M_u}\frac{u-iu_j+\frac12}{u-iu_j-\frac12}+u^L(u-2)^L\prod_{j=1}^{M_u}\frac{u-iu_j-\frac32}{u-iu_j-\frac12}\prod_{k=1}^{M_r}\frac{u-ir_k}{u-ir_k-1}\\ &+u^L(u-2)^L\prod_{n=1}^{M_v}\frac{u-iv_n-\frac12}{u-iv_n-\frac32}\prod_{k=1}^{M_r}\frac{u-ir_k-2}{u-ir_k-1}+u^L(u-1)^L\prod_{n=1}^{M_v}\frac{u-iv_n-\frac52}{u-iv_n-\frac32}\,.\end{aligned} \tag{3.14}$$

where the u_j, v_j and r_j are a set of Bethe roots associated with the SU(4) Dynkin diagram shown in figure 2. Since $\Lambda(u)$ is clearly a polynomial in u, the Bethe roots must satisfy a set of Bethe equations to cancel off the poles in (3.14),

$$\begin{aligned}\left(\frac{u_j+i/2}{u_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_u}\frac{u_j-u_k+i}{u_j-u_k-i}\prod_{k=1}^{M_r}\frac{u_j-r_k-i/2}{u_j-r_k+i/2}\\ 1 &= \prod_{k=1}^{M_r}\frac{r_j-r_k+i}{r_j-r_k-i}\prod_{k=1}^{M_u}\frac{r_j-u_k-i/2}{r_j-u_k+i/2}\prod_{k=1}^{M_v}\frac{r_j-v_k-i/2}{r_j-v_k+i/2}\\ \left(\frac{v_j+i/2}{v_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_v}\frac{v_j-v_k+i}{v_j-v_k-i}\prod_{k=1}^{M_r}\frac{v_j-r_k-i/2}{v_j-r_k+i/2}\,.\end{aligned} \tag{3.15}$$

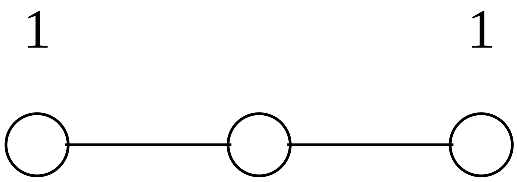

Figure 5: The SU(4) Dynkin diagram where the numbers indicate the Dynkin labels of the representation. The roots u_j are associated with one outer root, v_j with the other outer root, and r_j with the middle root.

The eigenvalues of $\bar{t}(u)$ can be found from the conjugation condition $\bar{\Lambda}(u) = \Lambda(2-u^*)$. We find:[3]

$$\begin{aligned}\bar{\Lambda}(u^*) \;=\; &(u-1)^L(u-2)^L \prod_{j=1}^{M_u} \frac{u-iu_j-\frac{5}{2}}{u-iu_j-\frac{3}{2}} + u^L(u-2)^L \prod_{j=1}^{M_u} \frac{u-iu_j-\frac{1}{2}}{u-iu_j-\frac{3}{2}} \prod_{k=1}^{M_r} \frac{u-ir_k-2}{u-ir_k-1} \\ &+u^L(u-2)^L \prod_{n=1}^{M_v} \frac{u-iv_n-\frac{3}{2}}{u-iv_n-\frac{1}{2}} \prod_{k=1}^{M_r} \frac{u-ir_k}{u-ir_k-1} + u^L(u-1)^L \prod_{n=1}^{M_v} \frac{u-iv_n+\frac{1}{2}}{u-iv_n-\frac{1}{2}}\,. \end{aligned} \tag{3.16}$$

Combining (3.14) with (3.16) and Taylor expanding at $u=0$, we find the momentum

$$e^{2iP} = \Lambda(0)\bar{\Lambda}(0) = \prod_{j=1}^{M_u} \frac{u_j+i/2}{u_j-i/2} \prod_{j=1}^{M_v} \frac{v_j+i/2}{v_j-i/2} \tag{3.17}$$

and energy, corresponding to the anomalous dimension γ,

$$E = \gamma = \lambda^2 \frac{d}{du} \ln(\Lambda(u)\bar{\Lambda}(u))\Big|_{u=0} = \lambda^2 \left(\sum_{j=1}^{M_u} \frac{1}{u_j^2+\frac{1}{4}} + \sum_{j=1}^{M_v} \frac{1}{v_j^2+\frac{1}{4}} \right). \tag{3.18}$$

The state with (K_u, K_r, K_v) Bethe roots belongs to the SU(4) representation with the Dynkin labels $[L-2K_u+K_r, K_u+K_v-2K_r, L-2K_v+K_r]$. Consequently, the excitation numbers must satisfy

$$2K_u \le L+K_r, \qquad 2K_v \le L+K_r, \qquad 2K_r \le K_u+K_v. \tag{3.19}$$

In (3.17) we see that the momentum carrying roots are the outer roots, which contrasts with the SU(4) spin chain found in $\mathcal{N}=4$ SYM which has the middle roots carrying the momentum [14]. We can make a few simple checks on the validity of the ansatz. First, we worked out the full set of solutions for $L=2$ in appendix C and showed that it matches with the spectrum of the Hamiltonian. There are also some subsectors which can be easily identified for the spin chain and in the Bethe ansatz equations. Let us choose a ground state operator

$$\mathrm{tr}[(Y^1 Y_4^\dagger)^L] \tag{3.20}$$

[3]Here we also use the fact that the Bethe roots are real or come in the complex conjugate pairs.

which is clearly symmetric and traceless in the Y^A and $Y_A^\dagger$. There are a few subsectors of SU(4) that are not mixed by anomalous dimension matrix. First there is an SU(2) $\times$ SU(2) subsector where the scalars are Y^1 or Y^2 and the adjoints are $Y_3^\dagger$ or $Y_4^\dagger$. For this sector $K_{i,i+1}$ is always zero, so we are left with two decoupled SU(2) chains on the even and odd sites. This corresponds to the absence of middle roots in the Bethe equations, and in this case the Bethe equations reduce to two decoupled Heisenberg spin chains with L sites each. The trace condition couples the two chains by enforcing $e^{i(P_1+P_2)} = 1$, where P_1 and P_2 are the momentum for the first and second Heisenberg chain. There is also an SU(3) sector where the scalars are Y^1, Y^2 or Y^3, but the conjugates remain as $Y_4^\dagger$. In this case the even sites are a trivial chain and the odd sites are part of an integrable SU(3) chain. This corresponds to the absence of one of the outer roots and the Bethe equations can be easily shown to reduce to that of an SU(3) spin chain. We could also construct a different SU(3) chain for the conjugate fields.

Since there are two types of roots that carry momentum, identifying the elementary magnons is a little problematic. To get some idea of what to expect, let us again start with the ground state operator in (3.20). On the string side, this should correspond to a point-like string on $R \times CP^3$ located at $(Z^1, Z^2, Z^3, Z^4) = (e^{i\omega t}, 0, 0, e^{-i\omega t})$ where ω is proportional to the energy and t is the world-sheet time which can be gauge fixed to the target space time. It is natural to expect the elementary magnons to be associated with excitations transverse to the motion. Four of the transverse directions correspond to rotations of Z^1 into Z^2 or Z^3, *or* $\bar{Z}_4$ into $\bar{Z}_2$ or $\bar{Z}_3$. In terms of the scalar fields, this takes one of the Y^1 into a Y^2 or Y^3 *or* one of the $Y_4^\dagger$ to a $Y_2^\dagger$ or $Y_3^\dagger$. If we choose the simple root vectors as $\alpha_1 = (1, -1, 0, 0)$, $\alpha_2 = (0, 1, -1, 0)$ and $\alpha_3 = (0, 0, 1, -1)$, then this corresponds to subtracting off α_1, $\alpha_1 + \alpha_2$, $\alpha_3 + \alpha_2$ or α_3 from the weights. Hence these elementary magnons are either a momentum carrying root or a momentum carrying root plus one middle root. The last transverse direction in CP^3 is a rotation of Z^1 into Z^4 and $\bar{Z}_4$ into $\bar{Z}_1$. On the gauge theory side this turns a Y^1 into a Y^4 *and* a $Y_4^\dagger$ into a $Y_1^\dagger$. The combination of roots that give this is $2\alpha_1 + 2\alpha_2 + 2\alpha_3$. But this actually produces two charged zero pairs, so there is a smaller excitation with half this much. Hence the last magnon has one each of the three types of roots. Since two of these roots carry momentum, one should think of this magnon as a bound pair of two of the other four magnons.

4. Extension to $OSp(2,2|6)$

In this section we consider the extension of the SU(4) spin chain to the full superconformal group, $OSp(2,2|6)$. The extension of the Bethe equations is analogous to the $\mathcal{N} = 4$ SYM$_4$ case in [16]. The bosonic subgroup is SO(2,3) $\times$ SO(6), the product group of the three dimensional conformal group and the R-symmetry group. The fermionic elements are the 12 supersymmetry generators $Q_\alpha^{[AB]}$ and the 12 superconformal generators $S_\alpha^{[AB]}$, where the A and B indices are anti-symmetrized.

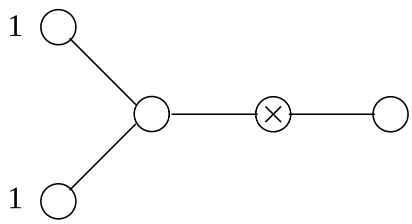

Figure 6: One choice for the $OSp(2,2|6)$ Dynkin diagram. The Dynkin labels are taken from the SU(4) spin chain.

Following Kac's classification, this is a D(2,3) algebra with a nonunique Cartan matrix

$$\begin{pmatrix} -2 & +1 & & & \\ +1 & & -1 & & \\ & -1 & +2 & -1 & -1 \\ & & -1 & +2 & \\ & & -1 & & +2 \end{pmatrix} \tag{4.1}$$

If we assume that the same bosonic roots carry the momentum then the Dynkin diagram, including the Dynkin labels, is that in figure 6. Note that one of the simple roots is fermionic and has invariant length 0.

Given this diagram and Cartan matrix and following the general recipe of [29], the Bethe ansatz for the $OSp(2,2|6)$ superalgebra is

$$\begin{aligned}
\left(\frac{u_j+i/2}{u_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_u} \frac{u_j-u_k+i}{u_j-u_k-i} \prod_{k=1}^{M_r} \frac{u_j-r_k-i/2}{u_j-r_k+i/2} \\
\left(\frac{v_j+i/2}{v_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_v} \frac{v_j-v_k+i}{v_j-v_k-i} \prod_{k=1}^{M_r} \frac{v_j-r_k-i/2}{v_j-r_k+i/2} \\
1 &= \prod_{k=1}^{M_r} \frac{r_j-r_k+i}{r_j-r_k-i} \prod_{k=1}^{M_u} \frac{r_j-u_k-i/2}{r_j-u_k+i/2} \prod_{k=1}^{M_v} \frac{r_j-v_k-i/2}{r_j-v_k+i/2} \prod_{k=1}^{M_s} \frac{r_j-s_k-i/2}{r_j-s_k+i/2} \\
1 &= \prod_{k=1}^{M_r} \frac{s_j-r_k-i/2}{s_j-r_k+i/2} \prod_{k=1}^{M_w} \frac{s_j-w_k+i/2}{s_j-w_k-i/2} \\
1 &= \prod_{k=1,k\neq j}^{M_w} \frac{w_j-w_k-i}{w_j-w_k+i} \prod_{k=1}^{M_s} \frac{w_j-s_k+i/2}{w_j-s_k-i/2}\,.
\end{aligned} \tag{4.2}$$

The five bosonic charges in $OSp(2,2|6)$ can be grouped as $(-D-S, -D+S; J_1, J_2, J_3)$, where D is the bare dimension, S is the spin and J_i are the three commuting R-charges in SO(6). The ground state operator in (3.20) has charges $(-L,-L;L,L,0)$ and the charges of the simple root vectors are

$$\begin{aligned}
\alpha_1 &= (0,0;0,1,-1)\,, & \alpha_2 &= (0,0;1,-1,0)\,, & \alpha_3 &= (0,0;1,-1,0) \\
\beta &= (1,-1;0,0,0) & \gamma &= (0,1;-1,0,0)\,,
\end{aligned} \tag{4.3}$$

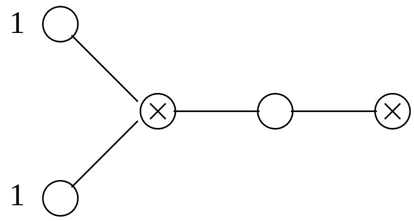

Figure 7: A different choice for the $OSp(2,2|6)$ Dynkin diagram with two fermionic roots.

where γ is a fermionic root and the signature is $(--+++)$. The elementary magnons are the four discussed in the previous section as well as four fermionic magnons. These last four have one momentum carrying root, either a u or a v, as well as an r and an s root. In addition the magnon may also include one w root. Hence an elementary fermionic magnon increases D by 1/2, increases or decreases S by 1/2, decreases J_2 by 1 and increases or decreases J_3 by 1. A covariant derivative does not correspond to an elementary magnon; instead this is a bound state of two fermionic roots. All such bound states contain one u and v root, two r and s roots, and either zero, one or two roots, corresponding to a spin of -1, 0 or $+1$. One can also see this another way: Unlike the $\mathcal{N}=4$ SYM$_4$ case, the SL(2) sector in the superconformal Chern-Simons is not a closed sector. In particular the combination $D_\mu Y_A^\dagger Y^B$ can mix into $\bar\psi^B\gamma_\mu\psi_A$, explicitly showing the two fermionic excitations.

Since the super Lie algebra has fermionic roots, the Dynkin diagram in figure 6 is not the only choice we can make. A different grading of roots can be found by grouping the charges as $(J_1; -D-S, -D+S; J_2, J_3)$ and choosing the simple roots as

$$\begin{aligned}
&\alpha_1 = (0;0,0;1,-1)\,, && \alpha_2 = (0,0;0,1,1) && \\
&\gamma_1 = (0;0,1;-1,0)\,, && \beta = (0;1,-1;0,0) && \gamma_2 = (1;-1,;0,0)\,.
\end{aligned} \tag{4.4}$$

Now the super Dynkin diagram is the one in figure 7 and the new Bethe equations are

$$\begin{aligned}
\left(\frac{u_j+i/2}{u_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_u}\frac{u_j-u_k+i}{u_j-u_k-i}\prod_{k=1}^{M_r}\frac{u_j-r_k-i/2}{u_j-r_k+i/2}\\
\left(\frac{v_j+i/2}{v_j-i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_v}\frac{v_j-v_k+i}{v_j-v_k-i}\prod_{k=1}^{M_r}\frac{v_j-r_k-i/2}{v_j-r_k+i/2}\\
1 &= \prod_{k=1}^{M_u}\frac{r_j-u_k-i/2}{r_j-u_k+i/2}\prod_{k=1}^{M_v}\frac{r_j-v_k-i/2}{r_j-v_k+i/2}\prod_{k=1}^{M_s}\frac{r_j-s_k+i/2}{r_j-s_k-i/2}\\
1 &= \prod_{k=1}^{M_s}\frac{s_j-s_k-i}{s_j-s_k+i}\prod_{k=1}^{M_r}\frac{s_j-r_k+i/2}{s_j-r_k-i/2}\prod_{k=1}^{M_w}\frac{s_j-w_k+i/2}{s_j-w_k-i/2}\\
1 &= \prod_{k=1}^{M_s}\frac{w_j-s_k+i/2}{w_j-s_k-i/2}\,,
\end{aligned} \tag{4.5}$$

where r and w are now fermionic roots. Of course, this system must be equivalent to the one in (4.2), which can be shown using the duality transformations in [30] (see also [31, 32]).

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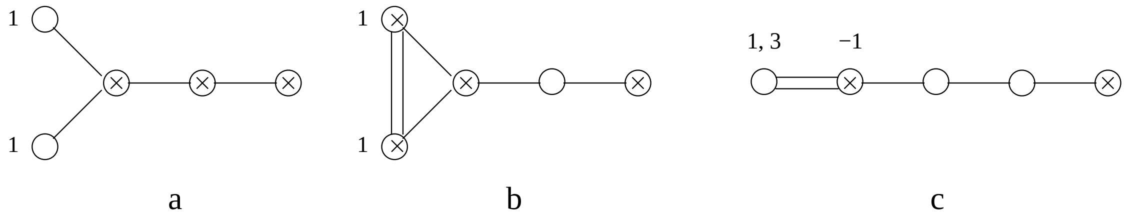


Figure 8: Other $OSp(2,2|6)$ Dynkin diagrams.

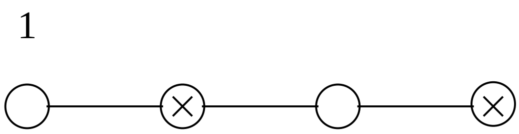
It is possible that this choice of basis is more amenable to higher loop generalizations. Figure 8 shows other bases for the simple roots, where the Bethe equations can all be connected through duality transformations. The duality transformation [30 − 32] on the middle node produces a double link between the momentum-carrying nodes in 8b, which are non-interacting in the original Dynkin diagram.[4] The last diagram in 8c is found by dualizing one of the momentum carrying nodes in 8b. The two weights over the left node signifies that both weights appear in the Bethe equations:

$$
\begin{aligned}
\left(\frac{u_j - i/2}{u_j + i/2}\,\frac{u_j - 3i/2}{u_j + 3i/2}\right)^L &= \prod_{k=1,k\neq j}^{M_u} \frac{u_j - u_k + 2i}{u_j - u_k - 2i} \prod_{k=1}^{M_v} \frac{u_j - v_k - i}{u_j - v_k + i} \\
\left(\frac{v_j + i/2}{v_j - i/2}\right)^L &= \prod_{k=1}^{M_u} \frac{v_j - u_k - i}{v_j - u_k + i} \prod_{k=1}^{M_r} \frac{v_j - r_k + i/2}{v_j - r_k - i/2} \\
1 &= \prod_{k=1,k\neq j}^{M_r} \frac{r_j - r_k - i}{r_j - r_k + i} \prod_{k=1}^{M_u} \frac{r_j - v_k + i/2}{r_j - v_k - i/2} \prod_{k=1}^{M_s} \frac{r_j - s_k + i/2}{r_j - s_k - i/2} \\
1 &= \prod_{k=1,k\neq j}^{M_s} \frac{s_j - s_k - i}{s_j - s_k + i} \prod_{k=1}^{M_r} \frac{s_j - r_k + i/2}{s_j - r_k - i/2} \prod_{k=1}^{M_w} \frac{s_j - w_k + i/2}{s_j - w_k - i/2} \\
1 &= \prod_{k=1}^{M_s} \frac{w_j - s_k + i/2}{w_j - s_k - i/2}\,.
\end{aligned}
\tag{4.6}
$$

The anomalous dimension for this choice of diagram is

$$
\gamma = \lambda^2 \left(2L + \sum_{j=1}^{M_u} \left[\frac{1}{u_j^2 + \frac{1}{4}} + \frac{3}{u_j^2 + \frac{9}{4}} \right] - \sum_{j=1}^{M_v} \frac{1}{v_j^2 + \frac{1}{4}} \right). \tag{4.7}
$$

We also note that $OSP(2,2|6)$ has an SU(2|3) subgroup with a diagram like



This is the same diagram one finds for the SU(2|3) subgroup of SU(2,2|4) in $\mathcal{N} = 4$ SYM$_4$ [33]. For higher loop calculations, one might expect to have the same set of Bethe

[4] We thank N. Beisert for pointing this out to us.

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equations in this sector as the $\mathcal{N}=4$ case [18], but with λ replaced by λ^2. However, the dressing factors in [20, 19] might need to be modified since the string action still contains an overall factor of $\sqrt{\lambda}$.

5. Summary and discussion

We have shown that the ABJM $\mathcal{N}=6$ super-Chern-Simons theory is integrable at two loops, the lowest nontrivial order. We also derived a set of Bethe equations for the spectrum of two-loop anomalous dimensions. In conjunction with classical integrability of the sigma-model on $AdS_4 \times CP^3$, the two loop integrability gives strong indications that the model is integrable at any coupling. It might then be solvable in the large-N limit using an all-orders Bethe ansatz. We believe that one can extend our results to higher loop orders along the lines of [17], and perhaps to construct the asymptotic Bethe ansatz equations at the non-perturbative level, as was been done for $\mathcal{N}=4$ super-Yang-Mills in four dimensions [18–20].

Even though we see no apparent relationship between $\mathcal{N}=6$ super-Chern-Simons and $\mathcal{N}=4$ SYM$_4$, the AdS_4/CFT_3 correspondence seems to be another instance where integrability plays an important role in the gauge/string duality .

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A. Contributions from fermion loops and gauge terms in Feynman diagrams

In this appendix we compute the contribution of fermion loops and gauge contributions to the spin chain Hamiltonian. The manifestly SU(4) invariant fermion couplings in the Lagrangian were computed in [11] and are of the form

$$\begin{aligned}\mathcal{L}_{YY\psi\psi} = -\frac{i}{2}\,\mathrm{tr}\Big[&Y_A^\dagger Y^A \bar{\psi}^B \psi_B - \bar{\psi}^B Y^A Y_A^\dagger \psi_B + 2\bar{\psi}^B Y^A Y_B^\dagger \psi_A - 2Y_B^\dagger Y^A \bar{\psi}^B \psi_A \\ &+\epsilon^{ABCD} Y_A^\dagger \psi_B^T \gamma_0 Y_B^\dagger \psi_D - \epsilon_{ABCD} Y^A \bar{\psi}^B Y^C \psi^{D*}\Big]\,.\end{aligned} \tag{A.1}$$

The two loop planar graphs with a fermion loop are shown in figure 2b and 2c. Both diagrams can lead to nontrivial interactions between the neighboring sites since SU(4) flavor is carried by the fermions. However, only the graph in 2b has a log divergence. The

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only possible interaction term is a contraction piece $K_{i,i+1}$ between neighboring sites, and only the the third term in the first line of (A.1) can contribute to it. The fourth term contributes to the conjugate diagram.

Concentrating on just the contraction piece, we find the following contribution to the operator renormalization between scalars i and $i+1$ coming from the counter term

$$Z^f_{i,i+1} = -(-1)\left(-i\,\frac{4\pi}{k}\right)^2 N^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\mathrm{Tr}\left[\frac{i}{\not p + i\epsilon}\frac{i}{\not q + i\epsilon}\right]\left(\frac{i}{p-q+i\epsilon}\right)^2 K_{i,i+1}\,. \tag{A.2}$$

The trace refers to the fermion trace for three dimensional Dirac fermions and the factor of (-1) is for the fermion loop. After a Wick rotation and writing $2\,p\cdot q = p^2+q^2-(p-q)^2$, we arrive at

$$Z^f_{i,i+1} = (4\pi\lambda)^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\frac{1}{p^2q^2(p-q)^2} - \frac{2}{p^2((p-q)^2)^2}\right] K_{i,i+1}\,. \tag{A.3}$$

The second term inside the brackets does not contribute to the anomalous dimension so we drop it. Dimensionally regulating the integral, inserting a small mass term μ to act as an infrared cutoff, and inserting a Feynman parameterization we get

$$\begin{aligned} Z^f_{i,i+1} &= (4\pi\lambda)^2\,\frac{1}{64\pi^3}\int_0^\infty \frac{d\rho}{\rho^{1-\varepsilon}}e^{-\rho\mu^2}\int_0^1 dx\int_0^x dy[x(1-x)+y(1-y)-xy]^{-3/2}K_{i,i+1} \\ &= \frac{1}{2}\lambda^2\varGamma(\varepsilon)\mu^{-2\varepsilon}K_{i,i+1}\,. \end{aligned} \tag{A.4}$$

Since $\varepsilon^{-1}\sim \ln\Lambda^2$, this contribution to the anomalous dimension from all neighboring sites is

$$\varGamma_f = \sum_{i=1}^{2L}\frac{d}{d\ln\Lambda}Z^f_{i,i+1} = \lambda^2\sum_i^{2L}K_{i,i+1}\,. \tag{A.5}$$

Diagrams containing gauge boson propagators are shown in figures 2d, 2e, 2f and 2g, but only 2d will contribute to the anomalous dimension. The gauge propagators are given by

$$\frac{2\pi}{k}\frac{p_\mu\epsilon^{\mu\nu\sigma}}{p^2}\,, \tag{A.6}$$

and only one of the SU(N) gauge bosons will contribute to the planar diagram (the other SU(N) contributes to the conjugate diagram.) Hence the conribution to the operator normalization between scalars i and $i+1$ from the diagram in 2d is

$$\begin{aligned} Z^g_{i,i+1} &= -(+i)\left(\frac{2\pi}{k}\right)^2 N^2\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}(2iq_\nu + ip_\nu)(2iq_{\nu'}+ip_{\nu'})\frac{p_\mu\epsilon^{\mu\nu}{}_\sigma}{p^2}\frac{p_{\mu'}\epsilon^{\mu'\sigma\nu'}}{p^2} \\ &\qquad\times\left(\frac{i}{p^2}\right)^2\frac{i}{(p+q)^2}K_{i,i+1}\,, \end{aligned} \tag{A.7}$$

where the factor of $(+i)$ comes from the four-point vertex. This then gives

$$Z^g_{i,i+1} = -(4\pi\lambda)^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^2q^2-(p\cdot q)^2}{p^4q^4(p+q)^2}\, K_{i,i+1}\,. \tag{A.8}$$

If we write

$$p^2q^2-(p\cdot q)^2 = \frac{1}{2}p^2q^2+\frac{1}{2}(p+q)^2(p^2+q^2)-\frac{1}{4}(p^4+q^4+(p+q)^4)\,, \tag{A.9}$$

only the first term will contribute to the log. Following the arguments for the fermion loop, we can quickly see that

$$Z^g_{i,i+1} = -\frac{1}{4}\lambda^2\Gamma(\varepsilon)\mu^{-2\varepsilon}K_{i,i+1} \tag{A.10}$$

and so this contribution to the anomalous dimension is

$$\Gamma_g = \sum_{i=1}^{2L}\frac{d}{d\ln\Lambda}Z^g_{i,i+1} = -\frac{1}{2}\,\lambda^2\sum_{i}^{2L}K_{i,i+1}\,. \tag{A.11}$$

The diagram in 2g is nonzero, but only has a linear divergence and no log divergence. The diagrams in 2e and 2f are zero because the momentum in the top scalar line is the same as the gauge momentum, and so they both have $\epsilon^{\mu\nu\sigma}p_\mu p_\nu$ factors.

Combining Γ_f and Γ_g, we get

$$\Gamma_f+\Gamma_g = \frac{1}{2}\,\lambda^2\sum_{i}^{2L}K_{i,i+1}\,, \tag{A.12}$$

precisely canceling the nearest neighbor term from the six-point graph.

B. Chiral primaries and spherical functions on CP^3

Any chiral primary operator, (1.3) with symmetric traceless $\chi^{B_1\dots B_L}_{A_1\dots A_L}$, defines a function on CP^3:

$$\chi(z,\bar z) = \chi^{B_1\dots B_L}_{A_1\dots A_L}z^{A_1}\dots z^{A_L}\bar z_{B_1}\dots\bar z_{B_L}\,, \tag{B.1}$$

where $z,\bar z$ are homogeneous coordinates constrained by $z^A\bar z_A = 1$, $z^A \sim \mathrm{e}^{i\varphi}z^A$, $\bar z_A \sim \mathrm{e}^{-i\varphi}\bar z_A$. The Laplace-Beltrami operator on CP^3 is the U(3) Casimir. In terms of the U(3) generators,

$$L^A_B = z^A\,\frac{\partial}{\partial z^B} - \bar z_B\,\frac{\partial}{\partial \bar z_A}\,, \tag{B.2}$$

the Laplacian is

$$\Delta = \frac{1}{2}\,L^A_BL^B_A. \tag{B.3}$$

It is easy to check that the function (B.1) is its eigenstate:

$$\Delta\,\chi(z,\bar z) = L(L+3)\chi(z,\bar z). \tag{B.4}$$

C. Dimension-two operators

In this appendix we explicitly diagonalize the Hamiltonian (B.4) for the spin chain of length 4, first by brute force, and then with the help of the Bethe ansatz equations. For the sake of generality we temporarily relax the trace condition (B.4). We will indicate which operators satisfy it but will compute the whole spectrum, including the states with non-zero total momentum.[5]

The length-four Hilbert space decomposes as $\mathbf{4}\otimes\bar{\mathbf{4}}\otimes\mathbf{4}\otimes\bar{\mathbf{4}} = \mathbf{1}^2\oplus\mathbf{15}^4\oplus\mathbf{20}\oplus\mathbf{45}\oplus\bar{\mathbf{45}}\oplus\mathbf{84}$. The $\mathbf{84}$ is the chiral primary with totally symmetric traceless wavefunction and zero energy:

$$\mathbf{84}: \ \chi^{(CD)}_{(AB)} - \text{traces}, \ \gamma_{\mathbf{84}} = 0, \ \mathrm{e}^{2iP} = 1. \tag{C.1}$$

The $\mathbf{45}$ and $\bar{\mathbf{45}}$ are symmetric in one pair of indices and anti-symmetric in the other. They do not correspond to any operators because of the trace condition. The permutation operator centered at the odd/even sites now yields a -1 and doubles the constant term in the Hamiltonian:

$$\begin{aligned} \mathbf{45}: \ &\chi^{(AB)}_{[CD]} - \text{traces}, \ \gamma_{\mathbf{45}} = 4\lambda^2, \ \mathrm{e}^{2iP} = -1, \\ \bar{\mathbf{45}}: \ &\chi^{[AB]}_{(CD)} - \text{traces}, \ \gamma_{\bar{\mathbf{45}}} = 4\lambda^2, \ \mathrm{e}^{2iP} = -1. \end{aligned} \tag{C.2}$$

The $\mathbf{20}$ is anti-symmetric in each pair of indices and the constant term is now doubled on all the sites:

$$\mathbf{20}: \ \chi^{[AB]}_{[CD]} - \text{traces}, \ \gamma_{\mathbf{20}} = 8\lambda^2, \ \mathrm{e}^{2iP} = 1. \tag{C.3}$$

The non-trivial mixing first occurs in the adjoint representation, the $\mathbf{15}$. Let us denote the four adjoint states by

$$\begin{aligned} |1\rangle_{\mathbf{15}}: \ &\chi^{CA}_{CB} - \text{trace}, \\ |2\rangle_{\mathbf{15}}: \ &\chi^{AC}_{CB} - \text{trace}, \\ |3\rangle_{\mathbf{15}}: \ &\chi^{AC}_{BC} - \text{trace}, \\ |4\rangle_{\mathbf{15}}: \ &\chi^{CA}_{BC} - \text{trace}. \end{aligned}$$

The Hamiltonian and momentum act in this basis as

$$\begin{aligned} \Gamma\,|n\rangle &= 5\lambda^2\,|n\rangle + \lambda^2\,|n+2\rangle \\ \mathrm{e}^{2iP}\,|n\rangle &= |n+2\rangle\,. \end{aligned} \tag{C.4}$$

The eigenstates $|1\rangle \pm |3\rangle$ and $|2\rangle \pm |4\rangle$ are doubly degenerate with the eigenvalues

$$\mathbf{15}: \ \gamma^{(\pm)}_{\mathbf{15}} = (5\pm 1)\lambda^2, \ \mathrm{e}^{2iP} = \pm 1. \tag{C.5}$$

The two singlets,

$$\begin{aligned} |1\rangle_{\mathbf{1}} &: \chi^{AB}_{AB}, \\ |2\rangle_{\mathbf{1}} &: \chi^{AB}_{BA}, \end{aligned}$$

[5] For length four, the shift operator e^{2iP} can have only two eigenvalues: $+1$ or -1.

Anomalous dimension	SU(4) representation
0	**84**
$2\lambda^2$	**1**
$6\lambda^2$	**15**
$6\lambda^2$	**15**
$8\lambda^2$	**20**
$10\lambda^2$	**1**

Table 1: The spectrum of operators at $L = 2$.

both have zero total momentum and mix according to

$$\Gamma|_{\mathbf{1}} = \lambda^2 \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}. \tag{C.6}$$

The eigenvalues are

$$\begin{aligned} \mathbf{1} : \gamma_{\mathbf{1}}^{(1)} &= 2\lambda^2, \;\; \mathrm{e}^{2iP} = 1, \\ \gamma_{\mathbf{1}}^{(2)} &= 10\lambda^2, \;\; \mathrm{e}^{2iP} = 1. \end{aligned} \tag{C.7}$$

The spectrum of dimension two operators is summarized in table 1.

Let us see how the Bethe equations (C.6) reproduce this spectrum. The conditions (C.6) admit the following root configurations (K_u, K_r, K_v):

$$\begin{aligned} \mathbf{84} &: \;\; (0,0,0) \\ \mathbf{45} &: \;\; (1,0,0) \\ \bar{\mathbf{45}} &: \;\; (0,0,1) \\ \mathbf{20} &: \;\; (1,0,1) \\ \mathbf{15} &: \;\; (1,1,1) \\ \mathbf{1} &: \;\; (2,2,2). \end{aligned}$$

For the configurations with only one u root or only one v root (the **45** and the $\bar{\mathbf{45}}$), the Bethe equations admit a unique solution: $u_1 = 0$ or $v_1 = 0$, whose energy (C.8) is $\gamma_{\mathbf{45}/\bar{\mathbf{45}}} = 4\lambda^2$, in agreement with (C.2). These states can be combined: $u_1 = 0 = v_1$, which yields the **20** with energy $\gamma_{\mathbf{20}} = 8\lambda^2$.

The Bethe equations for the **15** with $u_1 \equiv u$, $r_1 \equiv r$ and $v_1 \equiv v$ are

$$\left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right)^2 = \frac{u - r - \frac{i}{2}}{u - r + \frac{i}{2}}, \qquad 1 = \frac{r - u - \frac{i}{2}}{r - u + \frac{i}{2}} \frac{r - v - \frac{i}{2}}{r - v + \frac{i}{2}}, \qquad \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}}\right)^2 = \frac{v - r - \frac{i}{2}}{v - r + \frac{i}{2}}. \tag{C.8}$$

They have four solutions:

$$u = v = r = \pm\frac{1}{2}, \;\; \gamma_{\mathbf{15}}^{(-)} = 4\lambda^2; \qquad u = -v = \pm\frac{1}{2\sqrt{3}}, \;\; r = 0, \;\; \gamma_{\mathbf{15}}^{(+)} = 6\lambda^2, \tag{C.9}$$

which matches with (C.5).

The situation with singlets is more complicated. There is a regular solution with $u_1 = -u_2 \equiv u$, $r_1 = -r_2 \equiv r$, and $v_1 = -v_2 = u$ that satisfy

$$\frac{u+\frac{i}{2}}{u-\frac{i}{2}} = \frac{u-r-\frac{i}{2}}{u-r+\frac{i}{2}}\,\frac{u+r-\frac{i}{2}}{u+r+\frac{i}{2}}, \qquad 1 = \frac{r+\frac{i}{2}}{r-\frac{i}{2}}\left(\frac{r-u-\frac{i}{2}}{r-u+\frac{i}{2}}\,\frac{r+u-\frac{i}{2}}{r+u+\frac{i}{2}}\right)^2. \tag{C.10}$$

These equations have a unique solution:

$$u = \sqrt{\frac{3}{20}},\ r = \frac{1}{\sqrt{5}},\ \gamma_{\mathbf{1}}^{(2)} = 10\lambda^2. \tag{C.11}$$

The other singlet corresponds to a singular distribution of roots [34, 35]:

$$u_{1,2} = i\left(\pm\frac{1}{2} + \varepsilon \pm \delta\right) = v_{1,2}, \qquad r_1 \equiv r = -r_2, \tag{C.12}$$

which solves the Bethe equations in the limit $\varepsilon \to 0$ with $\delta \ll \varepsilon$, when both sides of the Bethe equations simultaneously turn to zero or to infinity. The balance of infinities determines δ in terms of ε:

$$\delta = \frac{r^2}{1+r^2}\,\varepsilon^2. \tag{C.13}$$

The middle-node equation is non-singular and gives:

$$r = \frac{i}{\sqrt{3}}. \tag{C.14}$$

In the energy (C.14) the $1/\varepsilon$ singularity cancels. It is important to keep the $O(\varepsilon^2)$ terms to get the finite part right:

$$\gamma_{\mathbf{1}}^{(1)} = 2\lambda^2 \lim_{\varepsilon\to 0}\left[\frac{1}{\frac{1}{4}-\left(\frac{1}{2}+\varepsilon-\frac{\varepsilon^2}{2}\right)^2} + \frac{1}{\frac{1}{4}-\left(\frac{1}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right)^2}\right] = 2\lambda^2, \tag{C.15}$$

which agrees with (C.7).

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