

Third family corrections to quark and lepton mixing in SUSY models with non-abelian family symmetry

To cite this article: Stefan Antusch *et al* JHEP05(2008)066

View the [article online](#) for updates and enhancements.

You may also like

- [Reconstructing Atiyah-Hitchin manifold in the generalized Legendre transform](#)
Masato Arai, Kurando Baba and Radu A. Iona
- [Two loop Kähler potential in -deformed = 4 SYM theory](#)
Simon J. Tyler
- [Gromov-Witten invariants and localization](#)
David R Morrison

Third family corrections to quark and lepton mixing in SUSY models with non-abelian family symmetry

Stefan Antusch

*Max-Planck-Institut für Physik (Werner-Heisenberg-Institut),
Föhringer Ring 6, D-80805 München, Germany
E-mail: antusch@mppmu.mpg.de*

Stephen F. King and Michal Malinský

*School of Physics and Astronomy, University of Southampton,
SO16 1BJ Southampton, U.K.
E-mail: sfk@hep.phys.soton.ac.uk, malinsky@phys.soton.ac.uk*

ABSTRACT: We re-analyse the effect of corrections from canonical normalisation of kinetic terms on the quark and lepton mixing angles. This type of corrections emerges, for example, from effective higher-dimensional Kähler potential operators in the context of locally supersymmetric models of flavour. In contrast to previous studies we find that the necessary procedure of redefining the fields in order to restore canonically normalised kinetic terms, i.e. canonical normalisation, can lead to significant corrections to the fermion mixing angles (as determined from the superpotential). Such potentially large effects are characteristic of flavour models based on non-Abelian family symmetries, where some of the possible Kähler potential (and superpotential) operators, in particular those associated with the third family, are only mildly suppressed. We investigate under which conditions the messenger sector of such flavour models generates such Kähler potential operators for which the canonical normalisation effects are sizeable, and under which conditions these operators may be absent and canonical normalisation effects are small. As explicit examples for potentially relevant CN effects, we will discuss the corrections to the CKM matrix element $|V_{cb}|$ as well as corrections to tri-bimaximal neutrino mixing.

KEYWORDS: Neutrino Physics, Beyond Standard Model.

Contents

1. Introduction	1
2. The Kähler potential and effects of canonical normalisation	3
2.1 Definitions and ambiguities	4
2.2 Effect of canonical normalisation on the Yukawa couplings	5
2.3 Perturbative prescription for the physical rotation matrices	6
3. Canonical normalisation corrections to quark sector mixing	8
3.1 Corrections to V_{cb} in classes of $SU(3)$ flavour models	9
4. Canonical normalisation corrections to lepton sector mixing	10
4.1 Corrections due to third family canonical rescaling	10
4.2 Canonical normalisation corrections to tri-bimaximal neutrino mixing	11
5. Canonical normalisation corrections in potentially realistic models	14
5.1 Corrections to tri-bimaximal mixing in a class of $SU(3)_f$ flavour models	14
5.2 Heavy $SU(2)_L$ doublet messengers & natural η^K suppression	16
6. Comparison to other corrections to fermion mixings	20
6.1 Renormalisation group corrections to tri-bimaximal neutrino mixing	20
6.2 Combined treatment of RG and canonical normalisation corrections	22
6.3 Charged lepton mixing corrections to tri-bimaximal neutrino mixing	22
6.4 Canonical normalisation corrections to lepton mixing sum-rules	23
7. Summary and discussion	25
A. Conventions - CKM & PMNS mixing matrices	26
B. Derivation of stable lepton mixing sum-rules	27

1. Introduction

The flavour problem of the Standard Model (SM), i.e. the question of the origin of the observed fermion masses and mixings, is one of the deepest mysteries in particle physics. Since the discovery of the small neutrino masses and large lepton mixings has added new aspects to this problem, it has received much attention. In addition to adding to the flavour problem, the discoveries in the neutrino sector have also inspired new approaches towards its solution. As the precision of the neutrino data has improved, it has become apparent that

lepton mixing is consistent with the so called Tri-bimaximal (TB) mixing pattern [1], and many models attempt to reproduce this as a theoretical prediction [2–10]. The essential starting point of many of such models is to invoke a non-Abelian family symmetry which spans all three families (like e.g. gauged $SO(3)$ or $SU(3)$, or their discrete subgroups such as A_4 or Δ_{27} [4–6]) and which is subsequently spontaneously broken by extra Higgs scalars. In addition to explaining the large observed lepton mixing from an underlying tri-bimaximal pattern in the neutrino sector, models of this type can also accommodate the experimental data on quark masses and mixings.

If supersymmetry (SUSY) is discovered at the LHC, the presence of a spectrum of superparticle masses and their mixings and CP phases would add further aspects to the flavour problem. Models of flavour which are capable of addressing also these issues are typically formulated in the supergravity (SUGRA) framework. In this context, another intriguing aspect of some classes of the non-Abelian family symmetry models initially build to explain the (approximate) tri-bimaximal neutrino mixing, is that they can also provide a solution to this SUSY flavour and CP problem [11]. In these classes of flavour models, in the exact flavour symmetry limit the Yukawa couplings vanish and the matter sector Kähler metric becomes proportional to the unit matrix. Consequently, the soft SUSY breaking sfermion mass matrices are universal at leading order at high energies. Only after spontaneous flavour symmetry breaking by the vacuum expectation values (VEVs) $\langle\phi\rangle$ of the flavons, the Yukawa couplings emerge from higher-dimensional operators involving flavon fields (and suppressed by powers of a messenger scale M). Their sizes can be expressed in terms of (powers of) expansion parameters $\varepsilon = \langle\phi\rangle/M$. In addition, after spontaneous flavour symmetry breaking effective higher-dimensional operators in the Kähler potential induce corrections to the universal sfermion mass matrices as well as corrections to the kinetic terms (which have standard canonical form in leading order).

Before any flavour theory of this type can be reliably interpreted, field transformations must be performed in order to return the kinetic terms back to canonical form. These field transformations, however, in general lead to modifications of the Yukawa couplings and thus to the fermion masses and mixings (compared to their values extracted from the initial superpotential). This rather technical but necessary procedure, to which we will refer to as canonical normalisation (CN) in the following, has been discussed in [12] and more recently in [11, 13, 14]. Although these possible effects have been to some extent addressed in the previous works [13] and [14], the conclusion of these studies has been that the effects on the mixing angles are too small to be relevant.

The motivation for re-visiting the effects of CN in this study is the observation that in classes of theories that predict TB mixing, especially those based on non-Abelian family symmetries spanning all three families of SM matter, certain Kähler potential operators can occur which are only very mildly suppressed. These operators lead to non-universal entries in the CN transformation matrices of order ε_3^2 , where $\varepsilon_3^2 = |\langle\phi_3\rangle|/M_3$, with typically $\varepsilon_3 \approx 0.5$ [11]. The reason for the appearance of this rather large “expansion parameter” is that the large third generation Yukawa couplings (in particular y_t) must originate from a effective vertex containing an insertion of (at least) one flavon field (here called ϕ_3). To accommodate for example a large y_t , typically a rather large parameter $\varepsilon_3 \approx 0.5$ is

introduced. For such large non-universality in the CN transformations, one can anticipate that their effects on fermion masses and mixings cannot be neglected anymore. In a recent study [15] focusing on the corrections to lepton sector mixing, it has been highlighted that such large third family wave-function corrections have to be included when comparing the model prediction of TB mixing in the neutrino sector with precision data of future neutrino oscillation facilities [16].

The main purpose of this paper is therefore to analyse in detail the possible impact of such potentially large CN effects on the quark and lepton mixing angles, encoded in the Cabibbo-Kobayashi-Maskawa (CKM) and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrices respectively. Another main question we will investigate in detail in this study is under which conditions the messenger sector of such flavour models generates these types of mildly suppressed Kähler potential operators and under which conditions these operators may be absent such that the corrections from canonical normalisation may be small. As explicit examples for potentially relevant CN effects, we will discuss the corrections to the CKM matrix element $|V_{cb}|$ as well as corrections to tri-bimaximal neutrino mixing. Regarding the tri-bimaximal mixing example, we go beyond the analysis of [15] by considering CN corrections in a realistic class of SU(3) flavour symmetry models and by providing additional details regarding the derivation of the CN results and of the procedure of combining CN corrections with corrections from renormalisation group (RG) running and Cabibbo-like charged lepton mixing contributions (i.e. regarding the there proposed stable mixing sum rule).

The paper is organised as follows: In the subsequent section we shall comment on some generalities of the canonical normalisation procedure focusing on ambiguities in the definition of the canonical normalisation transformation. We develop a perturbative technique to deal with effects of the canonical redefinition of fields in the Yukawa sector focusing in particular on its impact on the CKM and PMNS mixing parameters. Sections 3 and 4 are then devoted to a set of examples of the CN effects in the quark and lepton sectors respectively. In section 5 we present a more in-depth discussion of the expected magnitude of the CN corrections in a particular class of realistic SUSY flavour models based on non-Abelian family symmetry. In section 6 we compare the CN corrections to other typically relevant corrections. Section 7 concludes the paper. Some technical aspects of the discussion in the main body of the paper and a specification of the used conventions can be found in the appendices.

2. The Kähler potential and effects of canonical normalisation

Whenever the Kähler potential of a given SUSY model is nontrivial there are extra effects coming from the canonical normalisation procedure bringing the generic kinetic terms

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\tilde{f}} &= \partial_{\mu}\tilde{Q}_{i\alpha}^{*}(K_Q)_{ij}\partial^{\mu}\tilde{Q}_{\alpha j} + \partial_{\mu}\tilde{u}_i^{c*}(K_u)_{ij}\partial^{\mu}\tilde{u}_j^c + \partial_{\mu}\tilde{d}_i^{c*}(K_d)_{ij}\partial^{\mu}\tilde{d}_j^c + \dots, \\ \mathcal{L}_{\text{kin}}^f &= \overline{Q}_{i\alpha}(K_Q)_{ij}i\gamma^{\mu}\partial_{\mu}Q_{\alpha j} + \overline{u}_i^c(K_u)_{ij}i\gamma^{\mu}\partial_{\mu}u_j^c + \overline{d}_i^c(K_d)_{ij}i\gamma^{\mu}\partial_{\mu}d_j^c + \dots,\end{aligned}\quad (2.1)$$

(where $(K_f)_{ij}$ denotes the Kähler metric for the given scalar $\tilde{f}$, $\tilde{f}^c$ and fermionic f , f^c degrees of freedom) into the canonical form $K_f = \delta_{ij}(\tilde{f}_{\text{can}}^*)_i(\tilde{f}_{\text{can}})_j$ and $\delta_{ij}(\bar{f}_{\text{can}})_i(f_{\text{can}})_j$ respectively.

In a wide class of non-Abelian flavour models, the dominant contributions to K_{f,f^c} 's come from insertions of the flavon field associated to the third family Yukawas¹ (usually denoted by ϕ_3) yielding

$$K_{f,f^c} \approx \bar{f}_i \left[k_0^f \delta_{ij} + k_3^f \frac{1}{M_\psi^2} \langle \phi_3^\dagger \rangle_i \langle \phi_3 \rangle_j \right] f_j + \bar{f}_i^c \left[k_0^{f^c} \delta_{ij} + k_3^{f^c} \frac{1}{M_\chi^2} \langle \phi_3^\dagger \rangle_i \langle \phi_3 \rangle_j \right] f_j^c, \quad (2.2)$$

where $k_{0,3}^{f,f^c}$ are real constants and M_ψ and M_χ denote masses of the messenger fields relevant for the left- and right-chirality matter sectors respectively. If the flavon VEV $\langle \phi_3 \rangle$ is comparable to either M_ψ and/or M_χ , one can expect a potentially large deviations from the leading order universality (governed by the δ_{ij} factors above) in the relevant part of the Kähler metric.

It is important that one can hardly trim all these contributions to zero simultaneously by fiddling around with the messenger masses because there is no symmetry that could prevent every Yukawa sector relevant messenger from entering either K_f or K_{f^c} . This,² however, depends strongly on the character of the Yukawa operators, and can, in turn, single out a class of particular “Kähler-corrections-safe” flavour models; for more detailed discussion see section 5.2.

2.1 Definitions and ambiguities

Canonical normalisation consists in redefining the defining basis fields f and f^c so that the original (for instance scalar sector) kinetic terms $\mathcal{L}_{\text{kin}} = \partial \hat{f}^\dagger K_f \partial \hat{f} + \partial \hat{f}^{c\dagger} K_{f^c} \partial \hat{f}^c$ receive the canonical form $\mathcal{L}_{\text{kin}}^{\text{can}} = \partial f^\dagger \partial f + \partial f^{c\dagger} \partial f^c$. This is achieved³ by transforming the defining superfields by $\hat{f} \rightarrow P_f^{-1} \hat{f} \equiv f_{\text{can}}$ where P_f is a matrix bringing the relevant Kähler metric K_f into the diagonal form :

$$P_f^\dagger K_f P_f = \mathbb{1}, \quad \text{i.e.} \quad K_f = P_f^{\dagger-1} P_f^{-1}. \quad (2.3)$$

This can be easily done in two steps:

- First, one can always diagonalise the Hermitean Kähler metric by means of a unitary transformation $U_f K_f U_f^\dagger = K_f^D$ where K_f^D is a real diagonal matrix.

¹In the flavour models based on SU(3) family symmetry, the third family Yukawa couplings are governed by operators of the type $\frac{1}{M^2}(f \cdot \phi_3)(f^c \cdot \phi_3)H$ exploiting the triplet nature of matter of both chiralities f , f^c , while in the SO(3) theories the structure of the leading order operators typically looks like $\frac{1}{M}(f \cdot \phi_3)f_3^c H$ due to the singlet nature of the right-handed spinors. The dot product corresponds to the simplest bilinear invariants of the flavour symmetry under consideration.

²As we shall see, it is namely the left-handed sector (i.e. non-universalities in K_f) that could have large effects on quark and lepton mixing in the charged currents.

³In what follows, we shall focus on the left-chiral matter sector, i.e. we shall often give the results only for K_f ; the relevant formulae for K_{f^c} can be obtained upon replacing all the super-(sub-)scripts $f \rightarrow f^c$.

- Second, a diagonal rescaling by $\sqrt{K_f^D}^{-1}$ from both sides of $U_f K_f U_f^\dagger = K_f^D$ drives its r.h.s. to unity: $\left(\sqrt{K_f^D}\right)^{-1} U_f K_f U_f^\dagger \left(\sqrt{K_f^D}\right)^{-1} = \mathbb{1}$. Moreover, one can multiply this formula by any unitary matrix from the left (and its inverse from the right) with no effect on this unity matrix.

Thus, the most generic form of P_f reads

$$P_f = U_f^\dagger \left(\sqrt{K_f^D}\right)^{-1} \tilde{U}_f, \quad (2.4)$$

with the freedom to choose the unitary $\tilde{U}_f$ matrix arbitrarily. Thus, one can for instance have P_f Hermitean by choosing $\tilde{U}_f = U_f$ or exploit this freedom to bring the P_f into a triangular form as e.g. in [14].

Note on notation: in what follows, whenever appropriate we use hats to denote quantities in the defining basis (i.e. before canonical normalisation) while the unhatted symbols correspond to their physical counterparts, i.e. to quantities after the CN effects were already taken into account.

2.2 Effect of canonical normalisation on the Yukawa couplings

Lepton sector: suppose the original charged lepton and the light Majorana neutrino mass matrices are diagonalised⁴ by means of the biunitary transformations $\hat{V}_L^l \hat{M}_l \hat{V}_R^{l\dagger} = \hat{M}_l^D$ and $\hat{V}_L^\nu \hat{m}_\nu \hat{V}_L^{\nu T} = \hat{m}_\nu^D$ so that the lepton mixing matrix (before canonical normalisation) obeys $\hat{U}_{\text{PMNS}} = \hat{V}_L^l \hat{V}_L^{\nu\dagger}$. The effect of canonical normalisation on $\hat{M}_l$ and⁵ $\hat{M}_\nu$:

$$\hat{M}_l \rightarrow P_L^T \hat{M}_l P_{e^c} \equiv M_l \quad \text{and} \quad \hat{m}_\nu \rightarrow P_L^T \hat{m}_\nu P_L \equiv m_\nu \quad (2.5)$$

induces a relevant change on $\hat{V}_{L,R}^l \rightarrow V_{L,R}^l$ and $\hat{V}_L^\nu \rightarrow V_L^\nu$ so that $V_L^l M_l V_R^{l\dagger} = M_l^D$ and $V_L^\nu m_\nu V_L^{\nu T} = m_\nu^D$ is fulfilled, i.e.:

$$V_L^l P_L^T \hat{M}_l P_{e^c} V_R^{l\dagger} = M_l^D \quad \text{and} \quad V_L^\nu P_L^T \hat{m}_\nu P_L V_L^{\nu T} = m_\nu^D \quad (2.6)$$

should be satisfied. Then the *physical* lepton mixing matrix obeys (up to the rephasing bringing it into the standard PDG form [18]) $U_{\text{PMNS}} = V_L^l V_L^{\nu\dagger}$.

⁴The situation in the quark and lepton sectors is different: since the quark sector diagonalisation transformation is *bi-unitary*, one can always absorb the would-be phases of the diagonal entries of M_f^D by a suitable redefinition of V_L^f and V_R^f and get rid of all but one CP phase in the CKM matrix. This is not possible for Majorana neutrinos as there is only one unitary matrix in the relevant formula. This, in turn, gives rise to extra phase factors associated to PMNS mixing - the Majorana phases.

⁵Note that the P_{ν^c} actually does not enter the effective light neutrino matrix because it cancels among the right-handed components of the neutrino Yukawas and the inverse of the Majorana mass matrix in the seesaw formula $\hat{m}_\nu = M_\nu^D M_M^{-1} (M_\nu^D)^T$ [17].

Quark sector: the reasoning for the quark sector goes along the same lines as above - the original basis up and down-type Yukawa matrices $\hat{M}_{u,d}$ diagonalisable by biunitary transformations $\hat{V}_L^u \hat{M}_u \hat{V}_R^{u\dagger} = \hat{M}_u^D$ and $\hat{V}_L^d \hat{M}_d \hat{V}_L^{d\dagger} = \hat{M}_d^D$ (leading to $\hat{V}_{CKM} = \hat{V}_L^u \hat{V}_L^{d\dagger}$ before canonical normalisation) change upon canonical normalisation into

$$\hat{M}_u \rightarrow P_Q^T \hat{M}_u P_{u^c} \equiv M_u \quad \text{and} \quad \hat{M}_d \rightarrow P_Q^T \hat{M}_d P_{d^c} \equiv M_d \quad (2.7)$$

inducing a change on $\hat{V}_{L,R}^u \rightarrow V_{L,R}^u$ and $\hat{V}_{L,R}^d \rightarrow V_{L,R}^d$ so that

$$V_L^u P_Q^T \hat{M}_u P_{u^c} V_R^{u\dagger} = M_u^D \quad \text{and} \quad V_L^d P_Q^T \hat{M}_d P_{d^c} V_R^{d\dagger} = M_d^D. \quad (2.8)$$

The physical CKM matrix then obeys $V_{CKM} = V_L^u V_L^{d\dagger}$.

Irrelevance of $\tilde{U}_f$ matrices. It is easy to see that the arbitrary $\tilde{U}_f$ matrices in the definition of P_f do not play any role in either the mixing matrices U_{PMNS} , V_{CKM} or the physical spectra. Indeed, under any unitary change $\tilde{U}_f'$ of the relevant $\tilde{U}_f$ matrices in the definition (2.4), i.e. $P_f \rightarrow P_f \tilde{U}_f'$, the effects in (2.6) can be absorbed into redefinitions $V_L^l \rightarrow V_L^l \tilde{U}_L'^{\dagger}$ and $V_L^\nu \rightarrow V_L^\nu \tilde{U}_L'^{\dagger}$ so that (2.6) remains unaffected. However, $\tilde{U}_L'$ cancels in U_{PMNS} and the physical spectra remain intact, because the would-be effects of $\tilde{U}_L$ and $\tilde{U}_{e^c}$ matrices in (2.6) can be absorbed into the biunitary transformation revealing the spectrum of the charged lepton Yukawa matrix. Similarly, one can justify the irrelevance of the particular choice of P_Q and P_{u^c,d^c} for V_{CKM} and the quark sector spectra.

Exploiting the freedom in definition of $P_{f,fc}$. Thus, one can exploit the freedom in choosing $\tilde{U}_{f,fc}$ matrices in the definition of $P_{f,fc}$ to simplify the structure of (2.4) so that the $P_{f,fc}$ -factors are particularly easy to handle. The convenient choice is indicated by the fact that even if the original Kähler metric is just a slight perturbation of the unity matrix $\mathbb{1}$ (up to an irrelevant overall normalisation k_0^f), the diagonalisation matrix U_f in $P_f = U_f^\dagger (\sqrt{K_f^D})^{-1} \tilde{U}_f$ could still be large. The intention to write $(\sqrt{K_f^D})^{-1}$ as $(\mathbb{1} + \Delta K_f^D)/\sqrt{k_0^f}$, exploiting the limited departure of the Kähler metric spectrum from unity, then gives $P_f = (U_f^\dagger \tilde{U}_f + U_f^\dagger \Delta K_f^D \tilde{U}_f)/\sqrt{k_0^f}$ that could be brought to a particularly convenient form for $\tilde{U}_f = U_f$, and we can benefit from $P_f = (\mathbb{1} + U_f^\dagger \Delta K_f^D U_f)/\sqrt{k_0^f}$, i.e. hermiticity of $\Delta P_f \equiv U_f^\dagger \Delta K_f^D U_f$ and simplicity of $P_f = (\mathbb{1} + \Delta P_f)/\sqrt{k_0^f}$.

2.3 Perturbative prescription for the physical rotation matrices

For Hermitean P_{L,e^c} and P_{Q,u^c,d^c} and $P_{f,fc} = (\mathbb{1} + \Delta P_{f,fc})/\sqrt{k_0^{f,fc}}$ (assuming a small departure of K_f and K_{fc} from unity), one obtains from (2.6):

$$\begin{aligned} V_L^l (\mathbb{1} + \Delta P_L^T) \hat{M}_l (\mathbb{1} + \Delta P_{e^c}) V_R^{l\dagger} &= \sqrt{k_0^L k_0^{e^c}} M_l^D, \\ V_L^\nu (\mathbb{1} + \Delta P_L^T) \hat{m}_\nu (\mathbb{1} + \Delta P_L) V_L^{\nu T} &= k_0^L m_\nu^D \end{aligned} \quad (2.9)$$

for the lepton sector and from (2.8):

$$\begin{aligned} V_L^u(\mathbb{1} + \Delta P_Q^T) \hat{M}_u(\mathbb{1} + \Delta P_{u^c}) V_R^{u\dagger} &= \sqrt{k_0^Q k_0^{u^c}} M_u^D, \\ V_L^d(\mathbb{1} + \Delta P_Q^T) \hat{M}_d(\mathbb{1} + \Delta P_{d^c}) V_R^{d\dagger} &= \sqrt{k_0^Q k_0^{d^c}} M_d^D \end{aligned} \quad (2.10)$$

for the quarks. If all the (high-scale) physical spectra are sufficiently hierarchical,⁶ the smallness of $\Delta P_{f,f^c}$ factors ensures only small differences between the hatted and un-hatted diagonalisation matrices, i.e.

$$V_{L,R}^f = W_{L,R}^f \hat{V}_{L,R}^f, \quad (2.11)$$

where $W_{L,R}^f$ are small unitary rotations in the unity neighbourhood (up to a phase ambiguity to be discussed later):

$$W_{L,R}^f = \mathbb{1} + i\Delta W_{L,R}^f, \quad (2.12)$$

with $\Delta W_{L,R}^f$ denoting their Hermitean generators. One can disentangle the left-handed and right-handed rotations in formulae (2.9) and (2.10) by considering $M_f M_f^\dagger$:

$$\begin{aligned} W_L^l \hat{V}_L^l (\mathbb{1} + \Delta P_L^T) \hat{M}_l (\mathbb{1} + 2\Delta P_{e^c}) \hat{M}_l^\dagger (\mathbb{1} + \Delta P_L^*) \hat{V}_L^{l\dagger} W_L^{l\dagger} &= k_0^L k_0^{e^c} M_l^D M_l^{D\dagger}, \\ W_L^u \hat{V}_L^u (\mathbb{1} + \Delta P_Q^T) \hat{M}_u (\mathbb{1} + 2\Delta P_{u^c}) \hat{M}_u^\dagger (\mathbb{1} + \Delta P_Q^*) \hat{V}_L^{u\dagger} W_L^{u\dagger} &= k_0^Q k_0^{u^c} M_u^D M_u^{D\dagger}, \\ W_L^d \hat{V}_L^d (\mathbb{1} + \Delta P_Q^T) \hat{M}_d (\mathbb{1} + 2\Delta P_{d^c}) \hat{M}_d^\dagger (\mathbb{1} + \Delta P_Q^*) \hat{V}_L^{d\dagger} W_L^{d\dagger} &= k_0^Q k_0^{d^c} M_d^D M_d^{D\dagger}, \end{aligned} \quad (2.13)$$

which yields (from the three complex off-diagonal zero conditions) at the leading order:

$$\begin{aligned} (\Delta W_L^l)_{ij, i \neq j} &= \frac{i}{\hat{m}_j^{l2} - \hat{m}_i^{l2}} \left[(\hat{m}_i^{l2} + \hat{m}_j^{l2}) \left(\hat{V}_L^l \Delta P_L^T \hat{V}_L^{l\dagger} \right)_{ij} + 2\hat{m}_i^l \hat{m}_j^l \left(\hat{V}_R^l \Delta P_{e^c} \hat{V}_R^{l\dagger} \right)_{ij} \right], \\ (\Delta W_L^u)_{ij, i \neq j} &= \frac{i}{\hat{m}_j^{u2} - \hat{m}_i^{u2}} \left[(\hat{m}_i^{u2} + \hat{m}_j^{u2}) \left(\hat{V}_L^u \Delta P_Q^T \hat{V}_L^{u\dagger} \right)_{ij} + 2\hat{m}_i^u \hat{m}_j^u \left(\hat{V}_R^u \Delta P_{u^c} \hat{V}_R^{u\dagger} \right)_{ij} \right], \\ (\Delta W_L^d)_{ij, i \neq j} &= \frac{i}{\hat{m}_j^{d2} - \hat{m}_i^{d2}} \left[(\hat{m}_i^{d2} + \hat{m}_j^{d2}) \left(\hat{V}_L^d \Delta P_Q^T \hat{V}_L^{d\dagger} \right)_{ij} + 2\hat{m}_i^d \hat{m}_j^d \left(\hat{V}_R^d \Delta P_{d^c} \hat{V}_R^{d\dagger} \right)_{ij} \right], \end{aligned} \quad (2.14)$$

where the eigenvalues $\hat{m}_i^{f2}$ of the original $\hat{M}_f$ matrices can be at the leading order identified with the physical charged fermion masses and the overall normalisation factors k_0^{f,f^c} drop. Similarly, the neutrino sector corrections obey (replacing $\hat{M}_l \rightarrow \hat{m}_\nu$, $V_L^l \rightarrow V_L^\nu$, $V_R^l \rightarrow V_R^{\nu*}$ and $\Delta P_{e^c} \rightarrow \Delta P_L$ in the first formula above)

$$(\Delta W_L^\nu)_{ij, i \neq j} = \frac{i}{\hat{m}_j^{\nu2} - \hat{m}_i^{\nu2}} \left[(\hat{m}_i^{\nu2} + \hat{m}_j^{\nu2}) \left(\hat{V}_L^\nu \Delta P_L^T \hat{V}_L^{\nu\dagger} \right)_{ij} + 2\hat{m}_i^\nu \hat{m}_j^\nu \left(\hat{V}_L^\nu \Delta P_L^T \hat{V}_L^{\nu\dagger} \right)_{ji} \right]. \quad (2.15)$$

Due to the assumed hierarchy in the physical spectra, the first terms tend to dominate over the second (thus screening the ambiguity in the unknown structure of the right-handed rotations in the charged sector) and we shall often neglect the latter.

⁶This is certainly true for all charged matter fermion spectra; for neutrinos we shall stick to the hierarchical spectrum case from now on.

Notice that formulae (2.14), (2.15) provide only the off-diagonal entries of ΔW_L^f 's. However, this reflects the three phase ambiguity in defining the diagonalisation matrices W by means of relations like (2.13). Thus, it is not surprising that three parameters in ΔW_L^f 's remain unconstrained and can be in principle chosen arbitrarily with the only constraint coming from the required perturbativity of the W_L^f matrices (2.12). For simplicity, we shall put the diagonal entries of all W_L^f 's to zero keeping in mind the possible need for “standard” rephasing of the physical lepton mixing matrix. Another reason is that in the real case W become orthogonal and thus generated by antisymmetric purely imaginary ΔW_L^f 's. Thus, the ΔW_L^f matrices can be without loss of generality chosen in the form:

$$\Delta W_L^f = \begin{pmatrix} 0 & \Delta W_{L12}^f & \Delta W_{L13}^f \\ \Delta W_{L12}^{f*} & 0 & \Delta W_{L23}^f \\ \Delta W_{L13}^{f*} & \Delta W_{L23}^{f*} & 0 \end{pmatrix}, \quad (2.16)$$

with the off-diagonal entries given by formulae (2.14) and (2.15). With this at hand one can write the physical⁷ quark and lepton mixing matrices V_{CKM} and U_{PMNS} in term of the original ones $\hat{V}_{\text{CKM}}$ and $\hat{U}_{\text{PMNS}}$ as:

$$\begin{aligned} U_{\text{PMNS}} &= (\mathbb{1} + i\Delta W_L^l) \hat{U}_{\text{PMNS}} (\mathbb{1} - i\Delta W_L^{\nu\dagger}) = \hat{U}_{\text{PMNS}} + \Delta U_{\text{PMNS}}, \\ V_{\text{CKM}} &= (\mathbb{1} + i\Delta W_L^u) \hat{V}_{\text{CKM}} (\mathbb{1} - i\Delta W_L^{d\dagger}) = \hat{V}_{\text{CKM}} + \Delta V_{\text{CKM}}, \end{aligned}$$

with⁸

$$\Delta U_{\text{PMNS}} = i \left(\Delta W_L^l \hat{U}_{\text{PMNS}} - \hat{U}_{\text{PMNS}} \Delta W_L^{\nu\dagger} \right) + \dots, \quad (2.17)$$

$$\Delta V_{\text{CKM}} = i \left(\Delta W_L^u \hat{V}_{\text{CKM}} - \hat{V}_{\text{CKM}} \Delta W_L^{d\dagger} \right) + \dots. \quad (2.18)$$

Recall that in a particular model, all the ingredients are actually at hand - one can easily diagonalise the Hermitean Kähler metric to get the (conventionally) Hermitean P_f^{-1} factors (and from there ΔP_f 's) and the various $\hat{V}_{L,R}^f$ matrices in (2.14), (2.15) can be inferred in the same manner from the underlying model Yukawa couplings.

3. Canonical normalisation corrections to quark sector mixing

To illustrate the importance of CN corrections for the quark sector mixing we discuss as an example the dominant Kähler corrections to the V_{cb} CKM entry in the class of potentially realistic SU(3) setting with a large third family expansion parameter. Such large third family expansion parameter appears, e.g., in the models discussed in [11, 19, 20].

⁷From now on we shall always choose the free phases in $\hat{V}_{L,R}^f$ (i.e. work in a particular basis) so that the $\hat{V}_{\text{CKM}}$ and $\hat{U}_{\text{PMNS}}$ matrices are in their ‘standard’ form [18]. This, however, need not be the case after the CN corrections are taken into account and we shall comment on the phases later.

⁸Although $\Delta W_{L,R}^f$ are by definition Hermitean, we shall often keep the dagger in formulae like (2.17), (2.18) to help reader’s orientation in the text.

3.1 Corrections to V_{cb} in classes of SU(3) flavour models

For simplicity reasons, we shall focus on a *real* 2×2 case for the two heavy states only for a quasi-diagonal LH quark sector Kähler metric along the lines of [19] discussed in great detail in e.g. [11, 20]. Assuming that the expansion parameters in the Kähler sector coincide with those relevant for the superpotential (c.f. section 5.2 for a detailed discussion of this point), the relevant piece of the matter sector Kähler metric can be written as⁹

$$K_Q = k_0^Q \begin{pmatrix} 1 & \varepsilon^2 \\ \varepsilon^2 & 1 + \varepsilon_3^2 \end{pmatrix}, \quad (3.1)$$

which is diagonalised by means of $U_Q K_Q U_Q^\dagger = K_Q^D$, where:

$$U_Q \approx \begin{pmatrix} 1 & -\frac{\varepsilon^2}{\varepsilon_3^2} \\ \frac{\varepsilon^2}{\varepsilon_3^2} & 1 \end{pmatrix} \quad \text{and} \quad K_Q^D \approx k_0^Q \text{diag}(1, 1 + \varepsilon_3^2). \quad (3.2)$$

Adopting the Hermitean convention $\tilde{U}_Q = U_Q$, i.e. $P_Q = U_Q^\dagger (K_Q^D)^{-\frac{1}{2}} U_Q / \sqrt{k_0^Q}$, one obtains¹⁰

$$P_Q \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} 1 & -\frac{\varepsilon^2}{2} \\ -\frac{\varepsilon^2}{2} & 1 - \frac{\varepsilon_3^2}{2} \end{pmatrix}, \quad (3.3)$$

and the physical Yukawas obey (at the leading order)

$$\begin{aligned} \hat{Y}_u &\approx \begin{pmatrix} \varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon_3^2 \end{pmatrix} \rightarrow Y_u = (P_{1Q})^T \hat{Y}_u P_{u^c} \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} \varepsilon^2 & \varepsilon^2 - \frac{1}{2}\varepsilon_3^2\varepsilon^2 \\ \varepsilon^2 & \varepsilon_3^2 \end{pmatrix} P_{u^c}, \\ \hat{Y}_d &\approx \begin{pmatrix} \bar{\varepsilon}^2 & \bar{\varepsilon}^2 \\ \bar{\varepsilon}^2 & \varepsilon_3^2 \end{pmatrix} \rightarrow Y_d = (P_{1Q})^T \hat{Y}_d P_{d^c} \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} \bar{\varepsilon}^2 & \bar{\varepsilon}^2 - \frac{1}{2}\varepsilon_3^2\varepsilon^2 \\ \bar{\varepsilon}^2 & \varepsilon_3^2 \end{pmatrix} P_{d^c}, \end{aligned} \quad (3.4)$$

indicating non-negligible *additive* leading order corrections to 23 rotations in $V_L^{u,d}$, that, however, cancel at the leading order in the CKM mixing matrix.

The net effect eventually emerges from the next to leading order ratio of the 23 and 33 entries and can be readily obtained from the perturbative prescription (2.18) together with (2.15) provided:

$$\Delta P_Q = -\frac{1}{2} \begin{pmatrix} 0 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon_3^2 \end{pmatrix}, \quad \hat{V}_L^u \approx \begin{pmatrix} 1 & -\frac{\varepsilon^2}{\varepsilon_3^2} \\ \frac{\varepsilon^2}{\varepsilon_3^2} & 1 \end{pmatrix}, \quad \hat{V}_L^d \approx \begin{pmatrix} 1 & -\frac{\bar{\varepsilon}^2}{\varepsilon_3^2} \\ \frac{\bar{\varepsilon}^2}{\varepsilon_3^2} & 1 \end{pmatrix}, \quad (3.5)$$

⁹For sake of simplicity, we have chosen a particular shape of K_Q so that the numerical factors are simple.

¹⁰Note that for $\tilde{U}_Q = \mathbb{1}$ one receives $P_Q \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} 1 & \frac{\varepsilon^2}{\varepsilon_3^2} \\ -\frac{\varepsilon^2}{\varepsilon_3^2} & 1 - \frac{\varepsilon_3^2}{2} \end{pmatrix}$ instead with enhanced off-diagonal terms with respect to (3.3). As it was pointed out in [13], such a P_Q matrix can induce a potentially large deviation of the physical Yukawa matrices from their defining basis structure. However, as far as physical observables such as the CKM mixings are concerned, the individual relatively large 23 rotations arising in such case in both up and down sectors act against each other and leave only a subleading effect, which becomes almost trivial to infer upon adopting $\tilde{U}_Q = U_Q$.

In short, the ‘‘Hermitean’’ convention for P_Q ’s adopted here does not induce large fake corrections to the off-diagonal Yukawa couplings and the corresponding $V_L^{u,d}$ matrices.

(giving $\hat{V}_{\text{CKM}} \approx \begin{pmatrix} 1 & \frac{\bar{\epsilon}^2 - \epsilon^2}{\epsilon_3^2} \\ -\frac{\bar{\epsilon}^2 - \epsilon^2}{\epsilon_3^2} & 1 \end{pmatrix}$ and thus $\hat{V}_{cb} \approx \frac{\bar{\epsilon}^2 - \epsilon^2}{\epsilon_3^2}$ before canonical normalisation).

This yields at leading order $\Delta W_L^u \approx 0$ and from (2.15) $\Delta W_L^d \approx i \begin{pmatrix} 0 & \frac{\bar{\epsilon}^2 - \epsilon^2}{2} \\ -\frac{\bar{\epsilon}^2 - \epsilon^2}{2} & 0 \end{pmatrix}$. Therefore, (2.18) leads to:

$$\Delta V_{\text{CKM}} \approx -i \hat{V}_{\text{CKM}} \Delta W_L^{d\dagger} \approx \begin{pmatrix} 0 & \frac{\bar{\epsilon}^2 - \epsilon^2}{2} \\ \frac{\epsilon^2 - \bar{\epsilon}^2}{2} & 0 \end{pmatrix}, \quad (3.6)$$

so the CKM matrix changes after canonical normalisation into:

$$V_{\text{CKM}} \approx \begin{pmatrix} 1 & \frac{\bar{\epsilon}^2 - \epsilon^2}{\epsilon_3^2} (1 + \frac{1}{2} \epsilon_3^2) \\ \frac{\epsilon^2 - \bar{\epsilon}^2}{\epsilon_3^2} (1 + \frac{1}{2} \epsilon_3^2) & 1 \end{pmatrix}. \quad (3.7)$$

The physical value of the 23 quark-sector mixing is then modified to:

$$V_{cb} = \hat{V}_{cb} \left(1 + \frac{1}{2} \epsilon_3^2 \right) + \dots \quad (3.8)$$

As anticipated, there is a relatively large *multiplicative* correction due to the presence of the large expansion parameter associated to the third family canonical normalisation corrections, that was not appreciated in [14].

4. Canonical normalisation corrections to lepton sector mixing

In order to study the effects of Kähler corrections to a generic bi-large lepton sector mixing, one can not avoid the first generation anymore. Thus, in what follows, we consider the full 3×3 structure of the relevant mixing matrices as well as the matter sector Kähler metric.

4.1 Corrections due to third family canonical rescaling

Though the generic shape of the relevant piece of the Kähler metric (i.e. namely K_L as far the lepton sector is concerned¹¹) is rather complicated, in realistic cases one can expect the dominant effects coming from the leading non-universal contribution (2.2) governed by $\langle \phi_3 \rangle$. Thus, we shall first focus on the simplified setting where only the entries due to (2.2) are taken into account. Later on (in section 5.1), we shall compare the results obtained here with the full-fledged potentially realistic SU(3) model analysis to reveal that this is indeed a very accurate approximation.

In the present case, the lepton sector Kähler metric is given at leading order by:

$$K_L = k_0^L \left(\mathbb{1} + \frac{k_3^L}{k_0^L} \frac{\langle \phi_3^\dagger \rangle \langle \phi_3 \rangle}{M_K^2} \right) + \dots, \quad (4.1)$$

¹¹Recall that ΔP_{ec} is screened in (2.15) at the leading order and corrections due to $P_{\nu c}$ entirely cancel in the seesaw formula, c.f. (2.5).

with k_i^L denoting the relevant $\mathcal{O}(1)$ Wilson coefficients in (2.2), while M_K stands for a generic Kähler sector messenger mass. In models where the 33 Yukawa entries are (at least partly) generated by means of $SU(2)_L$ -doublet messengers (that in turn enter also the Kähler potential) M_K is around the scale of the relevant Yukawa-sector-active messengers (denoted by χ_i in section 5.2) and $\frac{\langle\phi_3\rangle}{M_K}$ is of the order of the Yukawa sector parameter ε_3 .

At the leading order, the lepton sector Kähler metric can be written in a matrix form:

$$K_L \approx k_0^L \left[\mathbb{1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix} \right] \quad \text{where} \quad \eta^K \equiv \frac{k_3^L}{k_0^L} \frac{|\langle\phi_3\rangle|^2}{M_K^2}. \quad (4.2)$$

Therefore, the P_L matrix is just:

$$P_L = \frac{1}{\sqrt{k_0^L}} \left[\mathbb{1} - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix} \right], \quad \text{and thus} \quad \Delta P_L = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix}. \quad (4.3)$$

In the next sub-section, we shall consider the canonical normalisation corrections models of (nearly) tri-bimaximal mixing in the lepton sector.

4.2 Canonical normalisation corrections to tri-bimaximal neutrino mixing

As an example for the impact of the potentially large third family CN corrections on lepton mixing, let us consider their effects on the pattern of exact tri-bimaximal lepton mixing. In many classes of flavour models this pattern of tri-bimaximal mixing emerges as a prediction of the neutrino sector [21]. These models are inspired by the proximity of the present neutrino oscillation data on PMNS matrix to the tri-bimaximal mixing matrix à la Harisson-Perkins-Scott [8], which has the form

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot P_M \quad \text{with} \quad P_M = \begin{pmatrix} e^{i\frac{\alpha_1}{2}} & 0 & 0 \\ 0 & e^{i\frac{\alpha_2}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.4)$$

where P_M is (so far) experimentally undetermined diagonal matrix encoding the two observable Majorana phase differences.

We shall first focus on the simplest setting and assume that the lepton mixing generated by the underlying family symmetry happens to be exactly tri-bimaximal (in the defining basis), i.e. $\hat{U}_{\text{PMNS}} = \hat{V}_L^l \hat{V}_L^{\nu\dagger} = U_{\text{TB}}$ and comes entirely from the neutrino sector [21], i.e. $\hat{V}_{L,R}^l \approx \mathbb{1}$ while $\hat{V}_L^{\nu\dagger} \approx U_{\text{TB}}$. In the canonical basis, $\hat{U}_{\text{PMNS}} = U_{\text{TB}}$ changes along (2.17) yielding:

$$U_{\text{PMNS}} = U_{\text{TB}} + i \left(\Delta W_L^l U_{\text{TB}} - U_{\text{TB}} \Delta W_L^{\nu\dagger} \right) + \dots, \quad (4.5)$$

and the correction matrices are given by (2.16) provided (2.14), (2.15). Taking into account the screening of the second terms $\propto 2\hat{m}_i\hat{m}_j/(\hat{m}_i^2 - \hat{m}_j^2)$ in formulae (2.14) and (2.15) in

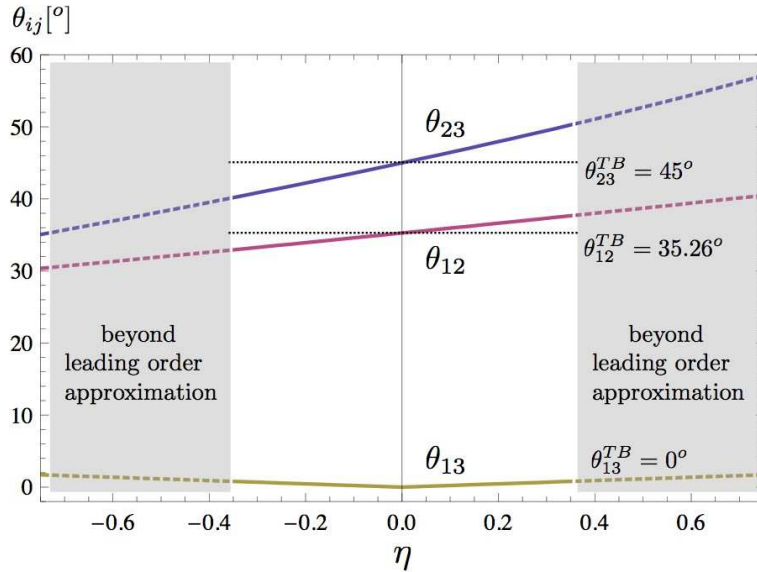


Figure 1: Corrections to the tri-bimaximal neutrino mixing due to canonical normalisation (taken from [15]). The η -parameter stands for $\eta = \eta^K$ from (4.2) for the canonical normalisation effects, $\eta = \eta^{\text{RG}}$ from (6.5) for the leading order running effects, or $\eta = \eta^K + \eta^{\text{RG}}$ if both CN and RG corrections are taken into account, c.f. (6.7). The shaded regions correspond to values of $|\eta|$ for which one can expect deviations from the leading-order perturbative results. The reactor mixing in this setup is seen to be rather stable with respect to the considered third family corrections.

case of the hierarchical neutrino spectrum, one obtains:¹²

$$\begin{aligned}
 (\Delta W_L^l)_{ij, i \neq j} &\approx \frac{i(\hat{m}_i^{l2} + \hat{m}_j^{l2})}{\hat{m}_j^{l2} - \hat{m}_i^{l2}} (\Delta P_L^T)_{ij} = 0, \\
 (\Delta W_L^\nu)_{ij, i < j} &\approx \frac{i(\hat{m}_i^{\nu 2} + \hat{m}_j^{\nu 2})}{\hat{m}_j^{\nu 2} - \hat{m}_i^{\nu 2}} \left(U_{\text{TB}}^\dagger \Delta P_L^T U_{\text{TB}} \right)_{ij} \approx i \left(U_{\text{TB}}^\dagger \Delta P_L^T U_{\text{TB}} \right)_{ij} + \dots \quad (4.6)
 \end{aligned}$$

that yields at the leading order

$$\Delta(U_{\text{TB}})_{ij, i < j} \approx -(U_{\text{TB}})_{ij} \left(U_{\text{TB}}^\dagger \Delta P_L^T U_{\text{TB}} \right)_{jj} \quad (\text{no summation over } j). \quad (4.7)$$

Remarkably enough, the corrections to the three matrix elements under consideration are (at the leading order) proportional to their values, c.f. formula (4.7) and thus, in particular, the canonical normalisation corrections to the reactor angle are cancelled by the 13 zero of U_{TB} . Second, the Majorana phases are irrelevant for the second bracket on the l.h.s. of formula (4.7) and enter only through the first term. Thus, the phase structure of the correction is identical to the phase structure of the original matrix element and there is no need for an additional rephasing.

Numerically, this leads for example to $(\Delta U_{\text{TB}})_{12} \approx \eta^K \frac{1}{6\sqrt{3}}$, $(\Delta U_{\text{TB}})_{23} \approx \eta^K \frac{1}{4\sqrt{2}}$ and $(\Delta U_{\text{TB}})_{13} \approx 0$ (up to irrelevant phase factors). The zero in the 13 correction, however,

¹²From now on, we shall always assume that the defining basis masses $\hat{m}_i^f$ coincide at the leading order with the corresponding physical quantities.

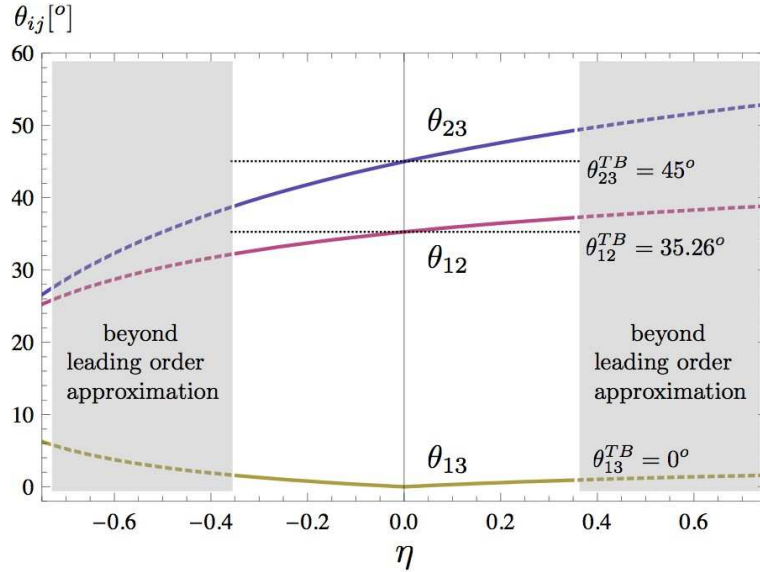


Figure 2: Corrections to the tri-bimaximal neutrino mixing (faint dashed lines) from canonical normalisation of the kinetic terms as a function of $\eta = \eta^K \equiv \varepsilon_3^2 \frac{k_3^L}{k_0^L}$ in a potentially realistic $SU(3)_f$ flavour model [11] discussed in detail in section 5.1. The displayed curves correspond to the leading order approximate results given by formulae (5.9). Remarkably enough, these results coincide for small η with those obtained by perturbative methods in the simplified setup discussed in section 4.2, thus demonstrating the crucial role played by the dominant 33-sector Kähler correction.

emerges only from the first term in the approximation (4.6) and gets lifted at the next-to-leading level. Indeed, employing the full-featured formula (2.15) one recovers (for hierarchical case):

$$\begin{aligned} \Delta(U_{\text{PMNS}})_{13} &= 2 \left[\frac{\hat{m}_2^\nu}{\hat{m}_3^\nu} (U_{\text{TB}})_{12} (U_{\text{TB}}^\dagger)_{23} + \frac{\hat{m}_1^\nu}{\hat{m}_3^\nu} (U_{\text{TB}})_{11} (U_{\text{TB}}^\dagger)_{13} \right] (\Delta P_L)_{33} (U_{\text{TB}})_{33} \\ &= -\frac{1}{3\sqrt{2}} \left(\frac{\hat{m}_2^\nu}{\hat{m}_3^\nu} e^{i\frac{\hat{\alpha}_2}{2}} - \frac{\hat{m}_1^\nu}{\hat{m}_3^\nu} e^{i\frac{\hat{\alpha}_1}{2}} \right) \eta^K, \end{aligned} \quad (4.8)$$

where the two phase-factors reflect the Majorana nature of the light neutrino masses. The last formula finally yields (assuming the first term in the bracket dominates):

$$\theta_{13} \approx |\eta^K| \frac{1}{3\sqrt{2}} \sqrt{\frac{\Delta m_\odot^2}{\Delta m_A^2}} \approx 4 \times 10^{-2} |\eta^K|. \quad (4.9)$$

All together, this gives at the leading order:

$$U_{\text{PMNS}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} - \eta^K \frac{1}{6\sqrt{6}} & \frac{1}{\sqrt{3}} + \eta^K \frac{1}{6\sqrt{3}} & 4 \times 10^{-2} |\eta^K| e^{-i\delta} \\ -\frac{1}{\sqrt{6}} + \eta^K \frac{1}{12\sqrt{6}} & \frac{1}{\sqrt{3}} - \eta^K \frac{2}{6\sqrt{3}} & \frac{1}{\sqrt{2}} + \eta^K \frac{1}{4\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \eta^K \frac{5}{12\sqrt{6}} & -\frac{1}{\sqrt{3}} - \eta^K \frac{1}{6\sqrt{3}} & \frac{1}{\sqrt{2}} - \eta^K \frac{1}{4\sqrt{2}} \end{pmatrix} \cdot P_M. \quad (4.10)$$

It can be easily checked that U_{PMNS} is unitary up to $\mathcal{O}(\eta^{K^2})$ terms. Exploiting the parametrisation of [22] one gets:

$$s_{12} = \frac{1}{\sqrt{3}}(1+s), \quad s_{23} = \frac{1}{\sqrt{2}}(1+a), \quad s_{13} = \frac{1}{\sqrt{2}}r, \quad (4.11)$$

and (comparing to (4.10)) the ‘TB-deviation’ parameters¹³ read:

$$r \approx 6 \times 10^{-2} |\eta^K|, \quad s = \frac{\eta^K}{6} \quad \text{and} \quad a = \frac{\eta^K}{4}. \quad (4.12)$$

We see, in particular, that θ_{13} is rather stable and that the atmospheric mixing is changing faster than the solar ($a = \eta^K/4$ while $s = \eta^K/6$), c.f. the shape of curves depicted in figure 1.

5. Canonical normalisation corrections in potentially realistic models

As discussed in the Introduction, it has been pointed out that the observed close-to tri-bimaximal lepton mixing, along with the main features of the quark and charged lepton sector observables, can be understood in frameworks with non-Abelian family symmetry ($\mathcal{F}$), that is spontaneously broken by the VEVs of three flavons $\phi_3, \phi_{23}, \phi_{123}$ (transforming as triplets under $\mathcal{F}$) pointing in particular directions in the family space. These flavon fields give rise to the Yukawa operators of the shape (in the case of SU(3) family symmetry, dropping superfield hats) [6]:

$$W_Y \approx f^i f^{cj} \frac{H}{M_f^2} \left[y_1^f (\phi_{123})_i (\phi_{23})_j + y_2^f (\phi_{23})_i (\phi_{123})_j + y_3^f (\phi_3)_i (\phi_3)_j + y_4^f (\phi_{23})_i (\phi_{23})_j \right]. \quad (5.1)$$

This approach requires the vacuum alignment in the SU(3) space of the form: $\phi_3 \sim (0, 0, 1)$, $\phi_{23} \sim (0, 1, 1)$, $\phi_{123} \sim (1, 1, 1)$, up to phases.

Note also that there is in principle at least two distinct types of messengers entering the formula (5.1), in particular those transmitting the SU(2)_L doublet nature of $f = Q, L$ to the Higgs VEV insertion point (for definiteness let’s call them $\chi_{Q,L}$), and the SU(2)_L-singlets propagating further the remaining SU(3)_c $\otimes$ U(1)_Y quantum numbers to $f^c = u^c, d^c, e^c$ and ν^c (to be called $\chi_{u,d,e,\nu}$), c.f. figure 4. However, for sake of simplicity, we shall use a generic symbol M_f for both these classes and come back to this distinction only upon getting to physical implications. Later in this section we shall address the question of topology of the underlying messenger sector Feynman graphs giving rise to the operators under consideration. We shall also discuss the relationship between these messengers and those which appear in the Kähler potential.

5.1 Corrections to tri-bimaximal mixing in a class of SU(3)_f flavour models

As an example, let us focus on the canonical normalisation corrections to the tri-bimaximal neutrino mixing in the classes of SU(3)_f flavour models considered in [11]. Taking into account the irrelevance of the canonical transformation P_{ν^c} in the see-saw formula for the light neutrino masses, the quantity of our main interest is the leading order Kähler metric for the lepton doublets K_L obeying:

$$K_L \approx k_0^L \mathbb{1} + \begin{pmatrix} \varepsilon_K^4 k_1^L & \varepsilon_K^4 k_1^L e^{i\phi_1} & \varepsilon_K^4 k_1^L e^{i\phi_2} \\ . & \varepsilon_K^2 k_2^L & \varepsilon_K^2 k_2^L e^{i\phi_3} \\ . & . & \varepsilon_K^2 k_3^L \end{pmatrix} + \dots, \quad (5.2)$$

¹³Recall that the 13 mixing can be always without loss of generality made positive by a suitable redefinition of the lepton sector Dirac CP phase.

where the subscript K in the expansion parameters ε_K and ε_{K3} indicates that the Kähler metric messenger masses M_K (entering through e.g. $\varepsilon_{K3} = |\langle\phi_3\rangle|/M_K$) may differ from those relevant for the Yukawa sector (5.1) and the dotted terms in (5.2) can be reconstructed from hermiticity. The P_L^{-1} matrix is obtained¹⁴ to leading order in $\varepsilon_K, \varepsilon_{K3}$ as:

$$P_L^{-1} = \begin{pmatrix} \sqrt{k_0^L} + \frac{\varepsilon_K^4 k_1^L}{2\sqrt{k_0^L}} & \frac{\varepsilon_K^4 k_1^L e^{i\phi_1}}{2\sqrt{k_0^L}} & \frac{\varepsilon_K^4 k_1^L e^{i\phi_2}}{\sqrt{k_0^L} + \sqrt{k_0^L + k_3^L \varepsilon_{K3}^2}} \\ \frac{\varepsilon_K^4 k_1^L e^{-i\phi_1}}{2\sqrt{k_0^L}} & \sqrt{k_0^L} + \frac{\varepsilon_K^2 k_2^L}{2\sqrt{k_0^L}} & \frac{\varepsilon_K^2 k_2^L e^{i\phi_3}}{\sqrt{k_0^L} + \sqrt{k_0^L + k_3^L \varepsilon_{K3}^2}} \\ \frac{\varepsilon_K^4 k_1^L e^{-i\phi_2}}{\sqrt{k_0^L} + \sqrt{k_0^L + k_3^L \varepsilon_{K3}^2}} & \frac{\varepsilon_K^2 k_2^L e^{-i\phi_3}}{\sqrt{k_0^L} + \sqrt{k_0^L + k_3^L \varepsilon_{K3}^2}} & \sqrt{k_0^L} + k_3^L \varepsilon_{K3}^2 \end{pmatrix} + \dots \quad (5.3)$$

Notice that due to the relatively large $\varepsilon_{3K} \sim 0.5$, the naive factorisation $P = \frac{1}{\sqrt{k_0}}(\mathbb{1} + \Delta P)$ (with $|\Delta P| = -\frac{1}{2}\Delta K_L \ll \mathbb{1}$ for $K_L \equiv k_0^L(\mathbb{1} + \Delta K_L)$) is violated in the third family due to higher power ε_{K3} -effects.

Charged lepton sector: inspecting the charged lepton Yukawa matrices in this class of models (c.f. [11]) before and after canonical normalisation, it can be seen that the charged lepton mixing angles themselves as well as the CN corrections are small. Therefore we can still treat the charged lepton mixing angles as only (CKM-like) small corrections to the neutrino sector dominated U_{PMNS} and we shall (first) focus on the neutrino sector.

Neutrino sector: in the class of models under consideration, the Majorana mass matrix $\hat{M}_M$ originates from operators which involve factors like $f^{ci} f^{cj} (\phi_{23})_i (\phi_{23})_j$ and $f^{ci} f^{cj} (\phi_{123})_i (\phi_{123})_j$. The relevant matrix structures read [6]:

$$\hat{Y}^\nu = \begin{pmatrix} 0 & B & C_1 \\ A & B e^{i\phi_1} + A e^{i\phi_1} & C_2 \\ A e^{i\phi_3} & B e^{i\phi_2} + A e^{i(\phi_1+\phi_3)} & C_3 \end{pmatrix}, \quad \hat{M}_M = \begin{pmatrix} M_A & M_A e^{i\phi_1} & 0 \\ M_A e^{i\phi_1} & M_A e^{2i\phi_1} + M_B & 0 \\ 0 & 0 & M_C \end{pmatrix}, \quad (5.4)$$

where the real positive entries in $\hat{M}_M$ satisfy $M_A < M_B < M_C$. In terms of the expansion parameters [11], the neutrino Yukawa matrix is given by

$$\hat{Y}^\nu = \begin{pmatrix} 0 & \varepsilon^3 y_1 & \varepsilon^3 y_1 e^{i\phi_3} \\ \varepsilon^3 y_2 & \varepsilon^3 (y_1 e^{i\phi_1} + y_2 e^{i\phi_1}) & \varepsilon^3 (y_1 e^{i(\phi_1+\phi_3)} + y_2 e^{i\phi_2}) \\ \varepsilon^3 y_2 e^{i\phi_3} & \varepsilon^3 (y_1 e^{i\phi_2} + y_2 e^{i(\phi_1+\phi_3)}) & y_3 \varepsilon_3^2 \end{pmatrix} + \dots, \quad (5.5)$$

which matches eq. (5.4) with $A = y_2^\nu \varepsilon^3$ and $B = y_1^\nu \varepsilon^3$, $\varepsilon \sim 0.05$. The CN transformation (2.5) then yields (since P_{ν^c} drops off the seesaw formula for the light Majorana neutrinos, we are free to choose for simplicity $P_{\nu^c} = \mathbb{1}$) $M_M = \hat{M}_M$ and:

$$Y^\nu = \begin{pmatrix} \mathcal{O}(\varepsilon^7) & \frac{\varepsilon^3 y_1}{\sqrt{k_0^L}} & \frac{\varepsilon^3 y_1 e^{i\phi_3}}{\sqrt{k_0^L}} \\ \frac{\varepsilon^3 y_2}{\sqrt{k_0^L}} & \frac{\varepsilon^3 (y_1 e^{i\phi_1} + y_2 e^{i\phi_1})}{\sqrt{k_0^L}} & \mathcal{O}(\varepsilon^2) \\ \frac{\varepsilon^3 y_2 e^{i\phi_3}}{\sqrt{k_0^L} \sqrt{1 + \varepsilon_{K3}^2 \frac{k_3^L}{k_0^L}}} & \frac{\varepsilon^3 (y_1 e^{i\phi_2} + y_2 e^{i(\phi_1+\phi_3)})}{\sqrt{k_0^L} \sqrt{1 + \varepsilon_{K3}^2 \frac{k_3^L}{k_0^L}}} & \frac{y_3 \varepsilon_3^2}{\sqrt{k_0^L} \sqrt{1 + \varepsilon_{K3}^2 \frac{k_3^L}{k_0^L}}} \end{pmatrix} + \dots \quad (5.6)$$

¹⁴Recall that relation (2.3) fixes the P -matrices only up to a global unitary transformation; as before we adopt the convention $\tilde{U}_L = U_L$ so that P 's are Hermitean, c.f. section 2.1.

In order to extract the mixing angles analytically from these matrices, it is convenient to transform Y^ν and M_M by means of a suitable non-singular matrix S [21, 11]:

$$Y^\nu \rightarrow Y'^\nu = Y^\nu S^{-1}, \quad M \rightarrow M' = S^{T-1} M S^{-1}, \quad M^{-1} \rightarrow M'^{-1} = S M^{-1} S^T, \quad (5.7)$$

(which again leaves the neutrino mass matrix invariant) to the case of a diagonal $M_M = \text{diag}(M_A, M_B, M_C)$ which corresponds to:

$$Y'^\nu = \begin{pmatrix} \mathcal{O}(\varepsilon^7) & \frac{\varepsilon^3 y_1}{\sqrt{k_0^L}} & \frac{\varepsilon^3 y_1 e^{i\phi_3}}{\sqrt{k_0^L}} \\ \frac{\varepsilon^3 y_2}{\sqrt{k_0^L}} & \frac{\varepsilon^3 y_1 e^{i\phi_1}}{\sqrt{k_0^L}} & \mathcal{O}(\varepsilon^2) \\ \frac{\varepsilon^3 y_2 e^{i\phi_3}}{\sqrt{k_0^L}} & \frac{\varepsilon^3 y_1 e^{i\phi_2}}{\sqrt{k_0^L}} & \frac{y_3 \varepsilon_3^2}{\sqrt{k_0^L}} \end{pmatrix} + \dots \quad (5.8)$$

Formulae for the corrected neutrino mixing angles: since the last transformation brought the neutrino Yukawa and Majorana matrices into a particular form along the lines of the Sequential Dominance setting [23], from eq. (5.8) we can directly read off the mixing angles (imposing $\phi_2 - \phi_1 = \phi_3 - \pi$) at leading order in m_2/m_3 (making use of the generic formulae given in [23]):

$$\tan \theta_{23}^\nu \approx \sqrt{1 + \eta^K}, \quad \tan \theta_{12}^\nu \approx \frac{1}{c_{23}^\nu + \frac{s_{23}^\nu}{\sqrt{1 + \eta^K}}}, \quad \theta_{13}^\nu \approx \frac{m_2}{m_3} \frac{(s_{12}^\nu)^2 |\eta^K|}{\sqrt{(1 + \eta^K)(2 + \eta^K)}}.$$

with $\eta^K \equiv k_3^L \varepsilon_{K3}^2 / k_0^L$. The η^K -behaviour of these relations is illustrated in figure 2. We see that this independent calculation confirms the findings of the previous sections for small η^K . To give a quantitative example we may take $\varepsilon_{K3} = 0.5$ and set the $\mathcal{O}(1)$ coefficients k_3^L and k_0^L to 1 yielding $\eta^K = 0.25$ and $\theta_{23}^\nu = 50.8^\circ$, $\theta_{12}^\nu = 38.3^\circ$ and $\theta_{13}^\nu = 1.1^\circ$, compared to the tri-bimaximal neutrino mixing predictions $\hat{\theta}_{23}^\nu = 45^\circ$, $\hat{\theta}_{12}^\nu = 35.26^\circ$ and $\hat{\theta}_{13}^\nu = 0^\circ$ before canonical normalisation.

5.2 Heavy $SU(2)_L$ doublet messengers & natural η^K suppression

Since the naive estimate of the CN effects above leads to non-negligible deviations from the TB-mixing in the lepton sector (in particular for relatively large $|\eta^K|$), let us sketch in brief the prospects of getting k_3^L/k_0^L (and thus η^K) naturally suppressed in the class of popular $SU(3)$ and $SO(3)$ flavour models.

Recall first that the k_0^L coefficient governs the “canonical”, i.e. renormalisable contribution in the Kähler $\propto \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L}$ (for scalars) while k_3^L emerges at higher order via operators like $\frac{1}{M_K^2} \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L} \phi_3^\dagger \phi_3$ only and therefore is sensitive to the relevant messenger sector masses. Second, due to the self-conjugated structure of this type of operators, any messenger ψ, ψ^c relevant for the Yukawa sector operators, i.e. with simultaneous couplings to flavon and matter superfields (like e.g. $\hat{L} \hat{\phi} \hat{\psi}^c$) necessarily enters the matter sector Kähler metric via effective operators of the form $\frac{1}{M_X^2} \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L} \phi^\dagger \phi$ because no symmetry forbids such structures, c.f. figure 3.

Since $SU(2)_L$ must remain intact upon flavour symmetry breaking, the messengers potentially affecting k_3^L must necessarily be $SU(2)_L$ -doublets, otherwise they can not couple

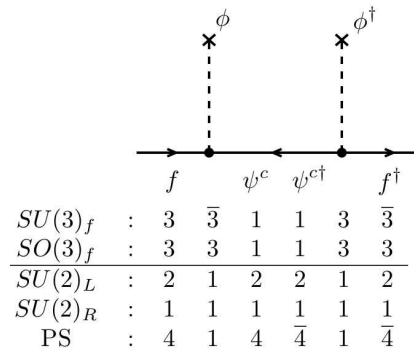


Figure 3: A typical tree level correction to the Kähler of LH fermions generating the effective k_i^L couplings driving the canonical normalisation corrections to the tri-bimaximal lepton mixing pattern.

to $\hat{L}\hat{\phi}$. In what follows, we shall namely check whether the $SU(2)_L$ -doublet part of the messenger sector (if any) in the popular models can be naturally made heavy compared to the $SU(2)_L$ -singlet messenger fields (transforming as $SU(2)_R$ -doublet in the PS approach).

Models with $SU(3)$ family symmetry. Starting with models based on $SU(3)$ family symmetry (or its discrete subgroups like Δ_{27}), the triplet nature of both matter chiralities f, f^c calls for a pair of antitriplet flavon insertions (up to the singular case of $\mathbf{3}_f \cdot \mathbf{3}_{f^c} \cdot \mathbf{3}_\phi$ -type contractions) so that the simplest Yukawa couplings have the internal structure depicted at figure 4 (for discussions of the messenger sector of $SU(3)$ models, see e.g. [24]).

The usual strategy in order to keep the particle content of a model minimal is to exploit just some of these topologies for all the Yukawa sector entries. Typically, the first alternative is chosen, because in such a case the spectra of the χ -type of messengers are sensitive to the large scale $SU(2)_R$ breaking providing for a bit more freedom in the Yukawa sector construction. This actually works rather well for all but the 33 Yukawa entries, that are preferred close to each other, at odds with the scaling properties of the other Yukawa entries (driven by expansion factors ε^2 or $\bar{\varepsilon}^2$ with $\varepsilon \sim 0.05$ and $\bar{\varepsilon} \sim 0.15$ for the up- and down-type sector respectively) and thus calling for extra contributions.

Such terms can then come from either an extra ϕ_3 -type flavon entering the graph of the same type (i.e. case 1 in figure 4) which has been exploited e.g. in the $SU(3)$ model by Varzielas-Ross [6] by means of the particular $SU(2)_R$ -structure of $\phi_3 = 1 \oplus 3$ (c.f. figure 5), or from a more complicated messenger sector with a left-handed “ ψ -type” messengers admitting the other (case 2,3 in figure 4) contributions to the 33 Yukawa coupling. However, with the latter choice, a relatively light “left-handed” messenger must be postulated, leading to the instability of the tri-bimaximal lepton sector mixing generated by a potentially large deviation from universality in the K_L part of the Kähler metric.

Thus, in order to avoid the potentially dangerous light $SU(2)_L$ -doublet messengers ψ one should base the effective Yukawa sector on the topologies of type 1 in figure 4 that, however, comes for the price of extending the third-family flavon sector along the lines of [6] and further complication in the vacuum alignment mechanism.

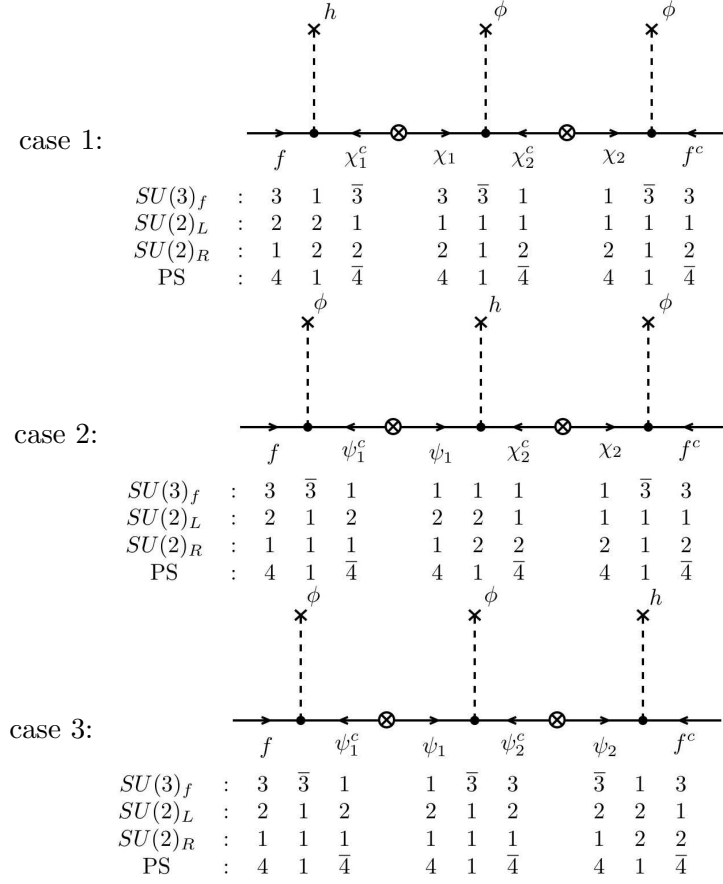


Figure 4: The three basic configurations of the messenger sector leading to the lowest order Yukawa sector effective operators in a typical $SU(3)$ flavour model. The position the SM Higgs VEV enters determines the $SU(2)_L \otimes SU(2)_R$ quantum numbers of the underlying messenger sector. We have used ψ_i for the $SU(2)_L$ doublets while χ_i for the $SU(2)_R$ doublets respectively.

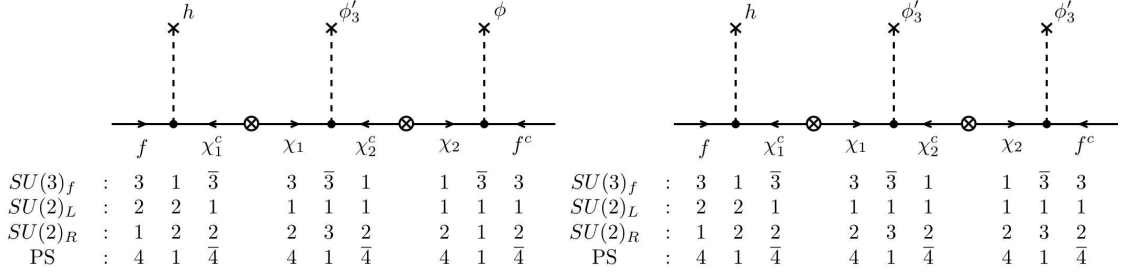


Figure 5: Generating a pair of extra contributions to the 33 Yukawa entries by means of an extra $SU(2)_R$ triplet of ϕ_3 -type flavon fields ϕ'_3 . Notice that the messenger sector is $SU(2)_L$ -singlet and thus does not trigger a potentially large violation of tri-bimaximal mixing in the lepton sector.

Models with $SO(3)$ family symmetry. The situation in models based on $SO(3)$ is slightly simplified by the fact that the basic nontrivial singlet structure can be built out of two rather than three triplets without complex conjugation. Thus, in order to get realistic

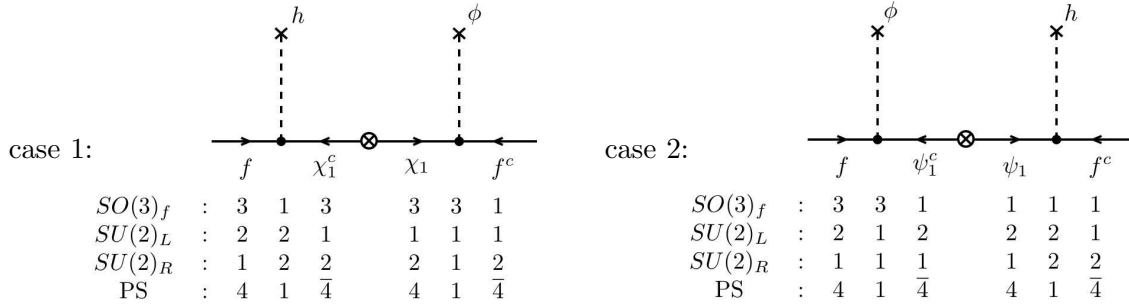


Figure 6: The two basic configurations of the messenger sector leading to the lowest order Yukawa sector effective operators in a typical $SU(3)$ flavour model. The position of the SM Higgs VEV determines the $SU(2)_L \otimes SU(2)_R$ quantum numbers of the underlying messenger fields. As before, we have used ψ_i for the $SU(2)_L$ doublets while χ_i for the $SU(2)_R$ doublets respectively. Note that the ψ_1 messenger in case 2 is usually “flavon-specific” and it is possible to forbid all the unwanted ϕ_{123}, ϕ_{23} type of insertions by just a proper choice of the messenger sector quantum numbers.

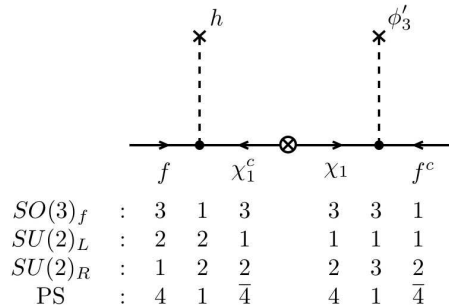


Figure 7: Extra ϕ_3 -type flavon ϕ_3' as a solution to the Yukawa 33 entry irregularity without the unwanted LH Kähler correction effects in an $SO(3)$ flavour model.

Yukawa patterns, only one chiral component (typically f) should transform as a triplet while the other as an $SO(3)$ -singlet [7, 29, 30].¹⁵ At the lowest level (in number of flavon insertions), we are left with only two basic options depicted in figure 6.

Again, one can utilise the right-handed messengers (i.e. doublets of $SU(2)_R$) to obtain most of the desired Yukawa structures, however, the above mentioned “irregularity” in the 33 entries calls for an extra contribution as in the $SU(3)$ case. Again, the basic options are either adding a left-handed (i.e. $SU(2)_L$ doublet) messenger sector fields along case 2 indicated at figure 6, c.f. [7], with potential impact on the left-handed Kähler corrections, or employ an extra ϕ_3 -type flavon, c.f. figure 7.

To conclude, the popular $SU(3)_f$ -based flavour models à la Ross and Varzielas [6, 19] do not in general suffer from large Kähler corrections to the lepton sector tri-bimaximal mixing pattern due to the mere absence of the potentially dangerous $SU(2)_L$ doublets in the messenger sector. On the other hand, the $SO(3)$ -class of models in versions [7] can lead

¹⁵This, however, leads to problems with universality of the right-handed soft masses in SUSY, that in potentially realistic setups must be addressed by further assumptions.

to substantial Kähler corrections because of employing a relatively light $SU(2)_L$ doublet in the messenger sector to resolve the 33 Yukawa issue. However, these models can be cured easily by invoking instead the “extra ϕ_3 -type flavon solution” with only $SU(2)_R$ -doublet light messengers entering the Kähler metric along the lines sketched above.

6. Comparison to other corrections to fermion mixings

In this section, we would like to set the CN corrections in context to other corrections to fermion mixing, focusing on the corrections in the lepton sector. We will first consider effects from renormalisation group (RG) running and then from charged lepton mixing contributions.

6.1 Renormalisation group corrections to tri-bimaximal neutrino mixing

The predictions for the Yukawa matrices arise at the scale of flavour symmetry breaking Λ_F , which we will assume to be close to the GUT scale (M_{GUT}). In order to test such predictions experimentally, the renormalisation group (RG) running between Λ_F and the electroweak scale M_Z has to be taken into account. In particular, if tri-bimaximal mixing is realised in the neutrino sector, deviations from this pattern are induced by RG running. The accurate calculation of such corrections requires evolving the effective neutrino mass matrix from $\Lambda_F \approx M_{\text{GUT}}$ to low energy using the β -functions for the energy ranges above and between the see-saw scales and below the mass scale of the lightest right-handed neutrino [25–27]. Numerically, this can be done conveniently using the software package REAP [28].

In what follows, we shall be interested mainly in estimating the size of the RG corrections in the case of a hierarchical neutrino spectrum in the MSSM, for which the running effects are comparatively small and where the leading logarithmic approximation works reasonably well. Note that due to the non-renormalisation theorem, only the radiative wave-function corrections contribute to the β -functions in supersymmetric theories and the RG corrections can be treated in a very similar fashion to the canonical normalisation corrections, as we will see explicitly below.

Following the spirit of section 4, let us consider the case when the wave-function renormalisation due to the 3rd family dominates. More explicitly, we will assume that the 33-elements govern both Y^e and Y^ν in the model basis (with diagonal M_M). This is the case, for instance, in the classes of non-Abelian flavour models discussed in [11] (and in the example given in section 6.4). Therefore, we will take: $Y^e \approx \text{diag}(0, 0, y_\tau)$ and $Y^\nu \approx \text{diag}(0, 0, y_{\nu_3})$. Above the mass threshold of the heaviest RH neutrino M_3 , the β -function for the effective neutrino mass matrix $m_\nu(\mu) = -v_u Y^\nu(\mu) M_M^{-1}(\mu) Y^{\nu T}(\mu)$ (where μ is the renormalisation scale and v_u is the VEV of the up-type Higgs doublet) reads:

$$16\pi^2 \mu \frac{d}{d\mu} m_\nu = \left(Y^e Y^{e\dagger} + Y^\nu Y^{\nu\dagger} \right) m_\nu + m_\nu \left(Y^{e*} Y^{eT} + Y^{\nu*} Y^{\nu T} \right) + \left(-\frac{6}{5} g_1^2 - 6g_2^2 + 2\text{Tr} Y^{\nu\dagger} Y^\nu + 6\text{Tr} Y^{u\dagger} Y^u \right) m_\nu, \quad (6.1)$$

where the last term is proportional to the unit matrix in flavour space. Below M_3 , the same β -function applies with $Y^\nu = 0$.

Keeping at leading order all terms (but m_ν) on the r.h.s. of (6.1) constant, one can integrate (6.1) analytically, yielding

$$\begin{aligned} m_\nu(M_Z) = & m_\nu(M_{\text{GUT}}) - m_\nu(M_{\text{GUT}}) \frac{1}{16\pi^2} \left[2\text{Tr } Y^{\nu\dagger} Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} \right. \\ & + \left(-\frac{6}{5}g_1^2 - 6g_2^2 + 6\text{Tr } Y^{u\dagger} Y^u \right) \ln \frac{M_{\text{GUT}}}{M_Z} \Big] \\ & - m_\nu(M_{\text{GUT}}) \frac{1}{16\pi^2} \left(Y^{e*} Y^{eT} \ln \frac{M_{\text{GUT}}}{M_Z} + Y^{\nu*} Y^{\nu T} \ln \frac{M_{\text{GUT}}}{M_3} \right) \\ & - \frac{1}{16\pi^2} \left(Y^{e\dagger} Y^e \ln \frac{M_{\text{GUT}}}{M_Z} + Y^{\nu\dagger} Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} \right) m_\nu(M_{\text{GUT}}) + \dots, \end{aligned}$$

which can be rewritten as (forgetting about the doubly-suppressed mixed terms):

$$m_\nu(M_Z) \approx P_{\text{RG}}^T m_\nu(M_{\text{GUT}}) P_{\text{RG}} \quad \text{with} \quad P_{\text{RG}} = r \mathbb{1} + \Delta P_{\text{RG}} + \dots, \quad (6.2)$$

where

$$\begin{aligned} r = & 1 - \frac{1}{16\pi^2} \left[\text{Tr } Y^{\nu\dagger} Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} + 3 \left(-\frac{1}{5}g_1^2 - g_2^2 + \text{Tr } Y^{u\dagger} Y^u \right) \ln \frac{M_{\text{GUT}}}{M_Z} \right], \\ \Delta P_{\text{RG}} = & -\frac{1}{16\pi^2} \left[Y^{e*} Y^{eT} \ln \frac{M_{\text{GUT}}}{M_Z} + Y^{\nu*} Y^{\nu T} \ln \frac{M_{\text{GUT}}}{M_3} \right]. \end{aligned} \quad (6.3)$$

Note that the r -factor in (6.2) is irrelevant for the lepton mixing, because at the leading order one can rewrite P_{RG} in the form

$$P_{\text{RG}} = r (\mathbb{1} + \Delta P_{\text{RG}}) + \mathcal{O} \left[\frac{1}{(16\pi^2)^2} \right] \text{ terms}, \quad (6.4)$$

but overall factors like r drop in formula (2.15).

As we mentioned, the leading order RG effect (6.2) has exactly the form of eq. (2.5), so both types of corrections, from RG running, as well as from canonical normalisation, can be treated on the same footing in this approximation. Furthermore, using eq. (6.1) and comparing eq. (6.2) with eq. (4.3), we find that there is again a single parameter governing the RG corrections to all the mixing angles given by:

$$\eta^{\text{RG}} = \frac{y_\tau^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_Z} + \frac{y_{\nu_3}^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_3}. \quad (6.5)$$

The quantitative predictions of the RG running effects can then be obtained from the relevant formulae for the CN corrections (4.8), (4.10), (4.12), (6.18), (6.20), upon swapping $\eta^K \leftrightarrow \eta^{\text{RG}}$. The last contribution (6.5) would be absent if $M_3 > M_{\text{GUT}}$.

We have cross-checked these results with the analytic approximations presented in [26] and found a perfect agreement for the considered case. In summary, with tri-bimaximal neutrino mixing at the GUT scale, the low scale parameters are given approximately by:

$$s_{12}^\nu(M_Z) = \frac{1}{\sqrt{3}} \left(1 + \frac{\eta^{\text{RG}}}{6} \right), \quad s_{23}^\nu(M_Z) = \frac{1}{\sqrt{2}} \left(1 + \frac{\eta^{\text{RG}}}{4} \right), \quad s_{13}^\nu(M_Z) \propto \eta^{\text{RG}} \frac{m_2}{m_3}. \quad (6.6)$$

6.2 Combined treatment of RG and canonical normalisation corrections

Finally, one can even subsume the effects of 3rd family dominated RG and CN corrections to tri-bimaximal mixing into a single physical parameter:

$$\eta = \eta^{\text{RG}} + \eta^{\text{K}} , \quad (6.7)$$

where η^{RG} is defined in (6.5) and η^{K} is given in section 4, c.f. eq. (4.2). In the following, we will apply this combined treatment to discuss CN and RG effects in the presence of charged lepton mixing corrections to tri-bimaximal mixing. Note that while the size of the RG effects depends mainly on $\tan \beta$ (which governs the size of y_τ) and on the $M_3 - M_{\text{GUT}}$ hierarchy, the size of the canonical normalisation corrections depends on the messenger sector as discussed in section 5.2.

6.3 Charged lepton mixing corrections to tri-bimaximal neutrino mixing

Assume that (in the basis in which $\hat{V}_L^{\nu\dagger} = U_{\text{TB}}$) there is a finite contribution to the lepton mixing matrix coming from the charged lepton sector, as it is actually common to many potentially realistic models of flavour employing unified gauge symmetries like Pati-Salam [31] or SO(10) [32]. The charged lepton sector mixing in such cases tends to copy the structure of $\hat{V}_L^d$ (up to Clebsch factors) that leads to a natural assumption about the structure of the $\hat{V}_L^l$ matrix (before the effects of canonical normalisation are taken into account):

$$\hat{V}_L^l \approx \begin{pmatrix} \hat{c}_{12}^l & \hat{s}_{12}^l e^{-i\hat{\rho}} & 0 \\ -\hat{s}_{12}^l e^{i\hat{\rho}} & \hat{c}_{12}^l & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (6.8)$$

where $\hat{s}_{12}^l$ is a small Cabibbo-like mixing (typically $\hat{s}_{12}^l \approx \lambda/3$) and $\hat{\rho}$ is a generic phase. In such a case, the exact tri-bimaximal structure of the (high-scale) lepton mixing matrix is lifted and one is left with (assuming as before $\hat{V}_L^{\nu\dagger} = U_{\text{TB}}$):

$$\hat{U}_{\text{PMNS}} = \hat{V}_L^l \hat{V}_L^{\nu\dagger} = \hat{V}_L^l U_{\text{TB}} , \quad (6.9)$$

up to a rephasing to the standard PDG form [18], which is needed due to the extra phase in (6.8). The charged lepton sector contribution (6.8) has multiple effects, in particular breaks the direct link between the “measured” (up to the renormalisation group running [26]) values of the lepton mixing parameters and the underlying purely neutrino sector rotations. However, due to the particular structure of $\hat{V}_L^l$ above (leading to just a mild alteration of the tri-bimaximal neutrino mixing pattern), a set of simple relations between the underlying neutrino and charged lepton sector mixings and $\hat{U}_{\text{PMNS}}$ can be obtained.

In particular, the original zero reactor angle is lifted by the 12 rotation in the charged lepton sector to:

$$(\hat{U}_{\text{PMNS}})_{13} = -\hat{s}_{23}^\nu \hat{s}_{12}^l e^{-i\hat{\rho}} \quad \Rightarrow \quad \hat{\theta}_{13} \approx \hat{s}_{23}^\nu \hat{s}_{12}^l = \frac{1}{\sqrt{2}} \hat{s}_{12}^l \quad (6.10)$$

(no phases enter because we are looking at a magnitude of the 13 term only), which in Georgi-Jarlskog type of unified models [33] (where $\hat{s}_{12}^l = \theta_C/3$ with θ_C denoting the quark-sector Cabibbo mixing) yields:

$$\hat{\theta}_{13} \approx \frac{\theta_C}{3\sqrt{2}}. \quad (6.11)$$

Second, there is an interesting phenomenologically testable sum-rule for the deviation of the solar angle from its exactly tri-bimaximal value $\theta_{\text{TB}} = 35^\circ 16'$ in the form [2]

$$\hat{\theta}_{12} = \theta_{12}^{\text{TB}} + \hat{\theta}_{13} \cos \hat{\delta}, \quad \text{i.e.} \quad \hat{s} = \hat{r} \cos \hat{\delta}, \quad (6.12)$$

where $\hat{\delta}$ stands for the Dirac CP phase in the lepton sector.¹⁶ An interested reader can find the derivation of formulae (6.10) and (6.12) in appendix B.

6.4 Canonical normalisation corrections to lepton mixing sum-rules

In view of results of section 4.2, let us discuss the stability of these formulae with respect to the effects of canonical normalisation. We shall again assume the (leading order) 33-sector non-universality in the corresponding Kähler metric (4.2). Remarkably enough, though $\hat{V}_L^l$ is nontrivial, the block-structure of K_f is such that $\hat{V}_L^l$ plays essentially no role in the leading-order formula (2.14) and one recovers (4.6) as in the simplest case discussed in the previous section. The canonical normalisation corrections to the lepton mixing matrix then obey:

$$\Delta U_{\text{PMNS}} = -i\hat{U}_{\text{PMNS}}\Delta W_L^{\nu\dagger}, \quad (6.13)$$

where $\hat{U}_{\text{PMNS}}$ is not equal to U_{TB} as in the simplest case, but $\hat{U}_{\text{PMNS}} = \hat{V}_L^l U_{\text{TB}}$.

Corrections to the charged-lepton-sector-induced 13 mixing: let us look first at the CN corrections induced in the simpler formula (6.10). There is no a-priori reason the 13 entry of U_{PMNS} should vanish as it was the case at the leading order in the purely tri-bimaximal setting (4.10). Indeed, we have:

$$\Delta(U_{\text{PMNS}})_{13} = -i(\hat{U}_{\text{PMNS}})_{11}(\Delta W_L^{\nu\dagger})_{13} - i(\hat{U}_{\text{PMNS}})_{12}(\Delta W_L^{\nu\dagger})_{23}, \quad (6.14)$$

giving at the leading order:

$$\begin{aligned} -i \sum_{i=1,2} (\hat{U}_{\text{PMNS}})_{1i} (\Delta W_L^{\nu})_{i3} &= \sum_i \sum_j (\hat{V}_L^l)_{1j} (U_{\text{TB}})_{ji} \left(U_{\text{TB}}^\dagger \Delta P_L^T U_{\text{TB}} \right)_{i3} \\ &\quad - \sum_j (\hat{V}_L^l)_{1j} (U_{\text{TB}})_{j3} \left(U_{\text{TB}}^\dagger \Delta P_L^T U_{\text{TB}} \right)_{33}. \end{aligned} \quad (6.15)$$

Due to unitarity and the shape (4.3) of ΔP_L , the first term on the r.h.s. of (6.15) is zero, while the latter yields:

$$\Delta(U_{\text{PMNS}})_{13} \approx (\hat{V}_L^l)_{12} (U_{\text{TB}})_{23} \frac{\eta^K}{4} = -\hat{s}_{12}^l e^{-i\hat{\rho}} \frac{\eta^K}{4\sqrt{2}}. \quad (6.16)$$

¹⁶This sum-rule can be easily derived (c.f. appendix B) from the *magnitude* of the 31 entry of $\hat{U}_{\text{PMNS}} = \hat{V}_L^l U_{\text{TB}}$ and thus is insensitive to the Majorana phases.

Notice that the Majorana phase structure of this correction is again the same like the phase structure of the defining basis 13 entry¹⁷ in (6.10) and thus the Dirac CP phase is stable under CN effects. Taking into account also the subleading correction $\propto \sqrt{\Delta m_\odot^2 / \Delta m_A^2}$ (which is of the same order as the term in eq. (6.16)) of the type (4.8), the last formula is extended to:

$$\Delta(U_{\text{PMNS}})_{13} \approx -\frac{\eta^K}{4\sqrt{2}} \left(e^{-i\hat{\rho}} \hat{s}_{12}^l + \hat{c}_{12}^l \frac{4}{3} e^{i\frac{\hat{\alpha}_2}{2}} \sqrt{\frac{\Delta m_\odot^2}{\Delta m_A^2}} \right), \quad (6.17)$$

where, as before, the α_2 phase accounts for the extra phase ambiguity due to the Majorana nature of the neutrinos, c.f. discussion of formula (4.8). In order to deduce the Kähler correction to the ‘induced’ 13 mixing (6.10), this result should be added to the r.h.s. of formula (6.10) leading to:

$$s_{13} \approx \left| \hat{s}_{12}^l \frac{1}{\sqrt{2}} \left(1 + \frac{\eta^K}{4} \right) - \hat{c}_{12}^l \frac{\eta^K}{3\sqrt{2}} e^{i\left(\frac{\hat{\alpha}_2}{2} + \hat{\rho}\right)} \sqrt{\frac{\Delta m_\odot^2}{\Delta m_A^2}} \right| \quad (6.18)$$

for the physical 13 mixing in U_{PMNS} . Notice that there is a slight ambiguity due to the phase factor in the second term, that can not be neglected with respect to the η^K -part of the first term therein. However, the smallness of the ‘charged-lepton-sector-induced’ reactor mixing angle $s_{13} \approx \frac{1}{\sqrt{2}} \hat{s}_{12}^l$ (which is typically $\frac{\theta_C}{3\sqrt{2}}$ for Georgi-Jarlskog type of flavour models corresponding to the first term above, c.f. formula (6.11) and the discussion around) is not disturbed by the effects of canonical normalisation.

Canonical normalisation corrections to $\hat{s} = \hat{r} \cos \hat{\delta}$: with the information about the θ_{13} stability at hand, one can infer the leading additive correction to the defining basis formula (6.12), that (swapping all the defining basis quantities for their physical counterparts) should read:

$$\theta_{12}^{\text{TB}} = \theta_{12} - \theta_{13} \cos \delta + f(\eta^K), \quad (6.19)$$

where $f(\eta^K)$ is a linear function of η^K vanishing for $\eta^K \rightarrow 0$, i.e. $f(\eta^K) = c\eta^K$ with a real proportionality factor c . As we have seen in the previous paragraph, the leading CN correction to the 13-mixing (6.18) is only multiplicative (6.18) and thus all the would-be corrections in (6.19) due to the η^K -sensitivity in θ_{13} or $\cos \delta$ are suppressed by θ_{13} . This means that in the $\theta_{13} \rightarrow 0$ limit in (6.19), one should recover the simple leading order θ_{12} scaling obtained in section 4.2. Thus, one gets $c = -\frac{1}{6\sqrt{2}}$, which gives at the leading order:

$$\theta_{12}^{\text{TB}} = \theta_{12} - \theta_{13} \cos \delta - \frac{\eta^K}{6\sqrt{2}}. \quad (6.20)$$

Formula (6.20) can be finally recast (using $a = \frac{\eta^K}{4}$ derived in section 4.2) into a sum-rule for *measurable quantities* a , s and r and δ only:¹⁸

$$s = r \cos \delta + \frac{2}{3} a. \quad (6.21)$$

¹⁷i.e. zero phase in the given global phase convention fixing the shape of P_M with a real 33 entry.

¹⁸We note that the sum rule of eq. (6.21) can be readily generalised to arbitrary $\hat{\theta}_{12}^\nu$ (but keeping $\hat{\theta}_{13}^\nu = 0^\circ$ and $\hat{\theta}_{23}^\nu = 45^\circ$ fixed) using $s_{12}^\nu = \hat{s}_{12}^\nu (1 + \eta^K (\hat{c}_{12}^\nu)^2 / 4)$ and repeating the derivation of appendix B.

This relation identifies a characteristic imprint of the canonical normalisation effects in the popular scheme where the charged leptons contribute in the Georgi-Jarlskog manner (i.e. only the 12 sector rotation is non-negligible) while the neutrino sector mixing is exactly tri-bimaximal. Note that in addition to the precision measurements required for testing the original sum rule [2], testing equation (6.21) requires an accurate measurement of the deviation from maximal atmospheric mixing [34].

The simple argument above can only be used to fix the shape of the leading order *additive* corrections in (6.19) and derive the main result (6.21), but does not, in general, provide any information on sub-leading corrections (entering either as multiplicative changes in small parameters or higher order effects in η^K) to (6.19). An interested reader can find a more detailed explicit derivation of (6.20) with a brief discussion of the shape of such subleading corrections in the appendix. Remarkably enough, this formula is *stable* also under radiative corrections due to the RG running (see in section 6.1), which makes it directly testable at future experimental facilities.

7. Summary and discussion

In summary, we have re-analysed the effect of canonical normalisation of kinetic terms on the quark and lepton mixing angles. In contrast to previous studies we have found that the effects can lead to significant corrections to the fermion mixing angles. Such potentially large effects are characteristic of flavour models based on non-Abelian family symmetries, where some of the possible Kähler potential (and superpotential) operators, in particular those associated with the third family, are only mildly suppressed. We have investigated under which conditions the messenger sector of such flavour models generates such Kähler potential operators for which canonical normalisation effects are sizeable, and under which conditions these operators may be absent and canonical normalisation effects are small. The quantitative significance of the canonical normalisation effects is clearly model dependent, and in order to address this we have provided a detailed discussion of the messenger sectors responsible for both the Kähler potential and the superpotential corrections in the class of SU(3) and SO(3) flavour models. For example in the SU(3) or Δ_{27} models [6], the left-handed messengers sector essentially decouples from the effective Yukawa couplings and the Kähler metric for the left-chirality matter fields is only subject to small corrections leading to $\eta^K \sim 0$ and thus negligible CN effects. On the other hand, in the SO(3) or A_4 models [7], the left-handed messengers ψ have been assumed to be quite light, in which case the wave-function effects of third family rescaling described in this paper are expected to be large with $\eta^K \sim \mathcal{O}(1)$.

We developed a general perturbative formalism which enables the CN effects in both quark and lepton sectors to be estimated. We then applied this formalism to explicit examples for potentially relevant CN effects. For example, we have discussed the corrections to the CKM matrix element $|V_{cb}|$ as well as corrections to tri-bimaximal neutrino mixing. In the quark sector we found that such canonical normalisation effects could imply a relatively large change in V_{cb} , that (although still only multiplicative) could be much larger than the estimates given previously in the literature where the possibility of a large third family

expansion parameter was not considered. Concerning leptons, we found that the physical effect of canonical normalisation is strongly amplified compared to the quark sector, because of the approximate tri-bimaximality of the solar and atmospheric mixings. On the other hand, the (comparatively) small reactor angle receives only sub-leading corrections.

We have also compared the CN corrections with other relevant corrections to predictions of flavour models. Regarding renormalisation group (RG) corrections in leading logarithmic approximation, we have expressed the effects in a form which allows a perturbative treatment analogous to the one used for our analytical estimates of the CN effects. We have shown how, in the case that third family effects dominate RG and CN corrections, both sorts of corrections can be subsumed into a single universal parameter at leading order. As application we have presented a detailed discussion of such third family effects on the lepton mixing sum rule $s = r \cos \delta$ [2] which emerges as a relation among lepton sector observables if the leading neutrino sector mixing is exactly tri-bimaximal and modified only by small (but relevant) Cabibbo-like charged lepton mixing contributions. In this sum rule s, r describe the deviations of solar and reactor mixing angles from their tri-bimaximal values, and δ is the observable Dirac CP phase in PDG parameterisation [18]. Assuming hierarchical neutrinos and taking into account both, CN and RG third family wave-function effects, we have discussed in detail how the stable version of the sum rule $s = r \cos \delta + \frac{2}{3}a$ [15] is derived (presenting additional details of the derivation, beyond the previous analysis). The additional parameter a in the stable sum rule accounts for the deviation of the atmospheric mixing angle from its (tri-bi)maximal value $\pi/4$ due to the combined third family CN and RG effects.

In conclusion, the main message of this paper is that in certain classes of models canonical normalisation effects, in particular those associated with the third family, may be larger than previously thought, leading to larger corrections to quark and lepton mixing angles than previously realised.

Acknowledgments

We acknowledge partial support from the following grants: PPARC Rolling Grant PPA/G/S/ 2003/00096; EU Network MRTN-CT-2004-503369; EU ILIAS RII3-CT-2004-506222; NATO grant PST.CLG.980066.

A. Conventions - CKM & PMNS mixing matrices

In general, the mixing matrix in the lepton sector, the PMNS matrix U_{PMNS} , is defined as the matrix appearing in the charged electroweak currents expressed in terms of lepton mass eigenstates. Denoting the charged lepton mass matrix by M_l and the light neutrino mass matrix by m_ν , the mass part of the matter sector lagrangian reads:

$$\mathcal{L} = -\bar{L}_L M_l l_R - \frac{1}{2} \bar{\nu}_L m_\nu \nu_L^c + \text{H.c.} . \quad (\text{A.1})$$

Performing the transformation from flavour to mass basis by

$$V_L^l M_l V_R^{l\dagger} = \text{diag}(m_e, m_\mu, m_\tau), \quad V_L^\nu m_\nu V_L^{\nu T} = \text{diag}(m_1, m_2, m_3), \quad (\text{A.2})$$

the PMNS matrix is given by

$$U_{\text{PMNS}} = V_L^l V_L^{\nu\dagger}. \quad (\text{A.3})$$

Here it is assumed implicitly that unphysical phases are removed by field redefinitions, and U_{PMNS} contains one Dirac phase and two Majorana phases.¹⁹

The standard PDG parameterisation of the PMNS matrix (see e.g. [18]) is:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} P_M, \quad (\text{A.4})$$

which is used in most analyses of neutrino oscillation experiments. Here δ is the Dirac CP violating phase which is in principle measurable in neutrino oscillation experiments, and $P_M = \text{diag}(e^{i\frac{\alpha_1}{2}}, e^{i\frac{\alpha_2}{2}}, 1)$ contains the two measurable Majorana phase differences α_1, α_2 . In the body of this manuscript we use this standard parameterisation also for $V_L^{\nu\dagger}$ and denote the corresponding mixing angles by θ_{ij}^ν , while the mixing angles θ_{ij} without superscript refer to the PMNS matrix.

B. Derivation of stable lepton mixing sum-rules

Let us recapitulate here the derivation of the sum-rules of our interest along the lines they were originally obtained in [2]. We shall for the moment forget about the canonical normalisation effects and drop all the hats in what follows. Later on in appendix A, we shall reiterate the same procedure carefully with all the potential sources of deviations due to canonical normalisation taken into account.

Perhaps the simplest method to obtain (6.10) and (6.12) consists in looking at particular elements of the lepton mixing matrix:²⁰

$$U_{\text{PMNS}} = V_L^l V_L^{\nu\dagger} = V_L^l U_{\text{TB}} \quad (\text{B.1})$$

and exploiting the fact that the particular shape (6.8) of V_L^l exposes unaltered the third row of the tri-bimaximal neutrino sector mixing in U_{TB} and also the 13 entry of U_{PMNS} receives a particularly simple form. Indeed, one easily obtains $|(U_{\text{PMNS}})_{3i}| = |(U_{\text{TB}})_{3i}|$ that (upon employing the standard parametrisation (A.4)) gives in particular:

$$\text{31 entry : } \left(s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} \right) e^{i\alpha_1/2} = s_{12}^\nu s_{23}^\nu \quad \text{up to a global phase,} \quad (\text{B.2})$$

and also:

$$\text{13 entry : } s_{13}e^{-i\delta} = -s_{12}^l e^{-i\rho} s_{23}^\nu \quad \text{up to a global phase.} \quad (\text{B.3})$$

¹⁹The latter are physical only in the case of Majorana neutrinos, for Dirac neutrinos the two Majorana phases can be absorbed as well.

²⁰Since we shall be looking on the *magnitude* of the matrix elements, the particular phase convention employed here is immaterial, but clearly must not be altered during the computation.

First, notice that a nonzero θ_{13} mixing is generated from a conspiracy between the 12 charged lepton sector mixing and θ_{23}^ν . In a wide class of models with a built-in Georgi-Jarlskog mechanism (leading typically to $\theta_{12}^l \approx \theta_C/3$ with $\theta_C \approx \lambda \approx 0.2$ denoting the Cabibbo CKM mixing governed by the down-type quark sector) one gets [35] $\theta_{13} \approx \frac{\theta_C}{3\sqrt{2}}$. Second, formula (B.2) subsequently leads to

$$s_{12}s_{23} - c_{12}c_{23}s_{13}\cos\delta = s_{12}^\nu s_{23}^\nu. \quad (\text{B.4})$$

Since the 23 sector mixing is stable under the perturbation (6.8), one can trade s_{23} and c_{23} in (B.4) for their TB values $\frac{1}{\sqrt{2}}$ while the r.h.s. gives $\frac{1}{\sqrt{6}}$. Expanding the left-hand side of (B.4) for small $s_{13} \approx \theta_{13}$ one gets:

$$s_{12} - c_{12}\theta_{13}\cos\delta = \frac{1}{\sqrt{3}}. \quad (\text{B.5})$$

The last step is to expand the physical θ_{12} around the tri-bimaximal value $\theta_{12} = \theta_{12}^{\text{TB}} + \Delta\theta_{12}$ which yields $s_{12} = \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}}\Delta\theta_{12}$ and $c_{12} = \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}}\Delta\theta_{12}$, leading to:

$$\sqrt{\frac{2}{3}}\Delta\theta_{12} - \theta_{13}\left(\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}}\Delta\theta_{12}\right)\cos\delta = 0. \quad (\text{B.6})$$

Forgetting about the doubly-suppressed $\theta_{13}\Delta\theta_{12}$ term on the l.h.s. of (B.6), we get:

$$\Delta\theta_{12} \approx \theta_{13}\cos\delta \quad \text{yielding} \quad \theta_{12} = \theta_{12}^{\text{TB}} + \theta_{13}\cos\delta, \quad (\text{B.7})$$

providing a simple estimate for the deviation of the solar mixing angle θ_{12} from its tri-bimaximal value $\theta_{12}^{\text{TB}} = 35^\circ 16'$ in terms of two other lepton sector measurables, namely the reactor mixing angle θ_{13} and the Dirac CP phase δ .

Corrections to the sum-rule $s = r \cos \delta$: suppose now that the assumptions made above and which lead in particular to formula (B.7) hold at the underlying flavour-model level, i.e. in the defining basis only. Thus, for sake of consistency with the notation used in the body of the manuscript, we shall re-equip all the relevant quantities therein with hats obtaining $\hat{\theta}_{12} = \theta_{12}^{\text{TB}} + \hat{\theta}_{13}\cos\hat{\delta}$ as only the leading order approximation to the physical (i.e. corrected) sum-rule, that should be written in terms of only unhatted quantities. The scope of this section is to see what happens once the effects of RG running and canonical normalisation are turned on.

Along similar lines as in section 6.4 one obtains first (utilising the perturbative procedure of section 2.3 for $\eta = \eta^K + \eta^{\text{RG}}$), c.f. equations (6.14)–(6.17):

$$\Delta(U_{\text{PMNS}})_{31} = \frac{\eta}{2}(U_{\text{TB}})_{31} [1 - |(U_{\text{TB}})_{31}|^2], \quad (\text{B.8})$$

and thus (up to the Majorana phase associated to the $(U_{\text{TB}})_{31}$ entry):

$$\Delta(U_{\text{PMNS}})_{31} = \frac{\eta}{2}\hat{s}_{12}^\nu\hat{s}_{23}^\nu [1 - |\hat{s}_{12}^\nu\hat{s}_{23}^\nu|^2] \equiv \hat{s}_{12}^\nu\hat{s}_{23}^\nu\Delta, \quad (\text{B.9})$$

where $\Delta \equiv \frac{\eta}{2} (1 - |\hat{s}_{12}^\nu \hat{s}_{23}^\nu|^2) = \frac{5}{12}\eta$. Notice that due to the phase structure of the $\Delta(U_{\text{PMNS}})_{31}$ correction (B.8), and in particular the $(U_{\text{TB}})_{31}$ term therein, the overall phase of the r.h.s. of (B.2) derived from (B.1) and the phase of $\Delta(U_{\text{PMNS}})_{31}$ coincide. This admits to write the analogue of relation (B.2) (derived now from $U_{\text{PMNS}} = \hat{U}_{\text{PMNS}} + \Delta U_{\text{PMNS}}$) in a simple form:

$$\left(s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} \right) e^{i\alpha_1/2} = \hat{s}_{12}^\nu \hat{s}_{23}^\nu (1 + \Delta) \quad \text{up to a global phase.} \quad (\text{B.10})$$

The next step (leading to (B.5) in the ‘unperturbed’ case) would be to trade the 23 rotations for their tri-bimaximal values $s_{23} = c_{23} = 1/\sqrt{2}$, that is completely plausible if there were no Kähler or RG corrections around, because the neutrino part of the 23-sector rotation in $U_{\text{PMNS}} = V_L^l V_L^{\nu\dagger}$ formula (provided both V_L^l are written as $U_{23}U_{13}U_{12}$ along the lines of [35]), hits the small charged-lepton correction only (upon being grouped together with 23-rotation in V_L^l), with just a negligible effect on the resulting physical 23 lepton sector mixing. However, turning η on, θ_{23} becomes actually quite η -sensitive even in the simplest case (c.f. section 4.2), and thus putting $s_{23} = c_{23} = 1/\sqrt{2}$ is not good enough.

Rather than that, we shall exploit the information²¹ obtained in section 4.2, see e.g. formula (4.10), to write (at the leading order):

$$s_{23} = \frac{1}{\sqrt{2}}(1 + a) = \frac{1}{\sqrt{2}} \left(1 + \frac{\eta}{4} \right) \quad \text{and thus} \quad c_{23} = \frac{1}{\sqrt{2}}(1 - a) = \frac{1}{\sqrt{2}} \left(1 - \frac{\eta}{4} \right), \quad (\text{B.11})$$

and from (B.10) then (since Δ is real):

$$s_{12} \frac{1}{\sqrt{2}}(1 + a) - c_{12} \frac{1}{\sqrt{2}}(1 - a) \theta_{13} \cos \delta = \frac{1}{\sqrt{6}}(1 + \Delta). \quad (\text{B.12})$$

Expanding again the physical θ_{12} around the tri-bimaximal value $\theta_{12} = \theta_{12}^{\text{TB}} + \Delta\theta_{12}$, i.e. $s_{12} = \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}}\Delta\theta_{12}$ and $c_{12} = \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}}\Delta\theta_{12}$ and neglecting the higher order terms in a , $\Delta\theta_{12}$ and θ_{13} , one receives:

$$\Delta\theta_{12} = \theta_{13} \cos \delta + \frac{1}{\sqrt{2}}(\Delta - a) = \theta_{13} \cos \delta + \frac{\eta}{6\sqrt{2}}, \quad (\text{B.13})$$

which is an analogue of formula (B.7). The sum-rule with the Kähler corrections taken into account then reads:

$$\theta_{12}^{\text{TB}} = \theta_{12} - \theta_{13} \cos \delta - \frac{\eta}{6\sqrt{2}}. \quad (\text{B.14})$$

Notice that in the $\hat{\theta}_{12}^l \rightarrow 0$ limit (causing $\hat{\theta}_{13} \rightarrow 0$ and thus due to (6.18) also $\theta_{13} \rightarrow 0$) one indeed reveals the leading order effect (4.12) in the solar mixing $s_{12} = \frac{1}{\sqrt{2}}(1 + \frac{\eta}{6})$ obtained in section 4.2, that in turn provides a non-trivial consistency check of relation (B.14).

²¹Those results, though being obtained for zero $\hat{\theta}_{12}^l$, provide a good leading order estimate of the atmospheric mixing η -behaviour and since the error due to the nonzero charged-lepton 12-sector mixing (hitting such a corrected 23 mixing) is the same (at the leading order) as in the “pure” case (i.e. without Kähler effects), it can be neglected as far as one looks for the *deviations* from the original sum-rule (B.7).

References

- [1] P.F. Harrison, D.H. Perkins and W.G. Scott, *Tri-bimaximal mixing and the neutrino oscillation data*, *Phys. Lett. B* **530** (2002) 167 [[hep-ph/0202074](#)];
P.F. Harrison and W.G. Scott, *Symmetries and generalisations of tri-bimaximal neutrino mixing*, *Phys. Lett. B* **535** (2002) 163 [[hep-ph/0203209](#)]; *Permutation symmetry, tri-bimaximal neutrino mixing and the S^3 group characters*, *Phys. Lett. B* **557** (2003) 76 [[hep-ph/0302025](#)];
an earlier related ansatz was proposed by L. Wolfenstein, *Oscillations among three neutrino types and CP-violation*, *Phys. Rev. D* **18** (1978) 958.
- [2] S.F. King, *Predicting neutrino parameters from $SO(3)$ family symmetry and quark-lepton unification*, *JHEP* **08** (2005) 105 [[hep-ph/0506297](#)];
I. Masina, *A maximal atmospheric mixing from a maximal CP-violating phase*, *Phys. Lett. B* **633** (2006) 134 [[hep-ph/0508031](#)];
S. Antusch and S.F. King, *Charged lepton corrections to neutrino mixing angles and CP phases revisited*, *Phys. Lett. B* **631** (2005) 42 [[hep-ph/0508044](#)];
S. Antusch, P. Huber, S.F. King and T. Schwetz, *Neutrino mixing sum rules and oscillation experiments*, *JHEP* **04** (2007) 060 [[hep-ph/0702286](#)].
- [3] P.H. Frampton, S.T. Petcov and W. Rodejohann, *On deviations from bimaximal neutrino mixing*, *Nucl. Phys. B* **687** (2004) 31 [[hep-ph/0401206](#)];
A. Dighe, S. Goswami and W. Rodejohann, *Corrections to Tri-bimaximal neutrino mixing: renormalization and Planck scale effects*, *Phys. Rev. D* **75** (2007) 073023 [[hep-ph/0612328](#)];
F. Plentinger and W. Rodejohann, *Deviations from tribimaximal neutrino mixing*, *Phys. Lett. B* **625** (2005) 264 [[hep-ph/0507143](#)];
R.N. Mohapatra and W. Rodejohann, *Broken μ - τ symmetry and leptonic CP-violation*, *Phys. Rev. D* **72** (2005) 053001 [[hep-ph/0507312](#)];
K.A. Hochmuth, S.T. Petcov and W. Rodejohann, $U_{\text{PMNS}} = U_\ell^\dagger U_\nu$, *Phys. Lett. B* **654** (2007) 177 [[arXiv:0706.2975](#)].
- [4] G. Altarelli, F. Feruglio and Y. Lin, *Tri-bimaximal neutrino mixing from orbifolding*, *Nucl. Phys. B* **775** (2007) 31 [[hep-ph/0610165](#)];
G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing, A_4 and the modular symmetry*, *Nucl. Phys. B* **741** (2006) 215 [[hep-ph/0512103](#)]; *Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions*, *Nucl. Phys. B* **720** (2005) 64 [[hep-ph/0504165](#)];
F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, *Tri-bimaximal neutrino mixing and quark masses from a discrete flavour symmetry*, *Nucl. Phys. B* **775** (2007) 120 [[hep-ph/0702194](#)];
C. Luhn, S. Nasri and P. Ramond, *Tri-bimaximal neutrino mixing and the family symmetry $Z_7 \rtimes Z_3$* , *Phys. Lett. B* **652** (2007) 27 [[arXiv:0706.2341](#)].
- [5] E. Ma, *Near tribimaximal neutrino mixing with $\Delta(27)$ symmetry*, *Phys. Lett. B* **660** (2008) 505 [[arXiv:0709.0507](#)]; *New lepton family symmetry and neutrino tribimaximal mixing*, *Europhys. Lett.* **79** (2007) 61001 [[hep-ph/0701016](#)]; *Supersymmetric $A_4 \times Z(3)$ and A_4 realizations of neutrino tribimaximal mixing without and with corrections*, *Mod. Phys. Lett. A* **22** (2007) 101 [[hep-ph/0610342](#)]; *Suitability of A_4 as a family symmetry in grand unification*, *Mod. Phys. Lett. A* **21** (2006) 2931 [[hep-ph/0607190](#)]; *Neutrino mass matrix from $\Delta(27)$ symmetry*, *Mod. Phys. Lett. A* **21** (2006) 1917 [[hep-ph/0607056](#)];
E. Ma, H. Sawanaka and M. Tanimoto, *Quark masses and mixing with A_4 family symmetry*, *Phys. Lett. B* **641** (2006) 301 [[hep-ph/0606103](#)];

- E. Ma, *Tribimaximal neutrino mixing from a supersymmetric model with A_4 family symmetry*, *Phys. Rev. D* **73** (2006) 057304 [[hep-ph/0511133](#)];
- B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M.K. Parida, *A_4 symmetry and prediction of $U(e3)$ in a modified Altarelli-Feruglio model*, *Phys. Lett. B* **638** (2006) 345 [[hep-ph/0603059](#)];
- E. Ma, *Tetrahedral family symmetry and the neutrino mixing matrix*, *Mod. Phys. Lett. A* **20** (2005) 2601 [[hep-ph/0508099](#)]; *Aspects of the tetrahedral neutrino mass matrix*, *Phys. Rev. D* **72** (2005) 037301 [[hep-ph/0505209](#)];
- S.-L. Chen, M. Frigerio and E. Ma, *Hybrid seesaw neutrino masses with A_4 family symmetry*, *Nucl. Phys. B* **724** (2005) 423 [[hep-ph/0504181](#)];
- E. Ma, *A_4 origin of the neutrino mass matrix*, *Phys. Rev. D* **70** (2004) 031901 [[hep-ph/0404199](#)].
- [6] I. de Medeiros Varzielas and G.G. Ross, *$SU(3)$ family symmetry and neutrino bi-tri-maximal mixing*, *Nucl. Phys. B* **733** (2006) 31 [[hep-ph/0507176](#)];
- I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Tri-bimaximal neutrino mixing from discrete subgroups of $SU(3)$ and $SO(3)$ family symmetry*, *Phys. Lett. B* **644** (2007) 153 [[hep-ph/0512313](#)]; *Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry*, *Phys. Lett. B* **648** (2007) 201 [[hep-ph/0607045](#)].
- [7] S.F. King and M. Malinsky, *A_4 family symmetry and quark-lepton unification*, *Phys. Lett. B* **645** (2007) 351 [[hep-ph/0610250](#)]; *Towards a complete theory of fermion masses and mixings with $SO(3)$ family symmetry and 5D $SO(10)$ unification*, *JHEP* **11** (2006) 071 [[hep-ph/0608021](#)].
- [8] P.F. Harrison and W.G. Scott, *Permutation symmetry, tri-bimaximal neutrino mixing and the S^3 group characters*, *Phys. Lett. B* **557** (2003) 76 [[hep-ph/0302025](#)]; *Symmetries and generalisations of tri-bimaximal neutrino mixing*, *Phys. Lett. B* **535** (2002) 163 [[hep-ph/0203209](#)];
- R.N. Mohapatra, S. Nasri and H.-B. Yu, *S^3 symmetry and tri-bimaximal mixing*, *Phys. Lett. B* **639** (2006) 318 [[hep-ph/0605020](#)];
- R.N. Mohapatra and H.-B. Yu, *Connecting leptogenesis to CP-violation in neutrino mixings in a tri-bimaximal mixing model*, *Phys. Lett. B* **644** (2007) 346 [[hep-ph/0610023](#)];
- M.-C. Chen and K.T. Mahanthappa, *CKM and tri-bimaximal MNS matrices in a $SU(5) \times {}^{(d)}T$ model*, *Phys. Lett. B* **652** (2007) 34 [[arXiv:0705.0714](#)];
- C.I. Low and R.R. Volkas, *Tri-bimaximal mixing, discrete family symmetries and a conjecture connecting the quark and lepton mixing matrices*, *Phys. Rev. D* **68** (2003) 033007 [[hep-ph/0305243](#)];
- X.-G. He, *A_4 group and tri-bimaximal neutrino mixing: a renormalizable model*, *Nucl. Phys. B* **168** (Proc. Suppl.) (2007) 350 [[hep-ph/0612080](#)];
- A. Aranda, *Neutrino mixing from the double tetrahedral group T'* , *Phys. Rev. D* **76** (2007) 111301 [[arXiv:0707.3661](#)].
- [9] A.H. Chan, H. Fritzsch, S. Luo and Z.-Z. Xing, *Deviations from tri-bimaximal neutrino mixing in type-II seesaw and leptogenesis*, *Phys. Rev. D* **76** (2007) 073009 [[arXiv:0704.3153](#)];
- Z.-Z. Xing, *Nontrivial correlation between the CKM and MNS matrices*, *Phys. Lett. B* **618** (2005) 141 [[hep-ph/0503200](#)];
- Z.-Z. Xing, H. Zhang and S. Zhou, *Nearly tri-bimaximal neutrino mixing and CP-violation from μ - τ symmetry breaking*, *Phys. Lett. B* **641** (2006) 189 [[hep-ph/0607091](#)];

- S.K. Kang, Z.-Z. Xing and S. Zhou, *Possible deviation from the tri-bimaximal neutrino mixing in a seesaw model*, *Phys. Rev. D* **73** (2006) 013001 [[hep-ph/0511157](#)];
- S. Luo and Z.-Z. Xing, *Generalized tri-bimaximal neutrino mixing and its sensitivity to radiative corrections*, *Phys. Lett. B* **632** (2006) 341 [[hep-ph/0509065](#)];
- M. Hirsch, E. Ma, J.C. Romao, J.W.F. Valle and A. Villanova del Moral, *Minimal supergravity radiative effects on the tri-bimaximal neutrino mixing pattern*, *Phys. Rev. D* **75** (2007) 053006 [[hep-ph/0606082](#)];
- N.N. Singh, M. Rajkhowa and A. Borah, *Deviation from tri-bimaximal mixings in two types of inverted hierarchical neutrino mass models*, *Pramana* **69** (2007) 533 [[hep-ph/0603189](#)];
- X.-G. He and A. Zee, *Minimal modification to the tri-bimaximal neutrino mixing*, *Phys. Lett. B* **645** (2007) 427 [[hep-ph/0607163](#)];
- N. Haba, A. Watanabe and K. Yoshioka, *Twisted flavors and tri/bi-maximal neutrino mixing*, *Phys. Rev. Lett.* **97** (2006) 041601 [[hep-ph/0603116](#)].
- [10] S. Antusch, L.E. Ibáñez and T. Macri, *Neutrino masses and mixings from string theory instantons*, *JHEP* **09** (2007) 087 [[arXiv:0706.2132](#)].
- [11] S. Antusch, S.F. King and M. Malinsky, *Solving the SUSY flavour and CP problems with SU(3) family symmetry*, [arXiv:0708.1282](#).
- [12] M. Leurer, Y. Nir and N. Seiberg, *Mass matrix models: the sequel*, *Nucl. Phys. B* **420** (1994) 468 [[hep-ph/9310320](#)];
- E. Dudas, S. Pokorski and C.A. Savoy, *Yukawa matrices from a spontaneously broken abelian symmetry*, *Phys. Lett. B* **356** (1995) 45 [[hep-ph/9504292](#)]; *Soft scalar masses in supergravity with horizontal U(1)_X gauge symmetry*, *Phys. Lett. B* **369** (1996) 255 [[hep-ph/9509410](#)];
- H.K. Dreiner and M. Thormeier, *Supersymmetric Froggatt-Nielsen models with baryon- and lepton-number violation*, *Phys. Rev. D* **69** (2004) 053002 [[hep-ph/0305270](#)];
- I. Jack, D.R.T. Jones and R. Wild, *Yukawa textures and the μ -term*, *Phys. Lett. B* **580** (2004) 72 [[hep-ph/0309165](#)].
- [13] S.F. King and I.N.R. Peddie, *Canonical normalisation and Yukawa matrices*, *Phys. Lett. B* **586** (2004) 83 [[hep-ph/0312237](#)].
- [14] S.F. King, I.N.R. Peddie, G.G. Ross, L. Velasco-Sevilla and O. Vives, *Kähler corrections and softly broken family symmetries*, *JHEP* **07** (2005) 049 [[hep-ph/0407012](#)].
- [15] S. Antusch, S.F. King and M. Malinsky, *Third family corrections to tri-bimaximal lepton mixing and a new sum rule*, [arXiv:0711.4727](#).
- [16] ISS PHYSICS WORKING GROUP collaboration, A. Bandyopadhyay et al., *Physics at a future neutrino factory and super-beam facility*, [arXiv:0710.4947](#).
- [17] P. Minkowski, $\mu \rightarrow e\gamma$ at a rate of one out of 1-billion muon decays?, *Phys. Lett. B* **67** (1977) 421;
- M. Gell-Mann, P. Ramond and R. Slansky, *Sanibel Talk*, CALT-68-709 (1979); *Complex spinors and unified theories*, in *Supergravity*, North Holland, Amsterdam (1979);
- T. Yanagida in *Horizontal gauge symmetry and masses of neutrinos*, in *Proceedings of the workshop on unified theory and baryon number of the universe*, KEK, Japan (1979);
- S.L. Glashow, *Quarks and leptons*, in *Cargese Lectures* (1979), M. Levy et al. eds., Plenum, New York U.S.A. (1980);
- R.N. Mohapatra and G. Senjanović, *Neutrino mass and spontaneous parity nonconservation*, *Phys. Rev. Lett.* **44** (1980) 912;

- J. Schechter and J.W.F. Valle, *Neutrino decay and spontaneous violation of lepton number*, *Phys. Rev. D* **25** (1982) 774.
- [18] PARTICLE DATA GROUP collaboration, W.M. Yao et al., *Review of particle physics*, *J. Phys. G* **33** (2006) 1.
- [19] I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry*, *Phys. Lett. B* **648** (2007) 201 [[hep-ph/0607045](#)].
- [20] M. Malinsky, *Tackling the SUSY flavour & CP problem — SUGRA versus SU(3)*, [arXiv:0710.2430](#).
- [21] The precise meaning of the statement that the TB mixing comes entirely from the neutrino sector is discussed for example in S.F. King, *Invariant see-saw models and sequential dominance*, *Nucl. Phys. B* **786** (2007) 52 [[hep-ph/0610239](#)].
- [22] S.F. King, *Parametrizing the lepton mixing matrix in terms of deviations from tri-bimaximal mixing*, *Phys. Lett. B* **659** (2008) 244 [[arXiv:0710.0530](#)].
- [23] S.F. King, *Atmospheric and solar neutrinos with a heavy singlet*, *Phys. Lett. B* **439** (1998) 350 [[hep-ph/9806440](#)]; *Atmospheric and solar neutrinos from single right-handed neutrino dominance and U(1) family symmetry*, *Nucl. Phys. B* **562** (1999) 57 [[hep-ph/9904210](#)]; *Large mixing angle MSW and atmospheric neutrinos from single right-handed neutrino dominance and U(1) family symmetry*, *Nucl. Phys. B* **576** (2000) 85 [[hep-ph/9912492](#)]; *Constructing the large mixing angle MNS matrix in see-saw models with right-handed neutrino dominance*, *JHEP* **09** (2002) 011 [[hep-ph/0204360](#)];
S. Antusch and S.F. King, *Sequential dominance*, *New J. Phys.* **6** (2004) 110 [[hep-ph/0405272](#)].
- [24] S.F. King and G.G. Ross, *Fermion masses and mixing angles from SU(3) family symmetry*, *Phys. Lett. B* **520** (2001) 243 [[hep-ph/0108112](#)]; *Fermion masses and mixing angles from SU(3) family symmetry and unification*, *Phys. Lett. B* **574** (2003) 239 [[hep-ph/0307190](#)].
- [25] P.H. Chankowski and Z. Pluciennik, *Renormalization group equations for seesaw neutrino masses*, *Phys. Lett. B* **316** (1993) 312 [[hep-ph/9306333](#)];
K.S. Babu, C.N. Leung and J.T. Pantaleone, *Renormalization of the neutrino mass operator*, *Phys. Lett. B* **319** (1993) 191 [[hep-ph/9309223](#)];
S.F. King and N.N. Singh, *Renormalisation group analysis of single right-handed neutrino dominance*, *Nucl. Phys. B* **591** (2000) 3 [[hep-ph/0006229](#)];
S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, *Neutrino mass operator renormalization revisited*, *Phys. Lett. B* **519** (2001) 238 [[hep-ph/0108005](#)]; *Neutrino mass operator renormalization in two Higgs doublet models and the MSSM*, *Phys. Lett. B* **525** (2002) 130 [[hep-ph/0110366](#)];
S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Neutrino mass matrix running for non-degenerate see-saw scales*, *Phys. Lett. B* **538** (2002) 87 [[hep-ph/0203233](#)];
S. Antusch and M. Ratz, *Supergraph techniques and two-loop β -functions for renormalizable and non-renormalizable operators*, *JHEP* **07** (2002) 059 [[hep-ph/0203027](#)].
- [26] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M.A. Schmidt, *Running neutrino mass parameters in see-saw scenarios*, *JHEP* **03** (2005) 024 [[hep-ph/0501272](#)];
S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Running neutrino masses, mixings and CP phases: analytical results and phenomenological consequences*, *Nucl. Phys. B* **674** (2003) 401 [[hep-ph/0305273](#)].

- [27] A. Dighe, S. Goswami and W. Rodejohann, *Corrections to tri-bimaximal neutrino mixing: renormalization and Planck scale effects*, *Phys. Rev. D* **75** (2007) 073023 [[hep-ph/0612328](#)];
A. Dighe, S. Goswami and P. Roy, *Radiatively broken symmetries of nonhierarchical neutrinos*, *Phys. Rev. D* **76** (2007) 096005 [[arXiv:0704.3735](#)].
- [28] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M.A. Schmidt, *Running neutrino mass parameters in see-saw scenarios*, *JHEP* **03** (2005) 024 [[hep-ph/0501272](#)].
- [29] S. Antusch and S.F. King, *From hierarchical to partially degenerate neutrinos via type-II upgrade of type-I see-saw models*, *Nucl. Phys. B* **705** (2005) 239 [[hep-ph/0402121](#)].
- [30] S.F. King, *Predicting neutrino parameters from SO(3) family symmetry and quark-lepton unification*, *JHEP* **08** (2005) 105 [[hep-ph/0506297](#)].
- [31] J.C. Pati and A. Salam, *Unified lepton-hadron symmetry and a gauge theory of the basic interactions*, *Phys. Rev. D* **8** (1973) 1240.
- [32] H. Georgi, *Particles and fields*, C.E. Carlson ed., A.I.P., U.S.A (1975);
H. Fritzsch and P. Minkowski, *Unified interactions of leptons and hadrons*, *Ann. Phys. (NY)* **93** (1975) 193.
- [33] H. Georgi and C. Jarlskog, *A new lepton-quark mass relation in a unified theory*, *Phys. Lett. B* **86** (1979) 297.
- [34] S. Antusch, P. Huber, J. Kersten, T. Schwetz and W. Winter, *Is there maximal mixing in the lepton sector?*, *Phys. Rev. D* **70** (2004) 097302 [[hep-ph/0404268](#)].
- [35] S. Antusch and S.F. King, *Charged lepton corrections to neutrino mixing angles and CP phases revisited*, *Phys. Lett. B* **631** (2005) 42 [[hep-ph/0508044](#)].