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# Non-perturbative superpotentials across lines of marginal stability

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**ABSTRACT:** We discuss the behaviour of non-perturbative superpotentials in 4d  $\mathcal{N} = 1$  type II compactifications (and orientifolds thereof) near lines of marginal stability, where the spectrum of contributing BPS D-brane instantons changes discontinuously. The superpotential is nevertheless continuous, in agreement with its holomorphic dependence on the closed string moduli. The microscopic mechanism ensuring this continuity involves novel contributions to the superpotential: As an instanton becomes unstable against decay to several instantons, the latter provide a multi-instanton contribution which reconstructs that of the single-instanton before decay. The process can be understood as a non-perturbative lifting of additional fermion zero modes of an instanton by interactions induced by other instantons. These effects provide mechanisms via which instantons with  $U(1)$  symmetry can contribute to the superpotential. We provide explicit examples of these effects for non-gauge D-brane instantons, and for D-brane gauge instantons (where the motions in moduli space can be interpreted as Higgsing, or Seiberg dualities).

**KEYWORDS:** Nonperturbative Effects, Brane Dynamics in Gauge Theories, D-branes, Intersecting branes models.

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## 1. Introduction

Non-perturbative effects in string theory are a key ingredient in the proper understanding of the theory, and in particular of the dynamics of compactifications to four dimensions. Already in the early times non-perturbative effects (in the form of strongly coupled field theory sectors) were considered to underlie moduli stabilization and supersymmetry breaking [1, 2]. The formal developments on euclidean brane instanton effects (see e.g. [3–6]), in particular D-brane instantons in type II compactifications (or F/M-theory duals) have led to a variety of (in some cases very explicit) applications to e.g. moduli stabilization [7–9] and the generation of perturbatively forbidden couplings [10, 11] (see also [12–14] and [15] for related applications). D-brane instantons in local D-brane models have also been explored, recovering field theory gauge instanton effects [16–20], and with realizations of old and new models of supersymmetry breaking [21–23]. Other formal aspects of D-brane instantons have been recently discussed in e.g. [24, 25].

In this paper we discuss an interesting formal aspect of non-perturbative superpotentials generated by instantons in string theory. In 4d  $\mathcal{N} = 1$  supersymmetric compactifications of string theory, non-perturbative contributions to the superpotential arise from brane instantons with two fermion zero modes, which are saturated by the  $d^2\theta$  superspace integration. These must necessarily be 1/2-BPS branes, so that the fermion zero modes are given by the Goldstinos of the two broken supersymmetries.

Hence, the non-perturbative superpotential depends on the precise list of BPS branes (satisfying certain additional constraints, like the absence of extra fermion zero modes) at a given point in moduli space. Now it is a well-known fact that the spectrum of BPS branes can jump discontinuously across lines of marginal stability [26–31]. Namely, in type IIA compactifications the spectrum of supersymmetric D2-brane instantons may jump as one moves in complex structure moduli space (with the geometric interpretation that a supersymmetric 3-cycle may split in two independent supersymmetric 3-cycles when the complex structure is changed); similarly for D-brane instantons in type IIB compactifications as one moves in Kahler moduli space.

It is therefore a natural question whether the non-perturbative superpotential is continuous across these lines of marginal stability. This is expected, given that superpotentials are protected quantities. In fact, an abrupt change in the superpotential would correspond to a non-holomorphic dependence on the moduli (since marginal stability walls are typically of codimension one), which is not compatible with supersymmetry. It turns out that the microscopic explanation of the continuity of the non-perturbative superpotential is related to a wealth of previously unnoticed surprises in D-brane instanton physics. We devote the present paper to uncovering them in a few illustrative examples, leaving a systematic discussion for future work.

The first interesting novelty is that multi-instanton processes can contribute to the non-perturbative superpotential. Consider an instanton A that contributes to the non-perturbative superpotential, and which reaches a line of marginal stability where it splits into two instantons B and C. Although the instantons B and C do not in general contribute to the non-perturbative superpotential by themselves, the 2-instanton process involving B

and C simultaneously does lead to a contribution to the superpotential. A key ingredient is that extra zero modes of the two individual instantons are saturated against each other, in such a way that only two fermion zero modes are left over for the combined system, see figure 2. Although multi-instanton processes have been extensively studied for  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  supersymmetric gauge theories (see [32] for a review), the possibility to have them generate non-perturbative superpotentials in  $\mathcal{N} = 1$  theories has not been considered in the past. Moreover, our result implies that the usual strategy to compute the non-perturbative superpotential by summing all contributions from suitable BPS instantons may miss important contributions, due to multi-instanton processes. We present several explicit examples of this phenomenon, for D-brane instantons with or without interpretation as gauge theory instantons.

A second interesting novelty arises from regarding the above 2-instanton process as a non-perturbative lifting of fermion zero modes. In considering the effective 4d interaction generated by say the instanton B, one needs to consider the possible interaction terms which may lift fermion zero modes (at the Gaussian level). The above mechanism corresponds to a non-perturbative contribution to the interactions of the fermions zero modes of the instanton B induced by the instanton C, so that the former can contribute to the superpotential. This will be more explicitly discussed in several examples.

A final interesting surprise is related to non-gauge D-brane instantons (these are standard D-brane instantons, but refer to them as non-gauge, or sometimes exotic, to distinguish them from D-brane instantons with gauge field theory interpretation). It is usually considered that, for a D-brane instanton in a perturbative type II model to contribute to the superpotential, it must have Chan-Paton symmetry  $O(1)$ , so that the orientifold projection eliminates some of the universal fermion zero modes (arising from an accidental  $\mathcal{N} = 2$  supersymmetry in the relevant open string sector). We however present several examples where instantons with  $U(1)$  symmetries contribute to the superpotential, with the extra zero modes being saturated by interactions in the instanton world-volume effective action.

As a last remark, the continuity of the non-perturbative superpotential, combined with string duality will lead to new interesting properties of non-perturbative superpotentials in F-theory compactifications across certain topology changing phase transitions, as we discuss in section 6.

Before entering the discussion, we would like to present the problem of the continuity of the non-perturbative superpotential in terms more familiar from the model building viewpoint, in the context of the recent approaches to use non-perturbative superpotentials to stabilize Kahler moduli in type IIB compactifications [7]. For concreteness, consider a compactification with a gauge sector arising from stack of D7-branes. In general, such configuration of branes is supersymmetric at a point (or locus)  $P$  in Kahler moduli space. At other points or loci  $Q$  in moduli space, the D7-branes have misaligned BPS phases and recombine to form bound states, which correspond to BPS branes at point  $Q$ . The field theory interpretation is that Kahler moduli couple as Fayet-Iliopoulos terms to the D-branes, which trigger processes of Higgsing/unHiggsing in the gauge theory. In any event, the gauge sector arising from the D7-branes is different at the points  $P, Q$ . Consider that the gauge sector at  $P$  develops a non-perturbative superpotential for the Kahler moduli,

such that the resulting scalar potential stabilizes the moduli at the point  $Q$ . If the non-perturbative superpotential would not be continuous across the line of marginal stability, we would find ourselves in the paradoxical situation that the minimum lies at a point where the original potential is no longer valid. Needless to say, such behavior would enormously complicate the problem of moduli stabilization. Happily, superpotentials are far better behaved quantities, which can be used universally all over moduli space.

In this paper we focus on non-perturbative superpotentials. On general grounds we expect that other quantities, such as higher derivative F-terms, arising from BPS instantons with additional fermion zero modes, are also continuous all over moduli space. We leave a systematic understanding for future work, and will be happy to constrain ourselves to the discussion of the continuity of the superpotential in a series of illustrative examples.

The paper is organized as follows. In section 2 we discuss some relevant background material on instantons, both gauge and non-gauge, and we introduce the geometric backgrounds we will consider. In section 3 we discuss the continuity of the superpotential for non-gauge instantons, explaining the role of multi-instanton processes. In section 4 we go on to discuss continuity and multi-instanton effects for gauge instantons in string theory. In particular we describe the continuity of the superpotential under Seiberg duality. In section 5 we study motions in moduli space that convert gauge instantons into non-gauge instantons, and vice versa. In section 6 we describe the dual realization of the processes we study in F and M theory. Section 7 contains our conclusions, and finally appendix A discusses some exotic geometric processes that evade the assumptions in this paper, and might lead to discontinuities in the superpotential.

## 2. Some background material

### 2.1 Instanton effects

In dealing with euclidean brane instantons in string theory compactifications, it is convenient to make some general classifications and distinctions, which are useful for future reference. For concreteness we focus on D-brane instantons, although the effects can arise from other brane instantons in dual pictures (a prototypical example are e.g. euclidean D3-brane instantons in type IIB on CY-threefolds described as M5-brane instantons in M-theory on CY-fourfolds [4]).

A first class of D-brane instantons corresponds to those whose internal structure is exactly the same as some of the 4d space filling branes in the background. Namely, in geometric setups, Euclidean  $Dp$ -branes wrapping the same  $(p+1)$ -cycle (and carrying the same world-volume gauge bundle) as some  $D(p+4)$ -branes in the background configuration (in more abstract CFT terms, they should be described by the same boundary state of the internal CFT). Such D-brane instantons correspond to gauge instantons on the corresponding 4d gauge sector, and thus reproduce non-perturbative effects arising from strong gauge dynamics.

#### 2.1.1 Superpotentials from gauge D-brane instantons

A prototypical case, which will appear in our examples, is the generation of the Affleck-

Dine-Seiberg superpotential

$$W = (N_c - N_f) \left( \frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}} \quad (2.1)$$

on a set of 4d space filling branes whose low-energy dynamics corresponds to  $SU(N_c)$  SQCD with  $N_f$  flavours (with dynamical scale  $\Lambda$ ; here  $M$  denotes the meson fields). For  $N_f = N_c - 1$  this arises from classical 4d instanton field configurations, and has been recovered from D-brane instantons in several instances with different levels of detail [33, 34, 13, 17]. For other values of  $N_f < N_c$ , it does not arise from classical 4d field configurations, and is obtained indirectly. Alternatively, it can be obtained by considering the theory on  $\mathbf{S}^1$ , where there exist suitable 3d classical field configurations (sometimes denoted calorons) leading to a 3d superpotential, which can be argued to survive in the 4d decompactification limit (with a microscopic description in terms of putative objects denoted “fractional instantons” or “merons”). The latter description fits perfectly with the string theory realization. Indeed, the computation of e.g. the euclidean D3-brane instanton superpotential in type IIB configurations with gauge sectors on D7-branes on 4-cycles is usually described by invoking compactification on a circle in order to use an M-theory dual. Upon compactification, one may use T-duality, leading to a picture where instantons are D-branes stretched along the circle direction, and gauge D-branes are pointlike on it. In this picture the superpotential is generated by “fractional” D-branes, which are suspended between the gauge D-branes and thus stretch only a fraction of the period along the circle direction [35]. Equivalently, in the dual M-theory picture, the gauge D7-branes turn into degenerations of the elliptic fibration, such that the fiber over the 4-cycle on the base is a sausage of 2-spheres. The ADS superpotential is generated by M5-branes which wrap the 4-cycle times a 2-sphere (leading to “fractional” objects, in the sense that the standard 4d gauge instanton corresponds to an M5-brane wrapping the whole fiber) [34]. We will often abuse language and regard the 4d ADS superpotential as generated by (fractional) instantons, although strictly speaking only the 3d ADS superpotential has such a microscopic description.

It is interesting to point out that many of the manipulations in the analysis of  $\mathcal{N} = 1$  supersymmetric field theories usually carried out in terms of the exact effective action, can be carried out microscopically in terms of the physics of the relevant (possibly fractional) instantons. For instance, an important point in working with gauge instantons in our examples below is the derivation, from the instanton physics viewpoint, of the matching of scales in processes like integrating out massive 4d matter etc. Let us describe this in a simple example. Consider an  $SU(N_c)$  theory with  $N_f < N_c$  flavours with mass (matrix)  $m$ , with dynamical scale  $\Lambda = (e^{-1/g^2})^{1/(3N_c - N_f)}$ . Consider the situation where we neglect the effect of the mass term on the instanton physics. Then the instanton feels  $N_f$  massless flavors and the non-perturbative dynamics is described by the effect of a  $\frac{1}{N_c - N_f}$ -fractional instanton, leading to the total superpotential

$$W = (N_f - N_c) \left( \frac{\Lambda^{3N_c - N_f}}{\det Q\tilde{Q}} \right)^{\frac{1}{N_c - N_f}} + m Q\tilde{Q} \quad (2.2)$$

We may want to use an alternative description where we include the effect of the mass terms from the start. From the spacetime viewpoint, we integrate out the massive flavours. From the instanton perspective, the instanton feels that the  $2N_f$  fermion zero modes  $\alpha, \beta$  associated to the flavors (in the D-brane picture, open strings stretched between the instanton brane and the flavor branes) are actually massive.<sup>1</sup> Integrating out the term  $S_{\text{inst}} = m\alpha\beta$  in the instanton action leads to a prefactor of  $\det m$  in the amplitude for the left-over  $\frac{1}{N_c}$ -fractional instanton. Therefore the superpotential is

$$W = N_c \left( \Lambda^{3N_c - N_f} \det m \right)^{\frac{1}{N_c}} \quad (2.3)$$

This is the standard  $\frac{1}{N_c}$ -fractional instanton amplitude for a SYM sector with effective scale  $\Lambda'$  defined by

$$\Lambda'^{3N_c} = \Lambda^{3N_c - N_f} \det m \quad (2.4)$$

Note that (2.3) in fact agrees with (2.2) upon integrating out the massive flavours in the latter. Also, the matching of scales is the familiar one in field theory.

For future reference, let us mention that D-brane instantons associated as above to 4d space filling D-branes, can lead to non-perturbative superpotentials despite the fact that there are 4 universal zero modes in the instanton-instanton open string sector. Indeed, two of these fermion zero modes have cubic couplings to the bosonic and fermionic zero modes in the mixed open string sector (strings stretched between the instanton and the gauge D-branes). Their role can be regarded as imposing the fermionic constraints to recover the ADHM instanton measure [16]. The two left-over fermion zero modes are Goldstinos of the  $\mathcal{N} = 1$  supersymmetry, and are saturated by the  $d^2\theta$  integration involved in the induced superpotential.

### 2.1.2 Non-gauge, “exotic” or “stringy” instantons

In general an euclidean D-brane instanton does not have the same internal structure as any gauge D-brane in the configuration. Such D-brane instantons do not have any known gauge field theory interpretation, and are thus dubbed “exotic” or “stringy” instantons. BPS instantons of this kind lead to superpotential terms only if they have two fermion zero modes, with additional fermion zero modes forcing multi-fermion insertions leading to higher F-terms as described below (additional fermions zero modes, with couplings to 4d chiral multiplets, are regarded here as non-zero modes, since they are lifted by background values of the latter; equivalently, integration over these zero modes leads to insertions of the 4d chiral multiplet in the induced superpotential). We are thus interested in stringy instantons with two fermion zero modes.

In the same way as for gauge instantons, there are 4 universal fermion zero modes in the instanton-instanton open string sector. However in this case, there are no bosonic zero modes which can lift the two non-goldstino modes. In the absence of other lifting

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<sup>1</sup>This follows from the fact that the massive flavours are open strings from the color to the flavor branes, and that gauge brane instantons wrap exactly on top of the color branes.



mechanisms (like closed string flux backgrounds), the only mechanism which can eliminate these extra modes in type II perturbative models is an orientifold projection. Therefore, only instantons invariant under the orientifold action, and with a Chan-Paton action leading to an  $O(1)$  symmetry, have two universal fermion zero modes, and have a chance of leading to a non-perturbative superpotential (of course if they do not have extra fermion zero modes in other sectors).

### 2.1.3 Higher F-terms from D-brane instantons

Besides D-brane instantons generating superpotentials, BPS D-brane instantons with additional fermion zero modes lead to higher F-terms in the effective action. These have been considered in [36, 37], and lead to operators with one insertion of  $\overline{D\Phi}$  for each additional fermion zero mode. Roughly speaking they have the structure

$$\int d^4x d^2\theta w_{\tilde{i}_1\tilde{j}_1\ldots\tilde{i}_n\tilde{j}_n}(\Phi) \overline{D\Phi}^{\tilde{i}_1} \overline{D\Phi}^{\tilde{j}_1} \ldots \overline{D\Phi}^{\tilde{i}_n} \overline{D\Phi}^{\tilde{j}_n} \quad (2.5)$$

where the tensor  $w(\Phi)$  depends holomorphically on the 4d chiral multiplets. The simplest situation is an instanton with two additional fermion zero modes, which is for instance realized for gauge instantons in  $N_f = N_c$  SQCD. The corresponding operator has the above structure with  $n = 1$  and implements the familiar complex deformation of the moduli space (in an intrinsic way, in the sense of the moduli space geometry).

The study of the interplay between non-perturbative higher F-terms and lines of marginal stability is beyond our scope in this paper, although we expect that it admits a similar microscopic description in terms of multi-instanton contributions after instanton splitting. In any event, even for the analysis of superpotential terms, such instantons will play an interesting role in some of our examples. We refer to these instantons as Beasley-Witten instantons.

## 2.2 A useful family of geometries

Here we describe a set of geometries which we use in several of our explicit examples below. They are non-compact geometries, but they suffice to study instanton effects and transitions as long as they involve just the local structure of compact cycles (see footnote 8 for one example where non-compactness is relevant to the discussion).

Let us consider the following class of local Calabi-Yau manifolds, described by.

$$\begin{aligned} xy &= \prod_{k=1}^P (z - a_k) \\ x'y' &= \prod_{k'=1}^{P'} (z - b'_{k'}) \end{aligned} \quad (2.6)$$

This kind of geometry is a particular case of those considered in [38]. It describes two  $\mathbf{C}^*$  fibrations, parametrized by  $x, y$  and  $x', y'$ , varying over the complex plane  $z$ , degenerating at the locations  $a_i, b_i$  respectively. In this geometry one can construct Lagrangian 3-cycles by considering segments joining pairs of degeneration points on the base, and fibering the

two  $\mathbf{S}^1$ 's in the two  $\mathbf{C}^*$  fibers. Segments joining pairs of  $a$ -type degenerations or pairs of  $b$ -type degenerations lead to 3-cycles with topology  $\mathbf{S}^2 \times \mathbf{S}^1$ . Segments joining  $a$ - and  $b$ -type degenerations lead to 3-cycles with topology  $\mathbf{S}^3$ . Let us introduce the notation  $[p_1, p_2]$  for the 3-cycle associated to the pair of degeneration points  $p_1, p_2$ , whatever their type.

Introducing the holomorphic 3-form

$$\Omega = \frac{dx}{x} \frac{dx'}{x'} dz \quad (2.7)$$

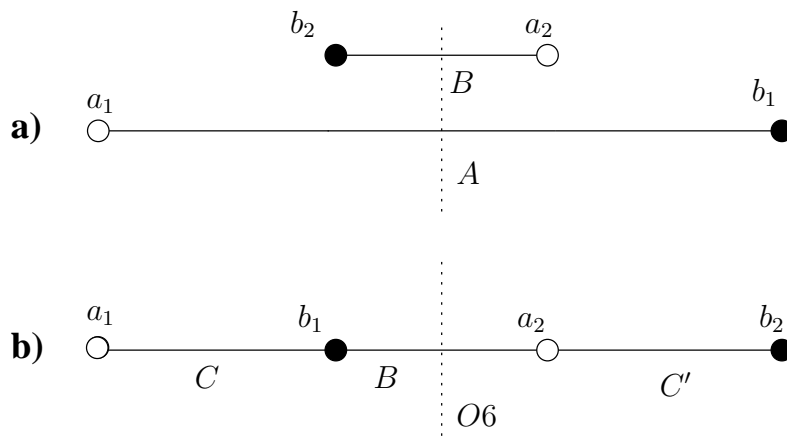
the 3-cycle  $[p_1, p_2]$  is calibrated by the form  $e^{i\theta}\Omega$ , where  $\theta$  is the angle of the segment  $[p_1, p_2]$  with the real axis in the  $z$ -plane. Namely  $\text{Im}(e^{i\theta}\Omega)|_{[p_1, p_2]} = 0$ , where  $|_{[p_1, p_2]}$  denotes restriction to the 3-cycle. Hence, segments which are parallel in the  $z$ -plane define 3-cycles which preserve a common supersymmetry. We will be interested in configurations where all degenerations are on (or near) the real axis.

We will consider stacks of 4d space filling D6-branes and/or euclidean D2-branes wrapping the different 3-cycles, and describe the non-perturbative superpotentials arising from these configurations. The open string modes and their interactions are easy to determine. For instance, each stack of  $N$  D6-branes on a 3-cycle leads to a  $U(N)$  gauge group in a vector multiplet of  $\mathcal{N} = 1$  supersymmetry for 3-cycles of  $\mathbf{S}^3$  topology, and of  $\mathcal{N} = 2$  supersymmetry for 3-cycles of  $\mathbf{S}^2 \times \mathbf{S}^1$  topology. The angle  $\theta$  introduced above determines the precise supersymmetry preserved by the corresponding set of branes. Also, two D6-branes wrapping two 3-cycles involving one common degeneration point lead to a vector-like pair of bi-fundamental chiral multiplets, arising from open strings in the intersection of 3-cycles (which is topologically  $\mathbf{S}^1$ , coming from the  $\mathbf{C}^*$  that does not degenerate at the intersection).

As discussed in [38] one can perform T-dualities along the two  $\mathbf{S}^1$  directions, and map the configuration to a Hanany-Witten setup of  $P$  NS-branes (along 012345) and  $P'$  NS'-branes (along 012389), with D4-branes (along 01236) suspended among them, in a flat space geometry with a noncompact  $x^6$  direction (in contrast to the usual Hanany-Witten configurations describing systems such as the conifold). The gauge theory content described above follows from the standard rules in this setup (see [39]). This picture also facilitates the computation of the superpotential, whose general discussion we skip, but which we present in our concrete example below.

### 3. Non-gauge D-brane instantons

In this section we consider “exotic” D-brane instantons (i.e. instantons arising from D-branes wrapping internal cycles different from those wrapped by the spacetime filling branes in the model). For simplicity we restrict ourselves to perturbative type IIA Calabi-Yau compactifications in the absence of fluxes. The aim of this section is to show the continuity of the non-perturbative superpotential across the lines of marginal stability for the instantons. We show that the microscopic mechanism underlying this continuity reveals interesting new properties of D-brane instanton physics, including multi-instanton processes and non-perturbative lifting of fermion zero modes.



**Figure 1:** Example of an  $O(1)$  instanton  $A$  (figure a) splitting into an  $O(1)$  instanton  $B$  and a  $U(1)$  instanton  $C$  and its image  $C'$  (figure b).

We have already mentioned in section 2.1.2 that in perturbative type II models (and in the absence of additional ingredients like 3-form fluxes), for instantons to have just the two fermion zero modes required to contribute to the superpotential they should be mapped to themselves under the orientifold action and have an  $O(1)$  Chan-Paton symmetry. This constrains the possible splittings of the instanton in walls of marginal stability, for instance an  $O(1)$  instanton cannot split into two  $O(1)$  instantons, as we show in appendix A. Still, there is enough freedom to have non-trivial splitting of instantons that contribute to the superpotential, as we now discuss in a simple example.

### 3.1 $O(1)$ instanton splitting as $O(1) \times U(1)$ instantons

#### 3.1.1 Configuration and marginal stability line

In this section we consider one simple example of an  $O(1)$  instanton  $A$ , which contributes to the non-perturbative superpotential, and can reach a line of marginal stability on which it splits as an  $O(1)$  instanton  $B$  and a  $U(1)$  instanton (described as a brane  $C$  and its image  $C'$ ).

Consider a geometry of the kind introduced in section 2.2, with two degenerations  $a_1, b_1$  located at  $z = \pm t/2$ , with  $t \in \mathbf{R}$ , and two degenerations  $a_2, b_2$  located at  $z = \pm s/2 + i\epsilon$ , with  $s, \epsilon \in \mathbf{R}$ , and  $s < t$  for concreteness, see figure 1. Namely

$$\begin{aligned} x^2 + y^2 &= (z + t/2)(z - s/2 - i\epsilon) \\ x'^2 + y'^2 &= (z + s/2 - i\epsilon)(z - t/2) \end{aligned} \quad (3.1)$$

Consider modding out the geometry by the orientifold action  $\Omega R(-1)^{FL}$ , where  $R$  is the antiholomorphic involution

$$z \rightarrow -\bar{z} \quad ; \quad (x, y) \leftrightarrow (\bar{x}', \bar{y}') \quad (3.2)$$

The set of fixed points defines an O6-plane along the imaginary  $z$  axis. This orientifold exchanges degenerations of  $a$  and  $b$  type. The parameters  $s, t, \epsilon$  belong to chiral multiplets associated to complex structure moduli invariant under the orientifold action. We choose the O6-plane charge such that it leads to O(1) Chan-Paton symmetry for D2-brane instantons on 3-cycles defined by horizontal segments crossing it.

For generic non-zero  $\epsilon$  there are two O(1) instantons in this configurations, corresponding to D2-branes on the segments  $[a_1, b_1]$  (denoted instanton A) and  $[b_2, a_2]$  (denoted instanton B). Each has just two fermion zero modes, and therefore leads to a contribution to the non-perturbative superpotential

$$W = f_1 e^{-T} + f_2 e^{-S} \quad (3.3)$$

where  $T, S$  are the closed string chiral multiplets whose real parts are given by the moduli  $t, s$  controlling the size of the wrapped 3-cycles. Here  $f_i$  are prefactors given by one-loop determinants, which depend on the Kahler moduli (but not on the complex structure moduli).

When  $\epsilon$  is taken to zero, the four degenerations align, and the instanton A reaches a line of marginal stability, and splits into an instanton of type  $B$ , and a U(1) instanton corresponding to a D2-brane on  $[a_1, b_2]$  and its orientifold image on  $[a_2, b_1]$  (denoted C and C' respectively). Since the complete superpotential should behave continuously in this motion in moduli space, there should be suitable instanton processes reproducing it. There are only two basic instantons, namely the O(1) instanton B on  $[b_2, a_2]$ , which indeed reproduces the  $e^{-S}$  term in (3.3), and the U(1) instanton C (with its image C'), which has four fermion zero modes and does not contribute to the superpotential. Hence, there is no instanton which reproduces the  $e^{-T}$  term. In analogy with the analysis in section (4) for gauge D-brane instantons, the resolution of the puzzle lies in understanding the mutual influence of different instantons, and can be understood in different ways as we now describe.

### 3.1.2 The 2-instanton process

In order to show that the 2-instanton process contributes to the superpotential, we have to discuss the structure of zero modes in the 2-instanton configuration, and how they are saturated. This will involve the saturation of additional zero modes due to higher order interactions on the instanton world-volume effective action.

Let us briefly describe the structure of zero modes in the different sectors. We refer to the instantons  $C, B$  as 1, 2 in this section.

- In the 11 sector (and its  $1'1'$  image), the open string sector feels a background with 8 supercharges, half of which are broken by the instanton. We have a U(1) gauge symmetry (although there are no gauge bosons), four bosonic zero modes  $x_1^\mu$  corresponding to the 4d translational Goldstones, and four fermionic zero modes  $\theta_1^\alpha, \tilde{\theta}_{1\dot{\alpha}}$ , corresponding to the Goldstinos. Note that the Lorentz symmetry under which these are chiral spinors is a global symmetry from the instanton volume viewpoint.
- The 22 sector is sensitive to the orientifold action and hence feels a background with 4 supercharges, half of which are broken by the instanton. The orientifold projection

truncates part of the spectrum, as compared with the above  $U(1)$  instanton case. There is an  $O(1) \equiv \mathbf{Z}_2$  gauge symmetry, and four bosonic zero modes  $x_2^\mu$ .

- Consider now the spectrum from open string stretching at the 12 intersection (and its image 1'2). Locally around it, the background admits 16 supersymmetries, half of which are broken by the D-branes. The massless modes thus form a hypermultiplet under the unbroken 8 supersymmetries. We have two complex bosonic zero modes  $\varphi_{12}, \varphi_{21}$ , with charges  $+1$  and  $-1$  under the  $U(1)_1$  gauge symmetry of the instanton 1, and four fermionic zero modes,  $\chi_{12}^\alpha, \chi_{21}^\alpha$ , with charges  $+1$  and  $-1$  under  $U(1)_1$ . Alternatively, these can be conjugated to  $\bar{\chi}_{21\dot{\alpha}}, \bar{\chi}_{12\dot{\alpha}}$ , with charges  $-1, +1$ . Let us call the chiral superfields in the hypermultiplet  $\Phi_{12}$  and  $\Phi_{21}$ .

Let us now describe the couplings of these modes on the volume of the instanton. They are analogous (upon dimensional reduction) to the couplings that would appear if we would have D6-branes instead of D2-branes. There is a first term which describes the mass terms of the open strings between the two instantons when they are separated in the 4d direction

$$S_{\text{kinetic}} = (x_1^\mu - x_2^\mu)^2 (|\varphi_{12}|^2 + |\varphi_{21}|^2) + i(x_1^\mu - x_2^\mu) \{ \bar{\chi}_{12} \sigma_\mu \chi_{12} - \bar{\chi}_{21} \sigma_\mu \chi_{21} \} \quad (3.4)$$

These terms are related to the couplings to gauge bosons in the D6-D6 system. There are also terms involving the neutral fermion zero modes  $\theta, \tilde{\theta}$  (analogous to the couplings to gauginos in the D6-D6 system), given by:<sup>2</sup>

$$S_\lambda = (\chi_{12}(\theta_1 - \theta_2))\varphi_{12}^* - (\chi_{21}(\theta_1 - \theta_2))\varphi_{21}^* + (\bar{\chi}_{12}\tilde{\theta})\varphi_{12} - (\bar{\chi}_{21}\tilde{\theta})\varphi_{21} \quad (3.5)$$

Notice that the combination  $\theta_1 + \theta_2$  is decoupled, and corresponds to the two Goldstinos of the combined two-instanton system. We also have a D-term potential (the same arising in a D6-D6-brane system):

$$S_D = (|\varphi_{12}|^2 - |\varphi_{21}|^2)^2 \quad (3.6)$$

Finally, there are quartic couplings involving the fields in the 12 sector. The local intersection preserves 8 supercharges, but the interaction is induced by effects that preserve only 4 supercharges (due to the different nature of degenerations at the intersection and adjacent to it). The interaction can be obtained from a superpotential of the form

$$W \simeq (\Phi_{12}\Phi_{21})^2 \quad (3.7)$$

This is in fact identical to the superpotential that would be obtained for D6-branes. The underlying reason is that both D2- and D6-branes have identical boundary states of the internal CFT (and a flip of DD to NN boundary conditions in the 4d part), thereby leading to essentially the same correlation functions.

Thus we obtain fermion-scalar interactions of the form

$$S_{\chi^2\varphi^2} = \chi_{12}\varphi_{21}\chi_{12}\varphi_{21} + 2\chi_{12}\chi_{21}\varphi_{12}\varphi_{21} + \varphi_{12}\chi_{21}\varphi_{12}\chi_{21} + \text{h.c.} \quad (3.8)$$

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<sup>2</sup>Similar couplings in the context of a D2-instanton intersecting its orientifold image have been described in [40].

and the F-term scalar potential

$$S_F = |\varphi_{21}\varphi_{12}\varphi_{21}|^2 + |\varphi_{12}\varphi_{21}\varphi_{21}|^2 \quad (3.9)$$

Let us now consider the role of this complete instanton effective action in the generation of a non-perturbative superpotential. Notice that the contribution to the superpotential is dominated by configurations of overlapping instantons, namely when  $x_1 - x_2 = 0$ , as follows. A large non-zero separation gives large masses to the open strings between the instantons (consistent with the equation (3.4)), so we can integrate out these fields and set their vevs to zero, making the couplings in (3.5) vanish. Then we cannot saturate the  $\theta_1 - \theta_2$  zero modes, and the integral vanishes. So let us focus for simplicity<sup>3</sup> in the case  $x_1 - x_2 = 0$ . In this case we have an instanton action given by  $S_{2\text{inst}} = S_\lambda + S_D + S_{\chi^2\varphi^2} + S_F$ . The pieces relevant for the saturation of zero modes will be  $S_\lambda$  and  $S_{\chi^2\varphi^2}$ . We can soak up  $(\theta_1 - \theta_2)$  by bringing down two insertions of  $(\chi_{12}(\theta_1 - \theta_2))\varphi_{12}^*$  from  $S_\lambda$ . Similarly we can soak up  $\tilde{\theta}$  by bringing down two insertions of  $(\bar{\chi}_{12}\tilde{\theta})\varphi_{12}$ . This also saturates the zero modes  $\chi_{12}, \bar{\chi}_{12}$ . The remaining zero modes  $\chi_{21}, \bar{\chi}_{21}$  can be soaked up by bringing down two insertions of  $\varphi_{12}\chi_{21}\varphi_{12}\chi_{21}$  from  $S_{\chi^2\varphi^2}$  and two insertions of its complex conjugate operator. Bringing everything together, and integrating over the (saturated) fermionic zero modes, we get the following 2-instanton contribution:

$$\int d^4x_+ d^2\theta_+ [d\varphi] \exp\{-S_D - S_F\} |\varphi_{12}|^4 \quad (3.10)$$

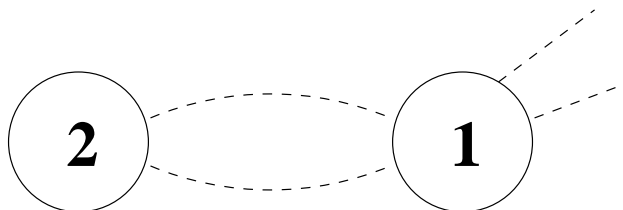
where  $x_+ = x_1 + x_2$ ,  $\theta_+ = \theta_1 + \theta_2$  are the surviving zero modes of the instanton. Note that the  $\varphi$  integral converges since there are no flat directions in the  $(\varphi_{12}, \varphi_{21})$  space, as is easily seen from the form of  $S_D$  and  $S_F$ . There are other similar contributions from other combinatorics of soaking up zero modes. The overall result is a non-zero contribution to the superpotential from the 2-instanton process.

The above mechanism is very similar to the lifting of accidental zero modes by world-volume interactions in other situations. For instance in the study of instanton effects on 4d  $\mathcal{N} = 4$  supersymmetric theories, where a world-volume 4-fermion interaction lifts fermion zero modes in groups of four (and allows multi-instanton processes contribute to the same 4d effective action terms as single-instanton ones). The analogy could be made much more explicit by integrating over the bosonic modes above, generating world-volume 4-fermion interactions. This is, to our knowledge, the first explicit realization of a similar mechanism in the computation of non-perturbative D-brane instanton superpotentials in  $\mathcal{N} = 1$  theories. Notice also the interesting fact that in such situations the usual recipe of adding the contributions from the individual instantons misses these new contributions.

The spacetime picture of the above mechanism is of the kind shown in figure 2, with two fermion zero modes of each instanton saturated against each other, and two left-over fermion zero modes.

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<sup>3</sup>In fact it is possible, and not much harder, to carry out the computation allowing for arbitrary  $x_1 - x_2$ . Namely one can perform the Gaussian integration over these bosonic zero modes, and conclude that the result is localized (with some exponentially vanishing tail) onto  $x_1 = x_2$ . We omit the detailed analysis since the conclusions are essentially unchanged, and the simplified discussion is enough to show that the 2-instanton process at hand provides a non-trivial contribution.



**Figure 2:** Schematic picture of a multi-instanton configuration contributing to the superpotential. A number of additional fermion zero modes are saturated against each other, due to interaction terms in the world-volume effective action of the 2-instanton system. The two left-over fermion zero modes are the Goldstinos of the overall BPS D-brane instanton system, and are saturated against the  $d^2\theta$  integration in the induced 4d effective action superpotential term.

As a last comment, note that the above system fits nicely with the concept of quasi-instanton as described in [32]. Namely the bosonic modes  $\varphi$  can be described as quasi-zero modes, and they parametrize a quasi-moduli space of quasi-instantons, in the sense that they correspond to a moduli space of instantons, which are lifted by a world-volume potential whose effects can be studied perturbatively in the value for the bosonic fields. Although strictly speaking such configurations do not correspond to BPS instantons, they can provide the dominant dynamical effect in the semiclassical approximation to certain quantities. Note that the additional Goldstinos (those associated to the supersymmetries preserved by BPS instantons) are not turned on in the first correction, and the effect of larger values for the bosonic fields is suppressed due to the exponential damping.

### 3.1.3 Non-perturbative lifting of zero modes of the U(1) instanton

One can interpret the appearance of the non-trivial contribution to the superpotential as the instanton 2 generating an effective interaction term for the additional zero modes of the instanton 1. Indeed the piece

$$\Delta S_{\text{inst}1} = \int d^2\theta_2 d^4\chi d^4\varphi \exp \left[ (\theta_1 - \theta_2) \varphi \chi + \tilde{\theta}_1 \bar{\varphi} \bar{\chi} + \chi^2 \varphi^2 + V(\varphi) \right] \quad (3.11)$$

of the integral above can be regarded as computing the non-perturbative contribution of the instanton 2 to the effective action of the instanton 1. The result corresponds to an effective mass term (of non-perturbative strength  $e^{-S}$ ) for the extra fermion zero modes of the instanton 1. Hence the amplitude of the instanton 1 is sketchily of the form

$$\begin{aligned} S_{4d} &\simeq \int d^4x d^2\theta d^2\tilde{\theta} \exp(-T_1 - e^{-S} \tilde{\theta}\tilde{\theta}) \\ &= \int d^4x d^2\theta e^{-S} e^{-T_1} = \int d^4x d^2\theta e^{-T} \end{aligned} \quad (3.12)$$

namely the appropriate superpotential term.

In section (4.2.2) we will provide yet another viewpoint regarding the non-perturbative lifting of fermion zero modes.

It is very interesting that U(1) instantons can contribute to non-perturbative superpotentials via this mechanism of non-perturbative lifting of the extra zero modes. We also

expect other instantons with additional universal fermion zero modes, like  $\text{Sp}(2)$  instantons, to similarly contribute under special circumstances. It would be interesting to use this mechanism to revisit the role of interesting  $\text{U}(1)$  and  $\text{Sp}(2)$  instantons in model building applications, like the instanton scan in [14]. In fact, multi-instanton processes can already arise in simple toroidal orientifolds (see [41] for an explicit  $\mathbb{T}^6/\mathbf{Z}_3$  example).

### 3.1.4 4d charged matter insertions

The bottom line of the above sections is that non-perturbative superpotentials for non-gauge D-brane instantons are continuous across lines of marginal stability. The microscopic instanton physics mechanism relies on the fact that additional zero modes in multi-instanton processes can be saturated by interactions, leaving only a few zero modes to be saturated by external insertions in 4d correlators. The initial instanton amplitude is thus fully reconstructed by a multi-instanton amplitude.

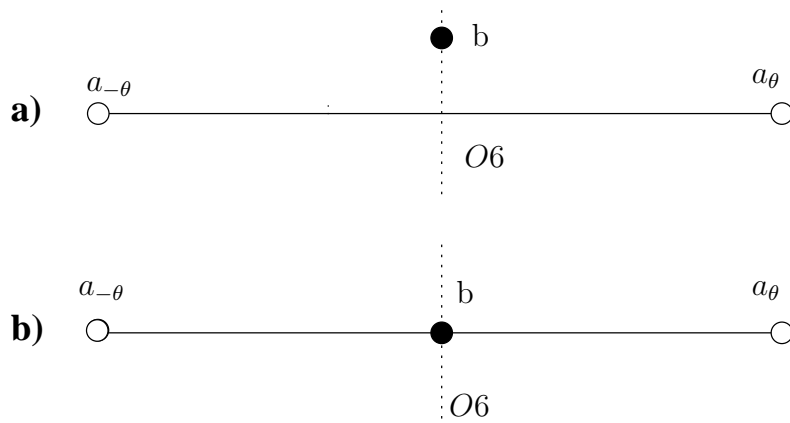
Let us comment on the situation where the initial instanton intersects some of the 4d space filling D-branes in the system. There are fermion zero modes charged under the 4d gauge group at those intersections. In order to contribute to the superpotential, these additional fermion zero modes should be coupled to operators involving the 4d charged matter fields, so that upon integration over them (or pulling down these interactions) one generates insertions of the 4d charged matter fields in the 4d effective superpotential as discussed in [42, 10, 11, 13]. The appearance of the same insertions in the multi-instanton amplitude at the line of marginal stability is easy to show: Notice that the homology charge of the contributing D-brane instanton system is preserved in the process of reaching the line of marginal stability. This ensures that the number of charged fermion zero modes is preserved in the process, and that the insertions of 4d fields are suitably generated. We refrain from delving into a more detailed discussion in concrete example, and prefer to move on.

## 3.2 $\text{O}(1)$ splitting as $\text{U}(1)$ instanton

In this section we would like to consider another possible splitting of an  $\text{O}(1)$  instanton across a line of marginal stability, in which it splits as a  $\text{U}(1)$  instanton and its image. In fact this kind of process was considered in [40], with the conclusion that such instantons cannot contribute to the superpotential due to the presence of additional zero modes. In fact our explicit example evades this no-go result: there exists an F-term interaction in the world-volume of the instanton (not considered in [40]) which lifts the additional fermion zero modes.

The geometry in this configuration is similar, but slightly different from those introduced in section 2.2. It is therefore better to introduce the configuration in terms of a type IIA Hanany-Witten setup. Consider a NS-brane along 012345, and two NS-branes along 0123 and rotated by angles  $\theta$  and  $-\theta$  in the planes 45 and 89 (so we denote them  $\text{NS}_\theta$  and  $\text{NS}_{-\theta}$ ). One can discuss the relevant part of the geometry by depicting the positions of the different branes in the  $z = x^6 + ix^7$  plane, as shown in figure 3. In our configuration, the NS-brane is located at  $z = i\epsilon$ , while the  $\text{NS}_{\pm\theta}$ -branes are located at  $z = \pm t$ , with  $t, \epsilon \in \mathbf{R}$ . We consider instantons arising from euclidean D0-branes suspended between the different





**Figure 3:** Configuration of an  $O(1)$  instanton splitting as a  $U(1)$  instanton (and its orientifold image). Interpreted as a HW setup, the dots  $b$ ,  $a_{\pm\theta}$ , denote the locations in the 67 plane for an unrotated NS-brane, and NS5-branes rotated by angles  $\pm\theta$  in the 4589 directions. Interpreted as D1-brane instantons in a threefold geometry, the dots  $a_{\pm\theta}$ ,  $b$  denote a projection of the degenerations loci of a  $\mathbf{C}^*$  fiber. D1-brane instantons wrap 2-cycles obtained by fibering the latter over segments defined by such degenerations, and are supersymmetric when the segments lie horizontally.

NS5-branes, thus corresponding to segments between the different NS5-brane locations in the  $z$ -plane. BPS instantons correspond to horizontal segments.

The above kind of configuration can be T-dualized using [43] into a type IIB geometry similar to those in section 2.2 (and similar to those studied in [44, 45]). As a complex variety, the geometry can be described as an unfolding of an  $A_2$  singularity

$$xy = u(u + \alpha v)(u - \alpha v) \quad (3.13)$$

with  $\alpha = \tan \theta$ . It can be regarded as a  $\mathbf{C}^*$  fibration over the  $(u, v)$  space, and degenerating at the loci  $u = 0$ ,  $u = \pm\alpha v$ . The directions  $u$ ,  $v$  are closely related to the directions 45 and 89 in the HW setup, and the degeneration loci correspond to the NS5-brane volumes in those directions. The geometry contains non-trivial 2-cycles, obtained by fibering the circle in the  $\mathbf{C}^*$  over a segment joining two degeneration loci. There are D-brane instantons arising from D1-branes wrapping these 2-cycles. The description of the geometry as a complex manifold provided in (3.13) does not encode the parameters  $\epsilon, t$ , which are Kahler parameters and control the lines of marginal stability of our instantons. We will rather use pictures like figure 3, which can be regarded as a depiction of the blow-up structure of the above geometry, or the representation of the 67 plane in the HW configuration. Since the spectrum of instanton zero modes and their interactions can be obtained from the latter using standard rules, we stick to this language, although it is straightforward to translate into the geometric one.

Let us introduce an O6-plane along 0123789 in the HW setup, which thus corresponds to a fixed line along the vertical axis on the  $z$ -plane. The O6-plane intersects the NS-brane (in an intersection preserving 8 supercharges) mapping it to itself, while it exchanges the

NS $_{\pm\theta}$ -branes. We choose the O6-plane charge such that it leads to O(1) Chan-Paton symmetry on instantons along horizontal segments crossing the O6-plane.

Consider the configuration for non-zero  $\epsilon$ , see figure 3a. The only BPS instanton is given by a D0-brane stretched between the NS $_{-\theta}$  and NS $_{\theta}$  branes. It has O(1) Chan-Paton symmetry and has just 2 fermion zero modes (for non-zero  $\theta$ ), and thus leads to a non-perturbative superpotential contribution  $W \simeq e^{-T}$ , with  $T$  the chiral multiplet with real part  $t$ .

Consider the configuration for  $\epsilon = 0$ , where the previous instanton reaches a line of marginal stability and splits into a U(1) instanton 1 (a D0-brane between the NS $_{-\theta}$  and the NS branes) and its orientifold image 1' (between the NS and NS $_{\theta}$  branes). At the Gaussian level, the instanton has many additional zero modes beyond the required set of two fermion zero modes, hence naively it would not contribute to the superpotential. However, it is easy to go through the analysis of zero modes and their interactions, and realize that the additional fermion zero modes are lifted. The argument is very similar to that in previous section, so our discussion is sketchy.

In the 11 sector of open strings with both endpoints on the instanton, there are four translational Goldstone bosonic zero modes  $x^{\mu}$ , and four fermionic zero modes, two of them  $\theta^{\alpha}$  associated to Goldstinos of the 4d  $\mathcal{N} = 1$ , and two  $\tilde{\theta}_{\dot{\alpha}}$  associated to the accidental enhancement to  $\mathcal{N} = 2$ . In the 11' sector of open strings between the instanton and its image, we have a hypermultiplet (given by the pair of chiral field  $\Phi$  and  $\Phi'$  in  $\mathcal{N} = 1$  language) of zero modes  $\varphi, \varphi', \chi_{\alpha}, \chi'_{\alpha}$  with U(1) charges  $\pm 2$  for unprimed/primed fields. The couplings between the 11 and 11' fields are

$$S = \tilde{\theta} (\varphi \bar{\chi} - \bar{\chi}' \varphi') \quad (3.14)$$

From the HW construction it is possible to derive that there are interactions among fields in the 11' sector. Given the amount of susy, it is possible to describe them by a superpotential  $W \simeq (\Phi \Phi')^2$ . Namely, there are scalar potential terms (involving also a D-term contribution)

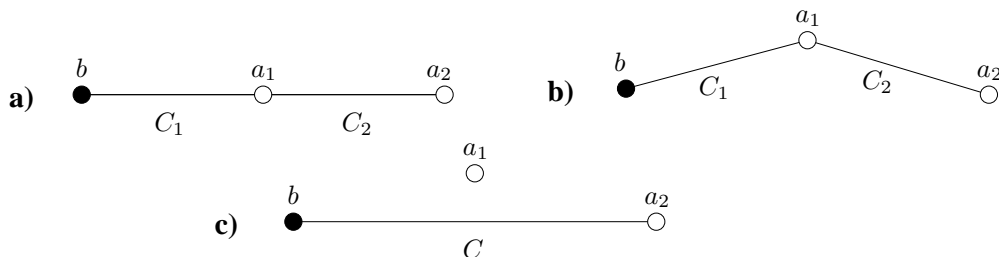
$$\begin{aligned} V_D &\simeq (|\varphi|^2 - |\varphi'|^2)^2 \\ V_F &\simeq |\varphi \varphi' \varphi|^2 + |\varphi' \varphi \varphi'|^2 \end{aligned} \quad (3.15)$$

and most importantly couplings to the 11' fermions

$$S_{\varphi\chi} \simeq \chi \chi \varphi' \varphi' + 2 \chi \varphi \chi' \varphi' + \varphi \varphi \chi' \chi' \quad (3.16)$$

As discussed in previous examples, all additional zero modes can be saturated by pulling down interaction terms from the instanton effective action. The only left-over fermion zero modes are the two Goldstinos  $\theta^{\alpha}$ , hence the U(1) instanton contributes to the superpotential. Note that in contrast with the previous examples, the lifting of zero modes of the U(1) instanton is purely perturbative (although is reminiscent of the non-perturbative lifting in previous section when regarded in the covering space).

Since the volume of the instanton and its image add up to the volume of the original O(1) instanton, the complete superpotential is continuous.



**Figure 4:** a) Marginally stable configuration. b) Moving  $a_1$  away from the horizontal axis renders the configuration nonsupersymmetric, so it can c) decay to a supersymmetric configuration by brane recombination.

## 4. Gauge D-brane instantons

Let us proceed to systems which are more familiar, namely configurations where the non-perturbative superpotential can be regarded as generated by gauge theory instantons. The idea is to consider a simple example of gauge sector with a non-perturbative superpotential, engineered via D-branes, and to consider its fate as one crosses a line of marginal stability. The general lesson of this example is the following. In this kind of setup, the crossing of lines of marginal stability in moduli space is basically described in terms of a Higgsing/unHiggsing in the field theory. Also, the dependence of the superpotential on the relevant moduli is encoded in the dynamical scales of the gauge factors associated to the 4d spacetime filling D-branes (since they control the gauge couplings). Thus the statement about the continuity of the superpotential across lines of marginal stability corresponds to the familiar matching of dynamical scales of a gauge theory in a Higgsing/unHiggsing process, at energies above and below the relevant vevs. Given this interpretation and the construction and discussion below, it is easy to find other examples of similar behaviour.

### 4.1 An example of $N_f < N_c$ SQCD non-perturbative superpotential

#### 4.1.1 Configuration, marginal stability, and the spacetime picture

Let us describe a system of D6-branes crossing a line of marginal stability in a geometry of the kind introduced in section 2.2. Consider the geometry in figure 4, having two  $a$ -type degenerations and one  $b$ -type degeneration, ordered as  $b, a_1, a_2$  from left to right along the real axis. We consider a set of  $N$  D6-branes wrapped on the 3-cycle  $C_1 = [b, a_1]$  and  $N$  D6-branes on  $C_2 = [a_1, a_2]$ . This configuration is supersymmetric as long as the degeneration  $a_1$  is aligned with the other two. Moving  $a_1$  away from the horizontal axis forces the D6-branes on  $C_1$  and  $C_2$  to misalign, and their tension increases. The system of branes can relax by forming a bound states, described by  $N$  D6-branes on the 3-cycle  $C = [b, a_2]$ . Namely, the locus in moduli space where  $a_1$  aligns with  $b, a_2$  corresponds to a line of marginal stability for a D6-branes on  $C$ , which become unstable against decay into D6-branes on  $C_1, C_2$ .

The above phenomenon of brane dynamics has a counterpart in classical gauge field theory. The system of  $N$  D6-branes on  $C_1, C_2$  leads to a  $U(N)_1 \times U(N)_2$   $\mathcal{N} = 1$  supersymmetric

gauge theory (see later for a discussion of the  $U(1)$  factors), with chiral multiplets  $Q, \tilde{Q}$  in the  $(\square_1, \bar{\square}_2), (\bar{\square}_1, \square_2)$ , and  $\Phi$  in the adjoint of  $SU(N)_2$ . There is a classical superpotential

$$W = \text{tr } Q\Phi\tilde{Q} \quad (4.1)$$

The parameters of the gauge theory are the gauge couplings  $g_i$ , and theta angles  $\theta_i$ , which are classically related to  $C_i$  by

$$T_i = \frac{1}{g_i^2} + i\theta_i = \frac{1}{g_s} \int_{C_i} \Omega + i \int_{C_i} A_3 \quad (4.2)$$

where  $A_3$  is the type IIA RR 3-form. In the quantum theory, these parameters are traded for dynamical scales

$$\Lambda_1 = \exp\left(-\frac{T_1}{2N}\right) \quad ; \quad \Lambda_2 = \exp\left(-\frac{T_2}{N}\right) \quad (4.3)$$

The change in the complex structure associated to moving  $a_1$  off the horizontal axis corresponds to turning on a Fayet-Iliopoulos parameter  $\xi$  for the difference of the two  $U(1)$ 's. This triggers a vev for the bi-fundamental flavours  $Q$  or  $\tilde{Q}$ , depending on the sign of  $\xi$ , and breaking the gauge group to the diagonal  $U(N)$ . Assuming that  $Q$  acquires the vev, the fields  $\Phi, \tilde{Q}$  become massive by the superpotential and disappear. We are left with a  $U(N)$  pure SYM gauge theory, with complex gauge coupling

$$T = T_1 + T_2 = \int_C \Omega + \int_C A_3 \quad (4.4)$$

This agrees with the picture of the D6-branes recombining into D6-branes wrapped on  $C$ .

It is worth noting that the  $U(1)$  generators have  $BF$  Stueckelberg couplings with closed string moduli, which make the gauge bosons massive, so the  $U(1)$  factors are really absent from the low energy effective theory. This modifies the above discussion very mildly. Namely, instead of turning on a FI parameter, the above transition can be regarded as moving along the baryonic branch of the  $SU(N)_1 \times SU(N)_2$  theory, to yield a pure  $SU(N)$  SYM theory.

We would now like to consider the non-perturbative superpotential in these two systems, and in showing that it is continuous across the line of marginal stability. Interestingly, the non-perturbative effects have a microscopic description in terms of D2-brane instantons on the relevant 3-cycles, along the lines described in section 2.1. In the discussion we stick to the description in gauge theory language. Also for convenience we use the description where the  $U(1)$ 's are not included in the low-energy dynamics.

Consider first the system of  $N$  D6-branes on  $C$ . Since it corresponds to a pure SYM theory, it confines and develops a gaugino condensate. There is a non-perturbative superpotential

$$W = \Lambda^3 = (e^{-T/3N})^3 \quad (4.5)$$

Consider now the situation when the instanton reaches the line of marginal stability. We consider the system of  $N$  D6-branes on  $C_1$  and  $N$  D6-branes on  $C_2$ , so we essentially

have to study the dynamics of the  $SU(N)_1 \times SU(N)_2$  gauge theory. Let us focus in the regime where  $\Lambda_1 \gg \Lambda_2$ , so the dynamics of  $SU(N)_1$  dominates.

In this case the  $SU(N)_1$  group confines first. It has  $N_f = N_c$ , so the instanton on  $C_1$  is a Beasley-Witten instanton, which induces a quantum deformation on the moduli space. Instead of using the intrinsic picture in moduli space and inducing an operator of the form (2.5), we prefer to work as usual in field theory analysis, by imposing the deformation by a quantum modified constraint. We describe the system in terms of mesons  $M$  and baryons  $B\tilde{B}$ , with superpotential:

$$W = \mu \Phi M + \mu^{-2N+2} X \left( \det M - B\tilde{B} - \Lambda_1^{2N} \right)$$

where we have introduced the scale  $\mu$  to keep the dimension of the operator in the superpotential invariant. This dynamical scale will be of the order of  $\Lambda_1$ , so we use it in what follows.

The F-term for  $\Phi$  enforces  $M = 0$ , and vice versa. The fields  $\Phi$  and  $M$  are massive, so we can integrate them out. We are left with a pure  $SU(N)_2$  SYM theory, with dynamical scale  $\Lambda_f$ , to be determined later. In addition we have the singlets  $X$ ,  $B$ ,  $\tilde{B}$ , with superpotential

$$W \simeq X \left( B\tilde{B} + \Lambda_1^{2N} \right)$$

The theory has a one-complex dimensional baryonic moduli space, but these singlets do not modify the theory otherwise.

The dynamical scale  $\Lambda_f$  is determined by the matching, in analogy with the discussion in (2.1.1), as

$$\Lambda_f^{3N} = \Lambda_2^N \Lambda_1^{2N} \quad (4.6)$$

and is in fact the same as the  $\Lambda$  introduced above.

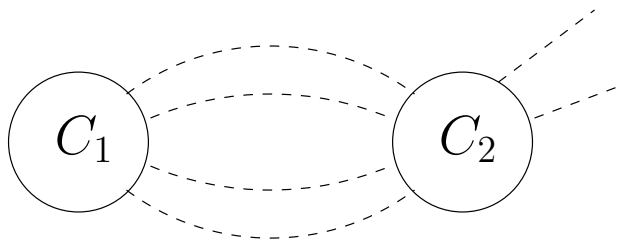
In this left-over  $SU(N)_2$  pure SYM theory, the effect of the (fractional) instanton on  $C_2$  is simply to develop a gaugino condensate non-perturbative superpotential

$$W = \Lambda_f^3 = \left( e^{-T/3N} \right)^3 \quad (4.7)$$

in agreement with (4.5).

This example provides a non-trivial and simple realization of the continuity of superpotentials across lines of marginal stability. The instanton wrapping  $C$  reaches the line of marginal stability, at which it splits into two BPS instantons, wrapping  $C_1$  and  $C_2$ . The instanton on  $C_1$  is of Beasley-Witten type and deforms the moduli space. The instanton on  $C_2$ , once the effect of the instanton on  $C_1$  is taken into account, induces a non-perturbative superpotential. The total effect neatly adds up to the effect of the single instanton on  $C$  before crossing the line of marginal stability.

For completeness, let us mention that the discussion with  $U(1)$ 's in the effective action is similar. There are no baryonic operators, so there are no fields left out after integrating out  $M$ ,  $\Phi$ . The one-dimensional moduli space is realized in this view in the closed sector, as the FI term for the relative  $U(1)$  corresponding to the position of  $a_1$  off the horizontal axis.



**Figure 5:** Schematic picture of the multi-instanton configuration discussed in the text.

## 4.2 Microscopic interpretation

In this section we discuss the microscopic interpretation of the continuity of the non-perturbative superpotential of the above configuration in terms of D-brane instanton physics.

### 4.2.1 The 2-instanton process

In analogy with the discussion for non-gauge instanton in section 3.1.2, and from the above discussion, it is clear that the superpotential contribution at the line of marginal stability arises from a two-instanton process, involving the instantons  $C_1$  and  $C_2$ . In fact, it is possible to compute the set of zero modes for the two-instanton system, and their interactions.

We skip the detailed discussion and just sketch the result. The contributions to the superpotential localize on configurations of instantons coincident in 4d. In addition the 3-cycle  $C_1$  is non-rigid, and there is a bosonic zero mode  $\phi$  parametrizing a branch where the instanton on  $C_1$  slides away from the D6-branes on  $C_1$ . Along this branch the configuration has additional zero modes  $\chi, \bar{\chi}$  (the partners of  $\phi$ ), which are not saturated. Hence the contributions to the superpotential localize at  $\phi = 0$ . At this point one can easily check that all fermion zero modes except for the two overall Goldstinos  $\theta_1 + \theta_2$  have non-trivial interactions, which can be pulled down to saturate the corresponding integrals.

The whole process can be described in spacetime in terms of a diagram 5. The instanton  $C_2$  has six unsaturated fermion zero modes, since it is a Beasley-Witten instanton with  $N_f = N_c$  (thus leading to two unsaturated fermion zero modes beyond the two  $\mathcal{N} = 1$  Goldstinos) and two additional fermion zero modes  $\chi, \bar{\chi}$  from being on a non-rigid cycle. The instanton  $C_1$  has four unsaturated fermion zero modes, since it is an  $N_f = N_c$  Beasley-Witten instanton. In the two-instanton process, one can generate interactions between the zero modes of the two instantons via the bosonic zero modes charged under both, which allow to contract four fermion zero modes, leading to an overall process with only two fermion zero modes.

### 4.2.2 Non-perturbative lifting of fermion zero modes

Along the lines of the discussion in section 3.1.3, we would like to improve on the additional viewpoint of the process as a lifting of fermion zero modes of the instanton  $C_2$  by a non-perturbative effect induced by the instanton  $C_1$ . In fact, for gauge instantons the mechanism can be posed in a much sharper setup. Consider a gauge instanton  $A$ , with field

configuration  $\mathcal{A}(x^\mu; \varphi, \psi)$ , as a function of the sets of bosonic and fermionic zero modes,  $\varphi, \psi$ . Here  $\mathcal{A}$  denotes the set of all 4d fields involved in the configuration. The classical effective action for the zero modes  $S_{\text{inst.}}(\varphi, \psi)$  can be obtained by replacing the instanton field configuration on the 4d action  $S_{4d}[\mathcal{A}] = \int d^4x \mathcal{L}[\mathcal{A}]$ , namely

$$S_{\text{inst}}(\varphi, \psi) = \int d^4x \mathcal{L}[\mathcal{A}(x^\mu; \varphi, \psi)] \quad (4.8)$$

From this point of view, any additional term in the 4d effective action  $\delta S_{4d}[\mathcal{A}]$  induces a corresponding term on the instanton effective action  $\delta S_{\text{inst}}(\varphi, \psi)$ .

$$\delta S_{\text{inst}} = \int d^4x \delta \mathcal{L}[\mathcal{A}(x^\mu; \varphi, \psi)] \quad (4.9)$$

When the additional term in the 4d effective action  $\delta S_{4d}$  is induced by another instanton  $B$ , the term  $\delta S_{\text{inst.}}(\varphi, \psi)$  can in a very precise sense be regarded as a non-perturbative interaction term for the zero modes of  $A$  induced by the instanton  $B$ . In particular interaction terms of this kind involving the fermion zero modes  $\psi$  of  $A$  are non-perturbatively lifted by the instanton  $B$ . Notice that the  $g_s$  dependence arranges in such a way that the total 4d effect is suppressed by the exponential factors of both instantons  $A$  and  $B$ , as in (3.12).

On general grounds, we may expect that an instanton  $B$  with  $k$  fermion zero modes induces a 4d F-term leading to contributions to  $S_{4d}$  with  $k$  4d fermion insertions. This will in general induce an interaction term  $S_{\text{inst.}}(\varphi, \psi)$  on the instanton  $A$  lifting  $k$  fermion zero modes. The spacetime picture of the process is a two-instanton process where  $k$  fermion zero modes of the two instantons are contracted against each other. In particular, in our gauge theory example, the instanton  $C_2$  induces a 4d effective operator corresponding to a 4-fermion F-term, which then induces a 4-fermion interaction term on the effective action for the zero modes of instanton  $C_1$ . Since the instanton  $C_1$  has six fermion zero modes, the lifting of four leaves only the two Goldstinos, so that there is a non-trivial contribution to the superpotential.

To conclude, we would like to add yet another equivalent, but related, viewpoint on the non-perturbative lifting of zero modes. The idea is based on a generalization of the analysis in section 4.3 of [36], which discussed the effects of a (perturbative) superpotential mass term on instantons with additional fermion zero modes. Consider an instanton  $A$  with  $k = 2n$  fermion zero modes beyond the two Goldstinos, and leading to a 4d higher F-term (2.5)

$$\int d^4x d^2\theta \mathcal{O}_w = \int d^4x d^2\theta w_{\bar{i}_1 \bar{j}_1 \dots \bar{i}_n \bar{j}_n}(\Phi) \mathcal{O}^{\bar{i}_1 \bar{j}_1} \dots \mathcal{O}^{\bar{i}_n \bar{j}_n} \quad (4.10)$$

with

$$\mathcal{O}_{\bar{i}\bar{j}} = \overline{D\Phi^{\bar{i}}} \overline{D\Phi^{\bar{j}}} \quad (4.11)$$

The operator  $\mathcal{O}_w$  is chiral (despite its appearance). In the presence of an additional superpotential  $W(\Phi)$ , the supersymmetry algebra is modified (since the fermion variations change,  $\delta\psi = F = -\overline{\partial W / \partial \Phi}$ ) and  $\mathcal{O}_w$  is no longer chiral. Still, since the instanton  $A$

remains BPS, it should induce an F-term. Indeed, in [36] it was argued that (for superpotential mass terms), there is a suitable deformation  $\tilde{\mathcal{O}}_w$  of  $\mathcal{O}_w$  which is chiral in the presence of the superpotential. The instanton amplitude is now given by

$$\int d^4x d^2\theta \tilde{\mathcal{O}}_w = \int d^4x d^2\theta w_{\tilde{i}_1\tilde{j}_1\ldots\tilde{i}_n\tilde{j}_n}(\Phi) \tilde{\mathcal{O}}^{\tilde{i}_1\tilde{j}_1} \ldots \tilde{\mathcal{O}}^{\tilde{i}_n\tilde{j}_n} \quad (4.12)$$

where, generalizing the result in [36],  $\tilde{\mathcal{O}}_{\tilde{i}\tilde{j}}$  has schematically the structure

$$\tilde{\mathcal{O}}^{\tilde{i}\tilde{j}} = \overline{D\Phi}^{\tilde{i}} \overline{D\Phi}^{\tilde{j}} + W^{\tilde{i}\tilde{j}} \quad (4.13)$$

Note that the total effect is that the instanton generate effective vertices not only with  $2n$  fermionic external legs, but also with  $2n - 2p$  fermionic external legs (with  $p$  taking several possible values, depending on the detailed structure of  $W$ ). The 4d interpretation is that  $2p$  fermionic legs have been soaked up by  $p$  insertions of the superpotential interaction.

In fact, one is lead to suspect a further generalization of the above argument. Consider the instanton  $A$  in the presence, not of a 4d superpotential term, but of a higher F-term (which could be of perturbative or non-perturbative origin). Consider the latter to be of the form

$$\delta S_{4d} = \int d^4x d^2\theta W_{\tilde{i}_1\tilde{j}_1\ldots\tilde{i}_m\tilde{j}_m}(\Phi) \overline{D\Phi}^{\tilde{i}_1} \overline{D\Phi}^{\tilde{j}_1} \ldots \overline{D\Phi}^{\tilde{i}_m} \overline{D\Phi}^{\tilde{j}_m} \quad (4.14)$$

Namely it leads to 4d interactions with  $2m$  4d fermions, and we assume  $m < n$ . Although we do not have a precise argument based on the supersymmetry algebra, we expect the amplitude of the instanton  $A$  to be modified in the presence of such term in the 4d action. Let us define  $\tilde{n} = n \bmod m$  and  $r = (n - \tilde{n})/m$ , hence  $n = rm + \tilde{n}$ . The instanton amplitude is expected to take the schematic form

$$\int d^4x d^2\theta w_{\{\tilde{i}_1\tilde{j}_1\}\ldots\{\tilde{i}_r\tilde{j}_r\}\tilde{p}_{\tilde{n}}} \mathcal{O}^{\{\tilde{i}_1\tilde{j}_1\}} \ldots \mathcal{O}^{\{\tilde{i}_r\tilde{j}_r\}} \overline{D\Phi}^{\tilde{k}_1} \overline{D\Phi}^{\tilde{p}_1} \ldots \overline{D\Phi}^{\tilde{k}_{\tilde{n}}} \overline{D\Phi}^{\tilde{p}_{\tilde{n}}}$$

where  $\{i_q, j_q\}$  denotes an  $m$ -plet of indices  $i_{q1}, j_{q1} \ldots i_{qm}, j_{qm}$ , and

$$\mathcal{O}^{\{\tilde{i}\tilde{j}\}} = \mathcal{O}^{\tilde{i}_1\tilde{j}_1\ldots\tilde{i}_m\tilde{j}_m} = \overline{D\Phi}^{\tilde{i}_1} \overline{D\Phi}^{\tilde{j}_1} \ldots \overline{D\Phi}^{\tilde{i}_m} \overline{D\Phi}^{\tilde{j}_m} + W^{\tilde{i}_1\tilde{j}_1\ldots\tilde{i}_m\tilde{j}_m} \quad (4.15)$$

The interpretation is that in the presence of the 4d F-term (4.14), the instanton with  $2n$  fermion zero modes can generate effective vertices with  $2n - 2m$  external fermionic legs, by having sets of  $2m$  fermionic legs soaked up by the F-term (4.14).

The above discussion can be carried out to the situation where the modification to the 4d action is induced by a second instanton  $B$  with  $2m$  fermion zero modes (which could be a gauge instanton or a non-gauge D-brane instanton). In the spacetime picture, we would have a multi-instanton process involving  $A$  and  $B$ , in which some of the fermionic external legs of the instanton  $A$  are soaked up by the 4d effective interaction induced by  $B$ . A simple example would be to consider the instanton  $B$  to have two fermion zero modes, so it generates a superpotential, thus fitting into the situation leading to (4.13). In fact, a particular case fitting within the analysis in [36] can be obtained by considering



the instanton  $B$  to be a non-gauge D-brane instanton inducing a superpotential mass term in the 4d action. Explicit examples of this have been considered e.g. in [18, 21, 41]. Our example of gauge theory instantons above corresponds to a more general situation of the kind (4.15), with the instantons  $A, B$  given by the instantons  $C_1, C_2$  (and  $n = 3, m = 2$ )

As a last remark, we expect processes with non-gauge instantons to admit a similar interpretation. Thus the contribution to the superpotential arising from the two-instanton process involving the  $U(1)$  and the  $O(1)$  instantons can be regarded as the 4d effective term induced by the  $U(1)$  instanton in the presence of the additional 4d interaction induced by the  $O(1)$  instanton.

### 4.3 Adding semi-infinite D-branes

It is interesting to consider some simple modifications of the above discussion in the presence of additional semi-infinite D6-branes sticking out of the  $\mathbf{C}^*$  degenerations. From the field theory viewpoint they correspond to the addition of extra flavours for some of the gauge factors. From the viewpoint of the instantons, they lead to additional fermion zero modes. In this section we consider a few possibilities

In the above situation we have focused on a case where the non-perturbative dynamics reduces to that of pure SYM. However, it is straightforward to modify the setup to SQCD with  $N_f$  flavours. It suffices to introduce a stack of  $N_f$  D6-branes wrapping the non-compact 3-cycle obtained from a horizontal semi-infinite line starting from the degeneration  $a_2$  (this can be regarded as a limit of infinite 3-cycle volume of a geometry with a second  $b$ -type degeneration, located on the far right of the figure). The above argument goes through, and implies the continuity of the non-perturbative superpotential across the line of marginal stability. Notice that in the particular case of  $N_f = N - 1$  the instantons under discussion are familiar gauge theory instantons.

Another straightforward addition of semi-infinite branes is to consider adding  $K$  D6-branes stretching from the  $a_1$  degeneration horizontally to the left infinity. Note that for the configuration in figure 4a, these D6-branes hit the  $b$  degeneration, so the configuration can be regarded as  $K$  D6-branes stretching along  $(-\infty, b]$ ,  $N + K$  on  $[b, a_1]$  and  $N$  on  $[a_1, a_2]$ . For the configuration in figure 4c, we have  $N$  D6-branes on  $[b, a_2]$  and a disconnected set of  $K$  D6-branes from left infinity to  $a_1$ .

It is easy to carry out an analysis similar to the above to derive the continuity of the superpotential. In the initial configuration, the gauge factor  $SU(N + K)$  has  $N_f = N_c$  and thus a Beasley-Witten instanton deforming its moduli space and forcing the gauge factor onto the baryonic branch. The adjoint of the  $SU(N)$  factor pairs up with some of the mesons and becomes massive, so the left over pure SYM theory develops a gaugino condensation superpotential. One recovers the same result from the instanton contribution in the final configuration figure 4c (upon matching of scales along the lines in section 2.1).

### 4.4 Gauge theory instantons and Seiberg duality

In this section we elaborate on an interesting point. It is a familiar fact that the realization of Seiberg duality in terms of the D-brane construction of gauge theories corresponds to

a motion in moduli space (in which D-branes typically break up and recombine) [38, 46–49] (see also [50–52] for other related approaches). Therefore they provide a large class of examples of motion across lines of marginal stability in which the non-perturbative superpotential is continuous.

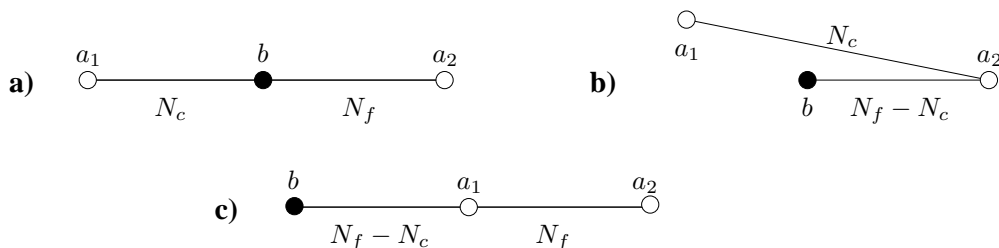
A comment is in order here. From field theory experience we know that Seiberg duality involves a non-trivial change of variables in the 4d chiral multiplets. We also know that tree-level superpotentials are crucial in matching properties of two Seiberg-dual theories. Both properties are related to the following fact. Seiberg dualities in the D-brane realization of field theories can be described as a motion between two points  $P$  and  $Q$  in moduli space, at each of which we have D-branes wrapped on cycles, whose sizes control the gauge couplings and thus the strength of instanton effects. This motion typically involves a region in moduli space larger than the radius of convergence of the instanton expansion at either point. In other words, the operation can also be described as a continuation past infinite coupling, in the sense that they can be obtained by shrinking a cycle  $C$  on which 4d space filling branes wrap and growing a cycle  $C'$  which is in the opposite homology class  $[C'] = -[C]$ . The point  $O$  where the cycle shrinks is strongly coupled from the viewpoint of the original instanton at  $P$ , but a different weakly coupled description is available at  $Q$  (and vice versa). The change of description has several effects, which we take into account implicitly in our discussions below:

- It relates the strengths of the instantons as  $e^{-T} = (e^{-T'})^{-1}$ , where  $T, T'$  control the sizes of  $C, C'$ . This underlies the fact that matching of scales in the Seiberg duality encodes the continuity of the superpotential as a function of the closed string moduli.
- It implies a non-trivial change of variables in the 4d chiral multiplets, hence the comparison of the superpotentials at  $P$  and  $Q$  requires expressing the open string 4d multiplets in terms of gauge invariant operators.
- It can map tree-level and non-perturbative superpotentials to each other. Thus the continuity applies to the full superpotential.

The D-brane realization of Seiberg duality for large classes of field theories thus provides a large class of examples of continuity of the non-perturbative superpotential across lines of marginal stability (with the appropriate change of variables for the charged matter fields). We restrict to the description of this phenomenon with simple examples, which are illustrative for this whole class.

Notice that it is easy to provide a D-brane realization of the original Seiberg duality [53] using the above geometries following [38], as we review now, see figure 6.

Consider a geometry with three aligned degenerations ordered as  $a_1, b, a_2$ , and introduce  $N_c$  D6-branes on  $[a_1, b]$  and  $N_f$  on  $[b, a_2]$ , with  $N_f \geq N_c$ , figure 6a. This describes the electric theory of  $SU(N_c)$  SQCD with  $N_f$  flavours, with a gauged flavour group. Now move up the degeneration  $a_1$ . The minimal energy configuration is obtained when  $N_c$  D6-branes



**Figure 6:** Realizing Seiberg duality in terms of D-branes: a) The electric configuration. b) We move  $a_1$  up a bit. The original branes are now nonaligned, so they recombine to minimize their tension. c) Finally moving  $a_1$  all the way to the middle position we get the magnetic dual theory.

recombine at  $b$ , so we have  $N_c$  D6-branes on the tilted<sup>4</sup> segment  $[a_1, a_2]$  and  $N_f - N_c$  on  $[b, a_2]$ , figure 6b. Now move  $a_1$  to the right and bring it down between  $b$  and  $a_2$ . The  $N_c - N_f$  D6-branes on  $[b, a_2]$  split, so we are left with  $N_f - N_c$  D6-branes on  $[b, a_1]$  and  $N_f$  on  $[a_1, a_2]$ . This describes the magnetic theory (again with gauged flavour group). Note that the gauging of the flavour group is just for the purposes of introducing configurations to be used later; a realization of the pure Seiberg duality can be obtained simply by sending the degeneration  $a_2$  to right infinity.

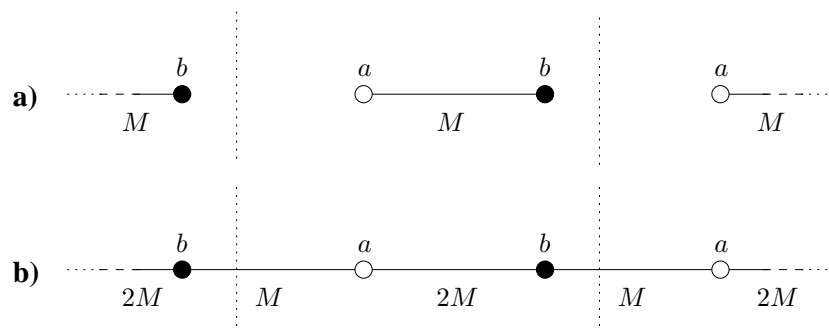
Clearly the possibility of embedding Seiberg dualities in terms of D-branes provides a huge class of examples of brane systems crossing walls of marginal stability. The continuity of the non-perturbative superpotential in these processes is automatically guaranteed by the field theory argument for the matching of scales, as discussed above. We will not delve into a more detailed discussion, and simply discuss some particular examples related to systems in other sections.

Let us focus on some particularly simple examples where the basic splitting processes of the D6-branes are of the kind analyzed in the previous section. Consider the situation with  $N$  D6-branes on  $[b, a_2]$  and no D6-branes on  $[a_1, b]$ . The  $a_1$  degeneration has no D6-branes attached, so moving it between the degenerations  $b, a_2$  is exactly the inverse process of the one in figure 4, studied in section 4.1.

For future convenience let us consider another example, now involving semi-infinite D6-branes. Consider the initial configuration with degenerations ordered as  $a_1, b, a_2$  and introduce  $N$  D6-branes on  $(-\infty, a_1)$ , no D6-branes on  $[a_1, b]$  and  $K$  D6-branes on  $[b, a_2]$ . As one moves the  $a_1$  degeneration between  $b, a_2$ , it drags the  $K$  semi-infinite D6-branes, which end up split in the final configuration. In the latter we have  $K$  D6-branes on  $(-\infty, b)$ ,  $N + K$  D6-branes on  $(b, a_1)$ , and  $N$  D6-branes on  $(a_1, a_2)$ . The splitting process of the semi-infinite D6-branes is exactly as in the last system of section 4.3, where we showed the continuity of the non-perturbative superpotential.

As a final example based on the configuration in the previous paragraph, let us consider a type IIA configuration mirror to D-branes at the conifold, and (one of the steps of)

<sup>4</sup>The tilting breaks supersymmetry in the intermediate steps of the argument; there are however simple modifications of the setup which allow to preserve supersymmetry throughout the process [38]. We skip their discussion since they will not be needed in our examples below.



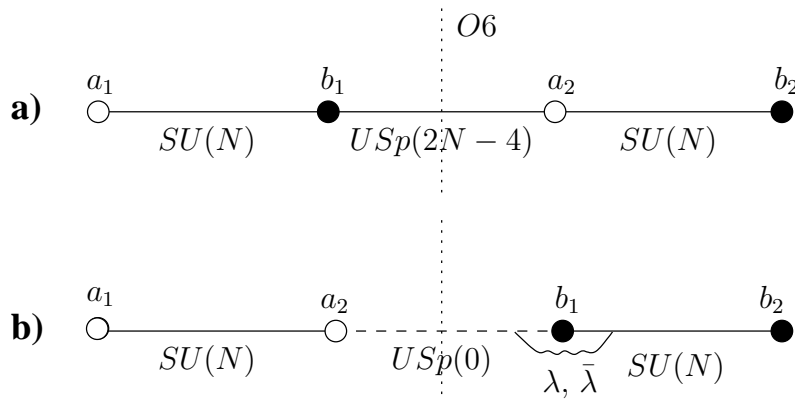
**Figure 7:** The periodic configuration dual to the conifold. The dotted vertical line denotes the period. a) Final step of the cascade. b) One step up in the cascade. We reach this point by moving all  $a$  degenerations one cell to the left.

the celebrated Klebanov-Strassler duality cascade [54]. Following [43, 55], a system of D-branes at a conifold can be realized in terms of D4-branes suspended (along a circle direction) between two rotated NS-branes. Equivalently, one can use an infinite periodic array of rotated NS-branes with suspended D4-branes. This systems can be mapped to one of our familiar double  $\mathbf{C}^*$ -fibration geometries by simply introducing a periodic array of degenerations  $\dots, a, b, a, b, \dots$ , with D6-branes on the finite segments, as shown in figure 7. This is equivalent to (but easier to visualize than) a double  $\mathbf{C}^*$  fibration over a cylinder, with one degeneration of each type.

Consider the configuration on figure 7a, with  $M$  D6-branes on the intervals of type  $[a, b]$ , and no D6-branes on those of type  $[b, a]$ . This describes the theory at the end of the duality cascade, and corresponds to  $SU(M)$  SYM, with a non-perturbative superpotential induced by a  $1/M$ -fractional instanton. Consider now the geometric operation that takes us one step up the cascade. This corresponds to moving the  $a$ -type degenerations once around the period, coming back to its original position in the periodically identified geometry but moving one period to the left in the covering space we are drawing. We do this in the same way as above: moving up the  $a$  singularity a bit, taking it one cell to the left, and finally returning it to its original vertical position. The resulting configuration is shown in figure 7b and contains  $M$  D6-branes on the  $[b, a]$  intervals and  $2M$  D6-branes on the  $[a, b]$  intervals. The geometric process, and in particular the splitting of branes, is exactly as that considered two paragraphs above, for  $K = N \equiv M$ . The continuity of the superpotential is easily derived, by showing (using the instanton interpretation of the field theory analysis in [56]) that the Beasley-Witten instanton of the  $SU(2M)$  theory (which has  $N_f = N_c$ ) deforms the moduli space of this theory and forces it into the baryonic branch, while the  $1/M$ -fractional instanton on the left over  $SU(M)$  theory (with scale suitably computed by matching) generates the superpotential.

## 5. Exotic instantons becoming gauge instantons

In the previous sections we have argued continuity of the non-perturbative superpotential



**Figure 8:** Prototypical example of a D-instanton effect being equivalent to a gauge theory effect via Seiberg duality. Figure a) shows the geometry leading to a  $USp \times SU$  gauge theory upon wrapping D6-branes on the appropriate 3-cycles. The dotted line denotes the orientifold plane. Figure b) shows the configuration after the motion in moduli space corresponding to Seiberg duality. There are no D6-branes in the 3-cycle  $[a_2, b_1]$ , but an instanton (dashed line) wrapped on it can contribute to the superpotential. We have indicated the charged fermionic zero modes  $\lambda, \bar{\lambda}$  between the D-brane instanton and the gauge D-brane.

for gauge and non-gauge D-brane instantons, in several examples. In this section we would like to consider a slightly more general situation where the nature of the instanton changes in the process of reaching lines of marginal stability. Namely a non-gauge D-brane instanton ends up as a gauge D-brane instanton after some motion in moduli space.

A prototypical situation where this takes place is in duality cascades [54] (see also e.g. [57–60]) of quiver gauge theories, in which one of the nodes of the quiver becomes eventually empty of 4d space filling branes. D-brane instantons which occupied this node change from gauge to non-gauge instantons in the motion in moduli space associated to the cascade. Since we are interested in studying contributions to the superpotential, one would need to consider cascades of orientifolded quiver gauge theories. In fact, this kind of analysis has been carried out in [61] in one particular example, focusing on the relevant part of the superpotential for the infrared theory. In section 5.2 we revisit the system in our language, and recover that the full superpotential is well-behaved in the process. Our analysis reproduces some pieces dropped in [61], which are irrelevant in the infrared, but are still part of the full superpotential of the theory.

Before revisiting the example of the duality cascade, let us consider the simplest case where a non-gauge D-brane instanton becomes a gauge theory effect.

### 5.1 Dualizing the $O(1)$ instanton

Let us consider a geometry of the kind in section 2.2, with an  $O6$ -plane, see figure 8. Let us wrap stack of D6-branes on the different 3-cycles corresponding to the configuration in figure 8a. The low energy dynamics of this configuration is given a  $SU(N) \times USp(2N-4)$

gauge theory, with quarks  $q \in (\square_{\text{SU}}, \square_{\text{USp}})$ ,  $\tilde{q} \in (\bar{\square}_{\text{SU}}, \square_{\text{USp}})$  and superpotential

$$W = q\tilde{q}q\tilde{q} \quad (5.1)$$

Let us focus on the strong dynamics for the  $USp$  theory.<sup>5</sup> As argued in [62], when the  $USp$  node becomes strongly coupled the theory has an effective description (corresponding to its Seiberg dual) in which the  $USp$  group confines completely, and the fundamental degrees of freedom are the mesons:

$$M_{\square} = q \cdot q ; \quad M_{\bar{\square}} = \tilde{q} \cdot \tilde{q} ; \quad M_{\text{Adj}} = q \cdot \tilde{q} \quad (5.2)$$

where we have expressed the mesons in terms of the electric fields,<sup>6</sup> and the dot denotes contraction in the  $USp$  indices, which antisymmetrizes the fields. The subindex denotes the representation of the  $SU(N)$  group under which the meson transforms. There is also a superpotential implementing the classical constraint between the mesons, which can be written as

$$W_0 = \text{Pf} \begin{pmatrix} M_{\square} & M_{\text{Adj}} \\ -M_{\text{Adj}} & M_{\bar{\square}} \end{pmatrix} \quad (5.3)$$

Adding the original superpotential in terms of the mesons we obtain

$$W = W_0 + M_{\bar{\square}} M_{\square} \quad (5.4)$$

We can solve the equations of motion for the massive mesons  $M_{\bar{\square}}$ ,  $M_{\square}$  just by setting them to zero. The resulting superpotential is then given by:

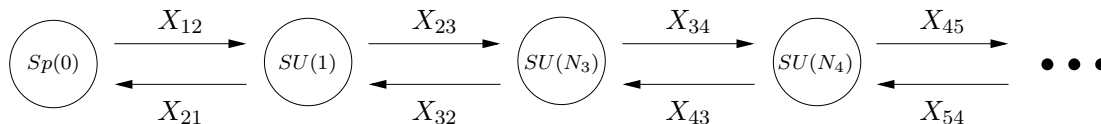
$$W = \text{Pf} \begin{pmatrix} 0 & M_{\text{Adj}} \\ -M_{\text{Adj}} & 0 \end{pmatrix} = \det M_{\text{Adj}}. \quad (5.5)$$

We can now perform a brane motion taking the configuration to that in figure 8b, where there is no brane stretching on the 3-cycle  $[a_2, b_1]$ . This result takes into account the brane creation effects due to the presence of the orientifold planes, as discussed in [38]. Despite the non-trivial change in the brane configuration, the superpotential is continuous. Namely the above superpotential is still generated, but now via an exotic  $O(1)$  instanton on  $[a_2, b_1]$  which can contribute.<sup>7</sup> The calculation in this case is simple. In figure 8b, the theory on the  $SU(N)$  brane is locally  $\mathcal{N} = 2$ , in particular it has an adjoint, which we identify with the adjoint meson of the gauge analysis (in both cases it parametrizes sliding

<sup>5</sup>There are additional non-perturbative effects from the  $SU(N)$  factor, which can also be followed along the transition below, in analogy with our examples above (in fact, they map to 2-instanton effects after the transition). We skip their discussion in order to emphasize the main point.

<sup>6</sup>We have omitted here the meson singlet under the  $SU(N)$ . In the stringy setup it will get a mass due to a coupling related by the  $\mathcal{N} = 2$  susy to the one giving mass to the  $U(1)$  gauge boson.

<sup>7</sup>Recall that the orientifold projection acts oppositely on 4d space filling D6-branes and D2-brane instantons, so an orientifold giving a  $USp$  gauge group will give a  $O(1)$  D-instanton. This works in the same way as for the perhaps more familiar D5-D9 system.



**Figure 9:** Relevant nodes of the quiver theory for the orbifolded conifold. We have indicated the ranks at the bottom of the cascade. There can be more  $SU(N)$  nodes to the right, ending with another  $USp(N)$  group.  $X_{ij}$  denotes the bifundamental from node  $i$  to node  $j$ .

the D6-branes along the two  $b$ -type degenerations, and their images along the  $a$ -type ones). The zero modes  $\lambda, \bar{\lambda}$  between the D2-brane instanton and the  $SU(N)$  brane couple to this adjoint via a term

$$S = \cdots + \lambda M_{\text{Adj}} \bar{\lambda} \quad (5.6)$$

in the instanton action (this has the same origin as the usual coupling between the adjoint and the flavors in  $\mathcal{N} = 2$  theories). Integrating over the fermionic zero modes gives us the determinant operator we found in equation 5.5. We thus recover the same kind of superpotential, with an exponential dependence on the closed string modulus associated to the 3-cycle defined by the degenerations  $b_1$  and  $a_2$ . Thus the result is continuous across the motion in moduli space, in which gauge and non-gauge instantons turn into each other.<sup>8</sup>

## 5.2 A duality cascade example

Let us proceed to the more complicated case of the duality cascade studied in [61], and show the continuity of the non-perturbative superpotential along a complicated chain of Seiberg dualities.

The theory under consideration is given by the quiver in figure 9, with gauge group at the bottom of the cascade given by  $USp(0) \times SU(1) \times SU(N_3) \times \cdots$  with  $N_3, \dots$  arbitrary. The superpotential is given by:

$$W = \sum_{i=1}^{N_{\text{factors}}} (-1)^i X_{i,i+1} X_{i+1,i+2} X_{i+2,i+1} X_{i+1,i}. \quad (5.7)$$

This theory can be easily realized in string theory by modding out an orbifold of the conifold [43] by a suitable orientifold action [63]. In terms of the geometries in section 2.2, we can consider a periodic array of degenerations  $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$  and introducing an orientifold quotient  $\Omega R(-1)^{F_L}$ , with  $R$  given by (3.2).

In terms of the geometrical setup, the cascade of Seiberg dualities simply amounts to a motion in moduli space, generalizing that discussed above. In this situation there are also some brane creation effects due to the presence of the orientifold planes, as discussed

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<sup>8</sup> One may try to discuss a similar configuration without the external degenerations and with semi-infinite branes. In this case the computation 8a gives a non-zero superpotential, while in 8b there are no dynamical mesons to help saturate the fermion zero modes of the instanton, hence there is no superpotential. The mismatch is related to the non-compact D-branes in the configuration (see comment at the beginning of section 2.2). Upon “compactification” by adding the external degenerations at a finite distance, one recovers the above full agreement.

in [38]. The configuration one step up in the cascade is given by the same quiver but with different ranks:

$$\mathrm{USp}(2N_2 - 4) \times \mathrm{SU}(N_2) \times \mathrm{SU}(N_4 - 1) \times \mathrm{SU}(N_4) \times \dots \quad (5.8)$$

In particular all nodes are occupied so there are no non-gauge instantons at this level.

The detailed gauge theory analysis of [61] for the initial configuration shows that the nonperturbative superpotential of the initial gauge theory configuration can be described in terms of the fields at the end of the cascade as

$$W_{\mathrm{np}}^{\mathrm{bottom}} = X_{23}X_{32} + \det(X_{34}X_{43}) + X_{23}X_{34}X_{43}X_{32} + \dots \quad (5.9)$$

where we have omitted some quartic terms of the same form as those in (5.7), which are tree level from the point of view of  $g_s$ , and thus not particularly interesting here. We rather focus on the first two terms, which are nonperturbative. Our aim is to recover them by studying possible D-brane instantons in the final configuration. Note that the determinant term was dropped as irrelevant in [61], since they were just interested in the infrared behaviour of the theory. We are interested in the continuity of the full superpotential so we should keep it, since it carries an implicit dependence on the closed string modulus controlling the corresponding cycle. For completeness, let us reproduce here a sketch of the gauge theory analysis done in [61]:

### 5.2.1 The gauge analysis

We will assume for simplicity a hierarchy of scales given by

$$\Lambda_1 \gg \Lambda_3 \gg \dots \gg \Lambda_2 \gg \Lambda_4 \gg \dots \quad (5.10)$$

We will choose the ranks in such a way that the bottom of the cascade is described by the quiver in figure 9. This can be achieved by choosing the following ranks:

$$N_1 = 2N_2 - 4 \quad ; \quad N_3 = N_4 - 1. \quad (5.11)$$

Due to the hierarchy of scales we have chosen, the first node to become strongly coupled is the  $USp$  one. This goes just as in section 5.1, and we end up with a  $\mathrm{USp}(0)$  group, some mesons  $M_{\mathrm{Adj}}$  charged in the adjoint of  $\mathrm{SU}(N_2)$ , and a nonperturbative superpotential:

$$W_{\mathrm{np}} = \det M_{\mathrm{Adj}} \quad (5.12)$$

Now we have to dualize the  $\mathrm{SU}(N_3)$  node. We have  $N_f = N_2 + N_4$ , so the dual description is in terms of a  $\mathrm{SU}(N_f - N_c = N_2 + 1)$  gauge group, and the dual quarks and mesons. The mesons get a mass due to the quartic terms in the superpotential 5.7, and they can be integrated out. Also, there is a mass coupling coming from the superpotential between  $M_{\mathrm{Adj}}$  and the meson  $M_2^{(3)}$  of  $\mathrm{SU}(N_3)$  charged under the adjoint of  $\mathrm{SU}(N_2)$ . The relevant term in the superpotential looks like:

$$W = \dots + \det M_{\mathrm{Adj}} + M_{\mathrm{Adj}} M_2^{(3)} + M_2^{(3)} q_{23} q_{32} \quad (5.13)$$



with  $q$  the dual quarks. Integrating the mesons out, we end up with a superpotential:

$$W = \cdots + \det q_{23}q_{32} + q_{23}q_{34}q_{43}q_{32} \quad (5.14)$$

where we have included a piece of the quartic superpotential that will play a role in a moment.

Going down in energy, eventually the  $SU(N_2)$  node will become strongly coupled. It has  $N_f = N_2 + 1$ , coming just from the third node, so the gauge group confines completely (let us call the resulting node “ $SU(1)$ ”, as in the stringy picture of the duality there is a single brane remaining). The description is in terms of mesons  $M_3^{(2)}$  in the adjoint of the third node and baryons  $B_3, \tilde{B}_3$  in the fundamental and antifundamental. There is a superpotential given by:

$$W = \cdots + B_3 M_3^{(2)} \tilde{B}_3 - \det M_3^{(2)} \quad (5.15)$$

When the second node confines the  $q_{23}, q_{32}$  quarks get confined into baryons and mesons. In particular, the superpotential 5.14 can be expressed as:

$$W = \cdots + B_3 \tilde{B}_3 + M_3^{(2)} q_{34} q_{43} \quad (5.16)$$

The last step in the chain of dualities, as far as the first three nodes are concerned, comes from dualizing the fourth node. This is important for our discussion as it gives a mass to  $M_3^{(2)}$  via the dual of the last coupling in eq. 5.16. After dualizing node 4, we end up with a superpotential:

$$W = \cdots + B_3 M_3^{(2)} \tilde{B}_3 - \det M_3^{(2)} + B_3 \tilde{B}_3 + M_3^{(2)} M_3^{(4)} - M_3^{(4)} X_{34} X_{43} \quad (5.17)$$

where we have denoted as  $X_{34}, X_{43}$  the dual quarks of node 4 charged under node 3. We see that the mesons of node 3 get massive as expected. Integrating them out, one gets:

$$W = \cdots + B_3 X_{34} X_{43} \tilde{B}_3 - \det X_{34} X_{43} + B_3 \tilde{B}_3 \quad (5.18)$$

which is the same as the one in (5.9) up to a relabeling of the baryons as  $X_{23}, X_{32}$ .

### 5.2.2 D-instanton effects at the bottom of the cascade

Let us now consider the final configuration, where the 4d space filling D6-brane configuration gives rise to a structure  $USp(0) \times SU(1) \times SU(N_3) \times \dots$  with  $N_3, \dots$ . There are two instantons which can contribute to the superpotential. There is a non-gauge D-brane instanton arising from the cycle corresponding to the node of the quiver with no 4d space filling branes. As argued in [61] and we now review, it leads to the first mass terms in (5.9). The instanton has  $O(1)$  symmetry and has two neutral fermion zero modes. In addition it has two fermion zero modes  $\alpha$  and  $\beta$  from the open strings going from the D-instanton to the  $SU(1)$  brane. The instanton action contains a coupling of the form  $\alpha X_{23} X_{32} \beta$ , arising from the same disk instantons that produce the terms in (5.7). Integrating over these fermionic zero modes, we get a mass contribution to the superpotential:

$$\begin{aligned} W &= \dots + \int d\alpha d\beta \alpha X_{23} X_{32} \beta \\ &= \dots + X_{23} X_{32}. \end{aligned} \quad (5.19)$$

There is another D-brane instanton which contributes to the non-perturbative superpotential, and which involves a somewhat novel effect. It corresponds to a D-brane instanton on the node with 4d group “SU(1)”. This instanton does not have a proper gauge theory interpretation, but still it shares some common features with gauge instantons. Namely, since it is a U(1) instanton, not mapped to itself by the orientifold action, it has four fermion zero modes. The two Goldstinos of  $\mathcal{N} = 1$  supersymmetry remain, while the two accidental  $\mathcal{N} = 2$  Goldstinos have non-trivial couplings with the bosonic and fermionic zero modes in the sector of open strings between the instanton and the SU(1)-brane. For gauge D-brane instantons, integration over these zero modes imposes the fermionic ADHM constraints [16], and reproduces the correct measure on instanton (super)moduli space. In the present setup, we lack an appropriate gauge theory interpretation for the coupling, but its effect of leading to the saturation of the additional fermion zero modes remains. We are therefore left with the two Goldstinos  $\theta^\alpha$  needed for contributing to the superpotential. We still need to saturate the charged zero modes going from the D-instanton to the SU( $N_3$ ) group, there are  $2N_c$  of these,  $N_c$  of each chirality. Let us call them  $\lambda_{23}$  and  $\lambda_{32}$ . They can be saturated via the same kind of quartic coupling  $\lambda_{23}Y_{34}Y_{43}\lambda_{32}$  as above. Expanding the instanton action we get a contribution to the superpotential:

$$\begin{aligned} W &= \dots + \int [d\lambda_{23}][d\lambda_{32}] \exp(\lambda_{23}Y_{34}Y_{43}\lambda_{32}) \\ &\simeq \dots + \epsilon^{i_1\dots i_{N_3}} \epsilon^{k_1\dots k_{N_3}} (Y_{34}Y_{43})_{i_1,k_1} \dots (Y_{34}Y_{43})_{i_{N_3},k_{N_3}} \\ &\simeq \dots + \det(Y_{34}Y_{43}). \end{aligned} \tag{5.20}$$

which correctly reproduces the second term in the nonperturbative superpotential 5.9.

We see that there is a beautiful agreement between both computations. Clearly, there are plenty of other systems where the agreement between the superpotential up in the cascade and at the lower steps can be checked. We leave this analysis for the interested reader.

## 6. Topology changing transitions in F-theory

In this section we comment on an intriguing implication of the continuity of the non-perturbative superpotential, when considering the F-theory viewpoint on non-perturbative effects on systems of D7-branes near lines of marginal stability. The process of D7-branes splitting/recombining corresponds to a topology changing transition in F/M-theory, along the lines of [64]. Our results therefore imply a non-trivial relation between the non-perturbative superpotentials on topologically different Calabi-Yau fourfolds.

We restrict to a simple local analysis of such D7-brane system, and of its F-theory lift. Consider the type IIB D7-brane realization of the D-brane configuration studied in section 4.1. There are two stack of D7-branes wrapped on two holomorphic 4-cycles  $C_1$  and  $C_2$ , intersecting over a complex curve  $\Sigma$ . It is possible to consider concrete examples of Calabi-Yau threefolds and 4-cycles with  $h_{2,0}(C_1) = 1$ ,  $h_{2,0}(C_2) = 0$ , which would fit our example, but it is not necessary to illustrate the main point. In fact, the basic idea is already present in a local model in a neighborhood of a point  $P$  in  $\Sigma$ . Using local complex coordinates  $z, w, u$  we have D7-branes on  $C_1$ , described locally by  $w = 0$  (and

$z, u$  arbitrary) and D7-branes on  $C_2$ , described locally by  $z = 0$  (and  $w, u$  arbitrary). The curve  $\Sigma$  is locally parametrized by  $u$ . In this local analysis, the direction  $u$  is an spectator and we can ignore it in the following (although it can lead to global obstructions in the compact model). Thus we have a system of D7-branes wrapped on the locus  $zw = 0$ .

The F-theory lift of this configuration is described by an elliptic fibration over the threefold, with degenerate fibers (due to pinching of a 1-cycle) over the 4-cycle wrapped by the D7-branes. We can also work locally near the pinching of the elliptic fiber, and describe the geometry as a  $\mathbf{C}^*$  fibration. For  $n, m$  D7-branes on the two different 4-cycles, the local description of the fourfold is thus given by the spectator direction  $u$  times the manifold

$$xy = z^n w^m \quad (6.1)$$

This kind of geometries were introduced in [43]. Let us focus on the simplest representative,  $n = m = 1$ , the conifold. In fact, the configuration corresponds to the resolved conifold, with the 2-cycle described as follows. The fiber on top of the intersection locus  $z = w = 0$  on the base degenerates into two 2-spheres touching at two points. The class of the 2-cycle corresponds to one of these 2-spheres (while the sum is the class of the fiber). For intersecting D7-branes, the F/M-theory lift corresponds to the limit of vanishing 2-cycle (and no background 2-form potential can be turned on). We are thus at the singular conifold limit, in which there are massless states [65] (arising from wrapped M2-branes in the M-theory picture). These are nothing but the open strings degrees of freedom between the two D7-brane stacks.

Consider now the D7-brane system away from the line of marginal stability. The D7-branes recombine into a single smooth one, wrapped on a 4-cycle which is a deformation of the above, namely  $zw = \epsilon$ . In the local model,  $\epsilon$  corresponds to a modulus, a flat direction for the fields arising at the intersection of the D7-branes. In the global model the flat direction is obstructed by a D-term condition, and the value of  $\epsilon$  is fixed by the closed string modulus moving us away from marginal stability. The F-theory lift of this configuration corresponds to the geometry

$$xy = zw - \epsilon \quad (6.2)$$

This describes the deformed conifold. This is expected, since the massless charged states have acquired a vev, thus triggering a topology changing transition [66]. The behaviour of the arbitrary  $n = m$  case is similar, using the deformation  $xy = (zw - \epsilon)^n$ .

The local analysis shows that the crossing of a line of marginal stability corresponds to a topology change in the F/M-theory fourfold. The continuity of the non-perturbative superpotential in this case implies a non-trivial matching between topologically different spaces.

It would be interesting to have a more microscopic derivation of this result. We conclude by mentioning a few key points to this aim. The relevant instanton in the IIB picture is a D3-brane wrapped on the 4-cycle which splits at the line of marginal stability. As emphasized, the continuity of the process requires a non-trivial contribution from a 2-instanton process in the intersecting D7-brane configuration. In the F/M-theory lift, the effect arises from an M5-brane instanton wrapping a 6-cycle which splits, and there should exist a non-trivial contribution to the superpotential arising from a 2-instanton process involving the

two M5-brane instantons wrapped on the two components of the split 6-cycle. Thus our analysis of superpotentials from multi-instantons should apply to M5-brane instantons on M-theory on CY fourfolds. Clearly this goes beyond the analysis in [4], since one would require a suitable generalization to M5-brane instantons on singular 6-cycles. In this respect, notice that one can rephrase the multi-instanton process as a non-perturbative lifting of zero modes of one M5-brane (A) by the effects of a second M5-brane (B). This is not inconsistent with the arguments in [4], which were based on counting fermion zero modes chiral with respect to the  $U(1)$  symmetry acting on the normal directions transverse to the M5-brane A. Indeed the second M5-brane B can induce couplings which violate this  $U(1)$  (which acts on directions along the volume of the M-brane B). Thus the non-perturbative lifting mechanism is powerful enough to allow the appearance of contributions from instantons which violate the celebrated arithmetic genus condition in [4]. Concrete examples of this are provided by suitable F/M-theory versions of the type II models studied in this paper.

## 7. Conclusions and outlook

In this paper we have studied the microscopic mechanisms via which D-brane instanton computations lead to non-perturbative superpotential continuous across moduli space. This understanding has revealed interesting surprises, including the interesting role of multi-instanton contributions to the superpotential, and its interpretation as non-perturbative lifting of fermion zero modes.

These results go in the direction that D-brane instanton effects are subtler, and more abundant, than hitherto considered. It would be interesting to revisit some of the models considered in the literature and look for additional contributing instanton processes of the kind we have introduced.

The computation of multi-instanton processes to the superpotential are involved, and require the precise knowledge of the zero mode interactions. It would be interesting to use the continuity of the non-perturbative superpotential to systematize or short-cut such computations. For instance, consider a set of BPS instantons  $\{C_i\}$  at a point  $P$  in moduli space. If these instantons can form an irreducible bound state  $C$  somewhere else in moduli space (at a point  $Q$ ), and if  $C$  has only two fermion zero modes, then in the theory at  $P$  there is a non-trivial multi-instanton process involving the instantons  $\{C_i\}$ . Similarly, if the instantons form a bound state, but it has more than two fermion zero modes, the corresponding superpotential (at  $Q$  and hence at  $P$ ) vanishes. This seemingly innocent statement is in fact very powerful. For instance it may be feasible to systematically construct instantons contributing to the superpotential at some tractable point in moduli space, and translate the corresponding instanton processes to the corresponding (possibly multi-)instanton processes in other regions. For instance, BPS instantons on type IIB models may be constructed as stable holomorphic gauge bundles in the large volume regime. Such contributing instantons could subsequently be translated into multi-instanton processes at other interesting points like orbifold limits or Gepner points.

The non-perturbative superpotential is an interesting quantity which is well-behaved all over moduli space, in a non-trivial way. It would be interesting to gain a deeper

understanding of the microphysics underlying this result in general, beyond the concrete examples we have analyzed. We expect further insights from more powerful approaches, for instance using the category of holomorphic D-branes, which is another interesting object with universal properties over moduli space. This category does not include the information about the stability conditions on D-branes, namely on the D-term contributions to the world-volume action. However our results suggest that the full superpotential is rather insensitive to the stability properties of individual BPS instantons: as soon as an instanton become unstable and decays into sub-objects, the latter can reconstruct the same contribution via a multi-instanton process.

We expect our results to shed light on the physics of non-perturbative superpotentials in string theory, both from the viewpoint of its formal properties, and for physical applications in concrete examples.

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## A. No splitting $O(1) \rightarrow O(1) \times O(1)$

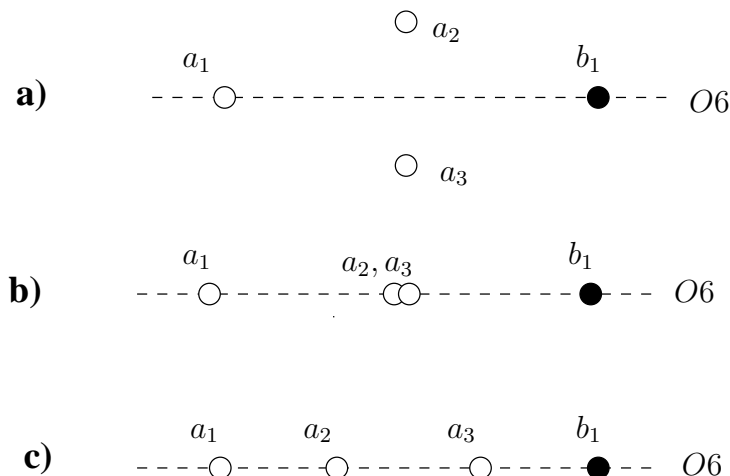
In the main text we have described examples where an  $O(1)$  instanton decays into a set of two instantons of  $U(1) \times O(1)$ , or a  $U(1)$  instanton and its orientifold image. In this appendix we show that it is not possible to have one  $O(1)$  instanton decay into two  $O(1)$  instantons (or more generally, that an instanton mapped to itself under the orientifold action cannot decay into two instantons invariant under the orientifold action). The argument is general, and applies to any BPS instanton, contributing to the superpotential or to higher-fermion F-terms.

The general argument that forbids such instantons from crossing such marginal stability lines goes as follows. In order to contribute to F-terms, the instanton must be BPS, so the cycle it wraps must be calibrated by  $e^{i\varphi}\Omega$  with some constant phase  $\varphi$ :

$$\arg \Omega|_{\Xi} = \pi\varphi. \quad (\text{A.1})$$

The phase  $\varphi$  of the calibration determines what is called in the  $\Pi$ -stability literature the *grading* of the brane, we will adopt this terminology here. This grading determines the supersymmetry conserved by the brane, and also the mass of the lightest bosonic string mode between the decay products  $\Xi_1$  and  $\Xi_2$  at the marginal stability line:

$$m^2 = \frac{1}{2}(\varphi_1 - \varphi_2). \quad (\text{A.2})$$



**Figure 10:** Splitting an instanton invariant under the orientifold action into two invariant instantons via a process involving a singular configuration.

When the gradings  $\varphi_1$  and  $\varphi_2$  of the decay products coincide  $\Xi_1$  and  $\Xi_2$  are mutually supersymmetric, the boson is massless, and we are on the marginal stability wall. Once we move slightly off the wall, the gradings will become different and bound state formation becomes possible. Whether we have a bound state or a stable superposition of two mutually non-supersymmetric branes depends on the sign of the boson mass: on one side of the wall it will become tachyonic, triggering bound state formation, while in the other side of the wall it will be massive, and the superposition of  $\Xi_1$  and  $\Xi_2$  is stable.

Here the main point of interest for our discussion is that the  $\Omega \rightarrow \overline{\Omega}$  action of the orientifold acts on these gradings as  $\varphi \rightarrow -\varphi$ , so invariant instantons must have integer grading. This obstructs the decay of the  $O(1)$  instanton into two  $O(1)$  factors: there is no question of continuity of the nonperturbative superpotential since the gradings of the branes are frozen by the orientifold.

It is possible to argue the same thing in a slightly different way. Imagine D6 branes wrapping the same cycles in the internal space as the instantons. In this case the process of brane recombination is typically seen as a Higgsing, triggered by a Fayet-Iliopoulos term. The interpretation of the discussion above in terms of the gauge theory living on the brane is that due to the orientifold, the gauge group on the brane gets reduced from  $U(1)$  to  $O(1)$ , and the Fayet-Iliopoulos is projected out. There is no continuous way of Higgsing  $O(1) \times O(1)$  to  $O(1)$ .

We conclude by pointing out that our argument above does not exclude other more exotic possibilities to split an instanton invariant under the orientifold action into two instantons invariant under the orientifold action. In fact, there is a simple example of such transition, which is related to those in [67], as we now describe. Consider a geometry of the kind considered in section 2.2, as shown in figure 10. The configuration includes an orientifold plane associated to the action  $\Omega R(-1)^{F_L}$ , with  $R$  given by

$$z \rightarrow \bar{z} \quad ; \quad (x, y) \rightarrow (\bar{y}, \bar{x}) \quad ; \quad (x' y') \rightarrow (\bar{y}', \bar{x}') \quad (\text{A.3})$$

or

$$z \rightarrow \bar{z} \quad ; \quad (x, y) \rightarrow (\bar{x}, \bar{y}) \quad ; \quad (x', y') \rightarrow (\bar{x}', \bar{y}') \quad (\text{A.4})$$

The two choices lead to orientifold planes whose projection on the  $z$ -plane is the horizontal axis. They act differently on the  $\mathbf{C}^*$  fibers, and lead to slightly different structures for the orientifold projection.<sup>9</sup> Namely the O6-plane defined by (A.3) is split when it encounters a  $\mathbf{C}^*$  degeneration, and it changes from  $\text{O6}^+$  to  $\text{O6}^-$  (and vice versa). The O6-plane defined by (A.4) is not split and has a fixed RR charge. This distinction will not be relevant for us, and for concreteness we focus on an orientifold of the kind (A.4), and choose the orientifold to lead to  $\text{O}(1)$  symmetries for D2-branes instantons.

Consider the transition shown in figure 10. We consider the fate of the  $\text{O}(1)$  instanton arising from a D2-brane on the 3-cycle  $[a_1, b_1]$ . As the two degenerations  $a_2, a_3$  approach the orientifold plane (in a way consistent with the orientifold action), we reach a singular configuration, figure 10b, where the  $\text{O}(1)$  instanton is split into two  $\text{O}(1)$  instantons. At this point a new branch emerges, where  $a_2, a_3$  can separate along the horizontal axis, and the original instanton is split into three  $\text{O}(1)$  instantons (for the orientifold action (A.3), the middle 3-cycle would lead to an  $USp$  instantons). It is thus possible to split  $\text{O}(1)$  instantons by a physical process, but which is in fact unrelated to (and more exotic than) lines of marginal stability. Indeed, notice that the transitions is not triggered by a Fayet-Iliopoulos parameter.

In fact, it is questionable that the transition has a simple description from the viewpoint of the instanton world-volume. Notice that the transition involves passing through a singular configuration, on which the 2-sphere of the 3-cycle  $[a_2, a_3]$  shrinks to zero size. This is an orientifold quotient of the singular CFT point of the  $A_1$  geometry, where the theory (at least before orientifolding) develops enhanced gauge symmetry, with additional massless gauge bosons arising from wrapped D2-branes. It is unlikely that the transition point admits a standard description from the viewpoint of the instanton world-volume. We thus expect that the non-perturbative superpotential can be discontinuous across this kind of transition. Indeed, in [67] similar transitions lead to discontinuous phenomena, like chirality changing phase transitions, not compatible with a local field theory description. It would be interesting to investigate these transitions in more detail, perhaps along the lines in [68].

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