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ABSTRACT: The three dimensional $\mathcal{N} = 2$ supersymmetric Chern-Simons theory coupled to matter fields, possibly deformed by a superpotential, give rise to a large class of exactly conformal theories with Lagrangian descriptions. These theories can be arbitrarily weakly coupled, and hence can be studied perturbatively. We study the theories in the large N limit, and compute the two-loop anomalous dimension of certain long operators. Our result suggests that various $\mathcal{N} = 2$ U(N) Chern-Simons theories coupled to suitable matter fields are dual to open or closed string theories in AdS_4, which are not yet constructed.

KEYWORDS: 1/N Expansion, Supersymmetric gauge theory, AdS-CFT Correspondence, Chern-Simons Theories.

Contents

1. Introduction

The AdS/CFT correspondence [1–4] has led to tremendous insight into gauge theory as well as non-perturbative descriptions of theories of gravity. Much progress has been made in understanding AdS_5/CFT_4 and AdS_3/CFT_2, where both the string theory side and the conformal field theory side can be solved in certain limits. (For a sample of recent advances see [5–13].) The AdS_4/CFT_3 correspondence, on the other hand, is much less understood. Two primary classes of examples of AdS_4/CFT_3 have been studied so far: (1) the duality between M-theory on $AdS_4 \times S^7$ (as well as $AdS_4 \times X_7$ with other Sasaki-Einstein seven-manifolds X_7) and the M2-brane CFT, which is believed to be the infrared fixed point of the three dimensional $\mathcal{N} = 8$ U(N) super-Yang-Mills theory [14]; (2) the duality between higher spin gauge theory in AdS_4 and the IR fixed point of three dimensional $O(N)$ model [15]. In the first example, it is difficult to calculate anything in the IR CFT from the gauge theory, which is strongly coupled. For instance, one does not even understand from the gauge theory perspective why the IR CFT has $N^{3/2}$ degrees of freedom, as predicted from the gravity dual. The gravity side, which involves M-theory, is also not completely formulated. In general we do not know how to go beyond supergravity and semi-classical M-branes. In the second example, one can do computations in the CFT in $1/N$ expansion and ϵ-expansion, but the gravity dual is again difficult to analyze. In both classes of examples, there is no adjustable parameter in the theory other than N itself.

It was pointed out in [16] that supersymmetric Chern-Simons theories, which are by themselves topological but may be coupled to matter fields that carry physical degrees of freedom, give rise to a natural class of classically conformal theories. Furthermore, since the Chern-Simons level is not renormalized up to a possible 1-loop shift, such theories are anticipated to be exactly conformal quantum mechanically, and hence potentially giving rise to an interesting new class of AdS_4/CFT_3 correspondences. The non-renormalization properties of $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Chern-Simons theories coupled to matter fields have been previously studied in [17–21].

More precisely, $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetric Chern-Simons theories coupled minimally to matter fields are exactly conformal, in the sense that there is no relevant or marginal quantum corrections to the classical action. The theories are labeled by the gauge group G, matter representation R, and the Chern-Simons level k. $1/k$ plays the role of the coupling constant, and in particular the theory can be made arbitrarily weakly coupled and can be analyzed in conventional perturbation theory. As we will show, the $\mathcal{N} = 3$ theory can be obtained as the IR fixed point of the $\mathcal{N} = 2$ theory perturbed by a certain superpotential. In fact, there is an even larger class of superconformal field theories, obtained from more general superpotential deformations of the $\mathcal{N} = 2$ CS-matter theory, all of which have (weakly coupled) Lagrangian descriptions.

The supersymmetric Chern-Simons theories coupled to fundamental or adjoint matter fields can often be engineered as the IR limit of the world volume theory on branes in string theory. One might try to find the gravity dual of CS-matter theories by studying the decoupling limit of such brane solutions. However, we do not know any example in which

such decoupling limit exists as a smooth AdS_4 solution in supergravity. It is conceivable that the AdS_4 dual of CS-matter theories, if described by a weakly coupled string theory, always has radius at string scale.

While we do not know a direct construction of the gravity dual, we can still study the large N limit of CS-matter theories perturbatively, and look for signatures of strings. It is a priori not obvious whether the usual lore of gauge/string duality applies in this case, since the Chern-Simons gauge field is effectively infinitely massive and does not carry propagating degrees of freedom by itself. Take as an example the $\mathcal{N} = 2$ $\mathrm{U}(N)$ Chern-Simons theory coupled to N_f fundamental matter fields. We expect this CFT to be dual to a theory of gravity in AdS_4, with $\mathrm{U}(N_f)$ gauge fields. One can consider twist-1 operators of the form

$$\bar{\phi} D_{(\mu_1} \cdots D_{\mu_n)} \phi \tag{1.1}$$

at large spin n, where ϕ is the scalar matter field and D_μ are gauge covariant derivatives. At weak 't Hooft coupling $\lambda \sim N/k$, the leading contribution to the anomalous comes in at two-loop. To this order we find that there are corrections of order $\mathcal{O}(\lambda^2)$, bounded in the $n \to \infty$ limit, and corrections of order $\mathcal{O}(\lambda^2 N_f/N)$, which grow like $\ln(n)$ at large n. The latter is the expected growth of the anomalous dimension of the operator dual to a classical string spinning in AdS_4. This sort of agreement was found in four dimensional $\mathcal{N} = 4$ super-Yang-Mills [22], where the $\ln(n)$ growth in fact holds to all orders in the 't Hooft coupling. This suggests that in the large N limit, with N_f/N finite, (1.1) could indeed be dual to a classical spinning open string in AdS_4, and that $\mathcal{N} = 2$ $\mathrm{U}(N)$ Chern-Simons theory with fundamental matter could be dual to a open string theory in AdS_4 (i.e. with N_f space filling D-branes). The radius of AdS_4 in string units will be a function of λ and N_f/N.

The above considerations extend naturally to theories with adjoint matter. It is natural to study spin chain descriptions of long operators in these theories. For example, the $\mathcal{N} = 3$ theory with one adjoint matter has $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_f$ global symmetry. There is a corresponding $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_f$ spin chain, which turns out to be non-integrable at two-loop. In general we do not expect the spin chain associated with CS-matter theories to be integrable. Although we do not know the precise holographic dual of this theory, our findings are compatible with a 7-dimensional supergravity dual at large 't Hooft coupling, with the geometry of the coset $OSp(3|4)/\mathrm{SO}(3,1)$. Indeed, the spectrum of protected operators consists of a tower of irreducible representations of $OSp(3|4) \times \mathrm{SU}(2)_f$ with spin (j,j) under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_f$ for each j, as expected for the KK-tower of modes on S^3. The analysis of the spin chain reinforces this idea: giant magnon-like excitations of large R-charge which carry spin $(j - \frac{1}{2}, j)$ have the same anomalous dimension as excitations of spin $(j, j - \frac{1}{2})$. This is the behavior expected for a giant magnon moving along the equator of the S^3.

This paper is organized as follows. In section 2 we recall the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Chern-Simons-matter theories, and the arguments for their non-renormalization properties. We will show that the $\mathcal{N} = 2$ theories deformed by superpotentials can flow to other weakly coupled superconformal fixed points, including $\mathcal{N} = 3$ ones. In section 3 we study the 't

Hooft limit of various CS-matter theories. For the abelian theory, we will solve the free energy at large N_f and study the thermodynamics of the theory. We will then discuss the operator spectrum of the $\mathrm{U}(N)$ Chern-Simons theory coupled to fundamental matter. In particular, we compute the two-loop anomalous dimension of twist-1 operators at large spin. We will also discuss supersymmetric Wilson loops and spin chain descriptions of long operators. Section 4 contains some comments on the possible string theory AdS_4 dual. Appendix A summarizes some details of the $\mathcal{N} = 3$ Lagrangian. Appendix B presents the saddle point analysis of the abelian theory in the large N_f limit. Appendix C contains a discussion of the free nonabelian theory on the sphere. In appendix D, we present the computation of the two-loop anomalous dimensions of some operators.

2. Supersymmetric Chern-Simons theories coupled to matter fields

2.1 $\mathcal{N} = 2$ Chern-Simons-matter theory

In this subsection we review the $\mathcal{N} = 2$ Chern-Simons theory coupled to matter fields [23, 24, 19, 20], as well as its non-renormalization properties. We shall start by describing the Lagrangian in superspace. The three dimensional $\mathcal{N} = 2$ vector superfield V consists of the gauge field A_μ, an auxiliary scalar field σ, a two-component Dirac spinor χ, and another scalar D. The superspace Lagrangian for abelian $\mathcal{N} = 2$ Chern-Simons theory, coupled to matter chiral superfield Φ, is given by

$$S_{Ab}^{\mathcal{N}=2} = \int d^3x \int d^4\theta \left(\frac{k}{4\pi} V\Sigma + \bar{\Phi} e^V \Phi \right) \tag{2.1}$$

where $\Sigma = \bar{D}^\alpha D_\alpha V$. When there are several matter fields Φ_i of different charges, it is convenient to absorb k into the charge and write the action as

$$S_{Ab}^{\mathcal{N}=2} = \int d^3x \int d^4\theta \left(V\Sigma + \bar{\Phi}_i e^{q_i V} \Phi_i \right) \tag{2.2}$$

The nonabelian $\mathcal{N} = 2$ Chern-Simons action is trickier to write in terms of the nonabelian vector superfield V:

$$S^{\mathcal{N}=2} = \int d^3x \int d^4\theta \left\{ \frac{k}{2\pi} \int_0^1 dt \mathrm{Tr} \left[V\bar{D}^\alpha (e^{-tV} D_\alpha e^{tV}) \right] + \bar{\Phi} e^V \Phi \right\} \tag{2.3}$$

where the matter field Φ is in an arbitrary representation R of the gauge group. Here "Tr" is normalized to be the trace in the fundamental representation when the gauge group is $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. We will denote by tr_R the trace taken in representation R. In component fields (Wess-Zumino gauge), the Chern-Simons action is simply

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A^3 - \bar{\chi}\chi + 2D\sigma \right) \tag{2.4}$$

The level k is quantized to be integer valued[1] to ensure invariance under large gauge transformations. We will often consider N_f $\mathcal{N} = 2$ chiral matter fields in some representation R,

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[1]or half-integer valued when there is a "parity anomaly" [25, 26], depending on the matter content. This subtlety is not essential for us as we will mostly work with weak coupling $k \gg 1$ and/or 't Hooft-like limits.

denoted by $\Phi^i = (\phi^i, \psi^i)$, with global $U(N_f)$ flavor symmetry. The coupling to the vector multiplet is the standard one,

$$S_{\text{matter}} = \int d^4\theta \sum_{i=1}^{N_f} \bar{\Phi}^i e^V \Phi^i = \int \sum_{i=1}^{N_f} \left(D_\mu \bar{\phi}^i D^\mu \phi^i + i\bar{\psi}^i \gamma^\mu D_\mu \psi^i - \bar{\phi}^i \sigma^2 \phi^i + \bar{\phi}^i D \phi^i \right.$$
$$\left. -\bar{\psi}^i \sigma \psi^i + i\bar{\phi}^i \bar{\chi} \psi^i - i\bar{\psi}^i \chi \phi^i \right). \tag{2.5}$$

The auxiliary fields σ and D are understood to act on (ϕ, ψ) in the representation R. Integrating out D sets $\sigma = -\frac{4\pi}{k}(\bar{\phi}^i T^a \phi^i) t^a$, where T^a are generators of the Lie algebra of the gauge group, normalized so that $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. Further integrating out $\chi, \bar{\chi}$ yields the action

$$S^{\mathcal{N}=2} = \frac{k}{4\pi} CS(A) + \int D_\mu \bar{\phi}^i D^\mu \phi^i - \frac{16\pi^2}{k^2}(\bar{\phi}^i T^a \phi^i)(\bar{\phi}^j T^b \phi^j)(\bar{\phi}^k T^a T^b \phi^k)$$
$$+i\bar{\psi}^i \gamma^\mu D_\mu \psi^i - \frac{4\pi}{k}(\bar{\phi}^i T^a \phi^i)(\bar{\psi}^j T^a \psi^j) - \frac{8\pi}{k}(\bar{\psi}^i T^a \phi^i)(\bar{\phi}^j T^a \psi^j). \tag{2.6}$$

This action is clearly classically marginal. We will now argue that it is in fact quantum mechanically exactly marginal, in the sense that there is no relevant or marginal quantum corrections to the action that cannot be absorbed into a redefinition of the fields.

When the matter field Φ transforms in a irreducible representation R, there is a $U(1)$ symmetry $\Phi \to e^{i\alpha}\Phi$. Since there is no anomaly for continuous global symmetry in three dimensions, this $U(1)$ symmetry holds in the quantum theory and forbids any superpotential involving Φ by holomorphy. Similarly, when the matter field lies in a reducible representation of the gauge group, there are $U(1)$ symmetries acting on each irreducible part of Φ, and the same argument forbids a dynamically generated superpotential.

It is well known that the Chern-Simons level k is not renormalized beyond a possible finite 1-loop shift [17].[2] The simplest way to argue this is that k is quantized to be integer valued in order for the path integral to be invariant under large gauge transformations [27]. Any quantum correction to k at 2-loop or higher order will be suppressed by $1/k$, which in general cannot be integer valued.

So the only possible quantum corrections to the classical Lagrangian is to the Kahler potential. This indeed happens. However, any corrections to the Kähler potential are either irrelevant in the IR or can be absorbed by a rescaling of Φ.[3] One might also worry about possible generation of Fayet-Iliopoulos term, when there are matter fields charged under the $U(1)$ part of the gauge field. Any dynamical FI parameter must be of the form $D_\alpha \bar{D}^\alpha(\cdots)$ in order to preserve gauge invariance, where $(\cdots)$ is a gauge invariant combination of the fields. Again, such terms are irrelevant in the IR. In conclusion, there cannot be any relevant or marginal correction to the classical Lagrangian (2.6), and hence the theory is exactly marginal.

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[2]See [18] for a discussion on the regularization dependence of the 1-loop shift.

[3]In Wilsonian effective action all corrections to the Kähler potential are non-singular at $\Phi = 0$, hence this argument is valid. This would not be the case for the 1PI effective action, since we would be integrating out massless fields, and the effective Kähler potential may well be singular at $\Phi = 0$.

The vanishing of two-loop beta function of the matter couplings has been explicitly shown in [19] for the abelian theory and [20] for the nonabelian theory.

As a consistency check of the existence of these conformal field theories, let us consider the example of a $\mathrm{U}(1)$ CS-matter theory with both positively and negatively charged matter fields (say of charges q and $-q$). The scalar potential takes the form

$$V(\phi, \tilde{\phi}) = \frac{q^4}{4}(|\phi|^2 - |\tilde{\phi}|^2)^2(|\phi|^2 + |\tilde{\phi}|^2) \tag{2.7}$$

A generic point on the Higgs branch moduli space is parameterized by nonzero ϕ. This moduli space cannot be lifted since no superpotential can be generated. Writing $\tilde{\phi} = \phi e^{\rho + i\theta}$, ρ has mass of order $q^2|\phi|^2$ and θ is a Goldstone boson which can absorbed by a gauge transformation. Similarly one fermion gets massive and the other remains massless. One obtains the quantum corrected metric on the moduli space by integrating out the massive fields. For example, the leading (two-loop) correction to the kinetic term for ϕ takes the form $q^4|\partial\phi|^2 \ln(|\phi|^2/\mu)$. Such corrections result in a cone-shaped metric near $\phi = 0$, reflecting the anomalous dimension of ϕ. The distance from the origin $\phi = 0$ to a generic point on the Higgs branch moduli space is finite, at least for small q. If a term like $\frac{\mu}{|\phi|^2}|\partial\phi|^2$ were generated, it would suggest that the CFT at $\phi = 0$ may not exist, but this doesn't seem to happen in the CS-matter theory, at least at weak coupling.

There may be caveats in our argument for the theory at strong coupling. In the $\mathcal{N} = 2$ theory, generally, the $\mathrm{U}(1)_R$ charge of ϕ gets renormalized, as we will discuss later. At strong coupling there could be dangerously irrelevant terms in the Kähler potential being generated, spoiling the above non-renormalization argument. We also see such possibility from the above analysis of the Higgs branch moduli space (when exists): if the wave function renormalization of ϕ is sufficiently large, the metric on the moduli space could be such that the origin is at infinite distance, and the CFT may cease to exist. It is not clear to us if this actually happens. The $\mathcal{N} = 3$ theory, which we will describe in the next subsection, does not have R-charge renormalization nor wave function renormalization. So one might expect better behavior of the theory at strong coupling.

2.2 $\mathcal{N} = 3$ Chern-Simons-matter theory

The maximally supersymmetric extension of Chern-Simons theory appears to be $\mathcal{N} = 3$ [28, 21], which can be coupled to hypermultiplet matter fields. In $\mathcal{N} = 2$ language the matter fields consists of a pair of chiral multiplets $(Q, \tilde{Q})$, transforming in conjugate representations of the gauge group. The action for the $\mathcal{N} = 3$ Chern-Simons matter theory takes the form[4]

$$S^{\mathcal{N}=2} = S_{CS}^{\mathcal{N}=2} + \int d^4\theta (\overline{Q}e^V Q + \tilde{Q}e^{-V}\overline{\tilde{Q}}) + \left[\int d^2\theta \left(-\frac{k}{4\pi}\mathrm{Tr}\Phi^2 + \tilde{Q}\Phi Q\right) + c.c.\right] \tag{2.8}$$

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[4]Our action may appear different from the nonabelian $\mathcal{N} = 3$ action of [29], but they are in fact the same. For example, the Chern-Simons part of the $\mathcal{N} = 3$ action in [29] contains the extra term $\mathrm{Tr}\bar{\phi}[\sigma, \phi]$. This term can be absorbed into the scalar potential of [29], and one recovers our F-term after redefining ϕ_1 and $\bar{\phi}_2$ in [29] as Q and $\tilde{Q}$, respectively.

Here Φ is an auxiliary chiral superfield in the adjoint representation, combined with V to give the $\mathcal{N}=4$ vector multiplet. The scalar component of Φ, which we denote by ϕ, combines with σ to form a triplet under the $\mathrm{SU}(2)_R$ symmetry. Similarly, F_Φ combines with the auxiliary field D to form a triplet.

We will often consider theories with N_f matter hypermultiplets in some irreducible representation R. There is $\mathrm{U}(N_f)$ global flavor symmetry. When R is a real representation (say the adjoint), the flavor symmetry is enhanced to $USp(2N_f)$. In appendix A we write the Lagrangian in a manifestly $\mathrm{SU}(2)_R \times USp(2N_f)$ invariant form.

The auxiliary field Φ may be simply integrated out, resulting in a superpotential

$$W = \frac{2\pi}{k}(\tilde{Q}T^a Q)(\tilde{Q}T^a Q). \tag{2.9}$$

In other words, the $\mathcal{N}=3$ Chern-Simons-matter theory is the same as $\mathcal{N}=2$ Chern-Simons-matter theory with matter fields $Q, \tilde{Q}$ and the superpotential (2.9). Note that the coefficient of (2.9) is fixed by $\mathcal{N}=3$ supersymmetry. Simialr non-renormalization argument as in the previous subsection also applies to the $\mathcal{N}=3$ theory. The $\mathcal{N}=3$ theory in fact has stronger non-renormalization property, since the $\mathrm{SU}(2)_R$ charge of the fields cannot be renormalized, in contrast to the $\mathcal{N}=2$ theory. For example, the gauge invariant meson operator $\tilde{Q}Q$ is a chiral primary, whose dimension is protected. It follows that there is no wave function renormalization for the matter fields [21]. In conclusion, the $\mathcal{N}=3$ Chern-Simons-matter theory is also exactly conformal.

From the point of view of the $\mathcal{N}=2$ theory, the appearance of a conformal fixed point at a finite deformation W is somewhat unexpected. Let us consider a more general superpotential,

$$W = \frac{\alpha}{2}(\tilde{Q}T^a Q)(\tilde{Q}T^a Q), \tag{2.10}$$

with α a non-negative real constant (one can always absorb the phase of α into a redefinition of the Q's). No other superpotential terms can be generated by standard arguments based on holomorphy and $\mathrm{U}(1)_R$ symmetry.[5] When $\alpha \gg 4\pi/k$ and $k \gg 1$, W dominates the interaction, and the theory in this limit is essentially the three dimensional Wess-Zumino model with superpotential W. This theory has a positive beta function, i.e. α decreases going to the IR. The leading two-loop RGE for α takes the form

$$\mu\frac{d\alpha^2}{d\mu} = \frac{b_0}{16\pi^2}\alpha^2\left[\alpha^2 - \left(\frac{4\pi}{k}\right)^2\right], \qquad b_0 > 0. \tag{2.11}$$

The coeffient b_0 can be determined from the beta function of the corresponding WZ model, $b_0 = \frac{2}{\dim R}\left[(\mathrm{tr}_R T^a T^b)^2 + \mathrm{tr}_R(T^a T^b T^a T^b)\right]$. This is because the theory has two conformal fixed points, at $\alpha = 0$ and $\alpha = 4\pi/k$, and it is not hard to see that the two-loop correction to α^2 is a quadratic function in α^2. Hence we learn that a small perturbation of the $\mathcal{N}=2$ theory ($\alpha = 0$) by the superpotential (2.10) flows to the $\mathcal{N}=3$ conformal fixed point ($\alpha = \frac{4\pi}{k}$). Knowing that $\alpha = 0, \frac{4\pi}{k}$ are exact fixed points, the 2-loop result (2.11) suggests that this RG flow holds in the full theory.

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[5]We should however be cautious with the standard non-renormalization arguments involving the gauge coupling, since in Chern-Simons theory the coupling k cannot be promoted to a dynamical field.

2.3 Chiral operators and chiral primaries

Let us first discuss the chiral operators in the *abelian* $\mathcal{N} = 2$ theory. The chiral operators are given by gauge invariant polynomials in ϕ's.[6] Since there is no superpotential, there is no relations in the chiral ring, these chiral operators are also chiral primaries. In theories where all matter fields have charges of the same sign, there are no chiral primaries. Let us consider the $N = 2$ theory with oppositely charged matter fields Q^i and $\tilde{Q}^i$, $i = 1, \cdots, N_f$. Consider a chiral primary $\mathcal{O} = f(Q, \tilde{Q})$, where f is a polynomial of homogeneous degree n. The conformal dimension of $\mathcal{O}$ is given by the unitarity bound, $\Delta = nq_R$, where q_R is the $\mathrm{U}(1)_R$ charge of Q and $\tilde{Q}$. Classically $q_R = \frac{1}{2}$, but quantum mechanically it is modified to be less than $\frac{1}{2}$. In fact, we have learned from (2.11) that the superpotential perturbation $(\tilde{Q}Q)^2$ is relevant, and the $\mathrm{U}(1)_R$ charge is renormalized at two-loop to

$$q_R = \frac{1}{2} - \frac{b_0}{8k^2} + \mathcal{O}\left(\frac{1}{k^4}\right). \tag{2.12}$$

This may be confirmed by directly computing the anomalous dimension of $\tilde{Q}Q$. It is also easy to check at two-loop that the anomalous dimension of $(\tilde{Q}Q)^n$ is n times that of $\tilde{Q}Q$, say to order N_f/k^2 (the $\sim n^2$ contributions cancel).

In a general nonabelian $\mathcal{N} = 2$ CS-matter theory with gauge group G and matter fields Φ_i in irreducible representation R_i, the chiral primaries are gauge invariant polynomials in the Φ_i's. Let us consider a few special cases:

1. $G = \mathrm{U}(N)$, $R_i = \mathbf{N}$, $i = 1, \cdots, N_f$. There are no chiral primaries (besides the identity operator) in this theory.

2. $G = \mathrm{SU}(N)$, $R_i = \mathbf{N}$, $i = 1, \cdots, N_f$. The chiral primaries only exist for $N_f \geq N$. They are (generated by) baryon operators, of the form

$$B_{i_1 \cdots i_{N_f-N}} = \epsilon_{i_1 \cdots i_{N_f-N} j_1 \cdots j_N} \epsilon^{a_1 \cdots a_N} \phi_{a_1}^{j_1} \cdots \phi_{a_N}^{j_N} \tag{2.13}$$

3. $G = \mathrm{U}(N)$, $R_i = \mathbf{N}$, $\tilde{R}_i = \overline{\mathbf{N}}$. The chiral primaries are mesons, $M_j^i = \phi_a^i \tilde{\phi}_j^a$.

4. $G = \mathrm{SU}(N)$, $R_i = \mathbf{adj}$, $i = 1, \cdots, N_f$. The chiral primaries are generated by "words", i.e. the trace of a string of Φ_i's in a particular order up to cyclic permutation, $\mathcal{O}_{i_1 \cdots i_n} = \mathrm{Tr}(\Phi_{i_1} \cdots \Phi_{i_n})$. Since there is no superpotential, the indices are not necessarily symmetrized. In the infinite N limit, all the traces are independent operators, and the number of chiral primaries of dimension n growths exponentially $\sim (N_f)^n$ for $N_f > 1$ (modulo the correction due to cyclic permutations). This is a peculiar feature. It is curious whether such chiral primaries could be dual to winding string-like objects in negatively curved spaces, a la [30].

Let us turn to the $\mathcal{N} = 3$ theory. In this case, there cannot be any quantum correction to q_R, since it is the $\mathrm{U}(1)$ part of the $\mathrm{SU}(2)$ R-symmetry. Hence there is no anomalous

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[6]Since $\Sigma = \bar{D}DV$ is already gauge invariant, the chiral operator $\bar{D}_\alpha \Sigma$ is a descendant and will be ignored. This is in contrast with four dimensional gauge theories, in which Σ doesn't exist.

dimension for the mesons $\tilde{Q}Q$. However, since there is the superpotential (2.9), there are nontrivial relations in the chiral ring. In particular, the operator $(\tilde{Q}Q)^2$ is no longer a chiral primary, and gets a positive anomalous dimension. This is of course expected since the perturbation by $(\tilde{Q}Q)^2$ is irrelevant in the $\mathcal{N}=3$ theory, as dicussed in the previous subsection.

Consider the example of $\mathcal{N}=3$ SU(N) theory with 1 adjoint hypermultiplet $(Q, \tilde{Q})$. The superpotential imposes chiral ring relations of the form

$$\text{Tr}([[\tilde{Q}, Q], Q] \cdots) \simeq 0, \qquad \text{Tr}([[\tilde{Q}, Q], \tilde{Q}] \cdots) \simeq 0. \qquad (2.14)$$

The chiral primaries are given by traces of strings of Q and $\tilde{Q}$ symmetrized with respect to permutation. The trace of a string of Q's and $\tilde{Q}$'s that is not symmetrized will in general acquire anomalous dimension, and may mix with other operators. The spectrum of such operators can be mapped to that of the Hamiltonian of an SU(2)$_f$ spin chain, as we will discuss later.

2.4 $\mathcal{N}=2$ superpotential deformations

Let us now consider the $\mathcal{N}=2$ Chern-Simons-matter theory, with matter fields Φ_i ($i = 1, \cdots, n$) in the representation R_i of the gauge group G. We can deform the theory by a superpotential

$$W = P(\Phi_1, \cdots, \Phi_n), \qquad (2.15)$$

P being a gauge invariant polynomial in the Φ_i's. We will consider classically marginal deformations, i.e. P of homogenous degree 4. We can write

$$P(\Phi_1, \cdots, \Phi_n) = \sum_{i,j,k,l} \alpha_{ijkl}(\Phi_i \otimes \Phi_j \otimes \Phi_k \otimes \Phi_l)^G \qquad (2.16)$$

where the superscript G means taking the G-invariant part of $R_i \otimes R_j \otimes R_k \otimes R_l$. When there is more than 1 singlet in the tensor product, there are more components of α_{ijkl}, which we suppress here. We would like to know where the theory flows to under the deformation by (2.15) in the UV.

Firstly, standard non-renormalization arguments forbid other superpotential terms from being generated. More precisely, if $\alpha_{ijkl} = 0$ (for *all* singlets in $R_i \otimes R_j \otimes R_k \otimes R_l$) in the classically superpotential, then α_{ijkl} stays zero in the quantum superpotential. Let us recall the argument. The theory has a classical U(1) R-symmetry, which we denote by U(1)$'$, under which Φ_i has charge $\frac{1}{2}$. We must distinguish this from what we call the quantum U(1)$_R$ symmetry, which is part of the superconformal symmetry of the IR theory, and the corresponding U(1)$_R$ charges of fields can be renormalized, as we have seen in the previous section. U(1)$'$ is nevertheless a good global symmetry of the quantum theory, and guarantees by holomorphy that the superpotential is quartic in the Φ_i's. To proceed, we shall promote α_{ijkl} to a neutral chiral superfield Z_{ijkl}. We have a U(1)^n symmetry, under which

$$\Phi_i \rightarrow e^{i\alpha_i}\Phi_i, \qquad Z_{ijkl} \rightarrow e^{-i(\alpha_i + \alpha_j + \alpha_k + \alpha_l)}Z_{ijkl}. \qquad (2.17)$$

This guarantees that the superpotential must be a sum of terms of the form $Z_{ijkl}(\Phi_i \otimes \Phi_j \otimes \Phi_k \otimes \Phi_l)^G$. However, when there are more than 1 singlets in the tensor product, our argument does not forbid other singlets to be generated, even if they are zero in the classical superpotential.

In supersymmetric Yang-Mills theories, it is usually possible to further argue non-renormalization of each coupling in the superpotential (modulo global anomalies say in four dimensions), by promoting the gauge coupling to a chiral superfield. This is not possible for the Chern-Simons level k, since it would otherwise violate (small) gauge invariance. So nothing protects the α_{ijkl}'s from $1/k$ corrections, and in fact, they do receive such corrections.

Suppose all α_{ijkl}'s are small. Then the RG flow is dictated by the anomalous dimension of $\mathcal{O}_{ijkl} = (\Phi_i \otimes \Phi_j \otimes \Phi_k \otimes \Phi_l)^G$. This operator is a chiral primary in the $\mathcal{N} = 2$ CS-matter theory (with no superpotential), and hence its dimension is determined in terms of its $\mathrm{U}(1)_R$ charge, $\Delta_{ijkl} = q_i^R + q_j^R + q_k^R + q_l^R$. The quantum corrected $\mathrm{U}(1)_R$ charge q_i^R of Φ_i appears to be always less than $\frac{1}{2}$. When $\bigoplus R_i$ is the sum of a (reducible) representation and its conjugate, the theory can be deformed into the $\mathcal{N} = 3$ theory and $q_i^R < \frac{1}{2}$ follows easily as in the previous subsections. We computed the two-loop correction to q_i^R explicitly in the theory with M adjoint matter fields in the planar limit in appendix D.1, and indeed find that the correction is negative. We expect $q_i^R < \frac{1}{2}$ to hold in general. Consequently, all the quartic superpotential terms are relevant perturbations, and will take the $\mathcal{N} = 2$ CS-matter theory away from $\alpha_{ijkl} = 0$.

On the other hand, if one or several α_{ijkl} are much bigger than $1/k$, at least when $k \gg 1$ these α_{ijkl}'s dominate and the theory is again approximated by the Wess-Zumino model, which has positive beta function and hence these α_{ijkl}'s decrease in the IR. In conclusion, α_{ijkl}'s are bounded as the theory flows to the IR.

Therefore, the $\mathcal{N} = 2$ theory deformed by the superpotential (2.16) must flow to some other nontrivial IR fixed point at finite α_{ijkl}, which will not be an $\mathcal{N} = 3$ theory in general. It is easy to write down the two-loop RG equations for the couplings,

$$\mu\frac{d\alpha_{ijkl}}{d\mu} = (q_i^R + q_j^R + q_k^R + q_l^R - 2)\alpha_{ijkl} + \frac{1}{16\pi^2}(B_i{}^r\alpha_{rjkl} + B_j{}^r\alpha_{irkl} + B_k{}^r\alpha_{ijrl} + B_l{}^r\alpha_{ijkr}) + \mathcal{O}(\alpha^5) \tag{2.18}$$

where B_i^j is due to the two loop wave function renormalization in the Wess-Zumino model with superpotential (2.16) (i.e. $k \to \infty$ limit), which is proportional to α and $\overline{\alpha}$.

Specializing to $\mathrm{U}(N)$ gauge theory with M adjoint matter fields Φ_i, and the superpotential deformation

$$W = \sum_{ijkl} \alpha_{ijkl}\mathrm{Tr}(\Phi_i\Phi_j\Phi_k\Phi_l). \tag{2.19}$$

For simplicity we will work in the planar limit, but the discussion generalizes easily to the non-planar case as well. We have $B_i{}^j = \frac{1}{2}N^2\alpha_{iklm}\overline{\alpha}^{jklm}$. By $\mathrm{U}(M)$ flavor symmetry of the $\mathcal{N} = 2$ theory, we can diagonalize the Hermitian matrix B_i^j to $B_i{}^j = \frac{1}{2}N^2c_i\delta_i^j$, with $c_i\delta_i^j = \alpha_{iklm}\overline{\alpha}^{jklm}$. All the matter fields Φ_i have the same renormalized R-charge q^R in the undeformed theory, and the two-loop RG equation has fixed points at (assuming $\alpha_{ijkl} \neq 0$
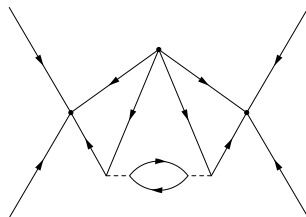
Figure 1: A four-loop contribution to the beta function of α_{ijkl}. The dotted lines are $\mathcal{N} = 2$ Chern-Simons gauge propagators. The solid lines are matter superfield propagators.

for some j, k, l)

$$\frac{1}{2} - q^R = \frac{N^2}{32\pi^2} c_i. \tag{2.20}$$

Or equivalently,

$$\alpha_{iklm}\overline{\alpha}^{jklm} = \left(\frac{1}{2} - q^R\right)\frac{32\pi^2}{N^2}\delta_i^j. \tag{2.21}$$

Note that the quantum correction to the R-charge is of order $\lambda^2 = (N/k)^2$, and hence the fixed point values of α_{ijkl} are of order $1/k$. When M is even, this includes the $\mathcal{N} = 3$ theory with $M/2$ adjoint matters. But there is clearly more, in fact, a continuous family $\mathcal{M}$ of fixed points. The question is whether they survive when higher loop corrections are included.

While we have not calculated the four-loop beta functions for α_{ijkl}, it appears that there are contributions of the form $\frac{N^4}{k^2}\alpha_{ijmn}\alpha_{klpq}\overline{\alpha}^{mnpq}$, from supergraphs such as the one in Fig 1. We do not see any reason why such contributions would cancel. These corrections will generate nontrivial RG flow along $\mathcal{M}$.

In general, since the two-loop fixed point values of α's are of order $1/k$, higher loop corrections will be suppressed by N/k. So at least for weak coupling $N/k \ll 1$, the full RG flow can be approximated by a flow on the manifold $\mathcal{M}$. Note that $\mathcal{M}$ is compact and smooth, and is freely acted by the $\mathrm{U}(M)$ flavor symmetry. The RG flow can then be represented by the flow according to a vector field on

$$\mathcal{M}_0 = \mathcal{M}/\,\mathrm{U}(M) \simeq V//\,\mathrm{U}(M), \tag{2.22}$$

where V is the parameter space of α_{ijkl}'s. For example, when M is even, the $\mathcal{N} = 3$ theory with $M/2$ adjoints corresponds to one point in $\mathcal{M}_0$. In general, the number of critical points (if discrete) of the vector field that dictates the RG flow, counted with sign, is given by the Euler characteristic $\chi(\mathcal{M}_0)$. So we learn that if only discrete RG fixed points (up to the $\mathrm{U}(M)$ action) survive when higher loop corrections are included, their number is at least $\chi(\mathcal{M}_0)$. In fact, the cohomology of $\mathcal{M}_0$ is generated by the Chern classes of the rank-M tautological bundle on $\mathcal{M}_0$, and hence $\chi(\mathcal{M}_0)$ is positive and grows with M.

The two-loop fixed point locus $\mathcal{M}$ has codimension M^2 in V. The superpotential W imposes chiral ring relations $\partial W/\partial \Phi_i \sim 0$, and in particular $\mathrm{Tr}(\Phi^i \frac{\partial W}{\partial \Phi_j})$ are M^2 descendants, while the remaining quartic chiral operators are in one-to-one correspondence with chiral

primaries. It follows that on $\mathcal{M}$, the two-loop renormalization of the $\mathrm{U}(1)_R$ charge vanishes. It would be interesting to know whether/how the $\mathrm{U}(1)_R$ charge is renormalized at the exact conformal fixed points near $\mathcal{M}$, when higher loop effects are included.

3. Large N limit

3.1 The abelian theory at large N_f

As a starter, we will consider the $\mathrm{U}(1)$ Chern-Simons-matter theories in the large N_f limit, i.e. $k, N_f \to \infty$, with $\lambda = 4\pi N_f/k = q^2 N_f$ kept finite (the 't Hooft coupling λ for the nonabelian theory will be defined differently). This limit of the $\mathrm{U}(1)$ theory is rather trivial. For example, finite dimensional operators only receives anomalous dimension to subleading order in $1/N_f$. Nevertheless, the theory can have nontrivial thermodynamics. It is a standard exercise to compute the free energy of the theory in the infinite N_f limit, by a saddle point approximation of the path integral. The details of the calculation can be found in appendix B. We shall discuss the main results here.

The $\mathrm{U}(1)$ $\mathcal{N} = 3$ theory with N_f charged hypermultiplets, as well as say $\mathrm{U}(1)$ $\mathcal{N} = 2$ CS-matter theory with N_f pairs of oppositely charged matter fields $(Q^i, \tilde{Q}^i)$, have rather trivial thermodyanmics at large N_f: their free energies at finite temperature are the same as that of the free field theory, at order $\mathcal{O}(N_f)$. One can easily convince himself/herself by examining the cancelation among the Feynman diagrams. It can also be confirmed through a saddle point analysis, as shown in appendix B.3. We argued earlier that the $\mathcal{N} = 2$ theory with $(Q, \tilde{Q})$ flows to the $\mathcal{N} = 3$ under a superpotential perturbation, and hence we expect that when $1/N_f$ corrections are included, the free energy of the former to be greater than the latter, as the number of degrees of freedom decreases under RG flow, although we have not checked this explicitly.

The $\mathcal{N} = 2$ $\mathrm{U}(1)$ theory with equally charged matter fields, on the other hand, has nontrivial free energy in the infinite N_f limit. The free energy of the theory in flat space, $F(T) = \frac{1}{A} \ln Z(T)$ (A being the spatial area), takes the form

$$F(T) = N_f T^2 (c_0(\lambda) + \mathcal{O}(1/N_f)) \tag{3.1}$$

The function $c_0(\lambda)$ can be computed from the saddle point approximation. It is an analytic function in λ except at $\lambda = 0$ due to an infrared divergence. This divergence is absent in the free energy of the theory on a sphere. $c_0(\lambda)$ decreases monotonously as λ increases, and asymptotes to a nonzero value in the $\lambda \to \infty$ limit. At weak 't Hooft coupling, it is given by

$$c_0(\lambda) = \frac{7\zeta(3)}{4\pi} + \frac{\lambda^2 (\ln \lambda)^3}{32\pi^2} + \cdots, \qquad \lambda \ll 1. \tag{3.2}$$

In the strong coupling limit, we find $c_0(\infty) \simeq 0.593071$. The function $c_0(\lambda)$ is plotted (at large coupling) in figure 2.

More generally, one can consider $\mathcal{N} = 2$ QED with N_f charged matter fields, with $\mathcal{N} = 2$ Chern-Simons term at level k. In the IR the Yang-Mills coupling is irrelevant, and the theory flows to the Chern-Simons-matter theory. On the other hand, we can take the
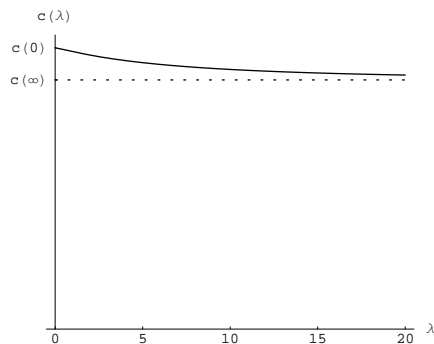
Figure 2: The free energy as a function of the 't Hooft coupling λ.

Chern-Simons coupling to infinity ($k/N_f \to 0$), and consider the strongly coupled IR fixed point of $\mathcal{N} = 2$ QED. The question is whether the $k/N_f \to 0$ limit and the IR limit are interchangeable. The answer appears to be yes, as far as the free energy is concerned. One can compute the low temperature free energy of $\mathcal{N} = 2$ QED in the large N_f limit by the saddle point approximation, and arrive at the same result as that of the $\mathcal{N} = 2$ abelian CS-matter theory at infinite λ.

It is a much more subtle (and important) question whether the infinite 't Hooft coupling limit of the $\mathrm{U}(N)$ $\mathcal{N} = 2$ or $\mathcal{N} = 3$ CS-matter theory (i.e. $k/N \to 0$) captures the IR fixed point of corresponding Yang-Mills theory with matter fields. The answer is not clear to us.

3.2 The $\mathrm{U}(N)$ theory: general remarks on the operator spectrum

Now let us turn to the $\mathrm{U}(N)$ Chern-Simons theory coupled to N_f fundamental matter fields. We will focus on the large N limit, with $\lambda = N/k$ and $c = N_f/N$ finite. The 't Hooft limit of this theory is much less trivial than the abelian theory. We will see that this theory has a number of features suggesting that it should be dual to a (non-critical) string theory in AdS_4. The radius of the AdS_4 scales like $R \sim \sqrt{N} l_{pl}$, and the string scale is finite compared to $1/R$ when the 't Hooft coupling is finite.

Let us examine the spectrum of gauge invariant operators, starting with the *free* theory. In the free theory there are infinitely many conserved currents of the form

$$J^{\bar{i}j}_{\mu_1\mu_2\cdots\mu_s} = \bar{\phi}^i D_{(\mu_1} D_{\mu_2} \cdots D_{\mu_s)} \phi^j + \cdots \tag{3.3}$$

where the scalar ϕ is in the fundamental representation of $\mathrm{U}(N)$. If there is an AdS_4 dual, these currents should be dual to spin s massless gauge fields in the bulk in the $\lambda \to 0$ limit [15]. The spectrum of gauge invariant operators in the free theory is analyzed in appendix C by studying the thermodynamics of the theory on a sphere. An expected transition from N_f^2 degrees of freedom at low temperature to NN_f degrees of freedom at high temperature is demonstrated explicitly.

In the interacting theory most of (3.3) will acquire anomalous dimensions. There will still be conserved currents, the stress-energy tensor $T_{\mu\nu}$, the $\mathrm{U}(1)_R$ current J^R_μ, and the

flavor current $J_{\mu\nu}^{U(N_f)}$, as well as fermionic currents related by supersymmetry. These are dual to the graviton, graviphoton, and $U(N_f)$ gauge fields in the bulk.

In the $\mathcal{N} = 2$ theory with fundamental matter, there are no chiral primaries of small dimensions, since gauge invariant operators of dimension less than N must involve matter fields in both fundamental and anti-fundamental representations, and hence must depend on both ϕ and $\bar{\phi}$. This suggests the absence of Kluza-Klein modes in the bulk. When $N_f \geq N$, there are baryonic chiral primary operators (2.13).[7] The bulk theory has 8 supersymmetries, and in particular the $U(N_f)$ gauge fields should lie in four dimensional $\mathcal{N} = 2$ vector multiplets. The latter also contains a complex scalar, which is dual to the operators

$$\bar{\phi}^i \phi_j, \qquad \bar{\psi}^i \psi_j + \frac{8\pi}{k}(\bar{\phi}^l \phi_l)(\bar{\phi}^i \phi_j). \tag{3.4}$$

By examining their supersymmetry variations, one can see that they lie in the same super-multiplet as the $U(N_f)$ flavor current. A surprising consequence is that the dimension of $\bar{\phi}^a \phi_b$ is protected to be 1, despite that it is not a chiral primary operator. This is checked explicitly at two-loop in appendix D.1. Recall that in the $\mathcal{N} = 2$ theory, the chiral primaries in fact have negative anomalous dimensions due to the renormalization of $U(1)_R$ charge.

In the $\mathcal{N} = 3$ theory there are in addition mesons [21], $M^i_j = \tilde{Q}^i_a Q^a_j$, which are dual to a complex scalar in the $\mathcal{N} = 4$ $U(N_f)$ gauge multiplet in AdS_4. There are two more $U(N_f)$ adjoint complex scalars in the four dimensional $\mathcal{N} = 4$ gauge multiplet. One of them is $\tilde{\psi}^i \psi_j + \frac{8\pi}{k}(\tilde{Q}^l Q_l)\tilde{Q}^i Q_j$, the other one consists of $\overline{Q}^i Q_j - \tilde{Q}^i \overline{\tilde{Q}}_j$ and $\overline{\psi}^i \psi_j - \tilde{\psi}^i \overline{\tilde{\psi}}_j + \cdots$.

In $\mathcal{N} = 2$ or $\mathcal{N} = 3$ theories with several *adjoint* matter fields, there are a lot more chiral primaries given by unsymmetrized traces of chiral fields, as described before. Their number grows exponentially in dimension. It would be interesting to understand its meaning in the holographic dual (if exists). However, this behavior is likely to be "non-generic", i.e the $\mathcal{N} = 2$ theory deformed by a generic superpotential will flow to an IR fixed point, where the number of chiral primaries may not have the exponential growth.

Since the Chern-Simons gauge field does not carry propagating degrees of freedom, the central charge of the theory with N_f fundamental flavors is expected to be of the form

$$c(\lambda) = N N_f f(\lambda), \tag{3.5}$$

where $f(\lambda)$ is some nonzero function, at least for sufficiently weak (but finite) 't Hooft coupling λ. Comparing this to the central charge expected for a theory of gravity in AdS_4, of radius $R \gg l_{pl}$, $c \sim (R/l_{pl})^2$, we expect

$$R \sim (N N_f)^{\frac{1}{2}} f(\lambda)^{\frac{1}{2}} l_{pl} \tag{3.6}$$

When N, N_f are large, we expect the AdS_4 dual to be a theory of gravity coupled to $U(N_f)$ gauge fields as well as massive fields, with large AdS_4 radius in Planck units. In the next section we will present evidence for the existence of semi-classical strings in this theory.

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[7]In the case of $U(N)$ gauge group with N_f fundamental matter, the $U(1)$ part only contribute to the anomalous dimensions to subleading order in $1/N_f$.
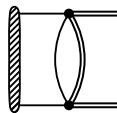
Figure 3: Mixing of $\mathcal{O}_{n,\Delta}$ with $\bar{\psi}\gamma_{(\mu_1}D_{\mu_2}\cdots D_{\mu_n)}\psi$. The solid lines are ϕ propagators and the double lines are ψ propagators.

3.3 Twist-1 operators

Let us now consider the twist-1 operators of the form

$$(\mathcal{O}_{n,\Delta})_i{}^j = \bar{\phi}_i D_{(\mu_1}\cdots D_{\mu_n)|}\phi^j \Delta^{\mu_1}\cdots\Delta^{\mu_n} \tag{3.7}$$

for large spin n, where Δ is an arbitrary null vector. The subscript | indicates the subtraction of traces in Lorentz indices. The analog of such operators in four dimensional $\mathcal{N}=4$ super-Yang-Mills theory in the large spin limit is dual to a folded semi-classical spinning string in AdS_5. We anticipate a similar interpretation in CS-matter theory: (3.7) should be dual to a long spinning open string in AdS_4 in the limit of large n. As noted in [22], the operators (3.7) are closely related to light-like open Wilson lines,

$$\bar{\phi}_i(x)\mathrm{P}e^{i\int_y^x A}\phi^j(y) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathcal{O}_{\mu_1\cdots\mu_n})_i{}^j(x-y)^{\mu_1}\cdots(x-y)^{\mu_n}. \tag{3.8}$$

The one-loop anomalous dimension of (3.7) vanishes trivially due to kinematics. There is a (regularization dependent) 1-loop shift of the Chern-Simons level k, and hence the 't Hooft coupling $\lambda = N/k$. Since there is no logarithmic divergence at 1-loop, this shift does not affect the anomalous dimension at 2-loop. One may need a more careful treatment of the regularization of the Chern-Simons propagator at higher than two loops.

There is another set of twist-1 operators, of the form $\bar{\psi}\gamma_{(\mu_1}D_{\mu_2}\cdots D_{\mu_n)|}\psi$, which may mix with (3.7), through diagrams such as the one in figure 3.

Such contribution will however be suppressed at large spin n. For example, the loop integral involved in figure 3 takes the form

$$\int d^3k\, d^3l \frac{(k\cdot\Delta)^n \slashed{l}}{(k^2)^2 l^2 (k+l-p)^2} \rightarrow \int d^3k \frac{(k\cdot\Delta)^n (\slashed{k}-\slashed{p})}{(k^2)^2 |k-p|} \tag{3.9}$$

After introducing a Feynman parameter and performing the integral over k, one ends up with

$$\sim (p\cdot\Delta)^{n-1}\slashed{\Delta}\ln\Lambda\int_0^1 dx\, n(1-x)x^{n-\frac{3}{2}}, \tag{3.10}$$

whose contribution to the operator mixing goes like $1/n$ at large n. In conclusion, we can ignore operator mixing in our computation.

We will now describe the computation of the leading two-loop anomalous dimension of (3.7). The leading contribution to the anomalous dimension comes from 2-loop diagrams as in figure 4, $(a)-(d)$.
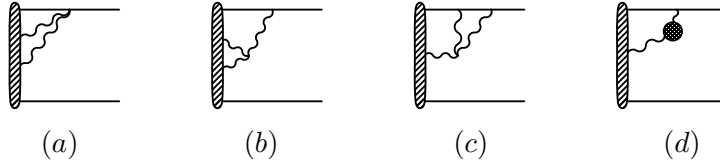
Figure 4: Two-loop contributions to the anomalous dimension of $J_{\mu_1 \cdots \mu_n}$.
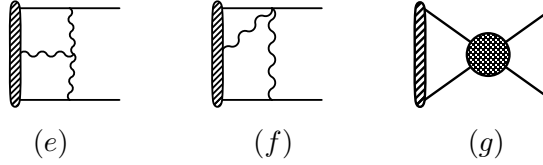
Figure 5: Some diagrams that are suppressed in the large spin limit.

The diagram (d) includes the 1-loop corrections to the gauge field propagator from a loop of the gauge field, ghost, and the matter fields. The gauge and ghost bubbles cancel, and only the matter field bubble contributes. These diagrams are computed in appendix D.2. We find that (b) vanishes,[8] (a) and (c) give contributions of order λ^2 that is finite in the $n \to \infty$ limit, whereas (d) gives a contribution of order $\lambda^2 N_f/N$ that grows like $\ln(n)$ in the large n limit. Diagrams that involve exchange of gauge fields or matter fields between the two external scalar lines, such as the ones in figure 5, are suppressed in the large n limit.

We find that the anomalous dimension of the operator $J_{\mu_1 \cdots \mu_n}$ is given to two-loop order by

$$\Delta - n - 1 = const \cdot \lambda^2 + \frac{N_f}{N}\frac{\lambda^2}{8}\ln(n) + \text{higher order corrections}, \qquad n \gg 1. \qquad (3.11)$$

Note that the above two-loop computation is essentially identical for $\mathcal{N} = 3$ theories, as well as theories with adjoint matter fields (for which the coefficient of $\ln(n)$ does not depend on N), since the vertices involving only matter fields do not contribute at this order.

Naively one might have expected higher loop corrections to give contributions to the anomalous dimension that grow like powers of $\ln(n)$. On the other hand, we have already seen nontrivial cancelations: one might have expected diagram (a) in figure 4 to grow with n, but in fact it doesn't. In four dimensional gauge theories, the $\ln(n)$ growth of the anomalous dimension of twist-2 operators at large spin n is well known, and is related to the cusp anomalous dimension of Wilson lines [31–33]. Namely, the Wilson line consisting of two straight pieces with a turn of (Lorentzian) angle γ acquires an anomalous dimension of the form $\Gamma_{\text{cusp}}(\gamma, \lambda) = \gamma\Gamma_{\text{cusp}}(\lambda) + \mathcal{O}(\gamma^0)$ in the large γ limit, where $\Gamma_{\text{cusp}}(\lambda)$, the cusp anomalous dimension, is a function of the coupling only. The coefficient of $\ln(n)$ in the

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[8]More generally, whenever two adjacent gauge field propagators coming out of $\mathcal{O}$ are joined by a cubic Chern-Simons vertex, the diagram is zero.

anomalous dimension of the twist-2 operator is in fact equal to $-2\Gamma_{\text{cusp}}(\lambda)$. The linearity of $\Gamma_{\text{cusp}}(\gamma, \lambda)$ in γ at large angles is argued in [31] by examining in a physical gauge the logarithmic divergences from the integration over momenta collinear to the (almost) light-like direction [34] along an edge of the Wilson line.

In Chern-Simons-matter theory in three dimensions, the matter-corrected gauge propagator is the sum of a piece that takes the same form as the Yang-Mills propagator in *four* dimensions in *position* space, i.e. proportional to $\eta_{\mu\nu}(x-y)^{-2}$ plus gauge dependent terms, and the pure Chern-Simons propagator, which in a physical gauge is a contact term in position space.[9] In fact, the YM-like piece of the matter-corrected gauge propagator in diagram $4(d)$ is responsible for the $\ln(n)$ growth of the two-loop anomalous dimension computed above. It is therefore plausible that a scaling argument similar to that of [31] should go through for Chern-Simons-matter theory as well. Hence we anticipate the $\ln(n)$ growth of the anomalous dimension of twist-1 operators to hold to all orders in perturbation theory.[10]

When $N_f/N \ll 1$, we expect the large n growth of the anomalous dimension to be of the form $\frac{N_f}{N} f(\lambda) \ln(n)$, for some function f. In [22] a classical folded closed string spinning in AdS space with energy E and spin J was considered, and it was found that in the large spin limit,

$$E - J = \frac{R^2}{\pi\alpha'} \ln\left(\frac{\alpha' J}{R^2}\right) + \cdots \tag{3.12}$$

The result for a spinning open string is similar, with α' replaced by $2\alpha'$. If the $\mathcal{N} = 2$ $\mathrm{U}(N)$ Chern-Simons-matter theory is dual to a string theory in AdS_4, we expect (for $N_f/N \ll 1$)

$$\frac{R^2}{\alpha'} \sim f(\lambda)\frac{N_f}{N} \tag{3.13}$$

On the other hand, by comparing the central charge in gravity with that of the gauge theory, we expect

$$R \sim (NN_f)^{\frac{1}{2}} \tilde{f}(\lambda) l_{pl} \tag{3.14}$$

for some other function $\tilde{f}(\lambda)$. Therefore,

$$\frac{\alpha'}{l_{pl}^2} \sim N^2 \frac{\tilde{f}(\lambda)^2}{f(\lambda)} \tag{3.15}$$

This is consistent with the expectation (say by examining three-point functions) that in the large N limit, the string coupling is independent of N_f.

Note that the $\mathrm{U}(N_f)$ gauge fields in the bulk also has coupling of order $1/N$ (times some function of λ), and hence (four-dimensional) 't Hooft coupling of order N_f/N. In the large N limit, with $N_f/N = c \ll 1$, the bulk theory is effectively weakly coupled, which justifies the above discussion. A naive attempt to get large AdS_4 radius in string units is to take N_f/N large. In this case, however, the bulk gauge theory becomes strongly

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[9]In the axial gauge, for instance, the pure CS propagator is given by $\epsilon_{ij}\mathrm{sgn}(x_3 - y_3)\delta^2(\vec{x} - \vec{y})$.

[10]Although, we do not understand the possible subtleties in the scaling argument involving the singular looking CS propagator in a physical gauge, which clearly deserves a more careful analysis.

coupled. This is particularly intriguing in the case of $\mathcal{N} = 3$ CS-matter theory, where the bulk theory contains four-dimensional $U(N_f)$ $\mathcal{N} = 4$ SYM coupled to $\mathcal{N} = 3$ supergravity, as well as infinitely many massive fields. As N_f/N increases, we expect the radius R to increase as well. In the limit of large N_f/N, the suitable description of the bulk theory may involve a further holographic dual to a five dimensional theory of gravity. This suggests that flavor singlets in the Chern-Simons-matter theory of the form

$$(\tilde{Q}^{i_1} Q_{i_2})(\tilde{Q}^{i_2} Q_{i_3}) \cdots (\tilde{Q}^{i_n} Q_{i_1}) \tag{3.16}$$

could be dual to closed strings in the five dimensional theory.

3.4 Supersymmetric Wilson loops

In this section we will describe supersymmetric Wilson loops in the theory. First consider the $\mathcal{N} = 2$ Chern-Simons theory. The supersymmetry transformations of the gauge field A_μ and auxiliary field σ are of the form

$$\delta_\epsilon A_\mu = \frac{i}{2}(\bar{\varepsilon}\gamma_\mu\chi - \bar{\chi}\gamma_\mu\varepsilon), \qquad \delta_\epsilon \sigma = \frac{i}{2}(\bar{\varepsilon}\chi - \bar{\chi}\varepsilon). \tag{3.17}$$

The Wilson line

$$P \exp\left[\int d\tau \left(A_\mu \dot{x}^\mu + \sigma |\dot{\vec{x}}|\right)\right] \tag{3.18}$$

locally preserves $1/2$ supersymmetries, whose supersymmetry parameters are solutions of $\gamma_\mu \frac{\dot{x}^\mu}{|\dot{\vec{x}}|}\varepsilon = -\varepsilon$. Globally, only the straight Wilson line can preserve half of the $\mathcal{N} = 2$ supersymmetries. Similarly, the straight Wilson line also preserves one half of the special supercharges of the superconformal algebra $OSp(2|4)$. More generally, any conformal transformation of the straight Wilson line (3.18) will preserve one half of the supersymmetries as well. As pointed out in [35], large conformal transformations in $\mathbf{R}^3$ that take a point on the Wilson line at finite distance to infinity will not preserve the expectation value of the BPS Wilson line.

In the pure Chern-Simons theory, it is well known that the Wilson loops are topological [36]. This is no longer the case when the Chern-Simons theory is coupled to matter fields. Note that the gauge field propagator (in Feynman gauge) in the 1PI effective action takes the form

$$\frac{\delta^{ab}}{1 - \Pi(k)^2}\left[\frac{\epsilon_{\mu\nu\rho}p^\rho}{2p^2} + \Pi(k)\frac{\delta_{\mu\nu} - p_\mu p_\nu/p^2}{p}\right] \tag{3.19}$$

where $\Pi(k)$ is due to loops involving the matter fields, a, b are gauge indices. k here is the Chern-Simons level, not to be confused with momentum. The structure of (3.19) is constrained by conformal invariance. From the one-loop matter bubble correction we have $\Pi(k) = -\frac{\pi N_f}{4k} + \mathcal{O}(\frac{1}{k^2})$. The Fourier transform of (3.19) to position space is

$$\langle A_\mu^a(x) A_\nu^b(y)\rangle \sim \frac{\delta_a{}^b}{1 - \Pi(k)^2}\left[\frac{\epsilon_{\mu\nu\rho}(x-y)^\rho}{2|x-y|^3} + \Pi(k)\left(\frac{\delta_{\mu\nu}}{|x-y|^2} - \partial_\mu\partial_\nu \ln|x-y|\right)\right] \tag{3.20}$$

We see that the second term on the r.h.s. of (3.20) takes the same form as the Yang-Mills propagator in *four* dimensions. Using the method of [35], it may be possible to compute

say the exact expectation value of circular Wilson loops by localizing the contributions to a point. (See [37] for a perturbative approach to Wilson loops in pure CS theory.) We leave this problem to future investigation.

There are also supersymmetric Wilson lines in the $\mathcal{N} = 3$ theory, of the form

$$P \exp\left[\int d\tau \left(A_\mu \dot{x}^\mu + s_i \dot{y}^i\right)\right] \tag{3.21}$$

where s_i's are the $SU(2)_R$ multiplet of auxiliary fields containing σ and Φ. $(x^\mu(\tau), y^i(\tau))$ parameterizes a path in $\mathbf{R}^3 \times \mathbf{R}^3$, where the second $\mathbf{R}^3$ is an internal space acted by the $SU(2)_R$. The condition for (3.21) to locally preserve supersymmetries is $|\dot{\vec{x}}| = |\dot{\vec{y}}|$. The preserved supersymmetry parameter ε_i are solutions of

$$\gamma_\mu \dot{x}^\mu \varepsilon_i + i\epsilon_{ijk} \dot{y}^j \varepsilon_k = 0. \tag{3.22}$$

A straight Wilson line of the form (3.21) with constant $\dot{y}^i$ preserves $1/3$ of the $\mathcal{N} = 3$ supercharges.

3.5 The anomalous dimension of a Wilson line with an angle

As recalled earlier, the $\ln(n)$ growth of the anomalous dimension of the twist-1 operator is related to the anomalous dimension of a Wilson line with a turn of angle γ, in the large γ limit. In this subsection we describe an intuitive physical picture of the latter, following [38]. Let us first consider the Euclidean version. Take a Wilson line consisting of two straight pieces, labelled by vectors u, v, joined at the origin. The turning angle θ is given by $\cos\theta = \frac{u \cdot v}{|u||v|}$. Since the Chern-Simons-matter theory is conformal, we can map the configuration conformally to one on $S^2 \times \mathbf{R}$, with the two straight pieces of the Wilson line at two points separated by the angle $\pi - \theta$ on the S^2, extending in $\mathbf{R}$. The anomalous dimension of the Wilson line is equivalent to the potential energy between a pair of quark and anti-quark on the S^2.

To leading order the potential energy is determined by the two point function of the gauge fields (3.20). The part that contributes is similar to the four-dimensional Yang-Mills propagator in position space, resulting in a potential energy

$$V(\pi - \theta) \propto \theta \cot\theta + const \tag{3.23}$$

Note that this should be interpreted as the potential due to a quark at $\theta = \pi$ and a uniform background charge on the S^2 that cancels the charge of the quark, due to the Gauss law constraint.

In Lorentzian signature, the angle is $\gamma = i\theta$, with $\cosh\gamma = \frac{u \cdot v}{|u||v|}$. The analytic continuation of the above conformal transformation brings $\mathbf{R}^{2,1}$ to $AdS_2 \times \mathbf{R}$, with the time direction of the AdS_2 compactified on a circle. The anomalous dimension of the Wilson loop, to leading order (one-loop in the matter-corrected gauge propagator), is given by

$$\Gamma(\gamma) \sim \gamma \coth\gamma + const \tag{3.24}$$
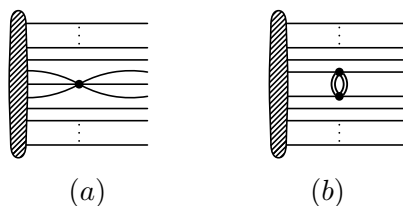
Figure 6: Two-loop diagrams contributing to the anomalous dimension of a string of Q's and $\tilde{Q}$'s.

Indeed, at large angle γ, the anomalous dimension is linear in γ. This essentially reproduces the result of the two-loop calculation of section 3.3. Less obviously, we expect the linearity at large γ to hold for the analytic continuation of the potential function on the S^2, to all orders in perturbation theory, based on arguments along the lines of [31].

3.6 The $\mathcal{N} = 3$ spin chain

A powerful technique that can be used to understand the operator spectrum is to map the dilatation operator to a spin chain Hamiltonian (see [39] for a review of the subject). In this subsection we consider the $\mathcal{N} = 3$ theory with one adjoint hypermultiplet matter $(Q, \tilde{Q})$. A basic chiral primary operator is $\mathrm{Tr}Q^J$. By acting on it with $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_f$ symmetry, we obtain a more general class of protected operators,

$$\mathrm{Tr}\left[(v^A Q_A(u))^J \right], \tag{3.25}$$

where $Q_A(u)$ are defined as in appendix A, u^a and v^A are doublets of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_f$ respectively. Among these there are symmetrized traces of Q's mixed with $\tilde{Q}$'s, which are $\mathcal{N} = 2$ chiral primaries. A perhaps less obvious class of protected operators are the symmetrized traces of Q's mixed with $\overline{\tilde{Q}}$'s.

The operators obtained by inserting a few $\tilde{Q}$ "impurities" in $\mathrm{Tr}(QQ\cdots Q)$, without symmetrization inside the trace, is not a chiral primary and will receive anomalous dimension, due to the superpotential $W = \frac{\pi}{k}\mathrm{Tr}([Q, \tilde{Q}]^2)$. Similarly, if $\overline{\tilde{Q}}$'s are inserted in $\mathrm{Tr}Q^J$, but not symmetrized, the operator is not protected and has anomalous dimension. We shall compute the two-loop anomalous dimension of such operators in the planar limit, assuming that the operator is very long and the impurities are far away from one another. The kind of diagrams that contribute are shown in figure 6.

Let us first consider $\tilde{Q}$-impurities, corresponding to excitations of the $\mathrm{SU}(2)_f$ spin chain, since Q and $\tilde{Q}$ are a doublet of $\mathrm{SU}(2)_f$. The operators of interest are of the form

$$\mathrm{Tr}(Q\cdots Q\tilde{Q}Q\cdots Q\tilde{Q}Q\cdots Q)$$

Diagram 6(a) may involve a sextic scalar coupling coming from the superpotential, of the form

$$\frac{4\pi^2}{k^2}\mathrm{Tr}\left([\tilde{Q},[\tilde{Q},Q]]\overline{[\tilde{Q},[\tilde{Q},Q]]} + [Q,[\tilde{Q},Q]]\overline{[Q,[\tilde{Q},Q]]} \right) \tag{3.26}$$

There are also sextic scalar interactions coming from the coupling to $\mathcal{N} = 2$ Chern-Simons gauge multiplet, $\mathrm{Tr}([\overline{Q}, \sigma][\sigma, Q] + [\overline{\tilde{Q}}, \sigma][\sigma, \tilde{Q}])$, but they do not in fact contribute to the

anomalous dimension in the planar limit. Similarly, the diagrams involving fermions (Fig 6 (b)) do not contribute in the planar limit either. It is a simple exercise to derive the two-loop spin chain Hamiltonian from (3.26), and we find

$$H_{(2)}^{\mathrm{SU}(2)_f} = \frac{\lambda^2}{4} \sum_i (4P_{i,i+1} - P_{i,i+2} - 6),$$ (3.27)

where $P_{i,j}$ is the operator that interchanges the i-th and j-th sites. This is the Hamiltonian of an $su(2)$ XXX spin-$\frac{1}{2}$ chain with next-to-nearest neighbor interactions. Such a spin chain is in fact *not* integrable. This suggests that the dual string worldsheet sigma model, if exists, will probably not be integrable either.

Let us now consider $\widetilde{\overline{Q}}$ impurities, corresponding to excitations of the SU(2)$_R$ spin chain. In this case, diagram (a) in Fig 6 gives rise to next-to-nearest neighbor interactions, $\sum P_{i,i+2}$, whereas diagram (b) gives rise to nearest neighbor interactions, $\sum P_{i,i+1}$. The resulting two-loop spin chain Hamiltonian $H_{(2)}^{\mathrm{SU}(2)_R}$ in fact takes the identical form as (3.27).

It follows from (3.27) that the spectrum of anomalous dimensions for operators in the SU(2)$_f$ or SU(2)$_R$ sector, in the limit of large J (length) and few impurities, is given by

$$\Delta - \frac{J}{2} = \frac{\lambda^2}{4} \sum_m \left[2 \sin\left(\frac{\pi l_m}{J} \right) \right]^4 + \mathcal{O}(\lambda^3), \quad l_m \in \mathbf{Z},$$ (3.28)

where l_m is the "momentum" of the m-th impurity. This is clearly a very different dispersion relation from that of string modes in ordinary pp-wave backgrounds [40].

In the SU(2)$_f$ × SU(2)$_R$ invariant notation of appendix A, $Q = q^{11}$, $\tilde{Q} = q^{21}$, and $\overline{Q} = -q^{12}$. There are two more basic impurities, given by insertions of fermions ψ_α^{11}. As we will partially justify, insertions of other fields such as $\overline{Q} = q^{22}$ in the infinitely long spin chain can be thought of as bound states of the four basic impurities. The superconformal group $OSp(3|4)$ has a subgroup SU(1|2) that leaves q^{11} invariant. The bosonic part of SU(1|2), U(1) × SU(2), is generated by $\Delta - J_R^3$ and the rotations in $\mathbf{R}^3$. The fermionic generators are the chiral supercharges Q_α and $\overline{S}^\alpha$.

The symmetrized traces of Q's with $\tilde{Q}$ or $\overline{Q}$ insertions, i.e. impurities with zero momenta, are protected because they sit in reduced representations of SU(1|2). In fact, at zero momentum, q^{21} is a singlet, whereas $(q^{12}, \psi_\alpha^{11} \sim Q_\alpha q^{12})$ form a fundamental representation of SU(1|2). A more general representation of SU(1|2) can be obtained by acting Q_α on a primary of anomalous dimension and spin (h, j). The resulting representation consists of U(1) × SU(2) content

$$(h, j) \oplus \left(h + \frac{1}{2}, j + \frac{1}{2} \right) \oplus \left(h + \frac{1}{2}, j - \frac{1}{2} \right) \oplus (h + 1, j + 1)$$ (3.29)

It is convenient to consider impurities in an infinitely long spin chain of q^{11}'s, to allow states with a single impurity of momentum p (finite traces always have total momentum zero by cyclicity). When there is nonzero momentum, the basic impurities are no longer in reduced representations of SU(1|2), and will acquire anomalous dimensions. Due to the superpotential of the $\mathcal{N} = 3$ theory, we have

$$Q_\alpha \psi_\beta^{11} \sim \epsilon_{\alpha\beta} \frac{1}{k} [[q^{11}, q^{21}], q^{11}]$$ (3.30)

In an infinitely long spin chain, the extra factors of q^{11} are unimportant, but the different terms in the commutators receive different phase factors $e^{2\pi ipn}$. Therefore, we have

$$Q_\alpha |\psi_\beta^{11}(p)\rangle \sim \epsilon_{\alpha\beta} \frac{1}{k} \left[2\sin(\pi p)\right]^2 |q^{21}(p)\rangle \tag{3.31}$$

Consequently, the impurities $(q^{12}, q^{21}, \psi_\alpha^{11})$ with nonzero momentum p sit in a long multiplet of SU(1|2). A somewhat unexpected consequence is that the SU(2)$_R$ impurity and the SU(2)$_f$ impurity have the same anomalous dimension, agreeing with our calculations of the two-loop spin chain Hamiltonian.

Unlike $\mathcal{N} = 4$ SYM, our basic impurities do not sit in any shortened representation of the supergroup, hence we cannot deduce the form of the exact anomalous dimensions to all order based on the deformed superalgebra for the infinite chain. Also note that the product of two representations corresponding to the basic impurities is a sum of irreducible representations (as opposed to one long representation in the case of $\mathcal{N} = 4$ SYM), the two impurity scattering S-matrix will be determined by several scattering phases. A detailed exploration of these topics is beyond the scope of the current paper.

3.7 The $\mathcal{N} = 2$ spin chain

Let us briefly describe the spin chain in the $\mathcal{N} = 2$ theory with one adjoint matter $\Phi = (\phi, \psi)$. A basic chiral primary is Trϕ^J. We will again consider infinitely long chain, with SU(1|2) symmetry. The basic impurities are $(\bar{\phi}, \psi_\alpha, \bar{\psi}_\alpha, D_\mu)$, forming a long multiplet of SU(1|2) (unlike the $\mathcal{N} = 3$ case, where such impurities are bound states of more elementary impurities). The action of the chiral supercharge Q_α on the fields are schematically

$$\begin{aligned} &Q_\alpha \bar{\phi} \sim \bar{\psi}_\alpha, \\ &Q_\alpha \bar{\psi}_\beta = 0, \\ &Q_\alpha \psi_\beta \sim -i\gamma^\mu D_\mu \phi + \frac{2\pi}{k}[[\phi, \bar{\phi}], \phi], Q_\alpha D_\mu \phi \sim (\gamma_\mu)_\alpha{}^\beta \frac{2\pi}{k}[\phi, \bar{\psi}_\beta], \end{aligned} \tag{3.32}$$

or in terms of impurities with momentum p,

$$\begin{aligned} &Q_\alpha |\bar{\phi}(p)\rangle \sim |\bar{\psi}_\alpha(p)\rangle, \\ &Q_\alpha |\bar{\psi}_\beta(p)\rangle = 0, \\ &Q_\alpha |\psi_\beta(p)\rangle \sim -i\gamma^\mu |D_\mu(p)\rangle + \frac{2\pi}{k}\left[2\sin(\pi p)\right]^2 |\bar{\phi}(p)\rangle, \\ &Q_\alpha |D_\mu(p)\rangle \sim (\gamma_\mu)_\alpha{}^\beta \frac{2\pi}{k}(e^{2\pi ip} - 1)|\psi_\beta(p)\rangle, \end{aligned} \tag{3.33}$$

One should be cautious that D_μ at zero momentum always have dimension 1, whereas ϕ itself has dimension q^R, renormalized to be less than 1/2.

The two-loop Hamiltonian of the chain made of ϕ's and $\bar{\phi}$'s takes the form

$$H_{(2)}^{\mathcal{N}=2} = \frac{\lambda^2}{4} \sum_i (3P_{i,i+1} - P_{i,i+2} - 4) + \text{(zero momentum contribution)} \tag{3.34}$$

where the zero momentum contribution gives the anomalous of the symmetrized traces. Note that in the continuum limit, the dispersion relation of the $\bar{\phi}$ impurity goes like

$\sin^2(\pi p)$, as familiar in ordinary pp-wave limits, in contrast to the $\sin^4(\pi p)$ behavior we saw for the $\mathcal{N} = 3$ theory.

As discussed in section 2, one can turn on a superpotential $W = \alpha \text{Tr}\Phi^4$, and flow to a different SCFT. This superpotential will kill all the chiral primaries of the form $\text{Tr}\phi^J$ except $\text{Tr}\phi^2$. Together with its descendants $\text{Tr}\phi\psi_\alpha$ and $\text{Tr}\psi_\alpha\psi_\beta\epsilon^{\alpha\beta}$, they are dual to an $\mathcal{N} = 2$ hypermultiplet in the bulk AdS_4.

4. Some preliminary attempts to construct the holographic dual

There are many ways of engineering three dimensional gauge theories with Chern-Simons coupling in string theory. We will describe some examples in this section, in attempt to find gravity duals of the Chern-Simons-matter system. However we have not been able to find AdS_4 dual in the supergravity regime.

4.1 $\mathcal{N} = 2$ U(N) theory with one adjoint

One can obtain $\mathcal{N} = 2$ U(N) Chern-Simons theory at level k coupled to an adjoint matter as the low energy limit of the world volume theory of N M5-branes wrapped on a special Lagrangian lens space $S^3/\mathbf{Z}_k$ in a Calabi-Yau 3-fold (say the cotangent bundle of $S^3/\mathbf{Z}_k$), at least for large k. The M-theory reduces to type IIA string theory compactified on a five-manifold involving an S^2, with k units of $F_{(2)}^{RR}$ flux on the S^2, and the M5-branes turn into D4-branes wrapped on the S^2. The RR-flux on the S^2 induces Chern-Simons coupling in the D4-brane world volume gauge theory. The gauge theory is three dimensional $\mathcal{N} = 2$ super-Yang-Mills with Chern-Simons coupling, coupled to an adjoint matter (corresponding to the 2 transverse coordinates of the D4-brane in $\mathbf{R}^{1,4}$), with no superpotential. At low energies the Yang-Mills coupling becomes irrelevant, and the theory flows to $\mathcal{N} = 2$ Chern-Simons-matter theory.

The gravity solutions of supersymmetric M5-branes wrapped on special Lagrangian cycles were solved in [41], with the sLag 3-cycle being either a quotient of S^3 or a quotient H_3 of the hyperbolic 3-space. It turns out that a smooth near horizon AdS_4 regime in the supergravity exists only when the M5-branes are wrapped on H_3, and not S^3 (nor its quotient). Since H_3 cannot be realized as a circle fibration over a Riemann surface, the corresponding M5-brane configuration is "intrinsically M-theoretic". In conclusion we do not find an AdS_4 dual to Chern-Simons theory with one adjoint matter in the supergravity regime. It is plausible, however, that when α'-corrections are included, there could be a smooth near-horizon limit of the S^2-wrapped D4-brane, involving an AdS_4 whose radius is at string scale. If such a solution were found, it would be presumably dual to $\mathcal{N} = 2$ CS coupled to 1 adjoint matter.

4.2 D2-branes in massive IIA theory

Another way of obtaining Chern-Simons coupling is to consider D2-branes in massive IIA theory, with $F_{(0)}^{RR} = k$. Fundamental matter fields can be introduced by adding D6 or D8-branes. Let us consider the system of N D2-branes and N_f D6-branes, with the D2 lying inside the world volume of D6 in flat space. There are 3 $\mathcal{N} = 2$ adjoint matter

fields Φ_i, $i = 1, 2, 3$, as well as fundamental and anti-fundamental matter fields $Q^j, \tilde{Q}^j$, $j = 1, \cdots, N_f$. Φ_1 corresponds to the 2 coordinates of the D2-brane transverse to the D6, where as $\Phi_{2,3}$ correspond to the transverse coordinates of the D2 within the world volume of D6. There is a superpotential of the form

$$W = \mathrm{Tr}\Phi_1[\Phi_2, \Phi_3] + \tilde{Q}^j \Phi_1 Q^j \qquad (4.1)$$

In the IR, it is conceivable that $Q^j, \tilde{Q}^j, \Phi_2, \Phi_3$ remain of dimension $\frac{1}{2}$, whereas Φ_1 becomes a dimension 1 field, so that W, of dimension 2, is marginal. The kinetic term for Φ_1 then becomes irrelevant. If we introduce a small deformation of the superpotential $\frac{1}{2}\epsilon\mathrm{Tr}\Phi^2$, so that the total superpotential is

$$W = \frac{1}{2}\epsilon\mathrm{Tr}\Phi_1^2 + \mathrm{Tr}\Phi_1[\Phi_2, \Phi_3] + \tilde{Q}^j \Phi_1 Q^j, \qquad (4.2)$$

we can then integrate out Φ_1 and obtain an equivalent superpotential

$$W = \frac{1}{2\epsilon}\mathrm{Tr}\left([\Phi_2, \Phi_3] + Q^j\tilde{Q}^j\right)^2 \qquad (4.3)$$

When $\epsilon \ll k$, the superpotential dominates the interaction due to Chern-Simons coupling, and acquires positive anomalous dimension (i.e. positive beta function). As in our previous discussion of $\mathcal{N} = 3$ Chern-Simons-matter theory, the theory with superpotential (4.3) flows to the point where the coefficient $1/\epsilon$ becomes π/k, and the theory becomes nothing but the $\mathcal{N} = 3$ $\mathrm{U}(N)$ Chern-Simons theory coupled minimally to an adjoint hypermultiplet (Φ_2, Φ_3) and N_f fundamental hypermultiplets $(Q^j, \tilde{Q}^j)$.

It is thus conceivable that the world volume theory of D2-D6 in massive IIA theory flows to the $\mathcal{N} = 3$ Chern-Simons-matter theory under the deformation given by a small mass term for Φ_1. Recall that the $\mathcal{N} = 2$ Chern-Simons-matter theory also flows to the $\mathcal{N} = 3$ theory, but in the opposite direction along the line of the coefficient of the superpotential. It would be interesting to study the near horizon geometry of D2-D6 system in massive IIA theory, and see if the above RG flow can be described in the gravity dual.

4.3 Hints from spin chains

The spin chain analysis in the previous section provides several useful hints on the holographic dual of $\mathcal{N} = 3$ CS coupled to one adjoint hypermultiplet. There is a tower of short representations of $OSp(3|4) \times \mathrm{SU}(2)_f$ generated by $\mathrm{Tr}Q^J$ with spin $(\frac{J}{2}, \frac{J}{2})$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_f$. This resembles the spectrum of KK modes of a 7-dimensional supergravity on $AdS_4 \times S^3$. The fact that the $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_f$ impurities have the same spectrum further reinforces the idea that the two $\mathrm{SU}(2)$'s should appear in the dual geometry in a symmetric fashion, as left and right rotations of an S^3. The supercoset $OSp(3|4)/\mathrm{SO}(3, 1)$ has $AdS_4 \times S^3$ as its bosonic part, and the correct symmetry group.

The $\mathcal{N} = 2$ CS with one adjoint matter Φ, deformed by the superpotential $\mathrm{Tr}\Phi^4$, should be dual to a theory of gravity in AdS_4, whose massless sector is $\mathcal{N} = 2$ supergravity coupled to a universal hypermultiplet. It might be possible that when the 't Hooft coupling is large, all other fields become infinitely massive, and only the supergravity sector remains.

We leave the detailed analysis of the holographic dual geometries to future investigation.

5. Concluding remarks

We found a surprisingly large class of three dimensional $\mathcal{N} = 2$ SCFTs with Lagrangian descriptions, whose couplings can be made arbitrarily weak, as superpotential deformations of $\mathcal{N} = 2$ Chern-Simons-matter theories. This allows perturbative understanding of the SCFTs in $1/k$. For $\mathrm{U}(N)$ theories, we can also study the $1/N$ expansion. We have seen evidences for the existence of a weakly coupled string theory dual in the large N limit, although we have not been able to construct the AdS_4 dual in a supergravity regime. It is possible that the holographic dual of these theories are described by string theory on AdS_4 whose radius is at string scale. It would be interesting to investigate the α' corrections in the brane constructions in type II string theories that give rise to Chern-Simons couplings, as well as exploring AdS_4 compactifications of massive IIA supergravity/string theory. In the example of $\mathcal{N} = 3$ theory, we have seen that certain sectors of long operators can be described by (non-integrable) spin chains. One may learn about the possible dual string sigma model from this.

The rich structure of RG flows among the superpotential deformations of $\mathcal{N} = 2$ CS-matter theories is intriguing. In the context of M-theory compactified on AdS_4 times a Sasakian-Einstein seven-manifold, there is also a rich structure of holographic RG flows, as explored in [42 – 44]. It would be interesting to see if there are connections between them. More ambitiously, one may wonder if there are connections of the CS-matter theories to the vast number of AdS_4 string vacua in flux compactifications (see for example [45]).

A. The $\mathcal{N} = 3$ Lagrangian

In this appendix we write the Lagrangian of $\mathcal{N} = 3$ Chern-Simons theory coupled to N_f hypermultiplets in a *real* representation R in component fields with manifest $USp(2N_f) \times \mathrm{SU}(2)_R$ symmetry. Let $a, b, \cdots$ be $\mathrm{SU}(2)_R$ indices, $A, B, \cdots$ indices for the fundamental representation of $USp(2N_f)$, and $\alpha, \beta, \cdots$ $\mathrm{SO}(2,1)$ spinor indices.

The components of the matter fields are scalars q^{Aa} and fermions ψ_α^{Aa}, with reality condition

$$(q^\dagger)_{Aa} = \omega_{AB}\epsilon_{ab}q^{Bb}, \bar{\psi}_{Aa}^\alpha = \omega_{AB}\epsilon_{ab}\epsilon^{\alpha\beta}\psi_\beta^{Bb}. \tag{A.1}$$

where ω_{AB} is a symplectic form. They are related to the fields $(Q, \tilde{Q})$ in the $\mathcal{N} = 2$ notation by

$$q^{A1} = (Q, \tilde{Q}), \qquad q^{A2} = (-\bar{\tilde{Q}}, \bar{Q}). \tag{A.2}$$

The on-shell supersymmetry transformations are

$$\delta^\alpha_{ab} A^m_{\beta\gamma} = \frac{4\pi}{k} q_{A(a} T^m \psi^A_{b)(\gamma} \delta^\alpha_{\beta)},$$

$$\delta^\alpha_{ab} q_{Ac} = \psi^\alpha_{A(a} \epsilon_{b)c}, \delta^\alpha_{ab} \psi^\beta_{Ac} = -i \nabla^{\alpha\beta} q_{A(a} \epsilon_{b)c} + \frac{4\pi}{k} (q_{Bc} T^m q^B_{(a}) T^m q_{b)A}. \tag{A.3}$$

Introducing auxiliary fields $s^m_{ab} = \frac{4\pi}{k} q_{A(a} T^m q^A_{b)}$, $\chi^m_{ab} = -\frac{4\pi i}{k} q_{A(a} T^m \psi^A_{b)}$, $\chi^m = -\frac{4\pi i}{k} q_{Aa} T^m \psi^{Aa}$, the Lagrangian can be written as

$$L = \frac{k}{4\pi} \left[CS(A) + \text{Tr} \left(D^{ab} s_{ab} - \frac{1}{2} \chi^{ab} \chi_{ab} + \chi\chi + \frac{1}{6} s^{ab} [s_{bc}, s^c{}_a] \right) \right]$$
$$+ \frac{1}{2} |\nabla_\mu q_{aA}|^2 + \frac{1}{2} q_{Aa} D^{ab} q^A{}_b - \frac{1}{4} |s_{ab} q^{Ac}|^2$$
$$+ \frac{i}{2} \psi_{aA} \gamma^\mu \nabla_\mu \psi^{aA} - \frac{1}{2} \psi^a_A s_{ab} \psi^{Ab} + i q_A{}^a \chi_{ab} \psi^{Ab} + i q_{Aa} \chi \psi^{Ab}. \tag{A.4}$$

To study BPS (chiral) operators one needs to pick a chiral supersymmetry generator $\delta^+ = u^a u^b \delta_{ab}$ for some twistor variable u^a. Then $Q_A(u) \equiv u^a q_{aA}$ are the chiral fields.

B. Thermodynamics of the abelian theory at large N_f

B.1 $\mathcal{N} = 2$ theory with equally charged matter in flat space

In this section we will consider the free energy of the $\mathcal{N} = 2$ theory with equally charged matter fields in flat space. Keeping the auxiliary fields in the vector super-multiplet, we can integrate out the matter fields, and compute the thermal partition function via the path integral

$$\int dA dD d\sigma d\chi d\bar{\chi} \exp \left[-\frac{N_f}{\lambda} \int (\omega_{CS}(A) + 2D\sigma - \bar{\chi}\chi) \right.$$
$$\left. -N_f \text{Str} \ln \begin{pmatrix} -D^\mu D_\mu + \sigma^2 + D & \bar{\chi} \\ chi & i\not{D} + \sigma \end{pmatrix} \right] \tag{B.1}$$

on $\mathbf{R}^2 \times S^1_\beta$. In the large N_f limit, the path integral by the saddle point contribution. We must be careful with the definition of this path integral: the superspace action (2.3) is linear in the auxiliary field D, and integration over D results in a delta functional that gives rise to a (Euclidean) action which is bounded from below. While keeping D in (B.1), we cannot naively minimize the action with respect to D, as it would obviously give a divergent answer. The correct prescription is to extremize the effective action with respect to D first, and then minimize the effective action with respect to the other fields.[11]

The saddle points involve translational invariant (as well as rotational invariant in the case of finite temperature) field configurations. So we can set $A_i = 0, \chi = 0$, and

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[11]To see this more explicitly, consider a potential function $V(D, x) = xD + f(x)$. "Integrating out" D sets the potential to $f(0)$. On the other hand, "integrating out" x amounts to a Legendre transform, $D = -f'(x)$. The potential $V(D, x(D))$ is extremized with respect to D at $x(D) = 0$. Whether it is a local maximum or minimum in D depends on the sign of $f''(0)$.

$D, \sigma, A_0 = \alpha$ to constants. We then have the very simple effective action in the constant modes D, σ, α, at finite temperature $T = 1/\beta$,

$$S_{\text{eff}} = N_f A \beta \left[-\frac{2D\sigma}{\lambda} + \beta^{-1} \sum_{n \in \mathbf{Z}} \int \frac{d^2 p}{(2\pi)^2} \ln \frac{p^2 + \left(\frac{2\pi}{\beta} n + \alpha\right)^2 + \sigma^2 + D}{p^2 + \left(\frac{2\pi}{\beta}\left(n + \frac{1}{2}\right) + \alpha\right)^2 + \sigma^2} \right] \quad \text{(B.2)}$$

We can dimensionally regularize the integral over spatial momenta, and then regularize the sum over the momentum along Euclidean time direction using zeta function regularization. The $1/\epsilon$ divergence cancel after the zeta function regularization, and we end up with

$$S_{\text{eff}} = N_f A \beta \left\{ -\frac{2D\sigma}{\lambda} - \frac{1}{4\pi\beta} \sum_{n \in \mathbf{Z}} \left[\left(\frac{2\pi n}{\beta} + \alpha\right)^2 + \sigma^2 + D \right] \ln \left[\left(\frac{2\pi n}{\beta} + \alpha\right)^2 + \sigma^2 + D \right] \right.$$
$$\left. + \frac{1}{4\pi\beta} \sum_{n \in \mathbf{Z}} \left[\left(\frac{2\pi(n + \frac{1}{2})}{\beta} + \alpha\right)^2 + \sigma^2 \right] \ln \left[\left(\frac{2\pi(n + \frac{1}{2})}{\beta} + \alpha\right)^2 + \sigma^2 \right] \right\} \quad \text{(B.3)}$$

To evaluate it, we shall use the formula (free energy of a harmonic oscillator)

$$\sum_{n \in \mathbf{Z}} \ln \left[x^2 + \left(\frac{2\pi n}{\beta} + \alpha\right)^2 \right] = \beta x + \ln(1 - e^{-\beta x + i\beta\alpha}) + \ln(1 - e^{-\beta x - i\beta\alpha}) \quad \text{(B.4)}$$

Integrating this, we obtain

$$\sum_{n \in \mathbf{Z}} \left[x^2 + \left(\frac{2\pi n}{\beta} + \alpha\right)^2 \right] \ln \left[x^2 + \left(\frac{2\pi n}{\beta} + \alpha\right)^2 \right]$$
$$= \frac{2\beta}{3} x^3 + 4 \sum_{k=1}^{\infty} \frac{1 + k\beta x}{\beta^2 k^3} e^{-k\beta x} \cos(k\beta\alpha) = \frac{2\beta}{3} x^3 + 4\beta^{-2} \mathcal{I}(\beta x, \beta\alpha) \quad \text{(B.5)}$$
$$= \frac{2\beta}{3} x^3 + \frac{2x}{\beta} \left[\text{Li}_2\left(e^{-\beta(x + i\alpha)}\right) + \text{Li}_2\left(e^{-\beta(x - i\alpha)}\right) \right] + \frac{2}{\beta^2} \left[\text{Li}_3\left(e^{-\beta(x + i\alpha)}\right) + \text{Li}_3\left(e^{-\beta(x - i\alpha)}\right) \right]$$

where we defined the function

$$\mathcal{I}(x, \alpha) = \sum_{k=1}^{\infty} \frac{1 + kx}{k^3} e^{-kx} \cos(k\alpha) = \sum_{k=1}^{\infty} \frac{\cos(k\alpha)}{k^3} + x^2 \ln \left| 2 \sin\left(\frac{\alpha}{2}\right) \right| - \frac{x^3}{6} + \frac{x^4}{32 \sin^2(\frac{\alpha}{2})} + \mathcal{O}(x^6) \quad \text{(B.6)}$$

Note that when $x = 0, \alpha = 0$, (B.5) reproduces the contribution to the free energy from a massless free field. The effective action is then

$$S_{\text{eff}} = -N_f A \beta \left\{ \frac{2D\sigma}{\lambda} + \frac{1}{6\pi} \left[(\sigma^2 + D)^{\frac{3}{2}} - |\sigma|^3 \right] + \frac{\mathcal{I}(\beta(\sigma^2 + D)^{\frac{1}{2}}, \beta\alpha) - \mathcal{I}(\beta|\sigma|, \beta\alpha + \pi)}{\pi\beta^3} \right\} \quad \text{(B.7)}$$

To separate the temperature dependence, we can rescale the variables $\tilde{\sigma} = \beta\sigma$, $\tilde{D} = \beta^2 D$, $\tilde{\alpha} = \beta\alpha$, and write

$$S_{\text{eff}} = -N_f A \beta^{-2} \left\{ \frac{2\tilde{D}\tilde{\sigma}}{\lambda} + \frac{1}{6\pi} \left[(\tilde{\sigma}^2 + \tilde{D})^{\frac{3}{2}} - |\tilde{\sigma}|^3 \right] + \frac{\mathcal{I}((\tilde{\sigma}^2 + \tilde{D})^{\frac{1}{2}}, \tilde{\alpha}) - \mathcal{I}(|\tilde{\sigma}|, \tilde{\alpha} + \pi)}{\pi} \right\} \quad \text{(B.8)}$$

We shall first minimize the effective action with respect to α. This is achieved simply at $\alpha = 0$. Define $\tilde{\rho} = (\tilde{\sigma}^2 + \tilde{D})^{\frac{1}{2}}$. The saddle point equations for $\tilde{\rho}$ and $\tilde{\sigma}$ are now simply

$$-\frac{4\pi}{\lambda}\tilde{\sigma}_* = \ln\left(2\sinh\frac{\tilde{\rho}_*}{2}\right), \frac{2\pi}{\lambda}(\tilde{\rho}_*^2 - 3\tilde{\sigma}_*^2) = \tilde{\sigma}_*\ln\left(2\cosh\frac{\tilde{\sigma}_*}{2}\right). \tag{B.9}$$

For $\lambda \ll 1$, this becomes

$$\tilde{\sigma}_* = -\frac{\lambda}{4\pi}\ln(\sqrt{3}\tilde{\sigma}_*) \sim \frac{\lambda|\ln\lambda|}{4\pi}, \quad \tilde{\rho}_* = \sqrt{3}\tilde{\sigma}_* \sim \frac{\sqrt{3}\lambda|\ln\lambda|}{4\pi}. \tag{B.10}$$

On the other hand, in the strong coupling limit $\lambda \to \infty$, the solution is

$$\tilde{\rho}_* = \ln\frac{3+\sqrt{5}}{2}, \quad \tilde{\sigma}_* = \frac{2\pi\tilde{\rho}_*^2}{\lambda\ln 2} \to 0. \tag{B.11}$$

The free energy is given by

$$F(\lambda, T) = \frac{N_f T^2}{\pi}\left\{\frac{2\pi(\tilde{\rho}_*^2 - \tilde{\sigma}_*^2)\tilde{\sigma}_*}{\lambda} + \frac{\tilde{\rho}_*^3 - \tilde{\sigma}_*^3}{6} + \sum_{k=1}^{\infty}\frac{(1+k\tilde{\rho}_*)e^{-k\tilde{\rho}_*} - (-)^k(1+k\tilde{\sigma}_*)e^{-k\tilde{\sigma}_*}}{k^3}\right\} \tag{B.12}$$

The free energy is analytic in λ except at $\lambda = 0$, and decreases monotonously as λ increases. We have

$$F(\lambda = 0, T) = N_f T^2 \frac{7\zeta(3)}{4\pi} \simeq 0.669597 N_f T^2,$$

$$F(\lambda = \infty, T) = \frac{N_f T^2}{\pi}\left[\frac{3}{4}\zeta(3) + \frac{1}{6}\left(\ln\frac{3+\sqrt{5}}{2}\right)^3 + \ln\left(\frac{3+\sqrt{5}}{2}\right)\mathrm{Li}_2\left(\frac{2}{3+\sqrt{5}}\right) + \mathrm{Li}_3\left(\frac{2}{3+\sqrt{5}}\right)\right]$$

$$\simeq 0.593071 N_f T^2. \tag{B.13}$$

At weak coupling, we can expand the free energy to first subleading order as

$$F(\lambda, T) = N_f T^2\left[\frac{7\zeta(3)}{4\pi} + \frac{\lambda^2(\ln\lambda)^3}{32\pi^2} + \cdots\right], \qquad \lambda \ll 1. \tag{B.14}$$

There exists another set of saddle point solutions at small λ, which have nonzero α. Their contribution to the free energy is always smaller than the saddle points described above. Note that the strong coupling limit result of the free energy in (B.13), apart from the contribution from free fermion, is the same as that of the IR fixed point of three dimensional $\mathrm{U}(N_f)$ model in the infinite N_f limit.

B.2 On the sphere

The free energy of $\mathcal{N} = 2$ Chern-Simons-matter theory on a sphere of radius $R = 1$ is given in the large N_f by the extremizing the function

$$-S_{\text{eff}}(\sigma, D, \alpha) = N_f\left[\frac{8\pi\beta}{\lambda}\tilde{D}\tilde{\sigma} - Z_B(\beta, \sqrt{\tilde{\sigma}^2 + \tilde{D}}, \alpha) + Z_F(\beta, |\tilde{\sigma}|, \alpha)\right] \tag{B.15}$$

– 28 –

where we have defined $D = \beta^2 \tilde{D}$, $\sigma = \beta \tilde{\sigma}$. Z_B and Z_F are given by

$$Z_B(\beta, x, \alpha) = \sum_{l=0}^{\infty} (2l+1) \left[\beta \sqrt{x^2 + \left(l + \frac{1}{2}\right)^2} + \ln\left(1 - e^{-\beta\sqrt{x^2 + (l+\frac{1}{2})^2} + i\alpha}\right) \right.$$

$$\left. + \ln\left(1 - e^{-\beta\sqrt{x^2 + (l+\frac{1}{2})^2} - i\alpha}\right) \right], \quad Z_F(\beta, x, \alpha) \tag{B.16}$$

$$= \sum_{l=1}^{\infty} 2l \left[\beta\sqrt{x^2 + l^2} + \ln\left(1 + e^{-\beta\sqrt{x^2 + l^2} + i\alpha}\right) + \ln\left(1 + e^{-\beta\sqrt{x^2 + l^2} - i\alpha}\right) \right].$$

After regularizing the divergent parts of the sums, we have

$$Z_B(\beta, x, \alpha) = \beta \sum_{l=0}^{\infty} (2l+1) \left(\sqrt{x^2 + \left(l + \frac{1}{2}\right)^2} - \left(l + \frac{1}{2}\right) - \frac{x^2}{2l+1} \right) + \mathcal{J}_B(\beta, x, \alpha),$$

$$Z_F(\beta, x, \alpha) = \beta \left[-\frac{x^2}{2} + \sum_{l=1}^{\infty} 2l \left(\sqrt{x^2 + l^2} - l - \frac{x^2}{2l} \right) \right] + \mathcal{J}_F(\beta, x, \alpha), \tag{B.17}$$

with

$$\mathcal{J}_B(\beta, x, \alpha) = \sum_{l=0}^{\infty} (2l+1) \left[\ln\left(1 - e^{-\beta\sqrt{x^2 + (l+\frac{1}{2})^2} + i\alpha}\right) + \ln\left(1 - e^{-\beta\sqrt{x^2 + (l+\frac{1}{2})^2} - i\alpha}\right) \right],$$

$$\mathcal{J}_B(\beta, x, \alpha) = \sum_{l=0}^{\infty} 2l \left[\ln\left(1 + e^{-\beta\sqrt{x^2 + l^2} + i\alpha}\right) + \ln\left(1 + e^{-\beta\sqrt{x^2 + l^2} - i\alpha}\right) \right]. \tag{B.18}$$

Note that the infinite sums in (B.17) are convergent. $\mathcal{J}_B$ and $\mathcal{J}_F$ are suppressed by powers of $e^{-\beta}$ in the low temperature limit. The free energy is now analytic in λ at $\lambda = 0$, as there is no infrared divergence.

We will now compute the free energy in the strict large N_f limit at low temperature. At the saddle point $\alpha = 0$. To leading order in $e^{-\beta}$, the saddle point equations for $\tilde{\sigma}$ and $\tilde{\rho} = \sqrt{\tilde{\sigma}^2 + \tilde{D}}$ are

$$\frac{8\pi}{\lambda} \tilde{\sigma}_* - 2e^{-\beta/2} + \frac{\pi^2}{4} \tilde{\rho}_*^2 + \mathcal{O}(e^{-\beta}) = 0,$$

$$\frac{8\pi}{\lambda} (\tilde{\rho}_*^2 - \tilde{\sigma}_*^2) - \tilde{\sigma}_* + \mathcal{O}(e^{-\beta}) = 0 \tag{B.19}$$

The solutions are

$$\tilde{\sigma}_* = \frac{\lambda}{4\pi(1 + \frac{\lambda^2}{256})} e^{-\beta/2}, \qquad \tilde{\rho}_* = \frac{\lambda}{4\sqrt{2}\pi\sqrt{1 + \frac{\lambda^2}{256}}} e^{-\beta/4}. \tag{B.20}$$

The free energy is then given by

$$F = N_f \left[2e^{-\beta/2} + \left(5 - \frac{\beta}{32\pi^2} \frac{\lambda^2}{1 + \frac{\lambda^2}{256}}\right) e^{-\beta} + \mathcal{O}(e^{-3\beta/2}) \right] \tag{B.21}$$

Figure 7: The solid lines and double lines represent the scalar and fermion respectively.

This expression is valid for all values of λ in the low temperature limit. The term of order $\beta e^{-\beta}$ in (B.21) can be reproduced by summing up diagrams as in figure 7. When higher order terms in $e^{-\beta}$ are included, diagrams involving the $|\phi|^6$ vertices also contribute.

Let us note that the saddle point approximation in the strict large N_f limit is not sufficient to extract the spectrum of low dimensional operators. In the free theory for instance, the lowest dimensional gauge invariant operator other than 1 is $\bar{\phi}^i \phi^j$, of dimension 1. There are N_f^2 such operators. They contribute to the partition function $N_f^2 e^{-\beta}$. In order to see this in the low temperature expansion of the partition function, we would need $N_f^2 e^{-\beta} \ll 1$. The saddle point approximation breaks down in this regime. For example, we must integrate out α in the full path integral in order to see that the partition function only sums up gauge invariant states. This becomes invisible in the saddle point approximation, where α is set to zero. In fact, a simple diagrammatics reveals that the anomalous dimensions of finite dimensional operators are subleading in $1/N_f$, and vanish in the infinite N_f limit.

B.3 $\mathcal{N} = 2$ theory with oppositely charged matter

The effective action for the saddle point is of the form

$$-\frac{\beta^2}{N_f A} S_{\text{eff}} = \frac{2D\sigma}{\lambda} + \frac{f_B(\sqrt{\sigma^2 + D}) + f_B(\sqrt{\sigma^2 - D}) - 2f_F(|\sigma|)}{\pi} \tag{B.22}$$

where f_B, f_F are given by

$$f_B(x) = \mathcal{I}(x, 0) + \frac{x^3}{6} = \sum_{k=1}^{\infty} \frac{1 + kx}{k^3} e^{-kx} + \frac{x^3}{6},$$

$$f_F(x) = \mathcal{I}(x, \pi) + \frac{x^3}{6} = \sum_{k=1}^{\infty} (-)^k \frac{1 + kx}{k^3} e^{-kx} + \frac{x^3}{6}. \tag{B.23}$$

as before. A subtlety is that, the saddle point of interest may lie in the regime $|D| > \sigma^2$, and we would need to consider $f_B(x)$ with imaginery x. This is not a serious problem. In fact, one can analytically continue $f_B(x)$, so that $f_B(x)$ is real for both real and imaginery x, and is an even function in x. The saddle point equations are now written as

$$-\frac{4\pi}{\lambda}\sigma = \ln \left| \frac{\sinh \frac{\sqrt{\sigma^2 + D}}{2}}{\sinh \frac{\sqrt{\sigma^2 - D}}{2}} \right|,$$

$$\frac{2\pi}{\lambda}D = \sigma \ln \left| \frac{\cosh^2(\frac{\sigma}{2})}{\sinh \frac{\sqrt{\sigma^2 + D}}{2} \sinh \frac{\sqrt{\sigma^2 - D}}{2}} \right|. \tag{B.24}$$

The only saddle point is $D = \sigma = 0$, and hence the free energy is identical to that of the free theory. For the $\mathcal{N} = 3$ theory, the only difference in the effective action (B.22) is that one replaces σ and D with $su(2)$-valued $\sigma_a \gamma^a$ and $D_a \gamma^a$, where γ^a are Pauli matrices, and take the trace over the doublet. The saddle points lead to the same result, i.e. the free energy of the $\mathcal{N} = 3$ theory is the same as that of the free theory to leading order in $1/N_f$.

C. Thermodynamics of the free $\mathrm{U}(N)$ theory on the sphere

In this appendix, we study the operator spectrum of the free theory ($\lambda = 0$) from the point of view of thermodynamics. We are essentially reproducing here some of the results of [46]. One might naively expect the thermodynamics to be trivial for the free theory. This would be the case in flat space, but is not the case on the sphere due to the restriction to gauge invariant states [47]. The partition function of the free theory reduces to a unitary matrix model of the form

$$\int [dU]_{\mathrm{U}(N)} e^{-S_{\mathrm{eff}}(U)} \tag{C.1}$$

where U is the holonomy of the gauge field along the thermal circle, $U = e^{i\alpha}$, α being the zero mode of A_0. The matrix model action is given by

$$S_{\mathrm{eff}}(U) = -2N_f \sum_{n=1}^{\infty} \frac{z_S(x^n) + (-)^{n+1} z_F(x^n)}{n} (\mathrm{tr} U^n + \mathrm{tr} U^{-n}), \tag{C.2}$$

where $x = e^{-\beta}$, $z_S(x)$ and $z_F(x)$ are the partition functions of a conformally coupled scalar particle and spin $1/2$ particle on the sphere,

$$z_S(x) = \sum_{l=0}^{\infty} (2l + 1) x^{l + \frac{1}{2}} = \frac{x^{\frac{1}{2}}(1 + x)}{(1 - x)^2},$$

$$z_F(x) = \sum_{l=1}^{\infty} 2l x^l = \frac{2x}{(1 - x)^2}. \tag{C.3}$$

Diagonalizing the unitary matrix $U = \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_N})$, the matrix integral becomes

$$\int \prod d\alpha_i \exp\left\{-\left[\sum_{i \neq j} \sum_{n=1}^{\infty} \frac{\cos(n(\alpha_i - \alpha_j))}{n} - 4N_f \sum_i \sum_{n=1}^{\infty} \frac{z_S(x^n) + (-)^{n+1} z_F(x^n)}{n} \cos(n\alpha_i)\right]\right\}$$

$$= \int \prod d\alpha_i \exp\left\{-\left[\sum_{i \neq j} \sum_{n=1}^{\infty} \frac{\cos(n(\alpha_i - \alpha_j))}{n} - 2N_f \sum_i \sum_{n=1}^{\infty} \frac{\cosh(\frac{n\beta}{2}) + (-)^{n+1}}{n \sinh^2(\frac{n\beta}{2})} \cos(n\alpha_i)\right]\right\} \tag{C.4}$$

In the large N limit, we can represent the eigenvalues by the eigenvalue density $\rho(\theta)$, with the property

$$\rho(\theta) \geq 0, \qquad \int_0^{2\pi} d\theta \rho(\theta) = 1. \tag{C.5}$$

The potential function in (C.4) now becomes

$$N^2 \sum_{n=1}^{\infty} \frac{1}{n} \left[\rho_n^2 - 2c\rho_n \frac{\cosh(\frac{n\beta}{2}) + (-)^{n+1}}{\sinh^2(\frac{n\beta}{2})} \right] \tag{C.6}$$

where $\rho_n = \int d\theta \rho(\theta) \cos(n\theta)$, and $c = N_f/N$. Defining the function

$$f_n(\beta) = \frac{\cosh(\frac{n\beta}{2}) + (-)^{n+1}}{\sinh^2(\frac{n\beta}{2})}, \tag{C.7}$$

we can rewrite (C.6) as

$$N^2 \sum_{n=1}^{\infty} \frac{(\rho_n - cf_n(\beta))^2 - c^2 f_n(\beta)^2}{n} \tag{C.8}$$

At low temperatures, the saddle point is given by $\rho_n = cf_n(\beta)$, and the free energy of the theory on the sphere is

$$F_{\text{low}}(\beta) = N_f^2 \sum_{n=1}^{\infty} \frac{f_n(\beta)^2}{n} \tag{C.9}$$

temperatures, the saddle points are very different since (C.5) severely constrains the ρ_n's. The free energy in the high temperature limit is simply given by the flat space result

$$F_{\text{high}}(\beta) \simeq N N_f \frac{7\zeta(3)}{\beta^2} \tag{C.10}$$

There is a transition from N_f^2 degrees of freedom at low temperature to NN_f degrees of freedom at high temperature.

At nonzero coupling, we can still integrate out all the matter fields while keeping the Chern-Simons auxiliary fields, and obtain a path integral of the form

$$\int dAdDd\sigma d\chi d\bar{\chi} \exp\left[-\frac{N}{\lambda} \int (\omega_{CS}(A) + 2D\sigma - \bar{\chi}\chi) \right.$$
$$\left. -N_f \text{Str} \ln \begin{pmatrix} -D^{\mu}D_{\mu} + \sigma^2 + D & \bar{\chi} \\ chi & i\slashed{D} + \sigma \end{pmatrix} \right] \tag{C.11}$$

where the "Str" in the second term in the effective action involves a trace in the fundamental representation of $U(N)$. However, we can no longer integrate out all but one field in the Chern-Simons multiplet. The saddle point approximation to the path integral (C.11) at large N is no longer valid, since there are $\sim N^2$ fields in (C.11) while the action is only multiplied by N.

D. Two-loop anomalous dimensions

D.1 The two-loop anomalous dimension of $\bar{\phi}\phi$

It was predicted that in $\mathcal{N} = 2$ CS-matter theory, the anomalous dimension of $\bar{\phi}^i \phi_j$ is zero, since it lies in the same supermultiplet as the $U(N_f)$ flavor current. It is nevertheless
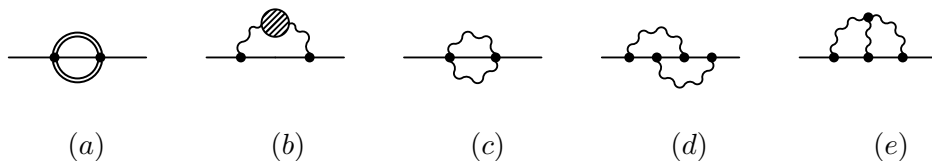
Figure 8: The solid and double lines stand for scalar and fermion propagators, respectively. The shaded bubble in (b) represents matter loops. The 1-loop gauge and ghost bubbles cancel.
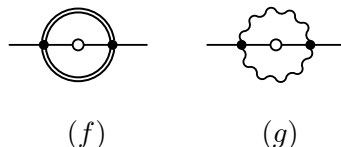
Figure 9: Two-loop renormalization of the operator $\bar{\phi}^i \phi_j$.

instructive to verify this through an explicit two-loop computation. The contributions come from wave function renormalization, as shown in Fig 8, as well as 1PI diagrams in Fig 9. We will work in Feynman gauge, regularizing the loop integrals with dimensional reduction method.

For simplicity we will do the calculation in the planar limit (for fundamental and adjoint matter), although the diagrams in Fig 8 and Fig 9 are not drawn as planar diagrams. The circle in Fig 9 represents the operator insertion of $\bar{\phi}^i \phi_j$. We have only shown diagrams with nonzero momentum integral. One must be careful with the planar combinatorics. This is most easily taken care of by keeping the propagators of auxiliary fields σ, D, χ, which have been integrated out in Fig 8 and 9.

Some relevant loop integrals are

$$\int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{1}{k^2 l^2 (k+l)^2} = \frac{\ln \Lambda}{16\pi^2}$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{2k \cdot l}{k^2 l^2 (p+k+l)^2} = \frac{p^2}{3} \frac{\ln \Lambda}{16\pi^2} + \cdots$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{(\epsilon_{\mu\nu\rho} p^\mu k^\nu l^\rho)^2}{k^2 l^2 (k+l)^2 (p+k)^2 (p+k+l)^2} = \frac{p^2}{6} \frac{\ln \Lambda}{16\pi^2} + \cdots \qquad \text{(D.1)}$$

where we omitted power divergent and finite terms. The wave function renormalization is computed to be

$$\delta Z = \left(\frac{2\pi}{k}\right)^2 \left(-\frac{5}{3} N N_f + \frac{8}{3} N N_f + \frac{1}{3} N^2 + 0 - \frac{4}{3} N^2\right) \frac{\ln \Lambda}{16\pi^2}$$

$$= \left(\frac{2\pi}{k}\right)^2 (N N_f - N^2) \frac{\ln \Lambda}{16\pi^2}, \qquad \text{(D.2)}$$

where the terms in the parenthesis in the first line come from diagrams $(a - e)$. The

contributions from the diagrams (f) and (g) to the anomalous dimension precisely cancel the correction due to δZ, confirming that $\bar{\phi}^i \phi_j$ has zero anomalous dimension at two-loop.

For the $\mathcal{N} = 2$ theory with M adjoint matters (in the planar limit), the wave function renormalization is

$$\delta Z = \left(\frac{4\pi}{k}\right)^2 \left[-\frac{1}{3}N^2(5M + 2) + \frac{8}{3}N^2M + \frac{1}{2}N^2 + \frac{4}{3}N^2 - \frac{2}{3}N^2\right] \frac{\ln \Lambda}{16\pi^2}$$

$$= \left(\frac{4\pi}{k}\right)^2 \left(N^2M + \frac{1}{2}N^2\right) \frac{\ln \Lambda}{16\pi^2}, \tag{D.3}$$

where the five terms in the first line come from diagrams $(a - e)$. The contribution from (f), (g) is

$$\left(\frac{4\pi}{k}\right)^2 \left[-N^2(M + 2) + \frac{3}{2}N^2\right] \frac{\ln \Lambda}{16\pi^2} \tag{D.4}$$

This precisely cancels (D.3), confirming that the anomalous dimension of $\text{Tr}(\bar{\Phi}^i \Phi^j)$ is zero. On the other hand, the operator $\text{Tr}(\Phi^i \Phi^j)$ is a chiral primary, whose anomalous dimension is entirely due to the renormalization of the $\text{U}(1)_R$ charge of Φ. For this operator, the contribution from (f) and (g) is

$$\left(\frac{4\pi}{k}\right)^2 \left(N^2M + \frac{3}{2}N^2\right) \frac{\ln \Lambda}{16\pi^2} \tag{D.5}$$

We find that the anomalous dimension of $\text{Tr}\Phi^a \Phi^b$ is

$$\Delta - 1 = -2(M + 1)\lambda^2, \tag{D.6}$$

or equivalently, $q_\Phi^R = \frac{1}{2} - (M + 1)\lambda^2$, at two-loop. When M is even, this result can be reproduced by simply comparing to the $\mathcal{N} = 3$ theory with $M/2$ adjoint hypermultiplets, as in described section 2.

D.2 Computation of the two-loop anomalous dimension of twist-1 operators

In this section we compute the two-loop diagrams of figure 4, for the anomalous dimension of twist-1 operators at large spin n.

The loop integral from diagram $4(a)$ is evaluated as follows:

$$\frac{1}{4}\sum_{i=0}^{n-2}\sum_{j=0}^{n-2-i} \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{\epsilon_{\mu\nu\alpha}k^\alpha \epsilon_{\rho\nu\beta}l^\beta \Delta^\mu \Delta^\rho}{k^2 l^2 (k+l+p)^2} (\Delta \cdot (k+l+p))^i (\Delta \cdot (l+p))^j (\Delta \cdot p)^{n-2-i-j}$$

$$= \frac{1}{4}\sum_{i,j} \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{((k-l)\cdot\Delta)((l-p)\cdot\Delta)}{k^2(k-l)^2(l-p)^2} (\Delta \cdot k)^i (\Delta \cdot l)^j (\Delta \cdot p)^{n-2-i-j}$$

$$= \sum_{i,j} \frac{2\pi^3}{4(2\pi)^6} \ln \Lambda \int_0^1 dx \int_0^{1-x} dy \frac{(\Delta \cdot p)^n}{(1-y)^{3/2}\left(x+y-\frac{x^2}{1-y}\right)^{3/2}} \left(\frac{x}{1-y} - 1\right)\left(\frac{x}{1-y}\right)^i$$

$$\times \left(\frac{y}{x+y-\frac{x^2}{1-y}} - 1\right)\left(\frac{y}{x+y-\frac{x^2}{1-y}}\right)^{i+j+1}$$

$$= \sum_{i,j} \frac{2\pi^3}{4(2\pi)^6} (\Delta \cdot p)^n \ln \Lambda \int_0^1 dx \int_0^{1-x} dy \frac{uv}{(xy)^{\frac{3}{2}}} (1-u)^{i+\frac{3}{2}} (1-v)^{i+j+\frac{3}{2}}, \tag{D.7}$$

where we have defined $u \equiv x/(1-y)$, $v \equiv y/(x + y - \frac{x^2}{1-y})$, and have thrown away power divergences. In the $n \to \infty$ limit, the integral is in fact finite, and is given by

$$\frac{1}{64\pi^2}(\Delta \cdot p)^n \ln \Lambda \tag{D.8}$$

The contribution from (b) is

$$\frac{1}{4} \sum_{i,j} \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{\epsilon_{\rho\mu\alpha}(l - k)^\alpha \epsilon_{\sigma\nu\beta}(p - l)^\beta \epsilon^{\rho\sigma\eta} \epsilon_{\tau\eta\gamma}(p - k)^\gamma \Delta^\mu \Delta^\nu (p + k)^\tau}{k^2(k - l)^2(p - l)^2(p - k)^2}$$
$$\times (\Delta \cdot k)^i (\Delta \cdot l)^j (\Delta \cdot p)^{n-2-i-j} \tag{D.9}$$

Let us examine the integral over l,

$$\int d^3l \frac{(l - k)^\alpha (p - l)^\beta (\Delta \cdot l)^j}{(k - l)^2(p - l)^2}$$
$$= -\int_0^1 d^3l \frac{(l + x(p - k))^\alpha (l + (1 - x)(k - p))^\beta (\Delta \cdot (l + xp + (1 - x)k))^j}{(l^2 + x(1 - x)(k - p)^2)^2}$$
$$= (\cdots)\delta^{\alpha\beta} + \Delta^\alpha(\cdots)^\beta + \Delta^\beta(\cdots)^\alpha \tag{D.10}$$

When contracted with $\epsilon^{\rho\sigma\eta}\epsilon_{\rho\mu\alpha}\epsilon_{\sigma\nu\beta}\Delta^\mu\Delta^\nu$, this is just zero. Hence the contribution from diagram (b) vanishes.

Diagram (c) is given by

$$\frac{1}{4} \sum_{i=0}^{n-1} \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{\epsilon_{\mu\nu\alpha}(k + l)^\alpha \epsilon_{\rho\sigma\beta} l^\beta \epsilon_{\eta\tau\gamma} k^\gamma \epsilon^{\rho\nu\eta}(2p + k)^\tau (2p + 2k + l)^\sigma \Delta^\mu}{k^2 l^2 (k + l)^2(p + k)^2(p + k + l)^2}$$
$$\times ((p + k + l) \cdot \Delta)^i (p \cdot \Delta)^{n-1-i} \tag{D.11}$$
$$= \sum_{i=0}^{n-1} \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{\epsilon_{\mu\nu\alpha}\epsilon_{\rho\sigma\beta}\epsilon_{\eta\tau\gamma}\epsilon^{\rho\nu\eta} l^\alpha (l - k)^\beta k^\gamma (p + k)^\sigma p^\tau \Delta^\mu}{k^2 l^2 (l - k)^2(p + k)^2(p + l)^2} ((p + l) \cdot \Delta)^i (p \cdot \Delta)^{n-1-i}$$

We will do the integral in two steps. First, the integral over k

$$\int d^3k \frac{(l - k)^\beta k^\gamma (p + k)^\sigma}{k^2(k - l)^2(k + p)^2}$$
$$= 2\int_0^1 dx \int_0^{1-x} dy \frac{(-k + (1 - x)l + yp)^\beta (k + xl - yp)^\gamma (k + xl + (1 - y)p)^\sigma}{[k^2 + x(1 - x)l^2 + y(1 - y)p^2 + 2xyp \cdot l]^3} \tag{D.12}$$

When multiplied by $\epsilon_{\mu\nu\alpha}\epsilon_{\rho\sigma\beta}\epsilon_{\eta\tau\gamma}\epsilon^{\rho\nu\eta} l^\alpha p^\tau$, the integral (D.12) simplies drastically to

$$\frac{3\pi^2}{2} \int_0^1 dx \int_0^{1-x} dy \frac{-x\delta^{\beta\gamma}l^\sigma + (1 - x)\delta^{\gamma\sigma}l^\beta}{\sqrt{x(1 - x)l^2 + y(1 - y)p^2 + 2xyp \cdot l}} \tag{D.13}$$

$$- 35 -$$

The full integral (D.11) is now

$$
\frac{3\pi^2}{(2\pi)^6}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}\sum_{i=0}^{n-1}\int_0^1 dx\int_0^{1-x} dy\int_0^1 dz\int_0^{1-z} dw(zx(1-x))^{-\frac{1}{2}}
$$

$$
\times\int d^3l\frac{(p\cdot\Delta)^{n-1-i}((p+l)\cdot\Delta)^i\left[-(l\cdot\Delta)(p\cdot l)+(p\cdot\Delta)l^2\right]}{\left[l^2+2(w+\frac{zy}{1-x})p\cdot l+(w+\frac{zy(1-y)}{x(1-x)})p^2\right]^{\frac{5}{2}}}
$$

$$
\longrightarrow\frac{3\pi^3}{(2\pi)^6}(p\cdot\Delta)^n\ln\Lambda\sum_{i=0}^{n-1}\int_0^1 dx\int_0^{1-x} dy\int_0^1 dz\int_0^{1-z} dw(zx(1-x))^{-\frac{1}{2}}
$$

$$
\times\left[2+(i-2)\left(w+\frac{yz}{1-x}\right)\right]\left(1-w-\frac{yz}{1-x}\right)^{i-1}\tag{D.14}
$$

This integral is also finite in the $n\to\infty$ limit, and is given by

$$
\frac{9\ln 2}{16\pi^2}(p\cdot\Delta)^n\ln\Lambda\tag{D.15}
$$

Diagram (d) involves the 1-loop correction to the gauge field propagator from the loop of gauge fields, ghosts, as well as the matter fields. The contributions from the gauge field loop and the ghost loop cancel. The former is given by

$$
\frac{1}{2}\int d^3l\frac{\epsilon_{\rho\mu\eta}\epsilon_{\nu\sigma\tau}\epsilon^{\sigma\rho\alpha}l_\alpha\epsilon^{\eta\tau\beta}(k+l)_\beta}{l^2(k+l)^2}=\frac{1}{2}\int d^3l\frac{l_\mu(k+l)_\nu+l_\nu(k+l)_\mu}{l^2(k+l)^2},\tag{D.16}
$$

where the latter is

$$
-\int d^3l\frac{(k+l)_\mu l_\nu}{l^2(k+l)^2}\tag{D.17}
$$

Indeed they cancel as expected from the pure Chern-Simons theory [48]. So far we have computed the 2-loop diagrams involving gauge interactions only, and found that the anomalous dimension of the twist-1 operator $J_{\mu_1\cdots\mu_n}$ to order λ^2 is bounded in the large spin n limit.

Now we consider the correction of order $\lambda^2 N_f/N$, coming from the matter loop in diagram (d). The correction to the gauge field propagator is

$$
\int\frac{d^3p}{(2\pi)^3}\frac{(2p+k)^\mu(2p+k)^\nu-\text{Tr}\left[\gamma^\mu\not{p}\gamma^\nu(\not{p}+\not{k})\right]}{p^2(p+k)^2}=\int\frac{d^3p}{(2\pi)^3}\frac{k^\mu k^\nu+2\delta^{\mu\nu}p\cdot(p+k)}{p^2(p+k)^2}\tag{D.18}
$$

After the contraction with $\frac{1}{4}\epsilon_{\rho\mu\alpha}\frac{k^\alpha}{k^2}\epsilon_{\nu\sigma\beta}\frac{k^\beta}{k^2}$, and dropping linear divergences, (D.18) becomes simply

$$
\frac{1}{32}\frac{\delta_{\rho\sigma}k^2-k_\rho k_\sigma}{(k^2)^{\frac{3}{2}}}\tag{D.19}
$$

– 36 –

The full integral is now

$$-\frac{1}{32}\sum_{i=0}^{n-1}\int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot\Delta)k\cdot(2p+k)-((2p+k)\cdot\Delta)k^2}{(k^2)^{\frac{3}{2}}(p+k)^2}((p+k)\cdot\Delta)^i(p\cdot\Delta)^{n-1-i}$$

$$\longrightarrow \frac{1}{32}\sum_{i=0}^{n-1}\int \frac{d^3k}{(2\pi)^3}\frac{(2p+k)\cdot\Delta}{(k^2)^{\frac{1}{2}}(p+k)^2}((p+k)\cdot\Delta)^i(p\cdot\Delta)^{n-1-i}$$

$$= \frac{1}{128\pi^2}(p\cdot\Delta)^n\ln\Lambda\sum_{i=0}^{n-1}\int_0^1 dx(1+x)x^{i-\frac{1}{2}}$$

$$\sim \frac{1}{128\pi^2}\ln(n)(p\cdot\Delta)^n\ln\Lambda \qquad (n\gg 1) \tag{D.20}$$

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