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The Zamolodchikov-Faddeev Algebra for $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

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Abstract: We discuss the Zamolodchikov-Faddeev algebra for the superstring sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$. We find the *canonical* $\mathfrak{su}(2|2)^2$ invariant S-matrix satisfying the standard Yang-Baxter and crossing symmetry equations. Its near-plane-wave expansion matches exactly the leading order term recently obtained by the direct perturbative computation. We also show that the S-matrix obtained by Beisert in the gauge theory framework does not satisfy the standard Yang-Baxter equation, and, as a consequence, the corresponding ZF algebra is twisted. The S-matrices in gauge and string theories however are physically equivalent and related by a non-local transformation of the basis states which is explicitly constructed.

Keywords: Integrable Field Theories, AdS-CFT Correspondence, Exact S-Matrix.

Contents

1. Introduction and summary

Unravelling the integrable properties of the $AdS_5 \times S^5$ string sigma model [1] and of the dual gauge theory [2, 3] allowed for the remarkable progress in understanding the spectra and interrelation of these theories. Most importantly, determination of the spectrum in the large volume (charge) limit was shown to rely on the knowledge of the S-matrix describing the scattering of world-sheet excitations, or, alternatively, excitations of a certain spin chain in the dual gauge theory [4 – 6].

It turned out that the form of the S-matrix is severely restricted by the requirement of invariance under the global symmetries of the model. It was argued in [6] and explicitly derived in [7], that the relevant symmetry algebra of the gauge-fixed off-shell string sigma model in the infinite volume limit is the centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ superalgebra. Requiring the S-matrix to be invariant under this symmetry algebra fixes its form essentially uniquely up to an overall phase [6, 8]. Constraints on this phase were established in [9] by demanding the S-matrix to satisfy the crossing relation, the common property of S-matrices in relativistic field theories. A physically relevant solution of the crossing relation, which successfully reproduces the known string theory data was conjectured in [10], building on the previous work [11]–[14]. Remarkably, the same solution for the crossing-symmetric phase was shown to arise from perturbative gauge theory [15]. A number of further important verifications has been made [16]–[19],[1] which supports the picture of the interpolating crossing-symmetric S-matrix from weak (gauge theory) to strong (string theory) coupling.

Bearing in mind all these highly exciting developments, a further question to ask is what is a proper characterization of elementary world-sheet excitations whose scattering is governed by the crossing-symmetric S-matrix? In other words, we would like to understand the structure of the Hilbert space of string theory arising in the infinite volume (charge) limit.

Although the gauge-fixed string theory we are interested in does not possess relativistic invariance on the world-sheet, it is useful to invoke an analogy with two-dimensional massive integrable models. The basic feature of these models is the existence of infinite number of commuting conservation laws which lead to the *factorized scattering* preserving both a number of particles and the set of their on-shell momenta [25]. The space of local operators is specified by the two-body S-matrix, and the Hilbert space of asymptotic states forms a representation of the Zamolodchikov-Faddeev (ZF) algebra [25, 26]. In particular, the asymptotic states diagonalize the local conservation laws. Thus, the ZF algebra provides an indispensable framework for treating massive integrable models. It also allows [27], through the bosonization procedure, the computation of the corresponding form-factors [28].

In this paper we discuss the ZF algebra for the superstring sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the string theory context the ZF algebra represents a natural generalisation of the free field oscillators which describe elementary excitations in the plane-wave limit [29] of the world-sheet theory. As interactions are switched on integrability ensures that the Fock structure of the Hilbert space is preserved. The effect of interactions is then taken into account by deforming the algebra of creation and annihilation operators with the help of a non-trivial scattering matrix. Thus, in momentum space the ZF algebra is generated by

[1] It is still unclear if the proposal of [15] correctly describes bound states present in string and gauge theories at large values of the 't Hooft coupling constant λ [20]. Then the resulting Bethe ansatz is supposed to be asymptotic and might not capture finite size corrections [21 – 24].

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the operators $A(p)$ and $A^\dagger(p)$ satisfying the following relations

$$A_1A_2 = \mathcal{S}_{12}A_2A_1\,, \quad A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathcal{S}_{12}\,, \quad A_1A_2^\dagger = A_2^\dagger \mathcal{S}_{21}A_1 + \delta_{12}\,,$$

where $\mathcal{S}_{12}$ is the two-body S-matrix of the model and δ_{12} is the delta-function depending on the difference of momenta of scattering particles.

We thus see that the construction of the ZF algebra relies on the knowledge of the two-body S-matrix. Moreover, the consistency of the ZF algebra relations implies that the S-matrix should obey the Yang-Baxter (YB) equation. In this paper we show that such an S-matrix consistent with the symmetries of the gauged-fixed string model does exist and we find its explicit form. This *canonical* $\mathfrak{su}(2|2)^2$ invariant S-matrix exhibits the following properties

- it obeys the standard Yang-Baxter equation

$$\mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12} = \mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23}$$

- it obeys the unitarity condition

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{21}(p_2,p_1) = \mathbb{I}$$

- it obeys physical unitarity condition which is sometimes referred to as "hermitian analyticity"

$$\mathcal{S}_{21}^\dagger(p_2,p_1) = \mathcal{S}_{12}(p_1,p_2)$$

- it obeys the requirement of crossing symmetry

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(p_1,p_2)\mathscr{C}_1\mathcal{S}_{12}(-p_1,p_2) = \mathbb{I}\,,$$

  where $\mathscr{C}$ is the charge conjugation matrix.

We will show that all these requirements on the S-matrix naturally follow from the consistency conditions of the ZF algebra. In particular, the requirement of crossing symmetry reflects the compatibility of the factorized scattering with the $\mathbb{Z}_4$-graded structure of the superalgebra $\mathfrak{psu}(2|2)$ and it results into a non-trivial equation on the overall phase of the S-matrix which is the same equation as derived in [9].

The S-matrix we consider depends on the string tension. We then show that expanding $\mathcal{S}$ in the large tension limit we recover the near plane-wave S-matrix which was recently obtained from the gauge-fixed string sigma-model in [30]. Moreover, we show that in accordance with [13], the overall phase of $\mathcal{S}$ is capable to incorporate the various gauge choices compatible with the symmetries of the string model.

We find it pretty remarkable that the sigma-model describing $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring admits the S-matrix which shares most of the usual properties of S-matrices arising in relativistic two-dimensional massive integrable models. The point is that in the dual gauge theory the spin chain which describes the local composite gauge-invariant operators is

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dynamic because the scattering process requires fluctuations of the length [31, 6]. Consequently, one would expect that the corresponding S-matrix would not obey the usual Yang-Baxter equation but rather a new dynamic, or "twisted" Yang-Baxter equation. Indeed, some examples of integrable systems are known which are naturally described in terms of a dynamic S-matrix satisfying a twisted version of the Yang-Baxter equation, see e.g. [32].

We have to bear in mind, however, that the form of the S-matrix and, consequently, its properties, do depend on the choice of the scattering basis. Performing "gauge" transformations on the two-particle scattering basis we transform the S-matrix as well. Note that in general we should allow for transformations of the two-particle scattering basis, which from the point of view of one-particle states, look mutually non-local. These transformations might significantly modify the properties of the S-matrix without changing the actual physical content. In particular, there could exist a preferable basis in the space of two-particle states in which the S-matrix exhibits the standard properties. In fact, it is precisely the basis provided by string theory in which we find $\mathcal{S}(p_1, p_2)$ with the nice properties listed above.

Our construction of the $\mathfrak{su}(2|2)$-invariant S-matrix is the conventional field-theoretic one. We start with the discussion of the usual ZF algebra and show that invariance of the ZF relations acting on the vacuum state under the action of the $\mathfrak{su}(2|2)$ symmetry algebra[2] translates into a certain invariance condition for the S-matrix which is essentially the same as in [8]. It is worthwile to note that due to the non-linear nature of the dynamical supersymmetry generators [7] these generators have a non-trivial braiding relations with the ZF operators. The braiding factors involve the operator of the world-sheet momentum $\mathbf{P} = \int \mathrm{d}p\; p\, A_i^\dagger(p) A^i(p)$ which also plays the role of the central charge for the symmetry algebra in question. Such a realization of non-linear symmetries is well known in the theory of integrable systems [33] and it has been recently emphasized in the context of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string sigma-model in [30].

In a generic two-particle basis the S-matrix, $S_{12} \equiv S(p_1, p_2)$, found from the $\mathfrak{su}(2|2)$-invariance condition satisfies the twisted Yang-Baxter equation whose form depends of the twist matrix F:

$$F_{23}(p_1) S_{23} F_{23}^{-1}(p_1) S_{13} F_{12}(p_3) S_{12} F_{12}^{-1}(p_3) = S_{12} F_{13}(p_2) S_{13} F_{13}^{-1}(p_2) S_{23}\,.$$

We find that this condition can be naturally interpreted as the consistency condition for the twisted ZF algebra based on the operator S-matrix $\widehat{S}_{12}$:

$$\widehat{S}_{12} = F_{12}(\mathbf{P}) S_{12} F_{12}^{-1}(\mathbf{P})\,.$$

We show, however, that there is a very special basis such that bosonic and fermionic ZF operators are transformed by supersymmetry generators in the same fashion, i.e. it is the basis which preserves a symmetry between two $\mathfrak{su}(2)$-factors of $\mathfrak{su}(2|2)$. We refer to this

[2]Since the symmetry algebra of the gauge-fixed string Hamiltonian is a sum of two copies of $\mathfrak{psu}(2|2)$ it is sufficed to discuss only one. Whenever it is impossible to treat the "chiral sectors" separately we will make a comment.

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basis as to the “string” one because in this basis the action of the symmetry generators is the same as implied by the gauge-fixed string sigma-model [7]. We find that in the string basis the S-matrix coincides with $\mathcal{S}(p_1,p_2)$, and, as was discussed above, satisfies the usual YB equation. We then show that the transformation of the ZF basis to a generic one involves the momentum operator $\mathbf{P}$, and as a result twists the ZF algebra and the corresponding YB equation. Finally, we also find a relation of $\mathcal{S}(p_1,p_2)$ to the one obtained in [6, 8].

Thus, we show that $AdS_5 \times S^5$ string sigma-model can be naturally embedded in the general framework of massive integrable systems. Of course, the S-matrix we find has also properties which are different from those in the relativistic integrable QFTs. In particular, it does not depend on the difference of rapidities of scattering particles which reflects a non-relativistic nature of our model. Then the crossing symmetry relation implies that $\mathcal{S}(p_1,p_2)$ cannot be a meromorphic function of the particle momenta [10, 19].

Concluding this section we note several open questions. First, it would be interesting to work out the details of the nested Bethe ansatz based on $\mathcal{S}(p_1,p_2)$. Second, it is desirable to construct realization of the symmetry generators of $\mathfrak{su}(2|2)$ via the oscillators of the ZF algebra. This would allow one to put the ZF algebra in the corner of the axiomatic construction. Since the S-matrix we found obeys most of the standard properties of the S-matrices in massive relativistic integrable systems it is tempting to use the ideas of the thermodynamic Bethe ansatz [35] to describe the finite size effects. Finally, it would be interesting to understand the relation of our approach to those of [36]–[38].

2. S-matrix and symmetries

2.1 S-matrix

In scattering theory, the S-matrix is a unitary operator, which we denote $\mathbf{S}$, mapping free particle *out*-states to free particle *in*-states in the Heisenberg picture. In Dirac notation, we define $|0\rangle$ as the vacuum quantum state. If $A_i^\dagger(p)$ is a creation operator, its hermitian conjugate is the vacuum annihilation operator:

$$A^i(p)|0\rangle = 0\,.$$

To describe the scattering process we introduce the *in*-basis and the *out*-basis as follows

$$\begin{aligned}
|p_1,p_2,\cdots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} &= A^\dagger_{i_1}(p_1)\cdots A^\dagger_{i_n}(p_n)|0\rangle\,, \quad p_1>p_2>\cdots>p_n\,,\\
|p_1,p_2,\cdots,p_n\rangle^{(out)}_{i_1,\ldots,i_n} &= A^\dagger_{i_n}(p_n)\cdots A^\dagger_{i_1}(p_1)|0\rangle\,, \quad p_1>p_2>\cdots>p_n\,.
\end{aligned}$$

Then in the scattering process the *in* state goes to the *out* state

$$|p_1,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} \to |p_1,\ldots,p_n\rangle^{(out)}_{i_1,\ldots,i_n}\,.$$

Both states belong to the same Hilbert space and we can expand initial states on a basis of final states. In particular, the two-particle *in* and *out* states are related as follows:

$$|p_1,p_2\rangle^{(in)}_{i,j} = \mathbf{S}\cdot|p_1,p_2\rangle^{(out)}_{i,j} = S_{ij}^{kl}(p_1,p_2)|p_1,p_2\rangle^{(out)}_{k,l}\,,$$

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or by using the explicit basis we can write

$$A_i^\dagger(p_1)A_j^\dagger(p_2)|0\rangle = \mathbf{S}\cdot A_j^\dagger(p_2)A_i^\dagger(p_1)|0\rangle = S_{ij}^{kl}(p_1,p_2)A_l^\dagger(p_2)A_k^\dagger(p_1)|0\rangle\,. \tag{2.1}$$

The usual ZF algebra then follows just by dropping $|0\rangle$ on both sides of the equation

$$A_i^\dagger(p_1)A_j^\dagger(p_2) = A_l^\dagger(p_2)A_k^\dagger(p_1)S_{ij}^{kl}(p_1,p_2)\,. \tag{2.2}$$

It is clear from this form of the algebra that in the absence of interaction it should be equal to the graded unit matrix, that is to the diagonal matrix with ± 1 depending on the statistics of the corresponding creation operator. In addition, in most of the two-dimensional integrable models the S-matrix is equal to the minus (usual!) permutation matrix for $p_1 = p_2$, because a two-particle state describes two solitons moving with momenta p_1 and p_2, and there is no two-soliton state with equal momenta.

Then, the consistency condition for the ZF algebra leads to the YB equation

$$S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}\,, \tag{2.3}$$

where we have used the standard tensor notations, see appendix A.1.

It is clear, however, that it is not the most general form of the ZF algebra, for example a tensor operator U leaving the vacuum invariant could appear on the rhs of eq. (2.2)

$$A_i^\dagger(p_1)A_j^\dagger(p_2) = S_{ab}^{kl}(p_1,p_2)A_n^\dagger(p_2)A_m^\dagger(p_1)U_{ij,kl}^{ab,mn}\,, \quad U_{ij,kl}^{ab,mn}|0\rangle = \delta_i^a\delta_j^b\delta_k^m\delta_l^n|0\rangle\,. \tag{2.4}$$

The appearance of the operator U modifies the consistency condition and the YB equation leading to the "twisted" ZF algebra and YB equation. We will see that this kind of modification occurs for strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ if one uses in the ZF algebra (2.4) the $\mathfrak{su}(2|2)^2$ spin chain S-matrix found in [6]. On the other hand, the operator U can also appear because of an inconvenient choice of the basis of the creation/annihilation operators in the ZF algebra. In that case, one can find a proper basis and untwist the twisted ZF algebra. We will see that this is what happens for strings in $\mathrm{AdS}_5\times\mathrm{S}^5$, and find the right and very natural from string theory point of view basis of the ZF algebra, and the *canonical* $\mathfrak{su}(2|2)^2$ invariant S-matrix satisfying the usual YB equation.

2.2 Symmetries

Let us assume that the Hamiltonian commutes with a set of symmetry algebra generators $\mathbf{J}^\mathbf{a}$ which are operators acting in the Hilbert space of states. Then the Hilbert space of states carries a linear representation of the symmetry algebra. This implies in particular that for generators preserving the number of particles

$$\begin{aligned} \mathbf{J}^\mathbf{a}\cdot|0\rangle &= 0\,, \\ \mathbf{J}^\mathbf{a}\cdot A_i^\dagger(p)|0\rangle &= J^{\mathbf{a}}{}_i^j(p)A_j^\dagger(p)|0\rangle\,, \\ \mathbf{J}^\mathbf{a}\cdot A_i^\dagger(p_1)A_j^\dagger(p_2)|0\rangle &= J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2)A_k^\dagger(p_1)A_l^\dagger(p_2)|0\rangle\,. \end{aligned} \tag{2.5}$$

Let us stress that the two-particle state $A_i^\dagger(p_1)A_j^\dagger(p_2)|0\rangle$ cannot in general be identified with $A_i^\dagger(p_1)|0\rangle\otimes A_j^\dagger(p_2)|0\rangle$ equipped with the standard action of the symmetry generators because in that case the representation constants would satisfy

$$J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2) = J^{\mathbf{a}}{}_i^k(p_1)\delta_j^l + \delta_i^k J^{\mathbf{a}}{}_j^l(p_2)\,.$$

The invariance condition for the S-matrix can be derived multiplying (2.1) by $\mathbf{J^a}$

$$\mathbf{J^a}\cdot A_i^\dagger(p_1)A_j^\dagger(p_2)|0\rangle = S_{ij}^{kl}(p_1,p_2)\,\mathbf{J^a}\cdot A_l^\dagger(p_2)A_k^\dagger(p_1)|0\rangle\,. \tag{2.6}$$

Computing the lhs and rhs of this equality, we see that the S-matrix must satisfy the following invariance condition

$$S_{kl}^{mn}(p_1,p_2)J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2) = J^{\mathbf{a}}{}_{lk}^{nm}(p_2,p_1)S_{ij}^{kl}(p_1,p_2)\,. \tag{2.7}$$

If we combine the symmetry generator structure constants in a matrix

$$J_{12}^{\mathbf{a}}(p_1,p_2) \equiv J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2)E_k^i\otimes E_l^j\,, \tag{2.8}$$

where E_k^i are the standard matrix unities, c.f. appendix 8.1, then the invariance condition can be written as

$$S_{12}(p_1,p_2)J_{12}^{\mathbf{a}}(p_1,p_2) = J_{21}^{\mathbf{a}}(p_2,p_1)S_{12}(p_1,p_2)\,. \tag{2.9}$$

Note that the condition reduces to the familiar one

$$[J^a\otimes\mathbb{I} + \mathbb{I}\otimes J^a, S_{12}] = 0\,, \tag{2.10}$$

only if $J^a(p) = J^{\mathbf{a}}{}_i^k(p)E_k^i$ is independent of p.

The form of $J_{12}^{\mathbf{a}}(p_1,p_2)$ depends on the symmetry algebra of a particular model. If the symmetry algebra is equipped with the structure of a bi-algebra, then the two-particle representation is given by the bi-algebra coproduct. If the symmetry algebra has a center then multi-particle representations are characterized by the values of the central elements, and a two-particle state may be considered as the tensor product of two one-particle states with in general different values of the central elements. Then, if a symmetry algebra is a Lie algebra, a two-particle representation is given by the standard coproduct, and the structure constants have the following form

$$J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2) = J^{\mathbf{a}}{}_i^k(p_1;c_1)\delta_j^l + (-1)^{\epsilon(i)\epsilon(\mathbf{a})}\delta_i^k J^{\mathbf{a}}{}_j^l(p_2;c_2)\,, \tag{2.11}$$

where c_i denote sets of central charges which characterize the "one-particle" representations, and $\epsilon(i)=0$ if $A_i^\dagger$ is boson, and $\epsilon(i)=1$ if $A_i^\dagger$ is fermion, and the same holds for $\epsilon(\mathbf{a})$. The structure constants $J^{\mathbf{a}}{}_i^k(p)$ in (2.5) of one-particle states correspond to definite values of the central charges, say, $c_i=0$. Since a two-particle state represents two asymptotic noninteracting particles, the central charges c_i may depend only on the coupling constants and the values of the momenta p_i of the particles.

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Two-particle structure constants of the form (2.11) which depend only on momenta of the particles can be derived by assuming the following commutation relations of the symmetry algebra generators with the creation operators

$$\mathbf{J}^{\mathbf{a}} A_i^\dagger(p) = J^{\mathbf{b}k}{}_m(p) A_k^\dagger(p) \Theta^{\mathbf{a}m}_{\mathbf{b}i}(p;\mathbf{P}) + (-1)^{\epsilon(i)\epsilon(\mathbf{a})} A_m^\dagger(p) \widetilde{\Theta}^{\mathbf{a}m}_{\mathbf{b}i}(p;\mathbf{P}) \mathbf{J}^{\mathbf{b}} \,. \tag{2.12}$$

Here $\mathbf{P}$ is the conserved world-sheet momentum of the model obeying the following simple commutation relation

$$\mathbf{P} A_i^\dagger(p) = A_i^\dagger(p)\left(\mathbf{P} + p\right) \,,$$

and the braiding operators $\Theta^{\mathbf{a}m}_{\mathbf{b}i}(p;\mathbf{P})$ and $\widetilde{\Theta}^{\mathbf{a}m}_{\mathbf{b}i}(p;\mathbf{P})$ should satisfy the following conditions

$$\begin{aligned} \Theta^{\mathbf{a}m}_{\mathbf{b}i}(\mathbf{P})|0\rangle &= \delta^{\mathbf{a}}_{\mathbf{b}} \delta^m_i |0\rangle \,, \\ J^{\mathbf{b}k}{}_m(p_1) A_k^\dagger(p_1) \Theta^{\mathbf{a}m}_{\mathbf{b}i}(p_1;\mathbf{P}) A_j^\dagger(p_2)|0\rangle &= J^{\mathbf{a}k}{}_i(p_1;c_1) A_k^\dagger(p_1) A_j^\dagger(p_2)|0\rangle \,, \\ A_m^\dagger(p_1) \widetilde{\Theta}^{\mathbf{a}m}_{\mathbf{b}i}(p_1;\mathbf{P}) \mathbf{J}^{\mathbf{b}} A_j^\dagger(p_2)|0\rangle &= J^{\mathbf{a}k}{}_j(p_2;c_2) A_i^\dagger(p_1) A_k^\dagger(p_2)|0\rangle \,. \end{aligned}$$

The last two conditions are equivalent to

$$J^{\mathbf{b}k}{}_m(p_1) \Theta^{\mathbf{a}m}_{\mathbf{b}i}(p_1;p_2) = J^{\mathbf{a}k}{}_i(p_1;c_1) \,, \quad \widetilde{\Theta}^{\mathbf{a}m}_{\mathbf{b}i}(p_1;p_2) J^{\mathbf{b}k}{}_j(p_2) = \delta^m_i J^{\mathbf{a}k}{}_j(p_2;c_2) \,.$$

It is clear that the braiding operators should also satisfy the consistency conditions that follow from the fact that the symmetry operators form an algebra. In what follows we restrict our attention to the following choice of (2.12) which is compatible with the ZF algebra we will discuss in next sections

$$\mathbf{J}^{\mathbf{a}} A_i^\dagger(p) = J^{\mathbf{b}k}{}_m(p) A_k^\dagger(p) \Theta^{\mathbf{a}m}_{\mathbf{b}i}(\mathbf{P}) + (-1)^{\epsilon(i)\epsilon(\mathbf{a})} A_i^\dagger(p) \mathbf{J}^{\mathbf{a}} \,, \tag{2.13}$$

where the only braiding operator $\Theta^{\mathbf{a}m}_{\mathbf{b}i}(\mathbf{P})$ depends only on the world-sheet momentum operator $\mathbf{P}$, and is equal to $\delta^{\mathbf{a}}_{\mathbf{b}}\delta^m_i$ if $\mathbf{J}^{\mathbf{a}}$ is bosonic.

Another simple choice of (2.12) is as follows

$$\mathbf{J}^{\mathbf{a}} A_i^\dagger(p) = J^{\mathbf{a}k}{}_m(p) A_k^\dagger(p) + (-1)^{\epsilon(i)\epsilon(\mathbf{a})} A_m^\dagger(p) \widetilde{\Theta}^{\mathbf{a}m}_{\mathbf{b}i}(p;\mathbf{P}) \mathbf{J}^{\mathbf{b}} \,. \tag{2.14}$$

In general, the corresponding (twisted) ZF algebra would not have the simplest form, and we will not be discussing this choice in detail.

3. The centrally extended $\mathfrak{su}(2|2)$ and its representations

The off-shell symmetry algebra of the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring was recently discussed in detail in [7] where it was shown that it consists of two copies of the centrally extended $\mathfrak{su}(2|2)$ algebra, both copies sharing the same central element which corresponds to the world-sheet light-cone Hamiltonian. To describe the ZF algebra we need to recall the structure of representations of the centrally extended $\mathfrak{su}(2|2)$ algebra. In our discussion we closely follow the one in [8].

3.1 Centrally extended $\mathfrak{su}(2|2)$ algebra

The centrally extended $\mathfrak{su}(2|2)$ algebra which we will denote $\mathfrak{su}(2|2)_{H,C}$, consists of the rotation generators $\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$ of the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ bosonic subalgebra, the supersymmetry generators $\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}^{\dagger}_a{}^\alpha$ and three central elements $\mathbf{H}$, $\mathbf{C}$ and $\mathbf{C}^\dagger$.

$$
\begin{aligned}
\Big[\mathbf{L}_a{}^b,\mathbf{J}_c\Big] &= \delta_c^b\mathbf{J}_a-\frac{1}{2}\delta_a^b\mathbf{J}_c\,, &\Big[\mathbf{R}_\alpha{}^\beta,\mathbf{J}_\gamma\Big] &= \delta_\gamma^\beta\mathbf{J}_\alpha-\frac{1}{2}\delta_\alpha^\beta\mathbf{J}_\gamma\,,\\
\Big[\mathbf{L}_a{}^b,\mathbf{J}^c\Big] &= -\delta_a^c\mathbf{J}^b+\frac{1}{2}\delta_a^b\mathbf{J}^c\,, &\Big[\mathbf{R}_\alpha{}^\beta,\mathbf{J}^\gamma\Big] &= -\delta_\alpha^\gamma\mathbf{J}^\beta+\frac{1}{2}\delta_\alpha^\beta\mathbf{J}^\gamma\,,\\
\{\mathbf{Q}_\alpha{}^a,\mathbf{Q}^{\dagger}_b{}^\beta\} &= \delta_b^a\mathbf{R}_\alpha{}^\beta+\delta_\alpha^\beta\mathbf{L}_b{}^a+\frac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbf{H}\,, & &\\
\{\mathbf{Q}_\alpha{}^a,\mathbf{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, &\{\mathbf{Q}^{\dagger}_a{}^\alpha,\mathbf{Q}^{\dagger}_b{}^\beta\} &= \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger\,.
\end{aligned}
\tag{3.1}
$$

Here in the first two lines we indicate how the indices c and γ of any Lie algebra generator transform under the action of the bosonic subalgebras generated by $\mathbf{L}_a{}^b$ and $\mathbf{R}_\alpha{}^\beta$. The supersymmetry generators $\mathbf{Q}_\alpha{}^a$ and $\mathbf{Q}^{\dagger}_a{}^\alpha$, and the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ are hermitian conjugate to each other: $(\mathbf{Q}_\alpha{}^a)^\dagger=\mathbf{Q}^{\dagger}_a{}^\alpha$. The central element $\mathbf{H}$ is hermitian and is identified with the world-sheet light-cone Hamiltonian. It was shown in [7] that the central element $\mathbf{C}$ is expressed through the world-sheet momentum $\mathbf{P}$ as follows

$$
\mathbf{C}=ig\,(e^{i\mathbf{P}}-1)e^{2i\xi}\,,\quad g=\frac{\sqrt{\lambda}}{4\pi}\,.
\tag{3.2}
$$

The phase ξ is an arbitrary function of the central elements, and reflects the obvious U(1) automorphism of the algebra (3.1): $\mathbf{Q}\to e^{i\xi}\mathbf{Q}$, $\mathbf{C}\to e^{2i\xi}\mathbf{C}$. The phase ξ can be fixed to be 0 by choosing a proper form of the supersymmetry generators. This is the choice we use throughout the paper. This simplifies the comparison with the explicit string theory computation of the S-matrix performed in [30]. It is worth mentioning, however, that to have the interpretation of a multi-particle state as the tensor product of fundamental representations, it is necessary to consider fundamental representations with arbitrary phases ξ, see the next section.

3.2 Fundamental representation

The fundamental representation of $\mathfrak{su}(2|2)_{H,C}$ is four-dimensional. We denote the corresponding vector space as $V(p,\zeta)$, and the basis vectors as $|e_M(p,\zeta)\rangle\equiv|e_M\rangle$, where the index $M=\{a,\alpha\}$, and $a=1,2$ are bosonic indices and $\alpha=3,4$ are fermionic ones. The parameters p and ζ are, for non-unitary representations, arbitrary complex numbers which parameterize the values (charges) of the central elements on this representation: $\mathbf{H}|e_M\rangle=H|e_M\rangle$, $\mathbf{C}|e_M\rangle=C|e_M\rangle$, $\mathbf{C}^\dagger|e_M\rangle=\overline{C}|e_M\rangle$. The canonical fundamental representation of bosonic generators is

$$
\mathbf{L}_a{}^b|e_c\rangle=|e_M\rangle L_{ac}^{bM}=\delta_c^b|e_a\rangle-\frac{1}{2}\delta_a^b|e_c\rangle\,,\qquad \mathbf{R}_\alpha{}^\beta|e_\gamma\rangle=|e_M\rangle R_{\alpha\gamma}^{\beta M}=\delta_\gamma^\beta|e_\alpha\rangle-\frac{1}{2}\delta_\alpha^\beta|e_\gamma\rangle\,,
$$

and the most general fundamental representation of supersymmetry generators is of the form

$$
\begin{aligned}
\mathbf{Q}_\alpha{}^a|e_b\rangle &= |e_M\rangle Q_{\alpha b}^{aM}=a\,\delta_b^a|e_\alpha\rangle\,, &\mathbf{Q}_\alpha{}^a|e_\beta\rangle &= |e_M\rangle Q_{\alpha\beta}^{aM}=b\,\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle\,,\\
\mathbf{Q}^{\dagger}_a{}^\alpha|e_\beta\rangle &= |e_M\rangle\overline{Q}_{a\beta}^{\alpha M}=d\,\delta_\beta^\alpha|e_a\rangle\,, &\mathbf{Q}^{\dagger}_a{}^\alpha|e_b\rangle &= |e_M\rangle\overline{Q}_{ab}^{\alpha M}=c\,\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle\,.
\end{aligned}
\tag{3.3}
$$

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Here a, b, c, d are complex numbers parametrizing the representation, and $Q^{aM}_{\alpha b}$, $Q^{aM}_{\alpha\beta}$ and so on are the representation structure constants, and the corresponding matrices Q^a_α and so on which appear in the invariance condition (2.7) for the S-matrix are constructed by using the matrix units defined in appendix A.1: $Q^a_\alpha = Q^{aM}_{\alpha N} e^N_M = Q^{aM}_{\alpha b} e^b_M + Q^{aM}_{\alpha\beta} e^\beta_M$.

The central charges of this representation are expressed through the parameters a, b, c, d as follows

$$H = ad + bc\,, \quad C = ab\,, \quad \overline{C} = cd\,, \tag{3.4}$$

and the consistency condition

$$ad - bc = 1 \tag{3.5}$$

has to be imposed to obtain a representation of the centrally extended $\mathfrak{su}(2|2)$.

For generic values of the parameters a, b, c, d the representation is non-unitary. To get a unitary representation one should impose the conditions $d = \bar{a}, c = \bar{b}$.

The parameters a, b, c, d depend on the string sigma model coupling constant g, and the world-sheet momenta p. To express this dependence it is convenient to introduce new parameters g, x^+, x^-, ζ, η as follows [8][3]

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{\zeta x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)\,. \tag{3.6}$$

The parameters $x^\pm$ satisfy the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \tag{3.7}$$

which follows from $ad - bc = 1$, and are related to the momentum p as

$$\frac{x^+}{x^-} = e^{ip}\,. \tag{3.8}$$

The values of the central charges can be found by using (3.4)

$$\begin{aligned} H &= 1 + \frac{2ig}{x^+} - \frac{2ig}{x^-} = 2igx^- - 2igx^+ - 1\,, \\ H^2 &= 1 + 16g^2 \sin^2\frac{p}{2}\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} C &= ig\zeta\left(\frac{x^+}{x^-} - 1\right) = ig\zeta\left(e^{ip} - 1\right)\,, \\ \overline{C} &= \frac{g}{i\zeta}\left(\frac{x^-}{x^+} - 1\right) = \frac{g}{i\zeta}\left(e^{-ip} - 1\right)\,. \end{aligned} \tag{3.10}$$

Comparing the expressions with the formula (3.2) derived in string theory [7], we see that the parameter ζ should be identified with $e^{2i\xi}$, and p with the value of the world-sheet momentum $\mathbf{P}$ on this representation.

[3] The parameter ζ in [8] should be rescaled as $\zeta \to -i\zeta$ to match our definition.

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The fundamental representation is completely determined by the parameters g, x^+, x^-, ζ. The parameter η just reflects a freedom in the choice of the basis vectors $|e_M\rangle$. For a non-unitary fundamental representation it can be set to 1 by rescaling $|e_M\rangle$ properly. From the string theory point of view it is not a natural choice of the basis because the supersymmetry generators act in a different way on bosons and fermions, and the explicit form of the generators found in [43, 7] shows that the action should be similar. The string theory symmetric choice corresponds to taking $\eta \sim \sqrt{\zeta}$.

3.3 Unitary representation

In string theory we are interested in unitary representations. A fundamental unitary representation is singled out by the conditions $c = \bar{b}, d = \bar{a}$, and p is real, and is given by choosing the parameters η and ζ to be

$$\zeta = e^{2i\xi}\,, \quad \eta = \sqrt{ix^- - ix^+}e^{i(\xi+\varphi)} \tag{3.11}$$

where φ and ξ are real parameters. Then the parameters a, b of the unitary representation are

$$a = \sqrt{g}\sqrt{ix^- - ix^+}e^{i(\xi+\varphi)} = \sqrt{g}\eta\,, \quad b = -\sqrt{g}\frac{\sqrt{ix^- - ix^+}}{x^-}e^{i(\xi-\varphi)}\,, \tag{3.12}$$

and the central charges are given by

$$H(p) = \sqrt{1 + 16g^2\sin^2(\frac{1}{2}p)}\,, \quad C(p,\xi) = ige^{2i\xi}(e^{ip} - 1)\,. \tag{3.13}$$

The phase of the parameter η has been chosen so that $\varphi = 0$ would correspond to the symmetric string theory choice we have discussed above. We will see in next sections that the S-matrix corresponding to the symmetric choice satisfies the usual YB equation, and its near-plane-wave expansion completely agrees with the leading-order string S-matrix computed in [30]. With this choice the ZF algebra for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ takes the usual form too.

On the other hand, if one makes the spin chain choice $\varphi = -\xi$ [6] then the corresponding spin chain S-matrix satisfies the twisted YB equation, and the ZF algebra has a more complicated structure which involves a twist operator depending on the world-sheet momentum $\mathbf{P}$.

4. The $\mathfrak{su}(2|2)$-invariant S-matrix

Since the manifest symmetry algebra of the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ consists of two copies of the centrally-extended $\mathfrak{su}(2|2)$, the creation operators $A^\dagger_{M\dot{M}}(p)$ carry two indices M and $\dot{M}$, where the dotted index is for the second $\mathfrak{su}(2|2)$. The n-particle states are obtained by acting by the creation operators on the vacuum

$$A^\dagger_{M_1\dot{M}_1}(p_1)\cdots A^\dagger_{M_n\dot{M}_n}(p_n)|0\rangle \equiv |A^\dagger_{M_1\dot{M}_1}(p_1)\cdots A^\dagger_{M_n\dot{M}_n}(p_n)\rangle\,. \tag{4.1}$$

For the purpose of this section we can think of $A^\dagger_{M\dot{M}}(p)$ as being a product of two creation operators $A^\dagger_{M\dot{M}}(p) = A^\dagger_M(p) \times A^\dagger_{\dot{M}}(p)$ and restrict our attention to the states created by $A^\dagger_M(p)$.

4.1 Two-particle states and the S-matrix

It is clear that a one-particle state $|A_M^\dagger(p)\rangle$ is identified with the basis vector $|e_M\rangle$ of the fundamental representation $V(p,1)$ of $\mathfrak{su}(2|2)_{H,C}$. Let us stress that we have to set the parameter ζ to 1, because we use the canonical form of the central charge $\mathbf{C}$ with $\xi = 0$

$$\mathbf{C} = ig\,(e^{i\mathbf{P}} - 1)\,, \quad \mathbf{C}|A_M^\dagger(p)\rangle = ig\,(e^{ip} - 1)|A_M^\dagger(p)\rangle\,. \tag{4.2}$$

We also set $\varphi = 0$ which for one-particle states describes both the string theory choice $\varphi = 0$, and the spin chain choice $\varphi = -\xi$.

Then the two-particle states created by $A_M^\dagger(p)$ should be identified with the tensor product of fundamental representations of $\mathfrak{su}(2|2)_{H,C}$

$$|A_{M_1}^\dagger(p_1)A_{M_2}^\dagger(p_2)\rangle \sim V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\,, \tag{4.3}$$

equipped with the canonical action of the symmetry generators in the tensor product. An important observation is that the parameters ζ_k cannot be equal to 1 [8]. The reason for that is very simple. Computing the central charge $\mathbf{C}$ on the two-particle state, we get

$$\mathbf{C}|A_{M_1}^\dagger(p_1)A_{M_2}^\dagger(p_2)\rangle = ig(e^{i(p_1+p_2)} - 1)|A_{M_1}^\dagger(p_1)A_{M_2}^\dagger(p_2)\rangle\,, \tag{4.4}$$

because $\mathbf{P}A_M^\dagger(p) = A_M^\dagger(p)(\mathbf{P}+p)$. On the other hand the value of the central charge on the tensor product of fundamental representations is just equal to the sum of their charges

$$\mathbf{C}\,V(p_1,\zeta_1)\otimes V(p_2,\zeta_2) = ig\,\big(\zeta_1(e^{ip_1}-1) + \zeta_2(e^{ip_2}-1)\big)\,V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\,. \tag{4.5}$$

Thus, we must have the following identity

$$e^{i(p_1+p_2)} - 1 = \zeta_1(e^{ip_1}-1) + \zeta_2(e^{ip_2}-1)\,, \tag{4.6}$$

and it cannot be satisfied if both ζ_1 and ζ_2 are equal to 1. In fact, it is easy to show that there are only two solutions to this equation for ζ_k lying on the unit circle

$$\{\zeta_1 = e^{ip_2}\,,\ \zeta_2 = 1\}\,, \quad \text{or} \quad \{\zeta_1 = 1\,,\ \zeta_2 = e^{ip_1}\}\,. \tag{4.7}$$

Both choices can be used to identify a two-particle state with the tensor product. It is readily seen that the first choice corresponds to the following rearrangement of the commutation relation of the central charge $\mathbf{C}$ with $A_M^\dagger(p)$

$$\mathbf{C}\,A_M^\dagger(p) = C(p)A_M^\dagger(p)\,e^{i\mathbf{P}} + A_M^\dagger(p)\,\mathbf{C}\,, \tag{4.8}$$

which is exactly of the form (2.13). The second choice, on the other hand, corresponds to another rearrangement of the commutation relation

$$\mathbf{C}\,A_M^\dagger(p) = C(p)A_M^\dagger(p) + e^{ip}A_M^\dagger(p)\,\mathbf{C}\,, \tag{4.9}$$

which is of the form (2.14).

In general there is no restriction on the choice of the parameters φ_i (or η_i) of the fundamental representations appearing in (4.3). However, in what follows we will be interested in the string theory choice $\varphi = 0$, and in the spin chain choice $\varphi = -\xi$. Both choices describe equivalent physical theories, but the string theory choice leads to the S-matrix satisfying the usual YB equation, and as a result to the conventional ZF algebra.

Thus, making the first choice in (4.7), we see that the invariance condition (2.9) for the $\mathfrak{su}(2|2)$ S-matrix takes the following form for bosonic generators L_a^b and R_α^β

$$S_{12}(p_1,p_2)\big(J\otimes\mathbb{I}+\mathbb{I}\otimes J\big)=\big(J\otimes\mathbb{I}+\mathbb{I}\otimes J\big)S_{12}(p_1,p_2)\,, \tag{4.10}$$

and for fermionic generators Q_α^a and $\overline{Q}_a^\alpha$

$$\begin{aligned}S_{12}(p_1,p_2)\big(J(p_1;e^{ip_2},\varphi_1)\otimes\mathbb{I}+\Sigma\otimes J(p_2;1,\varphi_2)\big) &=\\ \big(J(p_1;1,\tilde\varphi_1)\otimes\Sigma+\mathbb{I}\otimes J(p_2;e^{ip_1},\tilde\varphi_2)\big)S_{12}(p_1,p_2)\,,&\end{aligned} \tag{4.11}$$

where $J(p;\zeta,\varphi)$ denote the structure constants matrices of the fundamental representation parametrized by p, $\zeta=e^{2i\xi}$ and φ, see (3.3) and (3.6), and Σ is the diagonal matrix which takes care of the negative sign for fermions

$$\Sigma=\mathrm{diag}(1,1,-1,-1)\,. \tag{4.12}$$

These are the conditions that should be used to find the S-matrix. For the string symmetric choice leading to the canonical S-matrix satisfying the YB equation we choose

$$\textsc{String theory basis}: \qquad \varphi_1=\varphi_2=\tilde\varphi_1=\tilde\varphi_2=0\,, \tag{4.13}$$

and for the spin chain choice we have

$$\textsc{Spin chain basis}: \qquad \varphi_2=\tilde\varphi_1=0\,,\quad \varphi_1=-\frac{p_2}{2}\,,\quad \tilde\varphi_2=-\frac{p_1}{2}\,. \tag{4.14}$$

The form of the structure constants matrices $J(p;\zeta,\varphi)$ allows us to determine the commutation relations (2.13) of the symmetry operators with the creation and annihilation operators. It is convenient to use the matrix notations, i.e. to combine $A_M^\dagger$ and A_M into a row and column, respectively, and the symmetry algebra structure constants into matrices L_a^b, R_α^β, Q_α^a and $\overline{Q}_a^\alpha$, see (3.3). Then, for the string theory choice of the basis $\varphi_i=0$, the relations (2.13) for the centrally-extended algebra $\mathfrak{su}(2|2)$ can be written in the following simple form

$$\textsc{String theory basis}: \quad \begin{aligned}\mathbf{L}_a{}^bA^\dagger(p) &= A^\dagger(p)\,L_a^b+A^\dagger(p)\,\mathbf{L}_a{}^b\,,\\ \mathbf{R}_\alpha{}^\beta A^\dagger(p) &= A^\dagger(p)\,R_\alpha^\beta+A^\dagger(p)\,\mathbf{R}_\alpha{}^\beta\,,\\ \mathbf{Q}_\alpha{}^aA^\dagger(p) &= A^\dagger(p)\,Q_\alpha^a(p)\,e^{i\mathbf{P}/2}+A^\dagger(p)\,\Sigma\,\mathbf{Q}_\alpha{}^a\,,\\ \mathbf{Q}_a^{\dagger\,\alpha}A^\dagger(p) &= A^\dagger(p)\,\overline{Q}_a^\alpha(p)\,e^{-i\mathbf{P}/2}+A^\dagger(p)\,\Sigma\,\mathbf{Q}_a^{\dagger\,\alpha}\,,\end{aligned} \tag{4.15}$$

where L_a^b, R_α^β, Q_α^a and $\overline{Q}_a^\alpha$ are the symmetry algebra structure constants matrices of the one-particle representation with $\xi=\varphi=0$. Thus, for the string theory basis the braiding factors in (2.13) are just scalar operators $e^{\pm i\mathbf{P}/2}$.

In the case of the spin chain choice of the basis (4.14), the relations (2.13) take the following more complicated form

$$\textsc{Spin chain basis:}\qquad \begin{aligned} \mathbf{L}_a{}^b A^\dagger(p) &= A^\dagger(p)\, L_a^b + A^\dagger(p)\,\mathbf{L}_a{}^b\,, \\ \mathbf{R}_\alpha{}^\beta A^\dagger(p) &= A^\dagger(p)\, R_\alpha^\beta + A^\dagger(p)\,\mathbf{R}_\alpha{}^\beta\,, \\ \mathbf{Q}_\alpha{}^a A^\dagger(p) &= A^\dagger(p)\, Q_\alpha^a(p)\,\Theta(\mathbf{P}) + A^\dagger(p)\,\Sigma\,\mathbf{Q}_\alpha{}^a\,, \\ \mathbf{Q}^\dagger_a{}^\alpha A^\dagger(p) &= A^\dagger(p)\, \overline{Q}_a^\alpha(p)\,\overline{\Theta}(\mathbf{P}) + A^\dagger(p)\,\Sigma\,\mathbf{Q}^\dagger_a{}^\alpha\,, \end{aligned} \tag{4.16}$$

where the braiding factors $\Theta(\mathbf{P})$ and $\overline{\Theta}(\mathbf{P})$ are the diagonal matrices

$$\Theta(\mathbf{P}) = \mathrm{diag}(1, 1, e^{i\mathbf{P}}, e^{i\mathbf{P}})\,, \qquad \overline{\Theta}(\mathbf{P}) = e^{-i\mathbf{P}}\Theta(\mathbf{P})\,. \tag{4.17}$$

It is not difficult to check that the commutation relations (4.16) follow from (4.15) after the following change of the creation/annihilation operators

$$A^\dagger(p) \to A^\dagger(p)\, U(\mathbf{P})\,, \quad A(p) \to U^\dagger(\mathbf{P}) A(p)\,, \tag{4.18}$$

where

$$U(\mathbf{P}) = \mathrm{diag}(e^{\frac{i}{2}\mathbf{P}}, e^{\frac{i}{2}\mathbf{P}}, 1, 1)\,. \tag{4.19}$$

Let us stress that the transformation (4.18) is not unitary because as we will see soon it changes the commutation relation of the creation and annihilation operators, and, therefore, the form of the ZF algebra.

It is worthwhile to notice that the form of the two-particle structure constants matrices appearing in the invariance condition (4.11) allows us to reformulate the problem by using the Hopf algebra language similar to the considerations in [34], see appendix (A.4) for detail.

It is also interesting to mention that the (anti)-commutation relations (4.15) (and (4.16)) of the symmetry algebra generators with the creation and annihilation operators are written in the usual (anti)-commutator form. The only difference with the standard relations is in the appearance of the operator $e^{\pm i\mathbf{P}/2}$ on the r.h.s. of the relations (4.15). It is easy to see that one can get rid of the dependence on $e^{\pm i\mathbf{P}/2}$ in these relations by redefining the supersymmetry generators $\mathbf{Q}_\alpha{}^a$ and $\mathbf{Q}^\dagger_a{}^\alpha$ as follows

$$\mathbf{Q}_a{}^\alpha \to \mathbf{Q}_a{}^\alpha\, e^{i\mathbf{P}/2}\,, \quad \mathbf{Q}^\dagger_a{}^\alpha \to \mathbf{Q}^\dagger_a{}^\alpha\, e^{-i\mathbf{P}/2}\,. \tag{4.20}$$

Then the relations (4.15) for the redefined supersymmetry charges take the form of the braided (anti)-commutators

$$\begin{aligned} \mathbf{Q}_\alpha{}^a A^\dagger(p) - e^{-ip/2} A^\dagger(p)\,\Sigma\,\mathbf{Q}_\alpha{}^a &= A^\dagger(p)\, Q_\alpha^a(p)\, e^{-ip/2}\,, \\ \mathbf{Q}^\dagger_a{}^\alpha A^\dagger(p) - e^{ip/2} A^\dagger(p)\,\Sigma\,\mathbf{Q}^\dagger_a{}^\alpha &= A^\dagger(p)\, \overline{Q}_a^\alpha(p)\, e^{ip/2}\,. \end{aligned} \tag{4.21}$$

It is the form one usually discusses considering models with nonlocal charges [33]. The redefinition (4.20) changes the $\mathbf{P}$-dependence of the central charge $\mathbf{C}$:

$$\mathbf{C} \to ig\,(e^{i\mathbf{P}} - 1)e^{-i\mathbf{P}} = ig\,(1 - e^{-i\mathbf{P}})\,, \tag{4.22}$$

but it does not change the form of the S-matrix if one keeps the track of the additional phases. We will not be using this form of the commutation relations in this paper.

Finally, let us note that, if we think of vectors from $V(p;\zeta)$ as columns then, as can be seen from the formula (4.11), the S-matrix can be considered as the map

$$S_{12}(p_1,p_2): \qquad V(p_1,e^{ip_2})\otimes V(p_2,1)\to V(p_1,1)\otimes V(p_2,e^{ip_1})\,, \tag{4.23}$$

and if we think of vectors from $V(p;\zeta)$ as rows then the S-matrix can be considered as the opposite map

$$S_{12}(p_1,p_2): \qquad V(p_1,1)\otimes V(p_2,e^{ip_1})\to V(p_1,e^{ip_2})\otimes V(p_2,1)\,. \tag{4.24}$$

From this point of view the action of the S-matrix corresponds to exchanging the two possible choices of the parameters ζ_k of the two representations. Let us stress however, that no matter what interpretation we use, $S_{12}(p_1,p_2)$ is a 16×16 matrix acting in the 16-dimensional vector space of the two-particle states $|A^\dagger_{M_2}(p_2)A^\dagger_{M_1}(p_1)\rangle$, and as we discussed in section 2, if $p_1 = p_2$ the S-matrix reduces to the minus permutation.[4] Then, if the string coupling constant g goes to infinity the string sigma-model becomes free, and the ZF creation operators become the usual creation operators and just commute or anti-commute depending on the statistics, and, therefore, in this limit the S-matrix must be equal to the graded unity.

The S-matrix satisfying the invariance conditions (4.10) and (4.11) can be easily found up to an overall scalar factor, and its explicit form is given in appendix A.2, eq. (A.7).

4.2 Multi-particle states

Multi-particle states created by $A^\dagger_M(p)$ are correspondingly identified with the tensor product of fundamental representations of $\mathfrak{su}(2|2)_{H,C}$

$$|A^\dagger_{M_1}(p_1)\cdots A^\dagger_{M_n}(p_n)\rangle \sim V(p_1,\zeta_1)\otimes\cdots\otimes V(p_n,\zeta_n)\,, \tag{4.25}$$

equipped with the canonical action of the symmetry generators in the tensor product, and the parameters ζ_k have to satisfy the following identity

$$e^{i(p_1+\cdots+p_n)}-1=\sum_{k=1}^{n}\zeta_k(e^{ip_k}-1)\,. \tag{4.26}$$

In general, there are many different solutions to this equation. In our case, however, the choice of ζ_k is fixed by the commutation relations (2.13) and (4.8). One can easily see that the only solution compatible with (2.13) is

$$\zeta_1=e^{i(p_2+\cdots+p_n)}\,,\quad \zeta_2=e^{i(p_3+\cdots+p_n)}\,,\quad \ldots\,,\quad \zeta_{n-1}=e^{ip_n}\,,\quad \zeta_n=1\,. \tag{4.27}$$

On the other hand the only solution compatible with (2.14) and (4.9) is

$$\zeta_1=1\,,\quad \zeta_2=e^{ip_1}\,,\quad \ldots\,,\quad \zeta_{n-1}=e^{i(p_1+\cdots+p_{n-2})}\,,\quad \zeta_n=e^{i(p_1+\cdots+p_{n-1})}\,. \tag{4.28}$$

[4]Due to the definition of the S-matrix we are using, if $p_1 = p_2$ it is the minus permutation but not the graded permutation.

The multi-particle S-matrix just maps the vector space with the first choice of ζ_k to the (isomorphic) space with the second choice of ζ_k. For generic values of the parameters φ_i the factorizability condition for the S-matrix appears to be equivalent to a version of the YB equation which follows from a twisted Zamolodchikov algebra. The string theory choice $\varphi_i = 0$ is singled out because the corresponding string S-matrix satisfies the usual YB equation, and, therefore, can be used to construct the usual ZF algebra.

5. The Zamolodchikov-Faddeev algebra and the twisting

5.1 The Zamolodchikov-Faddeev algebra

The Zamolodchikov-Faddeev algebra for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ involves creation and annihilation operators which carry two indices M and $\dot{M}$ corresponding to the two centrally-extended $\mathfrak{su}(2|2)$ subalgebras in the symmetry algebra of the light-cone string theory. In what follows we use the indices $i, j, k, \ldots$ to denote the pairs of indices $(M, \dot{M})$, $(N, \dot{N}), \ldots$, and use the notations from appendix A.1.

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ string S-matrix that describes the scattering of two-particle states is a tensor product of two $\mathfrak{su}(2|2)$ S-matrices

$$\mathcal{S}_{ij}^{kl}(p_1, p_2) \equiv \mathcal{S}_{M\dot{M},N\dot{N}}^{K\dot{K},L\dot{L}}(p_1, p_2) = \mathcal{S}_{MN}^{KL}(p_1, p_2)\mathcal{S}_{\dot{M}\dot{N}}^{\dot{K}\dot{L}}(p_1, p_2)\,. \tag{5.1}$$

Here the $\mathfrak{su}(2|2)$ invariant S-matrix $\mathcal{S}_{MN}^{KL}(p_1, p_2)$ includes the necessary scalar factor, see appendix A.2

$$\mathcal{S}_{MN}^{KL}(p_1, p_2) = S_0(p_1, p_2) S_{MN}^{KL}(p_1, p_2)\,, \tag{5.2}$$

and we use the canonical S-matrix corresponding to the string symmetric choice $\varphi_i = 0$, because only under this choice of the parameters (up to trivial transformations) the S-matrix satisfies the normal YB equation.

Let us stress that even though the S-matrix is the tensor product the ZF algebra can be formulated only in terms of the two-index operators $A^\dagger_{M\dot{M}}(p)$ and $A^{M\dot{M}}(p)$ because it contains their commutation relations. One cannot think of the creation and annihilation operators as the products $A^\dagger_{M\dot{M}}(p) = A^\dagger_M(p) A^\dagger_{\dot{M}}(p)$ and $A^{M\dot{M}}(p) = A_M(p) A_{\dot{M}}(p)$ with the one-index operators forming independent $\mathfrak{su}(2|2)$ ZF algebras.[5]

The ZF algebra has the form

$$A_1 A_2 = \mathcal{S}_{12} A_2 A_1\,, \quad A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathcal{S}_{12}\,, \quad A_1 A_2^\dagger = A_2^\dagger \mathcal{S}_{21} A_1 + \delta_{12}\,. \tag{5.3}$$

Here we use the standard matrix notations, see appendix A.1 for details, in particular

$$\begin{aligned} A_1^\dagger A_2^\dagger &= \sum_{i,j} A_i^\dagger(p_1) A_j^\dagger(p_2)\, E^i \otimes E^j\,, \quad A_2^\dagger A_1^\dagger = \sum_{i,j} A_j^\dagger(p_2) A_i^\dagger(p_1)\, E^i \otimes E^j \\ \mathcal{S}_{12} &= \sum_{ijkl} \mathcal{S}_{ij}^{kl}(p_1, p_2)\, E_k^i \otimes E_l^j\,, \qquad \delta_{12} = \delta(p_1 - p_2) \sum_i E_i \otimes E^i\,. \end{aligned}$$

[5]However, all consistency conditions for the full S-matrix just follow from the same conditions for the $\mathfrak{su}(2|2)$ invariant S-matrix $\mathcal{S}_{MN}^{KL}(p_1, p_2)$.

Let us now recall the consistency conditions for the ZF algebra. The first one is the unitarity condition for the S-matrix

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{21}(p_2,p_1)=\mathbb{I}\,, \tag{5.4}$$

which follows by applying the commutation relation $A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathcal{S}_{12}$ twice. Then, the ZF algebra should have a unique basis of the lexicographycally ordered monomials (i.e. it has the Poincaré-Birkhoff-Witt property) and no new relations except (5.3) arise. This requirement leads to the YB equation. By using the two different ways of reordering $A_1^\dagger A_2^\dagger A_3^\dagger$ to $A_3^\dagger A_2^\dagger A_1^\dagger$, we get

$$\begin{aligned} A_1^\dagger A_2^\dagger A_3^\dagger &= A_3^\dagger A_2^\dagger A_1^\dagger \mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23}\,,\\ A_1^\dagger A_2^\dagger A_3^\dagger &= A_3^\dagger A_2^\dagger A_1^\dagger \mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}\,, \end{aligned}$$

and, therefore, derive the YB equation

$$\mathcal{S}_{23}(p_2,p_3)\mathcal{S}_{13}(p_1,p_3)\mathcal{S}_{12}(p_1,p_2)=\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{13}(p_1,p_3)\mathcal{S}_{23}(p_2,p_3)\,. \tag{5.5}$$

Let us mention that both the l.h.s. and r.h.s. of this equation represent the 3-particle scattering S-matrix, and the equation itself is the factorizability condition for the S-matrix. Realizing the $\mathfrak{su}(2|2)$ invariant S-matrix (A.7) as a 16×16 matrix it is a straightforward exercise to check that it satisfies both the unitarity condition (5.4) and the YB equation (5.5).

The ZF algebra also satisfies the physical unitarity condition that follows from the fact that the annihilation operators are hermitian conjugate of the creation operators. To derive the condition of the physical unitarity we find instructive to use explicit index notations. The commutation relations for the creation operators are of the form

$$A_i^\dagger(p_1)A_j^\dagger(p_2)=A_l^\dagger(p_2)A_k^\dagger(p_1)\mathcal{S}_{ij}^{kl}(p_1,p_2)\,.$$

Taking into account $(A_i^\dagger(p))^\dagger = A^i(p)$, we conjugate the relation above and change $p_1\leftrightarrow p_2\,,\ i\leftrightarrow j\,,\ k\leftrightarrow l$. Then we get the following commutation relations for the annihilation operators

$$A^i(p_1)A^j(p_2)=\mathcal{S}_{ji}^{*lk}(p_2,p_1)A^l(p_2)A^k(p_1)$$

According to our assumptions about the structure of the ZF algebra it must be equal to

$$A^i(p_1)A^j(p_2)=\mathcal{S}_{kl}^{ij}(p_1,p_2)A^l(p_2)A^k(p_1)\,.$$

Thus, the S-matrix must satisfy the following identity

$$\mathcal{S}_{ji}^{*lk}(p_2,p_1)=\mathcal{S}_{kl}^{ij}(p_1,p_2)\,,$$

which can be rewritten in the matrix form

$$\mathcal{S}_{21}^\dagger(p_2,p_1)=\mathcal{S}_{12}(p_1,p_2)\,. \tag{5.6}$$

This is the condition of the physical initarity. Taking into account the usual unitarity condition for S-matrix (5.4), we can write the physical unitarity condition (5.6) of the S-matrix in the following form:

$$\mathcal{S}^\dagger(p_1,p_2)\mathcal{S}(p_1,p_2) = \mathbb{I}.$$

It is easy to check numerically that this condition holds for the $\mathfrak{su}(2|2)$ invariant S-matrix (A.7) with η's satisfying the unitary representation conditions (3.11).

An important property of the ZF algebra is that it possesses an abelian subalgebra generated by the commuting operators of the form

$$\mathbf{I}_\omega = \int \mathrm{d}p\, \omega(p) A_i^\dagger(p) A^i(p)\,, \tag{5.7}$$

where $\omega(p)$ is an arbitrary function. In particular, the world-sheet momentum $\mathbf{P}$ and the Hamiltonian $\mathbf{H}$ belong to this subalgebra

$$\mathbf{P} = \int \mathrm{d}p\, p\, A_i^\dagger(p) A^i(p)\,, \quad \mathbf{H} = \int \mathrm{d}p\, H(p) A_i^\dagger(p) A^i(p)\,, \tag{5.8}$$

where $H(p)$ is the value of the Hamiltonian on a one-particle state with momentum p: $H(p) = \sqrt{1+16g^2\sin^2(\frac{1}{2}p)}$. It is not difficult to check that

$$\mathbf{I}_\omega\, A_i^\dagger(p) = A_i^\dagger(p)\big(\omega(p) + \mathbf{I}_\omega\big) \tag{5.9}$$

leading to the additivity property of the commuting integrals

$$\mathbf{I}_\omega\, A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|0\rangle = \left(\sum_{k=1}^{n}\omega(p_{i_k})\right) A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|0\rangle\,. \tag{5.10}$$

The existence of the family of the commuting integrals together with the additivity property of the integrals guarantees the integrability of the model.

5.2 Twisting the ZF algebra

As was mentioned in the previous section, the spin chain S-matrix corresponding to the choice $\varphi = -\xi$ in (3.11) does not satisfy the usual YB equation, and therefore cannot be used to construct the standard ZF algebra. In this subsection we show that the spin chain S-matrix appears in a twisted ZF algebra of the form (2.4) in such a way which leads to a modified YB equation satisfied by the S-matrix. The twisted Zamolodchikov algebra with the spin chain S-matrix is equivalent to the ZF algebra with the string theory S-matrix because they are related by a transformation of the creation/annihilation operators.

As we discussed in section 3.2, the parameter η just reflects our freedom in the choice of the basis of a fundamental representation. On the ZF algebra level this freedom is implemented by the following transformation of the creation/annihilation operators

$$A^\dagger(p) \to A^\dagger(p)\, U(\mathbf{P},p)\,, \quad A(p) \to U^\dagger(\mathbf{P},p) A(p)\,, \tag{5.11}$$

where $A^\dagger$ is a row, A is a column, and U is an arbitrary unitary matrix which can depend on the momentum p of the creation/annihilation operators, and on the world-sheet momentum operator $\mathbf{P}$. This transformation is not an automorphism of the ZF algebra. It is not difficult to see that under this transformation the ZF algebra can be again written in the same form (5.3) but with the following transformed operator-valued S-matrix

$$\mathcal{S}^U_{12}(p_1,p_2;\mathbf{P}) = U_2(\mathbf{P}+p_1,p_2)U_1(\mathbf{P},p_1)\mathcal{S}_{12}(p_1,p_2)U_2^\dagger(\mathbf{P},p_2)U_1^\dagger(\mathbf{P}+p_2,p_1)\,. \tag{5.12}$$

Obviously, the twisted ZF algebra is isomorphic to the original ZF one, and they describe one and the same physical system. The corresponding two-particle S-matrix $\mathcal{S}_{12}$ transforms, however, in a nontrivial way

$$\mathcal{S}^U_{12}(p_1,p_2) = U_2(p_1,p_2)U_1(0,p_1)\mathcal{S}_{12}(p_1,p_2)U_2^\dagger(0,p_2)U_1^\dagger(p_2,p_1)\,, \tag{5.13}$$

where we have taken into account that $\mathbf{P}|0\rangle = 0$. We see, therefore, that such a transformation allows one to introduce a number of unphysical parameters and spurious dependence of momenta p_i. It is clear that the transformed S-matrix S^U_{12} satisfies twisted YB equation with the twist determined by the matrix U.

To simplify the notations in the rest of this subsection we only consider the Zamolodchikov subalgebra with the $\mathfrak{su}(2|2)$ invariant S-matrix generated by the creation operators $A^\dagger_M(p)$. The generalization to the full ZF algebra with the string S-matrix is straightforward.

The transformation (5.11) can, in particular, be used to change the parameters η_i which appear in the two-particle S-matrix. For example, it is clear from the discussion in section 4, see (4.18), that to obtain the spin chain S-matrix corresponding to the choice of η with $\varphi = -\xi$, see (3.11) one should consider a diagonal matrix U of the following form

$$U(\mathbf{P},p) \equiv U(\mathbf{P}) = \mathrm{diag}(e^{\frac{i}{2}\mathbf{P}}, e^{\frac{i}{2}\mathbf{P}}, 1, 1)\,. \tag{5.14}$$

Note, that U has no dependence on p because we do not want to produce a trivial momentum dependent phase for η. To see that this transformation produces the desired effect we consider the corresponding change of a two-particle state

$$A_1^\dagger(p_1)A_2^\dagger(p_2)|0\rangle \to A_1^\dagger(p_1)U_1(p_2)A_2^\dagger(p_2)|0\rangle\,. \tag{5.15}$$

As we discussed in section 4, the two-particle state is identified with the tensor product of the fundamental representations $V(p_1, e^{ip_2}) \otimes V(p_2, 1)$. We see, therefore, that the transformation (5.15) implies a change of the basis of the vector space $V(p_1, e^{ip_2})$. One can easily check that this change means a multiplication of the constant a in (3.3) by the phase $e^{-\frac{i}{2}p_2}$ and, since $\xi_1 = \frac{p_2}{2}$, this leads to the spin chain choice of η_1 with the phase of η_1 equal to 0. There is no need to perform any change of the basis of the space $V(p_2, 1)$ because for this representation $\varphi_2 = \xi_2 = 0$. Finally, eq. (5.13) which relates the spin chain S-matrix to the canonical string S-matrix (A.7) with $\varphi = -\xi$ takes the following form

$$S^{\rm chain}_{12}(p_1,p_2) = U_2(p_1)S^{\rm string}_{12}(p_1,p_2)U_1^\dagger(p_2)\,. \tag{5.16}$$

Taking into account eq. (5.16), we see that the operator-valued S-matrix (5.12) can be cast in the following form

$$S_{12}^{\text{chain}}(p_1,p_2;\mathbf{P}) = F_{12}(\mathbf{P})S_{12}^{\text{chain}}(p_1,p_2)F_{12}^{-1}(\mathbf{P})\,, \tag{5.17}$$

where $F_{12} = U(\mathbf{P})\otimes U(\mathbf{P})$ is the twist matrix. The twisted Zamolodchikov algebra then takes the form

$$A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger S_{12}^{\text{chain}}(p_1,p_2;\mathbf{P})\,, \quad \mathbf{P}A_1^\dagger = A_1^\dagger(\mathbf{P}+p_1)\,, \tag{5.18}$$

leading to the following twisted YB equation[6] for $S_{12}^{\text{chain}}(p_1,p_2)$

$$F_{23}(p_1)S_{23}F_{23}^{-1}(p_1)S_{13}F_{12}(p_3)S_{12}F_{12}^{-1}(p_3) = S_{12}F_{13}(p_2)S_{13}F_{13}^{-1}(p_2)S_{23}\,, \tag{5.19}$$

where $S_{ij} \equiv S_{ij}^{\text{chain}}(p_i,p_j)$. Let us finally mention that, just as in the case of the usual ZF algebra, both the lhs and rhs of this equation represent the 3-particle scattering S-matrix, and the equation itself is the factorizability condition for the S-matrix. It is clear, however, that it is the S-matrix satisfying the standard YB equation that should be considered as the *canonical* $\mathrm{AdS}_5\times\mathrm{S}^5$ string S-matrix.

6. Crossing symmetry

It is well known that the S-matrix in relativistic quantum integrable systems satisfies an additional property called crossing symmetry. This relation allows one to express the S-matrix for the particle-antiparticle scattering via the S-matrix for scattering of two particles. From the point of view of the ZF algebra the crossing symmetry can be regarded as a certain algebra automorphism.

Though our string model does not possess the relativistic invariance on the world-sheet, it is still reasonable to require that the corresponding twisted ZF algebra has an additional invariance related to the particle-antiparticle symmetry of the theory. More precisely, we define the particle-to-antiparticle transformation as

$$A^\dagger(p)\to B^\dagger(p) = A^t(-p)\mathscr{C}(-p)\,, \quad A(p)\to B(p) = \mathscr{C}^\dagger(-p)A^{\dagger t}(-p)\,, \tag{6.1}$$

where $\mathscr{C}(p)$ is a "charge-conjugation" matrix which a priori may depend on p, and superscript t means the usual matrix transposition. We require this map to be an automorphism of the twisted ZF algebra for $p_1\neq p_2$. This means that if we first replace in the algebra relations A by B, and further use the formulas (6.1) to express B via A we should recover for A the same ZF algebra. Under the usual assumption $p_1>p_2$ the delta-function does not contribute which makes it possible to map by using (6.1) the exchange relations of $A(p_1)$ and $A(p_2)$ to that of $A(p_1)$ and $A^\dagger(p_2)$. Obviously, the requirement of the crossing symmetry should impose certain restrictions both on $\mathscr{C}(p)$ and on the S-matrix.

[6]The S-matrix constructed in the gauge theory framework in [6] is an operator S-matrix because besides being a matrix it also operates by inserting $Z^{\pm}$ symbols in an infinite chain of Z fields. As such, it satisfies the usual YB equation since the twisting is compensated by the effect produced by insertions. In our approach, however, we never deal with operator S-matrices.

Note that flipping the sign of p under the particle-to-antiparticle transformation (6.1) is also an immediate consequence of the fact that this map is an automorphism of the twisted ZF algebra. Indeed, this change of the sign is dictated by the compatibility of the particle-to-antiparticle transformation with the algebra relations

$$\mathbf{P}A^\dagger = A^\dagger(\mathbf{P}+p)\,, \quad \mathbf{P}A = A(\mathbf{P}-p)\,,$$

and the requirement that the symmetry algebra generators remain untouched by the transformation (6.1).

6.1 The particle-to-antiparticle transformation

To understand the implications of crossing symmetry we first note that the exchange relations (4.15) should remain invariant under the particle-to-antiparticle transformation (6.1), and this imposes sever restrictions on the form of $\mathscr{C}$. From the first two equations in (4.15) we find the following relations

$$\begin{aligned}\mathscr{C}(p)\,L_a^b &= -L_b^a\,\mathscr{C}(p)\,,\\ \mathscr{C}(p)\,R_\alpha^\beta &= -R_\beta^\alpha\,\mathscr{C}(p)\,,\end{aligned} \tag{6.2}$$

where we take into account that $(L_a^b)^\dagger = L_b^a$, and $(R_\alpha^\beta)^\dagger = R_\beta^\alpha$. These relations fix the form of $\mathscr{C}$ up to two coefficients

$$\mathscr{C}(p) = \begin{pmatrix} 0 & -c_1(p) & 0 & 0 \\ c_1(p) & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_2(p) \\ 0 & 0 & c_2(p) & 0 \end{pmatrix}.$$

The last two equations in (4.15) lead to the following relations

$$\begin{aligned} e^{i\frac{p}{2}}\mathscr{C}(p)\,\overline{Q}_a^\alpha(-p) &= -\left(\overline{Q}_a^\alpha(p)\right)^t\,\Sigma\mathscr{C}(p)\,,\\ \mathscr{C}(p)\;Q_\alpha^a(-p) &= -e^{i\frac{p}{2}}\left(Q_\alpha^a(p)\right)^t\,\Sigma\mathscr{C}(p)\;\;,\end{aligned} \tag{6.3}$$

where we take into account that $(Q_\alpha^a(p))^\dagger = \overline{Q}_a^\alpha(p)$. These relations not only fix the form of $\mathscr{C}$ but also determine how the coefficients a, b, c, d of the representation (3.3) transform under the change $p \to -p$. We find

$$\frac{c_2(p)}{c_1(p)} = \frac{b(-p)}{a(p)}e^{\frac{i}{2}p} = \frac{2\sin\frac{p}{2}}{\eta(p)\eta(-p)}\zeta(p)\,, \tag{6.4}$$

and

$$\frac{a(-p)}{b(p)} = -\frac{a(p)}{b(-p)}e^{-ip} = \frac{c(-p)}{d(p)}e^{-ip} = -\frac{c(p)}{d(-p)}\,. \tag{6.5}$$

It is not difficult to see that the relations (6.5) imply that the transformed coefficients satisfy the condition

$$a(-p)d(-p) - b(-p)c(-p) = 1\,,$$

and, therefore, they define a fundamental representation of the centrally-extended $\mathfrak{su}(2|2)$. This is the antiparticle representation, and one can read off the values of its central charges

$$H(-p) = -H(p)\,, \quad C(-p) = -C(p)e^{-ip}\,, \quad \overline{C}(-p) = -\overline{C}(p)e^{ip}\,.$$

In fact, it is straightforward to show that the relations (6.5) give the following transformation law for the coefficients $x^{\pm}$ and ζ under the particle-to-antiparticle transformation (6.1)

$$x^{\pm}(-p) = \frac{1}{x^{\pm}(p)}\,, \qquad \zeta(-p) = \zeta(p)\,. \tag{6.6}$$

Taking into account that for the unitary representation with the symmetric string theory choice $\eta(p) = \sqrt{ix^- - ix^+}\sqrt{\zeta}$, we simply get

$$\frac{c_2(p)}{c_1(p)} = \frac{\sin\frac{p}{2}}{\sqrt{-\sin^2\frac{p}{2}}} = -i\,\mathrm{sign}(p)\,, \tag{6.7}$$

i.e the ratio of the coefficients of the charge conjugation matrix is (almost) momentum independent. Note that the map (6.1) reflects the fact that $\mathfrak{su}(2|2)$ has the structure of the $\mathbb{Z}_4$-graded Lie algebra. Without loss of generality we can require the coefficient c_1 to be independent on p and satisfy $\bar{c}_1 c_1 = 1$. Then the matrix $\mathscr{C}$ takes the form

$$\mathscr{C}(p) = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -i\,\mathrm{sign}(p)\,\sigma_2 \end{pmatrix}\,, \tag{6.8}$$

where σ_2 is the Pauli matrix. Obviously, successive application of the map (6.1) four times gives the identity. This is nothing else as a reflection of the fact that $\mathfrak{su}(2|2)$ admits a structure of the $\mathbb{Z}_4$-graded Lie superalgebra, see e.g. [45].

6.2 Crossing symmetry condition

As was discussed in section 4.3, the S-matrix appears to be uniquely determined by symmetries up to a scalar factor which we call $S_0(p_1, p_2)$. A possible functional form of $S_0(p_1, p_2)$ is restricted by unitarity together with the requirement of crossing symmetry [9]. We therefore write the S-matrix obeying the condition of crossing symmetry in the form

$$\mathcal{S}_{12}(p_1, p_2) = S_0(p_1, p_2)\; S_{12}(p_1, p_2)\,,$$

where $S_{12}(p_1, p_2)$ is the S-matrix in the string basis; see appendix A.2.

If we assume that $p_1 > p_2$ then the map (6.1) is an automorphism of the ZF algebra provided the matrix $\mathcal{S}_{12}(p_1, p_2)$ obeys the following equations

$$\begin{aligned} &\mathscr{C}_1^{-1}(-p_1)\mathcal{S}_{12}^{t_1}(p_1, p_2)\mathscr{C}_1(-p_1)\mathcal{S}_{12}(-p_1, p_2) = \mathbb{I}\,, \\ &\mathscr{C}_2^{-1}(-p_2)\mathcal{S}_{21}^{t_2}(p_2, p_1)\mathscr{C}_2(-p_2)\mathcal{S}_{21}(-p_2, p_1) = \mathbb{I}\,. \end{aligned} \tag{6.9}$$

Here t_1 and t_2 mean the transposition in the first and second space, respectively, $\mathscr{C}_1 = \mathscr{C} \otimes I$, $\mathscr{C}_2 = I \otimes \mathscr{C}$ and $\mathscr{C}$ is the charge conjugation matrix (6.8). These two equations are in fact equivalent because the first one turns into the second one after applying the permutation and exchanging p_1 and p_2.

Substituting here the string S-matrix from appendix A.2 we indeed find that relations (6.9) produce the following equation for the scalar factor

$$S_0(-p_1,p_2)S_0(p_1,p_2) = \frac{1}{f(p_1,p_2)}\,, \tag{6.10}$$

where the function f was introduced in [9]

$$f(p_1,p_2) = \frac{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}\,. \tag{6.11}$$

Equation (6.11) is precisely the same as the one found by Janik [9]. However, in opposite to [9], our derivation does not refer to the $\mathfrak{su}(1|2)$ invariant S-matrix, rather we use the $\mathfrak{su}(2|2)$ S-matrix in the string basis. Moreover, since $\mathscr{C}$ is the usual charge conjugation matrix, the crossing equation (6.9) has the usual form known in relativistic models. Also, to derive the crossing equation we did not make any reference to a Hopf algebra structure of the symmetry algebra.

Finally, we remark that the crossing equation for the scalar factor is incompatible with the assumption that the S-matrix is an analytical function of p_1, p_2. This is reflected by the fact that the function $f(p_1,p_2)$ obeys the following properties

$$f(p_2,p_1)f(-p_1,p_2) = 1\,, \qquad f(p_2,p_1)f(p_1,-p_2) = 1\,,$$

which are incompatible with the assumption of both unitarity and analyticity.

In conclusion, we see that in the string basis the S-matrix not only satisfies the usual Yang-Baxter equation but it also admits a scalar factor which allows for the standard crossing symmetry condition.

7. Comparison with the near plane-wave S-matrix

In this section we compare the string theory S-matrix we used to define the ZF algebra to the S-matrix arising in the near plane-wave limit of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The later was recently computed in [30] by reading off the quartic interaction vertices of the string Lagrangian [43] obtained in the generalized uniform light-cone gauge[7] [44, 24]. As the result, the corresponding S-matrix, $\mathbb{S}^{\mathrm{KMRZ}}$, appears to have an explicit dependence on the gauge-fixing parameter a,[8] and it is written as[9]

$$\mathbb{S}^{\mathrm{KMRZ}} = \mathbb{I} + \frac{2\pi i}{\sqrt{\lambda}}\mathbb{T}\,,$$

[7]See [39]–[44] on an extensive work concerning the construction of the string Hamiltonian and finding near plane-wave corrections to energies of the plane-wave states.

[8]This parameter labeling different light-cone gauge choices should not be confused with parameter a in the fundamental representation.

[9]It is sufficient to compare only the $\mathfrak{su}(2|2)$ invariant S-matrices.

where $\mathbb{T}$ is a 16×16-matrix (see appendix A.5). It depends on ten non-trivial coefficients $\mathrm{A}, \cdots, \mathrm{L}$ which are functions of the momenta p_1 and p_2. According to [30], these coefficients read as[10]

$$\begin{aligned}
\mathrm{A} &= \frac{1}{4}\Big[(1-2a)(\epsilon(p_2)p_1-\epsilon(p_1)p_2)+\frac{(p_1-p_2)^2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\Big]\,,\\
\mathrm{B} &= -\mathrm{E} = \frac{p_1p_2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\,,\\
\mathrm{C} &= \mathrm{F} = \frac{1}{2}\frac{\sqrt{(\epsilon(p_1)+1)(\epsilon(p_2)+1)}(\epsilon(p_2)p_1-\epsilon(p_1)p_2+p_2-p_1)}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\,,\\
\mathrm{D} &= \frac{1}{4}\Big[(1-2a)(\epsilon(p_2)p_1-\epsilon(p_1)p_2)-\frac{(p_1-p_2)^2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\Big]\,,\\
\mathrm{G} &= \frac{1}{4}\Big[(1-2a)(\epsilon(p_2)p_1-\epsilon(p_1)p_2)-\frac{p_1^2-p_2^2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\Big]\,,\\
\mathrm{H} &= \mathrm{K} = \frac{1}{2}\frac{p_1p_2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\frac{(\epsilon(p_1)+1)(\epsilon(p_2)+1)-p_1p_2}{\sqrt{(\epsilon(p_1)+1)(\epsilon(p_2)+1)}}\,,\\
\mathrm{L} &= \frac{1}{4}\Big[(1-2a)(\epsilon(p_2)p_1-\epsilon(p_1)p_2)+\frac{p_1^2-p_2^2}{\epsilon(p_2)p_1-\epsilon(p_1)p_2}\Big]\,.
\end{aligned}$$

Here $\epsilon(p)=\sqrt{1+p^2}$ is the relativistic energy. To make a comparison of our string S-matrix $\mathcal{S}(p_1,p_2)$ with that of [30] we recall that the string theory choice of the parameters $\eta_{1,2}$ and $\tilde{\eta}_{1,2}$ in (A.7) is

$$\eta_1=\eta(p_1)e^{\frac{i}{2}p_2}\,,\qquad \eta_2=\eta(p_2)\,,\qquad \tilde{\eta}_1=\eta(p_1)\,,\qquad \tilde{\eta}_2=\eta(p_2)e^{\frac{i}{2}p_1}\,,$$

where $\eta(p)=\sqrt{ix^-(p)-ix^+(p)}$. Then, we also have to multiply the S-matrix in (A.7) by the scalar factor (A.9) which encodes the dynamical information and the gauge dependence on the parameter a. This gives us the string S-matrix we should compare with $\mathbb{S}^{\mathrm{KMRZ}}$

$$\mathcal{S}_{12}(p_1,p_2)=S_0(p_1,p_2)S_{12}(p_1,p_2)\,. \tag{7.1}$$

The near plane-wave expansion of the string S-matrix is constructed by rescaling the momenta as $p\to\frac{2\pi}{\sqrt{\lambda}}p$ and keeping in the expansion of $\mathcal{S}(p_1,p_2)$ around $\lambda=\infty$ the first two leading terms.

One can check that in the limit $\lambda\to\infty$ the matrix $\mathcal{S}(p_1,p_2)$ tends to the graded identity in accord with the discussion in section 2. On the other hand, $\mathbb{S}^{\mathrm{KMRZ}}$ goes to the identity in this limit. The reason for this discrepancy is in the different conventions used in [30] and this paper. Analyzing the definitions, one can see that the S-matrices should be related as follows

$$\mathbb{S}^{\mathrm{KMRZ}}(p_1,p_2)=\mathcal{P}P\Big(\mathcal{S}(p_1,p_2)\Big)^{-1}\Big|_{\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)}\,,$$

where P and $\mathcal{P}$ are the permutation and graded permutation matrices respectively; the product $\mathcal{P}P$ is the graded identity. We then find the perfect agreement between these S-matrices.

[10] The coefficient L presented here coincides with the one in the revised version of [30].

In [30] the S-matrix $\mathbb{S}^{\rm KMRZ}(p_1,p_2)$ was compared to the near plane-wave expansion of the S-matrix found by Beisert [6]. Upon a proper choice of the dressing factor the S-matrices appear to agree up to terms linear in momenta; this difference was attributed to the difference in the definition of the spin chain and string lengths arising from world-sheet excitations. In our present discussion we never refered to the gauge theory and we found a complete agreement between the S-matrix derived from symmetry principles and the one computed from the near plane-wave string Lagrangian. We note that it is the proper choice of η's in $S(p_1,p_2)$ which is ultimately responsible for this agreement.

We conclude this section by mentioning the relation of the spin chain S-matrix $S^{\rm chain}(p_1,p_2)$ (see appendix A.2) with that of Beisert [6], $S^B(p_1,p_2)$. Choosing $\tilde\eta_1=\eta_1$ and $\tilde\eta_2=\eta_2$ in (A.7), the relation is as follows

$$P\mathcal{P}S_{12}^{\rm chain}(p_2,p_1)\mathcal{P}=S_{12}^B(p_1,p_2)\,.$$

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A. Appendix

A.1 Notations

Throughout the paper we use the standard notations. The indices M and $\dot M$ run from 1 to 4, and the index i is a collective index to denote a set $\{M,\dot M\}$. Then we introduce rows $e^M\,,\ e^{\dot M}\,,\ E^i$ and columns $e_M\,,\ e_{\dot M}\,,\ E_i$ as

$$e^M=(0,\dots,1_M,\dots,0)\,,\quad e_M=(e^M)^\dagger\,,\quad e^M\cdot e_N=\delta^M_N \tag{A.1}$$

$$E^i\equiv E^{M\dot M}=e^M\otimes e^{\dot M}\,,\quad E_i\equiv E_{M\dot M}=e_M\otimes e_{\dot M}\,, \tag{A.2}$$

and matrix unities

$$e_M^N=e_M\otimes e^N\,,\quad E_k^i=E_k\otimes E^i\,. \tag{A.3}$$

Note that $E^kE^i_j=\delta^k_jE^i\,,\ E^i_jE_k=\delta^i_kE_j\,,\ E^i_jE^k_m=\delta^i_mE^k_j$. They are used to form columns and rows from the creation and annihilation operators, and matrices from the S-matrix, and symmetry algebra structure constants, e.g.

$$A^\dagger=\sum_i A_i^\dagger(p)\,E^i\,,\quad A=\sum_i A^i(p)\,E_i\,,\quad S_{12}=\sum_{ijkl}S_{ij}^{kl}(p_1,p_2)\,E_k^i\otimes E_l^j\,. \tag{A.4}$$

We also define the following matrix delta-function

$$\delta_{12} = \delta(p_1 - p_2) \sum_i E_i \otimes E^i\,, \tag{A.5}$$

and use the following convention

$$A_1^\dagger A_2^\dagger = \sum_{i,j} A_i^\dagger(p_1) A_j^\dagger(p_2)\, E^i \otimes E^j\,, \quad A_2^\dagger A_1^\dagger = \sum_{i,j} A_j^\dagger(p_2) A_i^\dagger(p_1)\, E^i \otimes E^j\,. \tag{A.6}$$

Finally, if A, B, C are either columns or rows with operator entries then in the notation $A_1 B_2 C_3$ the subscripts $1, 2, 3$ refer to the location of the columns and row, e.g. if $A = A^i(p_3)\, E_i\,,\ B = B_i(p_1) E^i\,,\ C = C_i(p_2) E^i$, then $A_1 B_3 C_2 = A^i(p_3) B_k(p_1) C_j(p_2) E_i \otimes E^j \otimes E^k$.

A.2 S-matrix

Up to a scalar factor the invariant S-matrix satisfying eqs. (4.10), (4.11) can be written in the following form

$$\begin{aligned}
S(p_1,p_2) =\ & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\Big)\\
&+\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\Big)\\
&-\Big(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\Big)\\
&+\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\Big(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\Big)\\
&+\frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde\eta_1}\Big(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\Big)\\
&+\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\frac{\eta_2}{\tilde\eta_2}\Big(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\Big)\\
&+i\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\Big(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\Big)\\
&+i\frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\Big(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\Big)\\
&+\frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\frac{\eta_2}{\tilde\eta_1}\Big(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\Big)\\
&+\frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\frac{\eta_1}{\tilde\eta_2}\Big(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\Big)
\end{aligned}\tag{A.7}$$

Here η_i depend on the momenta p_i and the parameters ξ_i, and for unitary representations they are given by $\eta = \sqrt{ix^- - ix^+}e^{i(\xi+\varphi)}$. In particular, for the string symmetric choice all $\varphi_i = 0$, and η_i are equal to

$$\textsc{String basis:}\quad \eta_1 = \eta(p_1)e^{\frac{i}{2}p_2}\,,\quad \eta_2 = \eta(p_2)\,,\quad \tilde\eta_1 = \eta(p_1)\,,\quad \tilde\eta_2 = \eta(p_2)e^{\frac{i}{2}p_1}\,,$$

where $\eta(p) = \sqrt{ix^-(p) - ix^+(p)}$. For the spin chain choice we have

$$\textsc{Spin chain basis:} \quad \eta_1 = \eta(p_1)\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde{\eta}_1 = \eta(p_1)\,, \quad \tilde{\eta}_2 = \eta(p_2)\,.$$

The string theory, $S^{\rm string}$, and spin chain, $S^{\rm chain}$, S-matrices are related as follows

$$S^{\rm chain}_{12}(p_1,p_2) = U_2(p_1) S^{\rm string}_{12}(p_1,p_2) U_1^\dagger(p_2)\,, \quad U(p) = {\rm diag}(e^{\frac{i}{2}p}, e^{\frac{i}{2}p}, 1, 1)\,. \tag{A.8}$$

Since the string theory S-matrix satisfies the usual YB equation, the spin chain S-matrix obeys the twisted YB equation with the twist determined by $U(p)$.

Let us now summarize the characteristic properties of the string S-matrix. Substituting in the supersymmetry generators $J(p,\zeta,\eta)$ the corresponding values of ζ and η from the string basis we see that the invariance condition for the string S-matrix takes the form

$$S(p_1,p_2)\Big(J(p_1)\otimes e^{\frac{i}{2}p_2} + \Sigma\otimes J(p_2)\Big) = \Big(J(p_1)\otimes \Sigma + e^{\frac{i}{2}p_1}\otimes J(p_2)\Big) S(p_1,p_2)\,,$$

where $J(p)$ are the supersymmetry generators $\{Q^a_\alpha\}$ with $\xi = \varphi = 0$, cf. eq. (3.11). From the explicit formula for the S-matrix we see that it depends not only on $x^\pm(p)$ but also on the exponential factors $e^{\frac{i}{2}p}$. While $x^\pm$ are periodic functions of p with the period 2π the exponential factors are not. As the result, the string S-matrix is not periodic under the shift $p \to p + 2\pi$, rather it exhibits the following monodromy properties

$$\begin{aligned} S_{12}(p_1, p_2 + 2\pi) &= -S_{12}(p_1,p_2)\Sigma_1\,, \\ S_{12}(p_1 + 2\pi, p_2) &= -\Sigma_2 S_{12}(p_1,p_2)\,, \end{aligned}$$

where $\Sigma_1 = \Sigma\otimes\mathbb{I}$ and $\Sigma_2 = \mathbb{I}\otimes\Sigma$. The compatibility of these equations with the invariance condition is guaranteed by the fact that all supersymmetry generators J obey the relation $\{\Sigma, J\} = 0$. Finally, we note that under the shift by 2π any of the three momenta p_1, p_2, p_3 entering the Yang-Baxter equation, this equation turns into itself because the S-matrix enjoys the following special property

$$[S_{12}(p_1,p_2), \Sigma\otimes\Sigma] = 0\,.$$

To describe the $\mathrm{AdS}_5\times\mathrm{S}^5$ light-cone string theory the S-matrix (A.7) has to be multiplied by the scalar factor, $S_0(p_1,p_2)$, of the following form [13]

$$S_0(p_1,p_2)^2 = \frac{x_2^+ - x_1^-}{x_1^+ - x_2^-}\,\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\, e^{i\theta(p_1,p_2)}\, e^{ia(p_1\epsilon_2 - p_2\epsilon_1)}\,. \tag{A.9}$$

Here the gauge-independent dressing phase $\theta(p_1,p_2)$ [4] is a two-form on the vector space of conserved charges $q_n(p)$

$$\theta(p_1,p_2) = \sum_{r=2}^{\infty}\sum_{n=0}^{\infty} c_{r,r+1+2n}(\lambda)\left(q_r(p_1)q_{r+1+2n}(p_2) - q_r(p_2)q_{r+1+2n}(p_1)\right)\,,$$

and a is the parameter of the generalized light-cone gauge [24], and $\epsilon_i = H(p_i)$ is the energy of a one-particle state. The gauge-dependent factor solves the homogeneous crossing equation [13].

The canonical $\mathfrak{su}(2|2)^2$ invariant S-matrix is the following tensor product

$$\mathcal{S}(p_1,p_2) = S_0(p_1,p_2)^2\, S(p_1,p_2)\otimes S(p_1,p_2)\,. \tag{A.10}$$

In the limit $p_1 = p_2$ the matrix $S_{12}(p_1,p_2)$ turns into $-P_{12}$, where P_{12} is the permutation, and the scalar factor $S_0(p_1,p_2)^2$ becomes -1. Thus, the full S-matrix becomes in this limit the minus permutation.

A.3 Symmetries from the Yang-Baxter equation

In this section we explain how the knowledge of the S-matrix satisfying the Yang-Baxter equation allows one to reconstruct the corresponding representation of the symmetry algebra which, in the present case, is the centrally extended $\mathfrak{su}(2|2)_{H,C}$. In our exposition we closely follow the general discussion due to Bernard and Leclair [42] which is also aimed to reveal the higher non-local symmetries of the S-matrix.

Let us introduce an associative algebra generated by the symbols $\mathrm{T}_{ij}(p,\mu)$, where $i,j=1,\ldots,4$, modulo the following relations

$$S_{12}(p_1,p_2)\mathrm{T}_1(p_1,\mu)\mathrm{T}_2(p_2,\mu) = \mathrm{T}_2(p_2,\mu)\mathrm{T}_1(p_1,\mu)S(p_1,p_2)\,. \tag{A.11}$$

Here the matrix spaces 1 and 2 are considered as auxiliary ones and we use the notation $\mathrm{T}_1 = \mathrm{T}\otimes\mathbb{I}$ and $\mathrm{T}_2 = \mathbb{I}\otimes\mathrm{T}$. The variable μ is a (spectral) parameter of the representation. The absence of cubic and higher order relations between the algebra generators is guaranteed by the Yang-Baxter equation for the S-matrix. A particular 4-dimensional representation of this algebra is provided by the S-matrix itself:

$$T_1(p_1,\mu) = S_{13}(p_1,p_3) \qquad \text{with} \qquad p_3 = \mu\,.$$

Upon expanding around a special point in the spectral parameter plane the algebra (A.11) is expected to produce the generators of both the local and non-local symmetries of the theory. In our model the only distinguished point corresponds to $p_3 = 0$. Thus, we have to understand the expansion of the Yang-Baxter equation around $p_3 = 0$.

Since the relation (3.7) is quadratic in x^+ and x^-, for $x^\pm$ as functions of p we find two different solutions. In the limit $p\to 0$ the first solution expands as

$$x^\pm = \frac{1}{gp} \pm \frac{i}{2g} + \Big(g - \frac{1}{12g}\Big)p + \cdots\,, \tag{A.12}$$

while the second one yields

$$x^\pm = -gp \mp \frac{ig}{2}p^2 + \cdots \tag{A.13}$$

In what follows we pick up the first solution because, in opposite to the second one, this solution implies that in the limit $p\to\ 0$ the variable $\eta = \frac{1}{\sqrt{g}}$ is real for a positive value of the coupling constant g, i.e. that the corresponding representation of $\mathfrak{su}(2|2)$ is unitary. Thus, taking into account eq. (A.12) the canonical $\mathfrak{su}(2|2)$-S-matrix simplifies in the limit $p_2\to 0$ to

$$S_{12}(p,0) = e^{-\frac{i}{2}p}\,\mathbb{I}\otimes(E_1^1+E_2^2) + \Sigma\otimes(E_3^3+E_4^4)\,.$$

If we put in the last formula $p = 0$ then the corresponding S-matrix turns into the graded unity. It is interesting to note that if we would take the limit $p_1 = p_2 = p$ first and then $p = 0$ we would obtain a different result which is the minus permutation. Thus, the order in which the limits are taken matters.

Consider now the Yang-Baxter equation

$$S_{12}(p_1, p_2)S_{13}(p_1, p_3)S_{23}(p_2, p_3) = S_{23}(p_2, p_3)S_{13}(p_1, p_3)S_{12}(p_1, p_2)$$

and expand it in power series around the special point $p_3 = 0$. The leading term of this expansion produces an equation

$$S_{12}(p_1, p_2)S_{13}(p_1, 0)S_{23}(p_2, 0) = S_{23}(p_2, 0)S_{13}(p_1, 0)S_{12}(p_1, p_2)$$

which is obviously satisfied because it is the Yang-Baxter equation considered for the fixed value of p_3. Next, we take the subleading equation, multiply its both sides from the left by $S_{13}^{-1}S_{23}^{-1}$ and make use of the Yang-Baxter equation. This gives the following relation

$$S_{12}\Big(S_{23}^{-1}S_{13}^{-1}\partial_3 S_{13}S_{23} + S_{23}^{-1}\partial_3 S_{23}\Big) = \Big(S_{13}^{-1}\partial_3 S_{13} + S_{13}^{-1}S_{23}^{-1}\partial_3 S_{23}S_{13}\Big)S_{12}\,, \quad \text{(A.14)}$$

where we used the shorthand notation

$$\partial_3 S_{13} \equiv \frac{\partial S_{13}(p_1, p_3)}{\partial p_3}|_{p_3=0}$$

and analogously for $\partial_3 S_{23}$. Equation (A.14) involves three matrix spaces. We project out the third space by multiplying both sides with a constant matrix Υ living in the third space and take the matrix trace over the third space. We get

$$\begin{aligned} &S_{12}\Big[\mathrm{Tr}_3\big(S_{23}^{-1}S_{13}^{-1}\partial_3 S_{13}S_{23}\Upsilon_3\big) + \mathbb{I}\otimes \mathrm{Tr}_3\big(S_{23}^{-1}\partial_3 S_{23}\Upsilon_3\big)\Big] = \\ &\qquad = \Big[\mathrm{Tr}_3\big(S_{13}^{-1}\partial_3 S_{13}\Upsilon_3\big)\otimes\mathbb{I} + \mathrm{Tr}_3\big(S_{13}^{-1}S_{23}^{-1}\partial_3 S_{23}S_{13}\Upsilon_3\big)\Big]S_{12}\,. \end{aligned} \quad \text{(A.15)}$$

The derivative of the S-matrix entering the last equation can be easily found if one takes into account the formulae

$$\frac{\partial x^{\pm}}{\partial p} = \frac{ix^{\pm 2}(1 - x^{\pm 2})}{(x^+ - x^-)(1 + x^+x^-)}\,, \qquad \frac{\partial \eta}{\partial p} = \frac{1}{2\eta}\frac{x^+ + x^-}{1 + x^+x^-}\,, \quad \text{(A.16)}$$

where $\eta = \sqrt{ix^- - ix^+}$. Evaluating (A.15) one finds that this equation is equivalent to the invariance condition (4.10) for the set of bosonic generators provided one makes a proper choice for the matrix Υ:

$$\begin{aligned} &L_a^b \leftrightarrow iE_a^b\,, \quad a \neq b, \quad a, b = 1, 2; \qquad && R_\alpha^\beta \leftrightarrow -iE_\alpha^\beta\,, \quad \alpha \neq \beta, \quad \alpha, \beta = 3, 4; \\ &L_1^1 = -L_2^2 \leftrightarrow \frac{i}{2}(E_1^1 - E_2^2)\,, && R_1^1 = -R_2^2 \leftrightarrow -\frac{i}{2}(E_3^3 - E_4^4)\,. \end{aligned}$$

Also, the Hamiltonian H is generated by $\Upsilon = i\Sigma$.

Now we would like to rederive the invarince condition (4.11) involving the supersymmetry generators. To this end we recall that S_{12} commutes with $\Sigma_1\Sigma_2$ so that we can rewrite equation (A.14) in the form

$$
\begin{aligned}
S_{12}\Big(S_{23}^{-1}S_{13}^{-1}\partial_3 S_{13}S_{23}\Sigma_1\Sigma_2+\Sigma_1 S_{23}^{-1}\partial_3 S_{23}\Sigma_2\Big) &= \\
&=\Big(S_{13}^{-1}\partial_3 S_{13}\Sigma_1\Sigma_2+S_{13}^{-1}S_{23}^{-1}\partial_3 S_{23}S_{13}\Sigma_1\Sigma_2\Big)S_{12}\,.
\end{aligned}
$$

Again we project out the third space in the last equation by multiplying both sides with a constant matrix Υ_3 and taking the trace over the third space:

$$
\begin{aligned}
S_{12}\Big[\mathrm{Tr}_3\big(S_{23}^{-1}S_{13}^{-1}\partial_3 S_{13}S_{23}\Upsilon_3\big)\Sigma_1\Sigma_2+\Sigma\otimes\mathrm{Tr}_3\big(S_{23}^{-1}\partial_3 S_{23}\Upsilon_3\big)\Sigma_2\Big] &= \\
=\Big[\mathrm{Tr}_3\big(S_{13}^{-1}\partial_3 S_{13}\Upsilon_3\big)\Sigma_1\otimes\Sigma+\mathrm{Tr}_3\big(S_{13}^{-1}S_{23}^{-1}\partial_3 S_{23}S_{13}\Upsilon_3\big)\Sigma_1\Sigma_2\Big]S_{12}\,.
\end{aligned}
$$

One can see that the last equation indeed reduces to

$$
S_{12}\big(Q^a_\alpha(p_1)\otimes e^{\frac{i}{2}p_2}+\Sigma\otimes Q^a_\alpha(p_2)\big)\;=\;\big(Q^a_\alpha(p_1)\otimes\Sigma+e^{\frac{i}{2}p_1}\otimes Q^a_\alpha(p_2)\big)S_{12}
$$

provided one makes the following choice of Υ: $Q^a_\alpha\leftrightarrow -iE^a_\alpha$, where $a=1,2$ and $\alpha=3,4$. Analogously, $Q^{\dagger\alpha}_a\leftrightarrow iE^\alpha_a$. Thus, we have shown that the symmetry algebra $\mathfrak{su}(2|2)_{H,C}$ (more precisely, its fundamental representation) arises upon expansion of the S-matrix around the special point $p=0$.

One can expand the Yang-Baxter equation further and find the fundamental representation for the higher (non-local) symmetries commuting with the S-matrix. We will not proceed with this expansion because, as it is clear from the our construction, the form of the S-matrix is fixed uniquely up to a scalar phase from the requirement of the $\mathfrak{su}(2|2)_{H,C}$-symmetry, and the additional nonlocal symmetries (in the fundamental representation) do not seem to lead to any additional equations for the scalar phase.

A.4 Hopf algebra interpretation

In this appendix we discuss a Hopf algebra interpretation[11] of the S-matrix invariance condition (4.11). The form of the comultiplication depends on the choice of the basis of the fundamental representation, and we restrict our attention to the symmetric string theory basis.

Fist of all, we stress that to define a nontrivial co-product one should consider not the centrally-extended super-Lie algebra $\mathfrak{su}(2|2)_{H,C}$ but a unitary graded associative algebra $\mathcal{A}$ generated by the even rotation generators $\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$, the odd supersymmetry generators $\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}^{\dagger\alpha}_a$ and two central elements $\mathbf{H}$ and $\mathbf{P}$ subject to the algebra relations (3.1) with the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ expressed through the world-sheet momentum $\mathbf{P}$ as follows

$$
\mathbf{C}=ig\,(e^{i\mathbf{P}}-\mathrm{id})\,,\quad \mathbf{C}^\dagger=-ig\,(e^{-i\mathbf{P}}-\mathrm{id})\,. \tag{A.17}
$$

[11] We would like to thank J. Plefka and F. Spill for an interesting discussion of the subject.

Then the comultiplication can be defined on the generators as follows

$$\begin{aligned}
\Delta(\mathbf{J}) &= \mathbf{J}\otimes \mathrm{id} + \mathrm{id}\otimes \mathbf{J} \quad \text{for any even generator}\,,\\
\Delta(\mathbf{Q}_\alpha{}^a) &= \mathbf{Q}_\alpha{}^a \otimes e^{i\mathbf{P}/2} + \mathrm{id}\otimes \mathbf{Q}_\alpha{}^a\,, \qquad\qquad (A.18)\\
\Delta(\mathbf{Q}_a^{\dagger\alpha}) &= \mathbf{Q}_a^{\dagger\alpha} \otimes e^{-i\mathbf{P}/2} + \mathrm{id}\otimes \mathbf{Q}_a^{\dagger\alpha}\,.
\end{aligned}$$

Here we use the graded tensor product, that is for any algebra elements a, b, c, d

$$(a\otimes b)(c\otimes d) = (-1)^{\epsilon(b)\epsilon(c)}(ac\otimes bd),$$

where $\epsilon(a) = 0$ if a is an even element, and $\epsilon(b) = -1$ if a is an odd element of the algebra $\mathcal{A}$. These comultiplication rules are equivalent (up to a twist and some redefinitions of the supersymmetry generators and the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$) to the ones discussed in [34]. In addition to the co-product, we also use the standard definitions of unit and co-unit.

Let us show that the comultiplication agrees with the form of the two-particle structure constants appearing in (4.11). Let V be the fundamental representation of the algebra $\mathcal{A}$. Obviously, it is isomorphic to the representation $V(p,1)$ of $\mathfrak{su}(2|2)_{H,C}$. The vector space V has a natural grading. The action of, say, the supersymmetry generators $\mathbf{Q}_\alpha{}^a$ on the tensor product $V\otimes V$ is given by the comultiplication (A.18)

$$\begin{aligned}
\Delta(\mathbf{Q}_\alpha{}^a)\cdot e_M\otimes e_N &= (\mathbf{Q}_\alpha{}^a\otimes e^{i\mathbf{P}/2} + \mathrm{id}\otimes \mathbf{Q}_\alpha{}^a)\cdot e_M\otimes e_N \qquad (A.19)\\
&= \mathbf{Q}_\alpha{}^a\cdot e_M\otimes e^{i\mathbf{P}/2}\cdot e_N + \Sigma\cdot e_M\otimes \mathbf{Q}_\alpha{}^a\cdot e_N\,.
\end{aligned}$$

Now one can recognize that the two-particle representation coincides with the one appearing on the l.h.s. of (4.11).

The antipode can be easily found by using (A.18). For any even generator including $\mathbf{H}$ and $\mathbf{P}$ it acts as

$$S(\mathbf{J}) = -\mathbf{J}\,,$$

while on supersymmetry generators it is defined as

$$S(\mathbf{Q}_\alpha{}^a) = -\mathbf{Q}_\alpha{}^a\, e^{-i\mathbf{P}/2}\,, \quad S(\mathbf{Q}_a^{\dagger\alpha}) = -\mathbf{Q}_a^{\dagger\alpha}\, e^{i\mathbf{P}/2}\,.$$

Finally, let us mention that the ZF algebra does not have a Hopf algebra structure.

A.5 The twisted Yang-Baxter equation

In this appendix we derive the most general form of the twisted YB equation.

Every fundamental representation V is parametrized by momentum p and the label ζ (and η which will not be always shown explicitly in the consideration below) which may be p-dependent. The invariant S-matrix is defined as

$$S_{12}(p_1,\zeta_1;p_2,\zeta_2): \qquad V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\to V(p_1,\widetilde{\zeta}_1)\otimes V(p_2,\widetilde{\zeta}_2)\,,$$

where the vector spaces V are considered as columns. Then, we must require the two tensor product spaces $V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)$ and $V(p_1,\widetilde{\zeta}_1)\otimes V(p_2,\widetilde{\zeta}_2)$ be isomorphic to each other and carry the same representation of the centrally extended $\mathfrak{su}(2|2)$. This implies that both spaces must have the same values of the central charges and satisfy the shortening conditions [8]. Then, one finds

$$\widetilde{\zeta}_1 = e^{ip_2}\zeta_2 \frac{(1-e^{ip_1})\zeta_1 + (1-e^{ip_2})\zeta_2}{e^{ip_1}\zeta_1 + e^{ip_2}\zeta_2 - e^{i(p_1+p_2)}(\zeta_1+\zeta_2)} \tag{A.20}$$

$$\widetilde{\zeta}_2 = e^{ip_1}\zeta_1 \frac{(1-e^{ip_1})\zeta_1 + (1-e^{ip_2})\zeta_2}{e^{ip_1}\zeta_1 + e^{ip_2}\zeta_2 - e^{i(p_1+p_2)}(\zeta_1+\zeta_2)} \tag{A.21}$$

In particular, we see that

$$\frac{\widetilde{\zeta}_2}{\widetilde{\zeta}_1} = \frac{e^{ip_1}\zeta_1}{e^{ip_2}\zeta_2}\,.$$

The S-matrix satisfies the invariance condition (4.11) which in our case takes the following form for bosonic generators

$$S_{12}\big(J(p_1;\zeta_1,\eta_1)\otimes\mathbb{I} + \mathbb{I}\otimes J(p_2;\zeta_2,\eta_2)\big) = \big(J(p_1;\widetilde{\zeta}_1,\tilde{\eta}_1)\otimes\mathbb{I} + \mathbb{I}\otimes J(p_2;\widetilde{\zeta}_2,\tilde{\eta}_2)\big)S_{12}\,,$$

and for fermionic generators

$$S_{12}\big(J(p_1;\zeta_1,\eta_1)\otimes\mathbb{I} + \Sigma\otimes J(p_2;\zeta_2,\eta_2)\big) = \big(J(p_1;\widetilde{\zeta}_1,\tilde{\eta}_1)\otimes\Sigma + \mathbb{I}\otimes J(p_2;\widetilde{\zeta}_2,\tilde{\eta}_2)\big)S_{12}\,,$$

where $\Sigma = \text{diag}(1,1,-1,-1)$ takes into account the fermionic statistics. These equations can be easily solved and lead to a unique (up to a scalar factor) solution. In terms of standard matrix unities this solution reads as

$$\begin{aligned}
S(p_1,\zeta_1;p_2,\zeta_2) = {} & a_1\Big(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\Big) \\
& + a_2\Big(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\Big) \\
& + a_3\Big(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\Big) \\
& + a_4\Big(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\Big) \\
& + a_5\Big(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\Big) \\
& + a_6\Big(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\Big) \\
& + a_7\Big(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\Big) \\
& + a_8\Big(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\Big) \\
& + a_9\Big(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\Big) \\
& + a_{10}\Big(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\Big)\,,
\end{aligned} \tag{A.22}$$

where the coefficients a_i are found to be

$$
\begin{aligned}
a_1 &= \frac{\big(x_1^+\zeta_1 - x_2^+\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^- x_2^-(\zeta_1+\zeta_2)\big)\eta_1\eta_2}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)\tilde\eta_1\tilde\eta_2}\,,\\
a_2 &= \frac{\big(x_1^- - x_1^+\big)\big(x_2^- - x_2^+\big)\big(x_1^+\zeta_1 + x_2^+\zeta_2\big)\big(x_1^+ x_2^-\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)\eta_1\eta_2}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)^2\tilde\eta_1\tilde\eta_2}\,,\\
a_3 &= -1\,,\\
a_4 &= \frac{\big(x_1^+ - x_1^-\big)\big(x_2^- - x_2^+\big)\big(x_2^-\zeta_1 + x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^- x_2^-(\zeta_1+\zeta_2)\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)}\,,\\
a_5 &= -\frac{\zeta_2\big((x_1^- x_2^- x_1^+ + x_2^- x_1^+ x_2^+ - x_1^- x_2^+ x_2^- - x_1^- x_1^+ x_2^+)\zeta_1 + x_1^- x_2^+(x_1^- - x_2^- - x_1^+ + x_2^+)\zeta_2\big)\eta_1}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)\tilde\eta_1}\,,\\
a_6 &= -\frac{\zeta_1\big(x_2^- x_1^+(x_1^- - x_2^- - x_1^+ + x_2^+)\zeta_1 + (x_1^- x_2^- x_1^+ + x_2^- x_1^+ x_2^+ - x_1^- x_2^+ x_2^- - x_1^- x_1^+ x_2^+)\zeta_2\big)\eta_2}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)\tilde\eta_2}\,,\\
a_7 &= \frac{\big(x_1^- - x_1^+\big)\big(x_2^- - x_2^+\big)\big(x_1^- x_2^- x_1^+ + x_2^- x_1^+ x_2^+ - x_1^- x_2^+ x_2^- - x_1^- x_1^+ x_2^+\big)\big(x_2^- x_1^+\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)\zeta_1\zeta_2}{x_1^- x_2^-\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)^2\tilde\eta_1\tilde\eta_2}\,,
\end{aligned}
$$

$$
\begin{aligned}
a_8 &= -\frac{x_1^- x_2^-\big(x_1^- - x_2^- - x_1^+ + x_2^+\big)\big(x_2^- x_1^+\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)\eta_1\eta_2}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^- x_2^-(\zeta_1+\zeta_2)\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)}\,,\\
a_9 &= \frac{\big(x_1^+ - x_1^-\big)\big(x_2^- x_1^+\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)\eta_2}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)\tilde\eta_1}\,,\\
a_{10} &= \frac{\big(x_2^+ - x_2^-\big)\big(x_2^- x_1^+\zeta_1^2 - x_1^- x_2^+\zeta_2^2\big)\eta_1}{\big(x_2^-\zeta_1 - x_1^-\zeta_2\big)\big(x_2^- x_1^+\zeta_1 + x_1^- x_2^+\zeta_2 - x_1^+ x_2^+(\zeta_1+\zeta_2)\big)\tilde\eta_2}\,.
\end{aligned}
$$

If $\zeta_1 = e^{ip_2}\,,\ \zeta_2 = 1$ then the S-matrix coincides with (A.7).

To derive the twisted YB equation for the S-matrix we start from the sequence of three spaces as

$$V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\otimes V(p_3,\zeta_3)\,.$$

We would like to bring it to the form $V(p_1,\widetilde{\widetilde{\zeta}}_1)\otimes V(p_2,\widetilde{\widetilde{\zeta}}_2)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3)$ by permuting the spaces with the help of the S-matrix. This can be done it two different ways. The first way is to perform the following three successive operations

$$S_{12}(p_1,\zeta_1;p_2,\zeta_2):\quad V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\otimes V(p_3,\zeta_3)\to V(p_1,\widetilde{\zeta}_1^\ell)\otimes V(p_2,\widetilde{\zeta}_2^\ell)\otimes V(p_3,\zeta_3)$$

$$S_{13}(p_1,\widetilde{\zeta}_1^\ell;p_3,\zeta_3):\quad V(p_1,\widetilde{\zeta}_1^\ell)\otimes V(p_2,\widetilde{\zeta}_2^\ell)\otimes V(p_3,\zeta_3)\to V(p_1,\widetilde{\widetilde{\zeta}}_1{}^\ell)\otimes V(p_2,\widetilde{\zeta}_2^\ell)\otimes V(p_3,\widetilde{\zeta}_3^\ell)$$

$$S_{23}(p_2,\widetilde{\zeta}_2^\ell;p_3,\widetilde{\zeta}_3^\ell):\quad V(p_1,\widetilde{\widetilde{\zeta}}_1{}^\ell)\otimes V(p_2,\widetilde{\zeta}_2^\ell)\otimes V(p_3,\widetilde{\zeta}_3^\ell)\to V(p_1,\widetilde{\widetilde{\zeta}}_1{}^\ell)\otimes V(p_2,\widetilde{\widetilde{\zeta}}_2{}^\ell)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3{}^\ell)$$

Here to find expressions for $\widetilde{\zeta}_i^\ell$ and $\widetilde{\widetilde{\zeta}}_i{}^\ell$ we need to use the formulas (A.20) and (A.21). Thus we have applied the following scattering operator

$$S_{23}(p_2,\widetilde{\zeta}_2^\ell;p_3,\widetilde{\zeta}_3^\ell)S_{13}(p_1,\widetilde{\zeta}_1^\ell;p_3,\zeta_3)S_{12}(p_1,\zeta_1;p_2,\zeta_2)$$

and ended up with the space

$$V(p_1,\widetilde{\widetilde{\zeta}}_1{}^\ell)\otimes V(p_2,\widetilde{\widetilde{\zeta}}_2{}^\ell)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3{}^\ell)$$

The second way to achieve the same final space is to perform another three successive operations

$$\begin{aligned}
&S_{23}(p_2,\zeta_2;p_3,\zeta_3): \quad V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)\otimes V(p_3,\zeta_3) \to V(p_1,\zeta_1)\otimes V(p_2,\widetilde{\zeta}_2^r)\otimes V(p_3,\widetilde{\zeta}_3^r)\\
&S_{13}(p_1,\zeta_1;p_3,\widetilde{\zeta}_3^r): \quad V(p_1,\zeta_1)\otimes V(p_2,\widetilde{\zeta}_2^r)\otimes V(p_3,\widetilde{\zeta}_3^r) \to V(p_1,\widetilde{\zeta}_1^r)\otimes V(p_2,\widetilde{\zeta}_2^r)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3{}^r)\\
&S_{12}(p_1,\widetilde{\zeta}_1^r;p_2,\widetilde{\zeta}_2^r): \ V(p_1,\widetilde{\zeta}_1^r)\otimes V(p_2,\widetilde{\zeta}_2^r)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3{}^r) \to V(p_1,\widetilde{\widetilde{\zeta}}_1{}^r)\otimes V(p_2,\widetilde{\widetilde{\zeta}}_2{}^r)\otimes V(p_3,\widetilde{\widetilde{\zeta}}_3{}^r)
\end{aligned}$$

Remarkably, $\widetilde{\widetilde{\zeta}}_i{}^\ell = \widetilde{\widetilde{\zeta}}_i{}^r$, and therefore in both cases we ended up with the same space. Thus, the exact form of the twisted Yang-Baxter equation is

$$\begin{aligned}
S_{23}(p_2,\widetilde{\zeta}_2^\ell;p_3,\widetilde{\zeta}_3^\ell)&S_{13}(p_1,\widetilde{\zeta}_1^\ell;p_3,\zeta_3)S_{12}(p_1,\zeta_1;p_2,\zeta_2) = \\
&= S_{12}(p_1,\widetilde{\zeta}_1^r;p_2,\widetilde{\zeta}_2^r)S_{13}(p_1,\zeta_1;p_3,\widetilde{\zeta}_3^r)S_{23}(p_2,\zeta_2;p_3,\zeta_3)\,.
\end{aligned} \tag{A.23}$$

If we choose the parameters ζ_i as follows

$$\zeta_3 = 1\,, \quad \zeta_2 = e^{ip_2}\,, \quad \zeta_1 = e^{i(p_2+p_3)}\,,$$

and the string theory choice of the parameters η_i then the twisted YB equation (A.23) becomes the usual one. If we choose the spin chain parameters η_i then we reproduce the twisted YB equation (5.19).

We conclude this appendix by noting that sometimes it is convenient to visualize the S-matrix as an explicit 16×16 matrix

$$S(p_1,\zeta_1;p_2,\zeta_2) \equiv \left(\begin{array}{cccc|cccc|cccc|cccc}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_1+a_2 & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & -a_7 & 0 & 0 & a_7 & 0\\
0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0\\
\hline
0 & -a_2 & 0 & 0 & a_1+a_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 & -a_7 & 0\\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0\\
\hline
0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0\\
0 & -a_8 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 & a_3+a_4 & 0 & 0 & -a_4 & 0\\
\hline
0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0\\
0 & a_8 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 & 0 & 0 & a_3+a_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3
\end{array}\right)$$

In section 5 the coefficients A, …, L were introduced to make a comparison with the near plane-wave S-matrix $\mathbb{S}_{12}^{\rm KMRZ}$ computed in [30]. These coefficients are expressed via the coefficients a_i above as follows

$$\begin{array}{lllll}
\mathrm{A} = a_1 + a_2\,, & \mathrm{B} = -a_2\,, & \mathrm{C} = -a_8\,, & \mathrm{D} = a_3 + a_4\,, & \mathrm{E} = -a_4\,,\\
\mathrm{F} = -a_7\,, & \mathrm{G} = a_5\,, & \mathrm{H} = a_{10}\,, & \mathrm{K} = a_9\,, & \mathrm{L} = a_6\,.
\end{array} \tag{A.24}$$

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Finally, we note that the permutation matrix P corresponds to the choice

$$a_1 = a_3 = a_9 = a_{10} = 1\,, \qquad a_2 = a_4 = -1\,, \qquad a_5 = a_6 = a_7 = a_8 = 0\,,$$

while the graded permutation $\mathcal{P}$ is given by

$$a_1 = a_4 = a_9 = a_{10} = 1\,, \qquad a_2 = a_3 = -1\,, \qquad a_5 = a_6 = a_7 = a_8 = 0\,.$$

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