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Separability of Hamilton-Jacobi and Klein-Gordon equations in general Kerr-NUT-AdS spacetimes

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Abstract: We demonstrate the separability of the Hamilton-Jacobi and scalar field equations in general higher dimensional Kerr-NUT-AdS spacetimes. No restriction on the parameters characterizing these metrics is imposed.

Keywords: Black Holes, Large Extra Dimensions.

Contents

1. Introduction

The study of the separability of the Hamilton-Jacobi and the corresponding scalar field equations in a curved spacetime has a long history. Robertson [1] and Eisenhart [2] discussed general conditions for such a separability in spaces which admit a complete set of mutually orthogonal families of hypersurfaces. An important class of 4-dimensional separable spacetimes, including several type D metrics, was found by Carter [3]. Carter also proved the separability of the Hamilton-Jacobi and the scalar field equation in the Kerr metric [4]. It was demonstrated in [5] that this separability follows from the existence of a Killing tensor. This result was generalized later, namely, it was shown that Killing and Killing-Yano tensors play an important role in the separability theory (see, e.g., [6 – 10]).

In the present paper we prove the separability of the Hamilton-Jacobi and scalar field equations in the general ($D \geq 4$) Kerr-NUT-AdS spacetimes [11]. These solutions were obtained as a generalization of the metrics for the rotating higher dimensional black holes with a cosmological constant [12 – 14], which, in their turn, are generalizations of the Myers-Perry solution [15]. There are several publications devoted to the separation of variables for this class of metrics. However, all the results obtained up to now assume either a restriction on the number of dimensions [16 – 18] or special properties of the parameters which characterize the solution [14, 19 – 24]. The separation of variables which we prove in this paper is valid in the general Kerr-NUT-AdS spacetime in any number of dimensions and without any restriction on the parameters of the metric. We also discuss the relation of the separation constants with the conserved quantities connected with the Killing-Yano and Killing tensors recently discovered for this class of the metrics [25 – 28].

2. Kerr-NUT-AdS metrics and their properties

Our starting point is the general higher dimensional Kerr-NUT-AdS metric obtained in [11]. The metric can be written using D coordinates x^a which naturally splits into two groups. Radial and latitude coordinates are denoted as x_μ and labelled by the Greek indices $\mu, \nu = 1, \dots, n$, $n = [D/2]$, i.e., $x_\mu = x^\mu$. Time and azimuthal coordinates $\psi_k = x^{n+1+k}$ are indexed by the Latin indices from the middle of the alphabet, $k, l = 0, \dots, m$, $m = D - n - 1$. We use the Einstein summation convention only for the indices $a, b, \dots$ running over all coordinates. For the convenience we also introduce $\varepsilon = D - 2n$. In these coordinates the metric and its inverse read[1]

$$ds^2 = \sum_{\mu=1}^{n} \left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \left(\sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2, \tag{2.1}$$

and

$$(\partial_s)^2 = \sum_{\mu=1}^{n} \left[Q_\mu (\partial_\mu)^2 + \frac{1}{Q_\mu U_\mu^2} \left(\sum_{k=0}^{m} (-x_\mu^2)^{n-1-k} \partial_k \right)^2 \right] - \frac{\varepsilon}{cA^{(n)}} (\partial_n)^2 \, . \tag{2.2}$$

Here,

$$\begin{aligned}
&Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2), \qquad c = \prod_{k=1}^{m} a_k^2, \\
&X_\mu = (-1)^{1-\varepsilon} \frac{1 + \lambda x_\mu^2}{x_\mu^{2\varepsilon}} \prod_{k=1}^{m} (a_k^2 - x_\mu^2) + 2M_\mu (-x_\mu)^{1-\varepsilon}, \\
&A_\mu^{(k)} = \sum_{\substack{\nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2, \qquad A^{(k)} = \sum_{\nu_1 < \dots < \nu_k} x_{\nu_1}^2 \dots x_{\nu_k}^2 \, .
\end{aligned} \tag{2.3}$$

The parameters (M_μ, a_k) are related to the mass, NUT parameters, and angular momenta, λ is proportional to the cosmological constant.

The metric (2.1) is an Einstein space obeying the equation

$$R_{ab} = (D - 1)\lambda g_{ab}. \tag{2.4}$$

It possesses $m + 1 = D - n$ Killing vectors ∂_k, as well as the principal Killing-Yano tensor of rank $(D - 2)$ which generates a full set of Killing-Yano and Killing tensors, making the geodesic motion completely integrable [26 − 28].

The aim of this paper is to demonstrate that in the coordinates x_μ, ψ_k, both the Hamilton-Jacobi and Klein-Gordon equations separate. To prove this we shall need a set of algebraic relations which are valid for quantities which enter the metric (2.1). It is useful to introduce quantities

$$U \equiv \prod_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{n} (x_\mu^2 - x_\nu^2) \, , \quad \Omega_\mu \equiv \frac{U}{U_\mu} \, , \tag{2.5}$$

[1] The form (2.1) of the metric is actually an analytical continuation related to the physical metric by a simple Wick rotation, see [11].

which satisfy the important identities

$$\sum_{\mu=1}^{n} x_\mu^{2(n-1)} \mathit{\Omega}_\mu = (-1)^{n-1} U \ , \tag{2.6a}$$

$$\sum_{\mu=1}^{n} x_\mu^{2k} \, \mathit{\Omega}_\mu = 0 \quad \text{for} \quad k = 0, \dots, n-2 \ , \tag{2.6b}$$

$$\sum_{\mu=1}^{n} \frac{1}{x_\mu^2} \, \mathit{\Omega}_\mu = \frac{U}{A^{(n)}} \ , \tag{2.6c}$$

$$\sum_{\mu=1}^{n} \frac{A_\mu^{(k)}}{x_\mu^2} \mathit{\Omega}_\mu = \frac{A^{(k)}}{A^{(n)}} U \quad \text{for} \quad k = 0, \dots, n-1 \ , \tag{2.6d}$$

and

$$\partial_\mu \mathit{\Omega}_\mu = 0 \ . \tag{2.7}$$

The first two identities follow from the fact that the matrix $B^\mu_{(k)} = (-x_\mu^2)^{n-1-k}/U_\mu$ is the inverse of $A_\mu^{(k)}$,

$$\sum_{k=0}^{n-1} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} A_\nu^{(k)} = \delta_\mu^\nu \ , \quad \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} A_\mu^{(l)} = \delta_k^l \tag{2.8}$$

(set $l = 0$ in the last expression), (2.6c) follows from (2.6a) by substitution $x_\mu \to 1/x_\mu$, and (2.6d) can be verified using (2.6c), (2.8) and $A_\mu^{(k)} = A^{(k)} - x_\mu^2 A_\mu^{(k-1)}$. The identity (2.7) is obvious.

The function U is simply related to the determinant of the metric

$$g = \det(g_{ab}) = \left(-cA^{(n)}\right)^\varepsilon U^2 \ . \tag{2.9}$$

3. Separability of the Hamilton-Jacobi equation

The Hamilton-Jacobi equation for geodesic motion on a manifold with metric g_{ab} has the form

$$\frac{\partial S}{\partial \lambda} + g^{ab} \, \partial_a S \, \partial_b S = 0 \ . \tag{3.1}$$

Here λ denotes an 'external' time which turns out to be an affine parameter of the corresponding geodesic motion. We want to demonstrate that in the background (2.1) the classical action S allows a separation of variables

$$S = -w\lambda + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{m} \Psi_k \psi_k \tag{3.2}$$

with functions $S_\mu(x_\mu)$ of a single argument x_μ.

Substituting (3.2) into the Hamilton-Jacobi equation (3.1) and multiplying by U introduced in (2.5), we obtain

$$\sum_{\mu=1}^{n} \mathit{\Omega}_\mu F_\mu = wU + \varepsilon \frac{\Psi_n^2}{c} \frac{U}{A^{(n)}} \ , \tag{3.3}$$

where F_μ is a function of x_μ only,

$$F_\mu = X_\mu S'^2_\mu + \frac{1}{X_\mu}\left(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\,\Psi_k\right)^2 . \tag{3.4}$$

Here, the prime denotes the derivative of S_μ with respect to its single argument x_μ. Thanks to the identities (2.6), the equation (3.3) is satisfied if the functions F_μ have the form

$$F_\mu = \sum_{k=0}^{m} c_k\,(-x_\mu^2)^{n-1-k} , \tag{3.5}$$

where c_k, $k = 1, \dots, n-1$ are arbitrary constants, $c_0 = w$, and the constant c_n, which is present only in odd number of dimensions, is related to Ψ_n as

$$c_n = -\frac{\Psi_n^2}{c} . \tag{3.6}$$

The condition (3.5) leads to equations for S'_μ

$$S'^2_\mu = -\frac{1}{X_\mu^2}\left(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\Psi_k\right)^2 + \frac{1}{X_\mu}\sum_{k=0}^{m} c_k\,(-x_\mu^2)^{n-1-k}, \tag{3.7}$$

which can be solved by quadratures. Notice that in odd dimensions there is an additional term in which c_n is not an independent constant, cf. eq. (3.6).

Thus we have shown that Hamilton-Jacobi equation (3.1) in the gravitational background (2.1) can be solved by the classical action S in the separated form (3.2) with S_μ satisfying (3.7). The solution contains D constants, namely $c_0 = w$, c_1, …, c_{n-1}, and Ψ_0, …, Ψ_m.

The gradient of S gives the momentum $p_a = \partial_a S$. Substituting our expression for S we obtain p_a in terms of the constants c_k and Ψ_k. These relations can be inverted. Clearly, $\Psi_k = p_k$ are constants linear in the momentum generated by Killing vectors. To evaluate c_k we rewrite (3.5) as

$$\frac{F_\mu}{U_\mu} - \varepsilon\frac{p_n^2}{cU_\mu x_\mu^2} = \sum_{k=0}^{n-1} c_k \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} . \tag{3.8}$$

It can be inverted using (2.8). Employing the expression for c_n with $\Psi_n = p_n$ and the identity (2.6d) we obtain

$$c_k = \sum_{\mu=1}^{n} A_\mu^{(k)}\frac{F_\mu}{U_\mu} - \varepsilon p_n^2\,\frac{A^{(k)}}{cA^{(n)}} , \tag{3.9}$$

where F_μ is given by (3.4) with p_μ and p_k substituted for S'_μ and Ψ_k, respectively.

We thus found that the constants c_k are quadratic in the momenta p_a (for example, for $k = 0$ we get $w = c_0 = g^{ab}p_ap_b$). It can be shown that they are the same as the constants introduced recently using the Killing-Yano tensor and that they are generated by second rank Killing tensors [28].

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4. Separability of the Klein-Gordon equation

The behavior of a massive scalar field Φ in the gravitational background g_{ab} is governed by the Klein-Gordon equation

$$\Box\Phi = \frac{1}{\sqrt{|g|}}\,\partial_a(\sqrt{|g|}g^{ab}\partial_b\Phi) = m^2\Phi. \tag{4.1}$$

This equation remains valid for the non-minimal coupling case as well. The term ξR is constant in the Einstein spaces and can be included into the definition of m^2.

Now, we demonstrate that the Klein-Gordon equation (4.1) in the background (2.1) allows a multiplicative separation of variables

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu)\prod_{k=0}^{m} e^{i\Psi_k\psi_k}. \tag{4.2}$$

This equation has the following explicit form

$$\sqrt{|g|}m^2\Phi = \sum_{\mu=1}^{n}\partial_\mu\left(\frac{\sqrt{|g|}}{U_\mu}X_\mu\partial_\mu\Phi\right) + \sum_{\mu=1}^{n}\frac{\sqrt{|g|}}{U_\mu X_\mu}\left(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}\partial_k\right)^2\Phi - \varepsilon\frac{\sqrt{|g|}}{cA^{(n)}}\partial_n^2\Phi\,. \tag{4.3}$$

Here we used the quasidiagonal property of the inverse metric g^{ab} and the fact that ∂_k are Killing vectors. We further notice that

$$\sqrt{|g|} \propto UP^\varepsilon, \quad P \equiv \prod_{\mu=1}^{n} x_\mu, \tag{4.4}$$

where "$\propto$" means equality up to a constant factor (which can be ignored in eq. (4.3)). Using the identities (2.6a), (2.6c), (2.7) and the definition of $\mathit{\Omega}_\mu$ we find that (4.3) gives

$$\begin{aligned}\sum_{\mu=1}^{n}\mathit{\Omega}_\mu&\left[(-1)^n m^2 x_\mu^{2(n-1)}\Phi + \partial_\mu(P^\varepsilon X_\mu\partial_\mu\Phi)/P^\varepsilon\right]\\ &+\sum_{\mu=1}^{n}\left[\frac{\mathit{\Omega}_\mu}{X_\mu}\left(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}\partial_k\right)^2\Phi - \frac{\varepsilon\mathit{\Omega}_\mu}{cx_\mu^2}\partial_n^2\Phi\right] = 0\,.\end{aligned} \tag{4.5}$$

Using the ansatz (4.2) we find

$$\partial_k\Phi = i\Psi_k\Phi, \quad \partial_\mu\Phi = \frac{R'_\mu}{R_\mu}\Phi, \quad \partial_\mu^2\,\Phi = \frac{R''_\mu}{R_\mu}\,\Phi, \tag{4.6}$$

and the Klein-Gordon equation (4.5) takes the form

$$\sum_{\mu=1}^{n}\mathit{\Omega}_\mu G_\mu\Phi = 0, \tag{4.7}$$

where G_μ is function of x_μ only,

$$G_\mu = (-1)^n m^2 x_\mu^{2(n-1)} + \frac{R'_\mu}{R_\mu}\left(X'_\mu + \varepsilon\frac{X_\mu}{x_\mu}\right) + X_\mu\frac{R''_\mu}{R_\mu}$$
$$-\frac{1}{X_\mu}\left(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}\Psi_k\right)^2 + \frac{\varepsilon\Psi_n^2}{cx_\mu^2}. \tag{4.8}$$

As earlier, the prime means the derivative of functions R_μ and X_μ with respect to their single argument x_μ. Employing the identity (2.6b) we realize that (4.7) is automatically satisfied when

$$G_\mu = \sum_{k=1}^{n-1} b_k(-x_\mu^2)^{n-1-k}\ , \tag{4.9}$$

where b_k are arbitrary constants.

Therefore we have demonstrated that the Klein-Gordon equation (4.1) in the background (2.1) allows a multiplicative separation of variables (4.2), where functions $R_\mu(x_\mu)$ satisfy the ordinary second order differential equations

$$\left(X_\mu R'_\mu\right)' + \varepsilon\frac{X_\mu}{x_\mu}R'_\mu - \frac{R_\mu}{X_\mu}\left(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\Psi_k\right)^2 - \sum_{k=0}^{m} b_k(-x_\mu^2)^{n-1-k}R_\mu = 0\ . \tag{4.10}$$

Here $b_0 = m^2$, $b_1, \dots, b_{n-1}$ are arbitrary separation constants. The constant b_n is present only in an odd number of spacetime dimensions and is related to the constants Ψ_n and c through

$$b_n = \frac{\Psi_n^2}{c}. \tag{4.11}$$

We expect that the separation constants are related to the Killing tensors obtained in [28].

5. Discussion

We demonstrated the separability of the Hamilton-Jacobi and the scalar field equations in the general (higher-dimensional) Kerr-NUT-AdS spacetime. For particle motion the separability implies that the corresponding equations of motion can be written in the first order form (3.7). In the Klein-Gordon case we obtained a set of ordinary second order differential equations (4.10). The problem to solve them is usually much simpler. Even when some of these equations cannot be solved in terms of known elementary or special functions, one can always use numerical methods. The numerical integration of ordinary differential equations can be performed very effectively.

In the present paper we established the separability property by 'brute force' — by writing the corresponding equations in a special coordinate system. As we already mentioned, the constants of separation are directly related to the existing complete set of the second rank Killing tensors [28]. It would be interesting to derive the separability property by starting with the general symmetry properties of the considered spacetime, using, for example, the results of [9].

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The separation of variables in the scalar field equation can be used for the study of different interesting problems. One of them is the calculation of the bulk Hawking radiation of higher dimensional rotating black holes. As it was shown by Teukolsky [29, 30] in the 4D Kerr metric, not only the scalar field equation allows separation of variables, but the equations of the other (massless) fields with non vanishing spin can also be decoupled and separated. An interesting question is whether the existing symmetry connected with a complete set of the Killing tensors in the general Kerr-NUT-AdS spacetime makes such a decoupling and separation of the higher spin fields equations possible.

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