Boundary conditions and dualities: vector fields in AdS/CFT

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ABSTRACT: In AdS, scalar fields with masses slightly above the Breitenlohner-Freedman bound admit a variety of possible boundary conditions which are reflected in the Lagrangian of the dual field theory. Generic small changes in the AdS boundary conditions correspond to deformations of the dual field theory by multi-trace operators. Here we extend this discussion to the case of vector gauge fields in the bulk spacetime using the results of Ishibashi and Wald [hep-th/0402184]. As in the context of scalar fields, general boundary conditions for vector fields involve multi-trace deformations which lead to renormalization-group flows. Such flows originate in ultra-violet CFTs which give new gauge/gravity dualities. At least for AdS_4/CFT_3, the dual of the bulk photon appears to be a propagating gauge field instead of the usual R-charge current. Applying similar reasoning to tensor fields suggests the existence of a duality between string theory on AdS_4 and a quantum gravity theory in three dimensions.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence.

Contents

1. Introduction

In the AdS/CFT correspondence, boundary conditions for bulk fields are related to the specification of the dual CFT [1–4]. In particular, small changes in the bulk boundary conditions correspond to deformations of the dual CFT Lagrangian. Bulk scalar fields in AdS$_{d+1}$ with mass in the range $-d^2/4 \leq m^2 < -d^2/4 + 1$ provide a particularly interesting example of this correspondence. As indicated by the work of Breitenlohner and Freedman [5, 6], such scalar fields admit a variety of possible boundary conditions. In particular, one may fix either the faster or slower falloff part of the scalar field at infinity.

The two resulting bulk theories correspond to two different dual CFTs, in which the field ϕ is dual to operators of dimensions Δ_- and $\Delta_+ = d - \Delta_-$ respectively, where $d/2 \geq \Delta_- > d/2 - 1$. In [7, 8], it was observed that a general linear boundary condition, relating the faster falloff part to the slower, corresponds to a double-trace deformation,

adding a term $f\mathcal{O}^2$ to the Lagrangian of the CFT. Starting from the Δ_- CFT, this is is a relevant deformation, which will produce an renormalization-group flow which is expected to end at the Δ_+ CFT in the IR; evidence for this picture has been obtained in [9 – 11]. Since a double-trace operator corresponds to a multiparticle state, the double-trace deformations in the CFT have also been related to worldsheet non-locality in the bulk string theory [12, 13].

In the present work, we conduct a similar analysis for vector fields. The possibility of general boundary conditions for vector gauge fields was first raised in [6] for $d = 3$. In a recent thorough analysis by Ishibashi and Wald [14], it was shown that for electromagnetic and gravitational perturbations in AdS spacetime, both the slow- and fast- falloff pieces of certain parts of the field are normalizable for $d = 3, 4, 5$; i.e., for bulk spacetime dimensions 4, 5 and 6. As a result, these fields admit general classes of boundary conditions. We investigate the dual CFT description of such general theories, focusing on the electromagnetic perturbations for simplicity. As in the scalar case, we will find different CFTs corresponding to fixing the faster and slower falloff pieces of the bulk field. Furthermore, a general local linear boundary condition corresponds to a deformation of the former theory by a relevant operator, generating a renormalization-group flow which should lead to the latter.

However, a number of interesting new features arise in the vector case. Some of these are associated with gauge invariance. In the slow falloff CFT, the operator dual to the bulk photon is a CFT gauge field instead of the more familiar R-symmetry current. As a result, a general boundary condition is dual to a field theory for which the gauge-invariant action is non-local, though it becomes local in the gauge picked out by the boundary condition. Other features have to do with the possibility of deforming only certain pieces of the gauge field, breaking Lorentz invariance as a result.

After posting the first version of this paper on the hep-th arxiv, we became aware of a body of literature with results overlapping those presented here for the case $d = 3$. In particular, the fact that 'conjugate' boundary conditions in AdS$_4$ are dual to a CFT$_3$ with a dynamical gauge theory was described in [15] and discussed further in [16 – 19]. Certain aspects of the general multi-trace deformations and renormalization group flows were discussed in [16 – 18], and these references also study higher spins for $d = 3$. For spin 1, the conjugate CFT is related to the quantum Hall effect [20]. For higher spins, there is a relation to higher spin theories in AdS$_4$; see [21, 22] for recent reviews. This earlier work focuses largely on the CFT point of view; our work provides a bulk counterpart, considers certain details required to yield a fully local theory, and addresses extensions to $d = 4, 5$, and the allowed boundary condition for $d = 2$.

We begin by carefully reviewing the analysis of the scalar case in section 2. We then address boundary conditions for vector gauge fields in section 3, drawing heavily on the results of [14]. In section 4, we develop our proposal for the dual CFT description. Some final remarks concerning both vector fields and extrapolations to tensor fields are contained in section 5.

2. Scalar fields: general linear boundary conditions and double-trace deformations

This section reviews the relation between boundary conditions for scalar fields and the associated deformations of the dual field theory. This correspondence was conjectured in [7, 8], derived in [23], and studied further in, e.g. [24–27]. Our treatment below is essentially a Lorentzian version of [23], extended in section 2.2 to the case of scalars with logarithmic behavior near the boundary of AdS. For simplicity, we use the familiar toy model of AdS/CFT in which the bulk theory is replaced by a real scalar test field ϕ in AdS_{d+1}.

2.1 Scalars with $m^2 > m^2_{\mathrm{BF}}$

As stated above, we consider a real scalar field which propagates on a fixed spacetime. We take this spacetime to be AdS_{d+1}, with AdS length scale $\ell = 1$. It is convenient to use coordinates such that the AdS_{d+1} metric is

$$ds^2 = g_{ab}dy^a dy^b = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega^2_{d-1}, \tag{2.1}$$

where $d\Omega^2_{d-1}$ is the round metric on the unit S^{d-1}.

Since we are interested in boundary conditions, we first describe the asymptotic behavior of the field. Suppose that our scalar is associated with a potential $V(\phi)$ with squared mass $m^2 = \frac{1}{2}V''(0)$. We restrict attention here to the case where the mass is close to, but slightly above, the Breitenlohner-Freedman bound [5, 6]:

$$-\frac{d^2}{4} + 1 > m^2 > -\frac{d^2}{4}. \tag{2.2}$$

For such values of m, one finds that all solutions to the equations of motion take the asymptotic form

$$\phi \to \frac{\alpha(x)}{r^{\lambda_-}} + \frac{\beta(x)}{r^{\lambda_+}}, \tag{2.3}$$

where x are coordinates on null infinity ($\partial\mathcal{M}$, also known as the conformal boundary) and where

$$\lambda_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}. \tag{2.4}$$

Note that (2.2) implies

$$2 > \lambda_+ - \lambda_- > 0. \tag{2.5}$$

The case $m^2 = -d^2/4$ involves various logarithmic terms and will be treated separately in section 2.2 below.

The boundary condition should be chosen to yield a well-defined phase space. This occurs when the symplectic structure is finite and the symplectic flux[1] through infinity vanishes, so that the symplectic structure is conserved.

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[1]The symplectic flux for a scalar field is proportional to the Klein-Gordon flux. See e.g. [28, 29], for general comments on symplectic structures and their role in quantization.

The mass range (2.2) is precisely the range for which all solutions (2.3) are normalizable with respect to the symplectic structure (see e.g. [30]). Thus, the only constraint is the requirement that the flux through infinity vanish. For two vectors $\delta_1\phi, \delta_2\phi$ tangent to the space of solutions, the symplectic flux through a region R of null infinity is

$$\omega_R(\delta_1\phi, \delta_2\phi) = (\lambda_+ - \lambda_-) \int_R \sqrt{\Omega}(\delta_1\alpha\delta_2\beta - \delta_1\beta\delta_2\alpha). \tag{2.6}$$

If our boundary condition is to force (2.6) to vanish for all regions R, then α must be an ultra-local function of β; i.e., $\alpha(x)$ can depend only on $\beta(x)$ at a point, and cannot depend on derivatives of β:

$$\alpha(x) = J_\alpha(x, \beta) \quad \text{or} \quad \beta(x) = J_\beta(x, \alpha). \tag{2.7}$$

Note that in each case, vanishing of (2.6) implies the existence of a potential $W_\alpha(\beta), W_\beta(\alpha)$ such that

$$\frac{1}{\sqrt{\Omega}}\frac{\delta W_\alpha}{\delta\beta(x)} = (\lambda_+ - \lambda_-)J_\alpha(x, \beta) \quad \frac{1}{\sqrt{\Omega}}\frac{\delta W_\beta}{\delta\alpha(x)} = -(\lambda_+ - \lambda_-)J_\beta(x, \alpha), \tag{2.8}$$

where the normalization factor $(\lambda_+ - \lambda_-)$ on the right-hand side was chosen for later convenience. One may further show that all such boundary conditions remain valid when the scalar field is coupled to gravity; see [31] for a general analysis and [32–37] for direct calculations. We recall the implications of various choices of such boundary conditions for AdS/CFT below.[2]

2.1.1 Fixing α

Because AdS is not globally hyperbolic, we must impose a boundary condition on the scalar field. Let us first suppose that one fixes the leading behavior by choosing some fixed function J_α on $\partial\mathcal{M}$ and imposing

$$\alpha(x) = J_\alpha(x), \quad \text{for } x \in \partial\mathcal{M}. \tag{2.9}$$

The coefficient $\beta(x)$ is then to be determined from the equations of motion and the initial conditions which, for the moment, we take to be given by specifying fixed values of ϕ on $\Sigma_\pm$:

$$\phi(x) = \phi_\pm(x), \quad \text{for } x \in \Sigma_\pm. \tag{2.10}$$

A valid action must be stationary on solutions. In particular, we wish the action to be stationary under all variations which preserve the boundary conditions (2.9) and (2.10). To this end, consider the action

$$S_{\alpha=const} = -\int_\mathcal{M} \left(\frac{1}{2}\partial\phi^2 + V(\phi)\right)\sqrt{-g} - \frac{1}{2}\lambda_- \int_{\partial\mathcal{M}} \sqrt{-h}\phi^2, \tag{2.11}$$

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[2]While it would not correspond to our usual notion of a local bulk theory, one could choose to require the integrated flux (2.6) to vanish only for a certain family of regions R. For example, if vanishing flux is required only for regions bounded by $t = constant$ surfaces then the boundary condition $J_\alpha(x, \beta)$ can be taken to be non-local in space (but still ultra-local in time), so long as $\frac{\delta J_\alpha(x)}{\delta\beta(y)}$ is an appropriately self-adjoint operator; i.e., so long as the potential W_α continues to exist. Such settings may also be of interest for AdS/CFT. Further generalizations should also be possible if one is willing to add extra boundary degrees of freedom.

where $\mathcal{M}$ denotes a region of AdS_{d+1} bounded to the past and future by Cauchy surfaces Σ_-, Σ_+, though we abuse notation by continuing to use $\partial\mathcal{M}$ to denote only the boundary at null infinity. As noted in [27], the action (2.11) is equivalent to the "improved action" advocated by Klebanov and Witten (see equation (2.14) of [30]) for configurations satisfying (2.3). In (2.11), h denotes the determinant of the (divergent) induced metric on null infinity.

We now compute variations:

$$\delta S_{\alpha=const} = \int_{\mathcal{M}} \sqrt{-g}\left(\nabla^2\phi - V'(\phi)\right)\delta\phi - \int_{\partial\mathcal{M}} \sqrt{-h}(n^a\partial_a\phi)\delta\phi - \lambda_- \int_{\partial\mathcal{M}} \sqrt{-h}\phi\delta\phi, \quad (2.12)$$

where n is the outward pointing unit normal to $\partial\mathcal{M}$ (i.e., with $n^a n^b g_{ab} = \pm 1$) and we have used (2.10) to show that the boundary terms at $\Sigma_\pm$ vanish. We have

$$\int_{\partial\mathcal{M}} \sqrt{-h}(n^a\partial_a\phi)\delta\phi = -\int_{\partial\mathcal{M}} \sqrt{\Omega}(r^{\lambda_+-\lambda_-}\lambda_-\alpha\delta\alpha + \lambda_-\alpha\delta\beta + \lambda_+\beta\delta\alpha),$$
$$\int_{\partial\mathcal{M}} \sqrt{-h}\phi\delta\phi = \int_{\partial\mathcal{M}} \sqrt{\Omega}(r^{\lambda_+-\lambda_-}\alpha\delta\alpha + \alpha\delta\beta + \beta\delta\alpha), \quad (2.13)$$

where Ω is the determinant of the metric on the unit S^{d-1} sphere, and we have neglected terms which vanish in the $r \to \infty$ limit. In particular, we have used the fact that $n^a\partial_a = (\sqrt{r^2+1})\partial_r = (r + O(r^{-1}))\partial_r$ and (2.5). As a result, one finds

$$\delta S_{\alpha=const} = \int_{\partial\mathcal{M}} \sqrt{-g}\left(\nabla^2\phi - V'(\phi)\right)\delta\phi + (\lambda_+ - \lambda_-)\int_{\partial\mathcal{M}} \sqrt{\Omega}\beta\delta\alpha. \quad (2.14)$$

Since (2.9) implies $\delta\alpha = 0$, we see that (2.11) indeed provides a valid variational principle for such boundary conditions. A similar calculation shows that under the same boundary condition the action $S_{\alpha=const}$ is also finite when the equations of motion hold.

Now, the variation of a path integral with respect to some family of deformations may be taken to define an operator. Furthermore, in the semi-classical limit, variations of the path integral are given by variations of the on-shell action. Consider then the operator $\mathcal{O}_\alpha$ in the dual CFT whose matrix elements are given in this approximation by the variation of the bulk on-shell action with respect to $J_\alpha(x)$:

$$\langle\mathcal{O}_\alpha\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\alpha=const}}{\delta J_\alpha} = (\lambda_+ - \lambda_-)\beta. \quad (2.15)$$

It is convenient to denote a generic matrix element by $\langle\mathcal{O}_\alpha\rangle$ and to leave implicit the specification of states between which the matrix element is computed.

The choice of states between which one computes the matrix element $\langle\mathcal{O}_\alpha\rangle$ determines the boundary conditions at $\Sigma_\pm$ and as well as additional boundary *terms* at $\Sigma_\pm$ which must be added to $S_{\alpha=const}$. For simplicity, we have suppressed such details here. As discussed in [38], the net result of adding the additional terms and altering the boundary conditions is that (2.14) is unchanged, though the solution on which (2.14) is evaluated depends on the choice of states.

2.1.2 Fixing β

For masses in the range (2.2), one may similarly consider a theory with boundary condition $\beta = J_\beta(x)$ [5, 6]. An appropriately stationary action for such theories is given by

$$S_{\beta=const} = -\int_{\mathcal{M}} \left(\frac{1}{2}\partial\phi^2 + V(\phi)\right)\sqrt{-g} + \int_{\partial\mathcal{M}} \sqrt{-h}\phi n_I^a \partial_a \phi + \frac{1}{2}\lambda_- \int_{\partial\mathcal{M}} \sqrt{-h}\phi^2$$

$$= S_{\alpha=const} - (\lambda_+ - \lambda_-)\int_{\partial\mathcal{M}} \sqrt{\Omega}\beta\alpha, \qquad (2.16)$$

for which we have

$$\delta S_{\beta=const} = \int_{\mathcal{M}} \sqrt{-g}\left(\nabla^2\phi - V'(\phi)\right)\delta\phi - (\lambda_+ - \lambda_-)\int_{\partial\mathcal{M}} \sqrt{\Omega}\alpha\delta\beta. \qquad (2.17)$$

In each such theory, there is an operator O_β associated with deformations of J_β:

$$\langle\mathcal{O}_\beta\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\beta=const}}{\delta J_\beta} = -(\lambda_+ - \lambda_-)\alpha. \qquad (2.18)$$

As conjectured in [30] and discussed in detail in [10], the bulk theory with $\beta = 0$ boundary conditions is dual to a CFT for which the generating functional for planar diagrams is related to that of the $\alpha = 0$ theory.

2.1.3 More general boundary conditions

Two particular classes of boundary conditions were considered above, defined by fixing either the value of α or β on $\partial\mathcal{M}$. We now wish to consider the more general boundary conditions (2.8), starting with the case defined by a potential $W_\alpha(\beta)$. From (2.14) we see that with the boundary condition (2.7) the original action $S_{\alpha=const}$ (2.11) is no longer stationary on solutions. The full action must be of the form

$$S_{W_\alpha} = S_{\alpha=const} + B(\alpha). \qquad (2.19)$$

On-shell, and for fixed boundary conditions at $\Sigma_\pm$, we clearly have

$$\delta S_{W_\alpha} = \int_{\partial\mathcal{M}} \sqrt{\Omega}\left[(\lambda_+ - \lambda_-)\beta\delta\alpha + \frac{1}{\sqrt{\Omega}}\frac{\delta B}{\delta\alpha}\delta\alpha\right], \qquad (2.20)$$

so we must choose B to satisfy

$$\frac{\delta B}{\delta\alpha} = -(\lambda_+ - \lambda_-)\beta\sqrt{\Omega}. \qquad (2.21)$$

Let us now ask about the field theory dual of the bulk theory defined by the general boundary condition (2.7). The action of this theory will differ from the action $S_{\alpha=0}^{FT}$ of the $\alpha = 0$ CFT by some term ΔS^{FT}. One may calculate how such a theory is related to the $\alpha = 0$ CFT by considering a continuous deformation along the one-parameter family of boundary conditions $\alpha = \lambda J(x, \beta)$ for $\lambda \in [0, 1]$. The argument below is essentially a Lorentzian version of the argument of [23].

Suppose that one deforms some such boundary condition by a small amount $\delta\lambda$. We may compute the corresponding deformation $\delta S^{FT} = \partial_\lambda S^{FT}\delta\lambda$ of the dual field theory

action using the AdS/CFT version [38] of the Schwinger variational principle [39–41] to compute the matrix element of $\partial_\lambda S^{FT}$ between two states $|\psi_1\rangle, |\psi_2\rangle$. Let us define $\hat{W}_{\alpha,\lambda}(\psi_1, \psi_2) := \langle\psi_1|(S^{FT}_\lambda - S^{FT}_{\alpha=0})|\psi_2\rangle$. The Schwinger principle relates the variation of the inner product $\langle\psi_1|\psi_2\rangle$ element to the variation of the action as follows:

$$\partial_\lambda \hat{W}_{\alpha,\lambda}(\psi_1, \psi_2) := \langle\psi_1|\partial_\lambda S^{FT}|\psi_2\rangle = -i\partial_\lambda\langle\psi_1|\psi_2\rangle = \partial_\lambda S^{\text{AdS}}_{\psi_1\psi_2}, \qquad (2.22)$$

where the function $S^{\text{AdS}}_{\psi_1\psi_2}$ is built from the action S_{W_α} (2.19), together with the bulk wave functions corresponding to the states $|\psi_1\rangle, |\psi_2\rangle$. Furthermore, the boundary conditions for the variation are such that $S^{\text{AdS}}_{\psi_1\psi_2}$ on the right-hand side of (2.22) is to be evaluated on the particular solution which causes all $\Sigma^\pm$ boundary terms in $\delta S^{\text{AdS}}_{\psi_1\psi_2}$ to vanish [38]. This is just the condition that the classical solution considered is the proper stationary point of the path integral to approximate matrix elements between $|\psi_1\rangle$ and $|\psi_2\rangle$.

As a result, (2.22) is given just by the terms in δS_{W_α} on $\partial\mathcal{M}$:

$$\partial_\lambda \hat{W}_{\alpha,\lambda} = \partial_\lambda B + \int_{\partial\mathcal{M}} \sqrt{\Omega}(\lambda_+ - \lambda_-)\beta\partial_\lambda\alpha. \qquad (2.23)$$

Functionally differentiating this relation with respect to β yields:

$$\partial_\lambda\frac{\delta}{\delta\beta}\hat{W}_\alpha = \partial_\lambda\frac{\delta B}{\delta\beta} + \sqrt{\Omega}(\lambda_+ - \lambda_-)\partial_\lambda\alpha + \int_{\partial\mathcal{M}} \sqrt{\Omega}(\lambda_+ - \lambda_-)\beta\partial_\lambda\frac{\delta\alpha}{\delta\beta} = \sqrt{\Omega}(\lambda_+ - \lambda_-)\partial_\lambda\alpha,$$
$$(2.24)$$

where in the last step we have used (2.21) and the rule $\frac{\delta B}{\delta\beta} = \int_{\partial\mathcal{M}} \frac{\delta B}{\delta\alpha(x)}\frac{\delta\alpha(x)}{\delta\beta}$.

When acting on α, the derivative with respect to λ produces two types of terms: those associated with the explicit variation of the form of the boundary condition (2.7) which relates α to β as well as an "implicit" change resulting from a possible change in the value of β itself. The point here is that β is in general evaluated at some point between Σ_- and Σ_+, and so must be determined from the fixed boundary conditions at $\Sigma_\pm$ via the λ-dependent dynamics. As a result, we see that $\hat{W}_{\alpha,\lambda}(\psi_1, \psi_2) = W_{\alpha,\lambda}(\beta)$ for a function $W_{\alpha,\lambda}$ whose explicit form satisfies a version of (2.24) in which the right-hand side is understood to represent only the explicit change in the form of α. Integrating from $\lambda = 0$ to $\lambda = 1$, and using $\alpha_{\lambda=0} = 0$ and $W_{\alpha,\lambda=0} = 0$ then yields

$$\frac{1}{\sqrt{\Omega}}\frac{\delta W_{\alpha,\lambda=1}}{\delta\beta} = (\lambda_+ - \lambda_-)\alpha, \qquad (2.25)$$

so that $W_{\alpha,\lambda=1}$ is just the potential W_α in (2.8) which was guaranteed to exist by (3.5). The result (2.25) gives a version of the relation from [7, 8] consistent with the normalizations of (2.15).

Using large N factorization, we see from (2.15) that

$$\Delta S^{\text{FT}} = W_\alpha\big|_{\beta=\frac{1}{\lambda_+ - \lambda_-}\mathcal{O}_\alpha} + \mathcal{O}(1/N), \qquad (2.26)$$

since the matrix elements of the left and right-hand sides agree between any two states $|\psi_1\rangle, |\psi_2\rangle$, up to $1/N$ corrections.

Similarly, one may show that the field theory action differs from that of the $\beta = 0$ CFT by the term

$$S^{\rm FT} - S^{\rm FT}_{\beta=0} = W_\beta\big|_{\alpha=\frac{-1}{\lambda_+ - \lambda_-}\mathcal{O}_\beta} + \mathcal{O}(1/N), \tag{2.27}$$

where W_β satisfies

$$\frac{1}{\sqrt{\Omega}}\frac{\delta W_\beta}{\delta\alpha} = -(\lambda_+ - \lambda_-)\beta. \tag{2.28}$$

2.2 Saturating the Breitenlohner-Freedman bound

Let us now consider the case saturating the Breitenlohner-Freedman bound, where the asymptotic behavior is

$$\phi \to \frac{\alpha(x)\ln r}{r^{d/2}} + \frac{\beta(x)}{r^{d/2}}. \tag{2.29}$$

In analogy with (2.11), consider the action

$$S_{\alpha=0} = -\int_{\mathcal{M}} \left(\frac{1}{2}\partial\phi^2 + V(\phi)\right)\sqrt{-g} - \frac{1}{2}\lambda_- \int_{\partial\mathcal{M}}\sqrt{-h}\phi^2, \tag{2.30}$$

for which we find

$$\delta S_{\alpha=0} = -\int_{\mathcal{M}}\sqrt{\Omega}\alpha(\ln r\delta\alpha + \delta\beta). \tag{2.31}$$

We see that $S_{\alpha=0}$ yields a satisfactory variational principle only for the boundary condition $\alpha = 0$.

To fix α to some other value ($\alpha = J_\alpha(x)$), we can use

$$S_{\alpha=J_\alpha} = S_{\alpha=0} + \int_{\partial\mathcal{M}}\sqrt{\Omega}\beta J_\alpha. \tag{2.32}$$

Performing the usual calculation then yields

$$\langle\mathcal{O}_\alpha\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\alpha=const}}{\delta J_\alpha} = \beta. \tag{2.33}$$

Furthermore, if we deform the $\alpha = 0$ theory to a theory with boundary conditions $\alpha = J(x,\beta)$ satisfying (3.5), the arguments of section (2.1.3) lead to the conclusion that the action of the dual field theory has been deformed by the addition of $W_\alpha(\mathcal{O}_\alpha)$ where

$$\frac{1}{\sqrt{\Omega}}\frac{\delta W_\alpha}{\delta\beta} = \alpha. \tag{2.34}$$

In the same way, considering deformations of the $\beta = 0$ theory yields

$$\langle\mathcal{O}_\beta\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\beta=const}}{\delta J_\beta} = -\alpha, \tag{2.35}$$

and

$$\frac{1}{\sqrt{\Omega}}\frac{\delta W_\beta}{\delta\alpha} = -\beta. \tag{2.36}$$

However, in this case the $\beta = 0$ theory is not precisely conformal [7]. Instead, it has a logarithmic behavior associated with the $\ln r$ in (2.29).

3. Boundary conditions for vector fields

In section 2 above, we reviewed the freedom of choosing boundary conditions for scalar fields. It is natural to expect that similar choices of boundary conditions are allowed for spinors, vectors, and tensor fields in AdS_{d+1} with similar interpretations in terms of deformations of the dual field theory. In the scalar case, the range (2.2) of masses for which such boundary conditions are allowed depends on the dimension d. One expects similar results for higher spin fields but, for the vector and tensor case, we note that one particular value of the mass (zero, in the obvious convention) will be associated with gauge invariance. Thus, if one focuses on either vector gauge fields or the linearized graviton, one expects general boundary conditions to be allowed only for certain dimensions d. In fact, such boundary conditions exist for $d = 3, 4, 5$, though only for $d = 3$ will they preserve Lorentz invariance.

For simplicity, we focus here on case of a vector field A_μ satisfying the source-free Maxwell equation

$$\nabla_\nu F^{\mu\nu} = 0, \tag{3.1}$$

though the tensor case is clearly of interest as well. From our perspective, the fundamental question is what boundary conditions turn the space of solutions to (3.1) into a well-defined phase space. Any such setting leads to a well-defined (though not necessarily renormalizable) framework for perturbative quantization [28, 42, 43]. In particular, we ask under what boundary conditions is the symplectic structure both finite and conserved, meaning that no symplectic flux flows outward through the AdS boundary $\partial\mathcal{M}$.

3.1 Symplectic flux through $\partial\mathcal{M}$

Let us first consider the symplectic flux through a region $R \subset \partial\mathcal{M}$ of null infinity. We will take our Maxwell field to have the usual symplectic structure:

$$\omega_R(\delta_1 A, \delta_2 A) = -\int_R \sqrt{-h}\, n^\mu (\delta_1 A^\nu \delta_2 F_{\mu\nu} - \delta_2 A^\nu \delta_1 F_{\mu\nu}). \tag{3.2}$$

One might also consider modifying (3.2) by the addition of boundary terms. We will not do so here, though one should bear in mind that allowing such generalizations will enlarge the space of allowed boundary conditions derived below.

Introducing indices $I, J, K \ldots$ which run over directions in $\partial\mathcal{M}$, it is clear that the flux (3.2) vanishes whenever the pull-back A_I to $\partial\mathcal{M}$ of A_μ is appropriately related to the projection F^I to $\partial\mathcal{M}$ of

$$F^\nu := -\frac{\sqrt{-h}}{\sqrt{\Omega}} n_\mu F^{\mu\nu} = -r^d n_\mu F^{\mu\nu}, \tag{3.3}$$

where the factor of $-r^d$ is chosen to simplify later expressions. That is, we wish to impose either

$$A_I = J_{A_I}(x, F|_{\partial\mathcal{M}}) \quad \text{or} \quad F^I = J_{F^I}(x, A|_{\partial\mathcal{M}}), \tag{3.4}$$

where

$$\frac{\partial J_{A_I}}{\partial F^J} \quad \text{and} \quad \frac{\partial J_{F^I}}{\partial A_J} \tag{3.5}$$

must be symmetric in order for ω_R to vanish. The symmetry conditions (3.5) are just the integrability conditions for the boundary conditions (3.4) to be specified in terms of potentials W_α, W_β such that

$$J_{A_I} = -\frac{1}{\sqrt{\Omega}}\frac{\delta W_A}{\delta F^I}, \quad \text{or} \quad J_{FI} = \frac{1}{\sqrt{\Omega}}\frac{\delta W_F}{\delta A_I}. \tag{3.6}$$

Since the boundary conditions (3.4) are local on $\partial\mathcal{M}$ one expects that these theories are fully local. In particular, one expects that the advanced and retarded Green's functions $G^\pm(x, y)$ vanish unless x and y are connected by a causal curve.

Before proceeding, let us make a few observations about the effects of gauge symmetry and charge conservation. In (3.6), we considered W_A to be some fixed functional of an arbitrary vector field F^I on the boundary. However, due to charge conservation, F^I is divergence-free on-shell:

$$\mathcal{D}_I F^I = 0, \tag{3.7}$$

where $\mathcal{D}^I$ is the covariant derivative on the boundary. Thus, if one instead considers W_A as a functional of the on-shell fields, the variations of F^I are constrained by (3.7) and the functional derivatives (3.6) are ill-defined. However, the ambiguity is just that associated with the gauge freedom; under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$ we have $J_{A,I} \to A_I + \partial_I\Lambda$. Similarly, due to (3.7), we must have $\mathcal{D}_I J_{FI} = 0$ on shell. Thus, on shell and when the boundary condition holds, W_F must be equal (up to boundary terms at $\Sigma_\pm$) to some gauge-invariant functional of A_I.

3.2 Normalizability and boundary conditions

We now turn to the question of normalizability of the modes with respect to the symplectic structure. A related normalizability criterion was analyzed in [14] by Ishibashi and Wald, whose results will be of central use below. The results of [14] are stated in terms of a decomposition of the vector field A_μ into vector and scalar parts with respect to some $\mathrm{SO}(d)$ symmetry in AdS_{d+1}, which we now recall.

3.2.1 Preliminaries

We begin by introducing notation in order to recall the results of [14] and to reformulate these results in a more transparent form. One notes that spheres invariant under the $\mathrm{SO}(d)$ symmetry foliate the spacetime, and that the spheres themselves can be labelled by the coordinates y^a, $a = 0, 1$ with $y^0 = t, y^1 = r$. It is convenient to introduce an associated two-dimensional metric

$$\widehat{ds}^2 = \hat{g}_{ab}dy^a dy^b = -(r^2 + 1)dt^2 + \frac{dr^2}{r^2 + 1}, \tag{3.8}$$

with metric-compatible covariant derivative $\hat{\nabla}_a$, and Levi-Civita tensor ϵ_{ab} satisfying $\epsilon_{rt} = 1$. On the unit sphere S^{d-1}, we introduce coordinates z^i, $i = 1 \ldots d-1$, and we take the metric and covariant derivative on the unit sphere to be Ω_{ij}, D_i.

It is useful to introduce orthonormal bases of scalar and vector eigenmodes of the Laplacian on S^{d-1}, satisfying

$$(D^2 + k_S^2)\mathbb{S}_{k_S} = 0, \tag{3.9}$$

$$\int_{S^n} \mathbb{S}_{k_S}\mathbb{S}_{k'_S} = \delta_{k_S,k'_S}, \tag{3.10}$$

$$(D^2 + k_V^2)\mathbb{V}_{i,k_V} = 0, \quad \Omega^{ij}D_i\mathbb{V}_{j,k_V} = 0, \tag{3.11}$$

$$\int_{S^n} \mathbb{V}_{i,k_V}\mathbb{V}_{j,k'_V}\Omega^{ij} = k_V^2\delta_{k_V,k'_V}, \tag{3.12}$$

where $D^2 = \Omega^{ij}D_iD_j$. The normalization (3.12) differs from the one used in [14], but is useful to display certain parallels between the vector and scalar parts.

Using the above bases, one can decompose A_μ into a vector and scalar part with respect to SO(d):

$$A_\mu = A_\mu^V + A_\mu^S, \tag{3.13}$$

where

$$A_\mu^V dx^\mu = \sum_{k_V} \phi_{V,k_V}\mathbb{V}_{i,k_V}dz^i, \tag{3.14}$$

and

$$A_\mu^S dx^\mu = \sum_{k_S} A_{ak_S}\mathbb{S}_{k_S}dy^a + A_{k_S}D_i\mathbb{S}_{k_S}dz^i. \tag{3.15}$$

Gauge transformations affect only the scalar part; the gauge-invariant information in the scalar parts is contained in a scalar mode ϕ_{S,k_S} defined by[3]

$$\nabla_a\phi_{S,k_S} = \epsilon_{ab}r^{d-3}(\nabla^b A_{k_S} - A_{k_S}^b). \tag{3.16}$$

We emphasize here that $\phi_{S,k_S}, \phi_{V,kV}, A_{k_S}$ depend only on the y^a coordinates; that is, they are fields only on the two-dimensional quotient space $AdS_{d+1}/\,\mathrm{SO}(d)$. In [14], it was found that for these two scalars fall off at infinity as

$$\phi_{V,k_V} = \alpha_{V,k_V}r^0 + \beta_{V,k_V}r^{2-d} + \mathcal{O}(r^{-2}) + \mathcal{O}(r^{-d}), d \neq 2 \tag{3.17}$$

$$\phi_{S,k_S} = \begin{cases} \alpha_{S,k_S}r^{d-4} + \beta_{S,k_S}r^0 + \mathcal{O}(r^{-2}) + \mathcal{O}(r^{d-6}) & \text{for} \quad d \neq 4 \\ \beta_{S,k_S} + \alpha_{S,k_S}\ln r + \mathcal{O}(r^{-2}\ln r) & \text{for} \quad d = 4 \end{cases}. \tag{3.18}$$

Note that there are no vector modes for $d = 2$, as all vector harmonics with non-zero angular momentum on S^1 are the gradients of scalars.

Equations (3.17) and (3.18) are the main results we take from [14], but it will be useful to summarize these results in a somewhat more local and covariant form. To this end we construct fields $\alpha_S, \beta_S, \alpha_i, \beta_i$ on the boundary from the modes $\alpha_{S,k_S}, \beta_{S,k_S}, \alpha_{V,k_V}, \beta_{V,k_V}$ as follows:

$$\alpha_S(z^i,t) := \sum_{k_S} \alpha_{S,k_S}\mathbb{S}_{k_S}, \quad \beta_S(z^i,t) := \sum_{k_S} \beta_{S,k_S}\mathbb{S}_{k_S},$$

$$\alpha_i(z^i,t) := \sum_{k_V} \alpha_{V,k_V}\mathbb{V}_{i,k_V}, \quad \beta_i(z^i,t) := \sum_{k_V} \beta_{V,k_V}\mathbb{V}_{i,k_V}. \tag{3.19}$$

---

[3]Note that such scalar modes are defined only for on-shell field configurations; the form on the right-hand side is closed as a consequence of the equation of motion (3.1).

Similarly, we introduce

$$\phi_S := \sum_{k_S} \phi_{S,k_S} \mathbb{S}_{k_S}, \quad \text{and the "pure gauge" field} \quad A(z^i, t, r) := \sum_{k_S} A_{k_S}(t,r)\mathbb{S}_{k_S}, \quad (3.20)$$

so that we may write

$$\phi_S = \begin{cases} \alpha_S r^{d-4} + \beta_S r^0 + \mathcal{O}(r^{-2}) + \mathcal{O}(r^{d-6}) & \text{for} \quad d \neq 4 \\ \alpha_S \ln r + \beta_S r^0 + \mathcal{O}(r^{-2}\ln r) & \text{for} \quad d = 4 \end{cases}, \quad (3.21)$$

$$A_i = D_i A + \alpha_i(z^i,t)r^0 + \beta_i(z^i,t)r^{2-d} + \mathcal{O}(r^{-2}), \quad (3.22)$$

$$A_t = \partial_t A + r^{5-d}\hat{\nabla}_r \phi_S = \partial_t A + c_S(d)\alpha_S + \mathcal{O}(r^{2-d}) + \mathcal{O}(r^{-2}), \quad (3.23)$$

and

$$A_r = \partial_r A + r^{1-d}\hat{\nabla}_t \phi_S, \quad (3.24)$$

where

$$c_S(d) = \begin{cases} (d-4) & \text{for } d \neq 4 \\ 1 & \text{for } d = 4 \end{cases}. \quad (3.25)$$

Furthermore, note that $F_{ab} = \epsilon_{ab}F$ where

$$F = -(1/2)\epsilon^{ab}F_{ab} = -\hat{\nabla}_a r^{3-d}\hat{\nabla}^a \phi_S = D^2 \phi_S r^{1-d}, \quad (3.26)$$

and where the last step follows from the equation of motion for ϕ_S (eq. (67) from [14]). Thus we may write

$$F^t = -r^d n_\mu F^{\mu t} = \begin{cases} D^2(\alpha_S r^{d-4} + \beta_S) + \mathcal{O}(r^{d-6}) + \mathcal{O}(r^{-2}) & \text{for} \quad d \neq 4 \\ -D^2(\alpha_S \ln r + \beta_S) + \mathcal{O}(r^{-2}\ln r) & \text{for} \quad d = 4 \end{cases}, \quad (3.27)$$

and

$$\begin{aligned} F^i &= -r^d n_\mu F^{\mu i} = -\Omega^{ij}\left[\hat{\nabla}_t D_j \phi_S + r^{d-2} n^\mu \hat{\nabla}_\mu (A_j - D_j A)\right] \\ &= \begin{cases} \Omega^{ij}\left(r^{d-4}\hat{\nabla}_t D_j \alpha_S + \hat{\nabla}_t D_j \beta_S - (2-d)\beta_j\right) + \mathcal{O}(r^{-2}) + \mathcal{O}(r^{d-6}) & \text{for } d \neq 4 \\ \Omega^{ij}\left(\hat{\nabla}_t D_j \alpha_S \ln r + \hat{\nabla}_t D_j \beta_S - (2-d)\beta_i\right) + \mathcal{O}(r^{-2}\ln r) & \text{for } d = 4 \end{cases} \end{aligned} \quad (3.28)$$

These results summarize the asymptotic behavior of the gauge field and form the cornerstone of the normalizability analysis below and in [14].

3.2.2 Normalizability of the symplectic structure

The most familiar AdS/CFT boundary conditions for a vector field are to fix A_I on the boundary [2]. From (3.14), (3.22), (3.23) we see that this corresponds to fixing α_i, α_S, and also the "pure-gauge" field A. This is true even for $d = 2, 3$, where β_S is the *slower* fall-off part of ϕ_{Sk_S}. This alone is enough to make one suspect that more general boundary conditions should be available, and to motivate a general study.

As stated above, a boundary condition of the form (3.6) will be allowed whenever it renders the symplectic structure finite. Computing the symplectic structure on a hypersurface Σ defined by $t = constant$ using (3.22), (3.23), (3.24), and the fact that the vector modes are divergence-free on S^{d-1}, we find

$$
\begin{aligned}
\omega_\Sigma(\delta_1 A, \delta_2 A) = & -\int_\Sigma \sqrt{q}\, t^\mu (\delta_1 A^\nu \delta_2 F_{\mu\nu} - \delta_2 A^\nu \delta_1 F_{\mu\nu}) \\
= & -\int_\Sigma \sqrt{\Omega} d^{d-1}z dr\, r^{d-5} \Omega^{ij} (\delta_1 \alpha_i + \delta_1 \beta_i r^{2-d}) \hat{\nabla}_t (\delta_2 \alpha_i + \delta_2 \beta_i r^{2-d}) \\
& -\int_\Sigma \sqrt{\Omega} d^{d-1}z dr\, r^{1-d} (\hat{\nabla}_t \delta_1 \phi_S)(D^2 \delta_2 \phi_S)] \\
& +\int_{\partial\Sigma} \sqrt{\Omega} d^{d-1}z\, \delta_1 A \delta_2 F^t + (1 \leftrightarrow 2) + \text{finite}, \quad\quad\quad (3.29)
\end{aligned}
$$

where t^μ is the unit normal to Σ and q is the determinant of the metric on Σ. In (3.29), the terms implicit in "finite" come from the higher order corrections in (3.21), (3.28) and are explicitly finite for $2 \leq d \leq 6$, which will be the cases of primary interest.

For the vector modes, the inner product studied in [14] agrees with (3.29) up to a factor of the mode frequency ω. For the scalar modes, the inner product agrees up to a factor of ω and a factor of k_S^2. Thus, the desired normalizability results are directly related to those of [14]:

- **d $\leq$ 1:** Since the bulk spacetime dimension is ≤ 2, there are no propagating modes for A_μ. This case is trivial.

- **d = 2:** There are no vector modes, and the the β_{S,k_S} modes fail to be normalizable. We therefore choose to fix $\beta_S = J_{\beta_S}(x)$ for all α_S. From (3.27) we see that for $d = 2$ the contribution of α_S to F^I vanishes at $\partial\mathcal{M}$. Thus, fixing β_S is equivalent to imposing $F^I\big|_{\partial\mathcal{M}} = J_{F^I}(x)$, where J_{F^I} is independent of the dynamical fields. We must also keep the pure-gauge field A from growing too quickly at infinity. This is easily accomplished by imposing the gauge condition $\Omega^{ij} \mathcal{D}_i A_j = \mathcal{O}(r^2)$.

- **d = 3:** All modes $\alpha_S, \beta_S, \alpha_i, \beta_i$ are normalizable so long as the pure-gauge field A is finite on $\partial\mathcal{M}$. Thus, any boundary condition of the form (3.4) is allowed.

- **d = 4 or 5:** The α_{V,k_V} modes fail to be normalizable and must be fixed. From (3.22) we see that, up to gauge transformations, this is equivalent to imposing $A_i\big|_{\partial\mathcal{M}} = J_{A_i}(x)$, where J_{A_i} is independent of the dynamical fields.

  If one considers only the integral over Σ in (3.29), then all scalar modes are normalizable. However, because F^t is divergent for $d = 4, 5$, there is a potential for the final term involving the pure gauge field A to alter this conclusion. We remove this possibility by noting that the above boundary condition on A_i fixes A on the boundary and by also imposing the gauge condition $\Omega^{ij} \mathcal{D}_i (A_j - J_{A_j}) = \mathcal{O}(1/r)$. We may then use any boundary condition of the form

$$
A_t = \frac{1}{\sqrt{\Omega}} \frac{\delta W_{A_t}}{\delta F^t} \quad \text{or} \quad F^t = -\frac{1}{\sqrt{\Omega}} \frac{\delta W_{F^t}}{\delta A_t}, \quad\quad\quad (3.30)
$$

where W_{A_t} is the integral of a local function of F^t alone or W_{F^t} is the integral of a local function of A_t alone.

As noted above, F^t is divergent for general values of α_S. Nonetheless, we may display the above boundary conditions in a manifestly finite form by introducing the quantity $F^I_{\beta_S=0}$, defined by setting $\beta_{S,k_S} = 0$ in the mode expansion (3.27), (3.28) of F^I. We also introduce $F^I_{\beta_S\ only} := F^I - F^I_{\beta_S=0}$ which is finite on $\partial\mathcal{M}$. We may then reformulate (3.30) as

$$A_t = \frac{1}{\sqrt{\Omega}}\frac{\delta W_A}{\delta F^t_{\beta_S\ only}} \quad \text{or} \quad F^t_{\beta_S\ only} = -\frac{1}{\sqrt{\Omega}}\frac{\delta \tilde{W}_F}{\delta A_t}, \tag{3.31}$$

where $\tilde{W}_F = W_F + F^t_{\beta_S=0}A_t$. Choosing W_A to be a finite function of $F^t_{\beta_S\ only}$ or choosing $\tilde{W}$ to be a finite function of A_t results in a well-defined boundary condition.

- **$d \geq 6$:** Neither the α_{V,k_V} modes nor the α_{S,k_S} modes are normalizable. We must impose $A_I|_{\partial\mathcal{M}} = J_{A_I}(x)$, with J_{A_I} is independent of the dynamical fields.

Ishibashi and Wald studied the case of linear boundary conditions in detail, and obtained interesting results as to which boundary conditions yield stable bulk theories. In contrast, our desire is to understand the general boundary condition above in terms of deformations of the dual field theory. We turn to this question in section 4 below.

4. Dual CFT description

For a scalar field with α completely fixed by the boundary condition, the expectation value of the operator dual to deformations of α is given by $(\lambda_+ - \lambda_-)\beta$. The dimension of this operator is thus related to the scaling of β in the bulk spacetime. Similarly, if we fix the value of β, the dimension of the operator associated with variations of β is related to the scaling of α in the bulk spacetime.

Here we study the corresponding relations and the details of the operators dual to a vector gauge field. At least for $d = 3$, we expect to have two operators $\mathcal{O}_{A,}{}^I$ and $\mathcal{O}_{F,I}$ dual to variations of A_I and F^I respectively. Now, under a scaling $r \to \Lambda r$, the components of the gauge field scale as $A_I \to A_I$, while $F^I \to \Lambda^{1-d}F^I$. Thus, $\dim \mathcal{O}_{A,}{}^I = \dim F^I = d-1$, which has the right dimension to represent a conserved current.

On the other hand, $\dim \mathcal{O}_{F,I} = \dim A_I = 1$. We note that this agrees with the results of [17] obtained by CFT methods. At first, this may seem like a surprisingly low dimension. Indeed, the dimension of local vector-like observables in a unitary CFT is bounded below by $d-1$ (see e.g. [44]). The natural conclusion [15, 17] is that $\mathcal{O}_{F,I}$ is not strictly a local *observable*, but instead represents a U(1) vector gauge field in the CFT.

The details of this picture are discussed below. We present bulk actions appropriate to each of the boundary conditions stated in section 3 and discuss the corresponding implications for the dual field theory. In order to neglect certain additional terms which contribute in higher dimensions, we restrict attention to the case $2 \leq d \leq 5$, which encompasses the most interesting cases identified above. The generalization to higher dimensional cases is

straightforward. We proceed in parallel with our treatment of the scalar field in section 2, first reviewing the case where one fixes A_I or F^I alone, and then considering more general boundary conditions.

4.1 Fixing A_I on the boundary

As noted in section (3.2), for $d \geq 3$ we may choose the familiar boundary condition

$$A_I = J_{A_I}(x), \tag{4.1}$$

where J_{A_I} independent of any dynamical fields. For this boundary condition, consider the action

$$S_{A=const} = -\frac{1}{4} \int_{\mathcal{M}} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \int_{\partial\mathcal{M}} \sqrt{-h} n_\mu A_\nu F^{\mu\nu}_{\beta_S, \beta_V = 0}, \tag{4.2}$$

where $F^{\mu\nu}_{\beta_S, \beta_V = 0}$ is constructed (in analogy with $F^{\mu\nu}_{\beta_S = 0}$ above) by setting $\beta_{S,k_S} = \beta_{V,k_V} = 0$ in the mode expansion of $F^{\mu\nu}$ for all k_S, k_V. We also define the analogous $F^I_{\beta_S, \beta_V = 0}$.

From (3.23), (3.22), (3.27), and (3.28), it is clear that $F^I_{\beta_S, \beta_V = 0}$ is a local function (on the boundary) of $A_I|_{\partial\mathcal{M}}$ and its derivatives. As a result, under a general variation which fixes boundary conditions at $\Sigma_\pm$, we find

$$\delta S_{A=const} = \int_{\partial\mathcal{M}} \sqrt{\Omega} F^I_{\beta\ only} \delta A_I, \tag{4.3}$$

where $F^I_{\beta\ only} = F^I - F^I_{\beta_S, \beta_V = 0}$ and we have used the equations of motion for the background. Clearly, (4.3) vanishes when the variation preserves (4.1). The corresponding dual operator $\mathcal{O}^I_{A,}$ satisfies

$$\langle \mathcal{O}_{A,}{}^I \rangle = \frac{1}{\sqrt{\Omega}} \frac{\delta S_{A=const}}{\delta A_I} = F^I_{\beta\ only}. \tag{4.4}$$

Of course, conservation of this current follows from gauge invariance, and it is natural to introduce the notation $j^I = \mathcal{O}_{A,}{}^I$. This is the familiar AdS/CFT duality for vector fields [2].

4.2 Fixing F^I on the boundary

For $d = 2$ and $d = 3$, we have seen that an allowed boundary condition is to set

$$F^I = J_{F^I}(x), \tag{4.5}$$

where J_{F^I} is independent of any dynamical fields. From (3.27), (3.28) we see that, for such values of d, the condition (4.5) fixes β_{S,k_S} and β_{V,k_V} but leaves α_{S,k_S} and α_{V,k_V} unconstrained. For $d = 2$ this in fact the *only* allowed boundary condition in our class.

For the boundary condition (4.5), consider the action

$$S_{F=const} = -\frac{1}{4} \int_{\mathcal{M}} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \int_{\partial\mathcal{M}} \sqrt{-h} n_\mu A_\nu F^{\mu\nu}. \tag{4.6}$$

Under a general variation which fixes boundary conditions at $\Sigma_\pm$, we find

$$\delta S_{F=const} = -\int_{\partial\mathcal{M}} \sqrt{\Omega} A_I \delta F^I, \tag{4.7}$$

where we have used the equations of motion for the background. The result (4.7) vanishes as required when the variation preserves (4.1). The corresponding dual operator $\mathcal{O}_{F,I}$ satisfies

$$\langle \mathcal{O}_{F,I} \rangle = \frac{1}{\sqrt{\Omega}} \frac{\delta S_{F=const}}{\delta F^I} = -A_I + \partial_I \Lambda. \tag{4.8}$$

Here Λ is an arbitrary function on $\partial\mathcal{M}$ introduced to take account of the fact that, since (4.8) uses the on-shell action, variations of F^I are constrained to satisfy $\mathcal{D}_I F^I = 0$. Thus, functional derivatives with respect to F^I are inherently ambiguous. This ambiguity strongly suggests that $\mathcal{O}_{F,}{}^I$ is itself a vector gauge field in the dual theory. For $d = 3$, this conclusion was reached previously in [15, 19] using related path-integral reasoning. As observed in [17], the well-defined (i.e., gauge invariant) part of $\mathcal{O}_{F,}{}^I$ is inherently a non-local operator and is thus not subject to the bound $\Delta \geq d - 1$ on the dimension of local vector operators.

Recall that for $d = 2, 3$ the engineering dimension of a vector gauge field is $0, 1/2$. In contrast, $\dim \mathcal{O}_{F,I} = 1$, so the anomalous dimension of this operator is 1 for $d = 2$ and $1/2$ for $d = 3$. From this point of view, it is no surprise that there is no $F^I = 0$ CFT for $d > 4$; such theories would necessarily contain operators with negative anomalous dimension. The case $d = 4$ is clearly marginal, and the $F^I = 0$ theory fails to be conformal due to the logarithmic behavior at large r.

4.3 More general boundary conditions

For $d = 3$ we may consider any boundary conditions (3.4) determined by some W_A or W_F. A general class of boundary condition (3.30) is also available in $d = 4, 5$. There we cannot consider the theory as a deformation of the $F^I = 0$ theory (which does not exist), but it does make sense to define the theory through any functional $W_A = W_{A_t} + \int \sqrt{\Omega} J_{A_i} F^i$, where W_{A_t} is an integral of a local function of F^t.

Let us therefore consider (in $d = 3, 4, 5$) such a boundary condition as a deformation of the $A_I = constant$ theory via the action

$$S_{W_A} = S_{A=const} + B_A(A|_{\partial\mathcal{M}}). \tag{4.9}$$

It is clear that for this action is to be stationary on solutions we must have

$$\frac{1}{\sqrt{\Omega}} \frac{\delta B_A}{\delta A_I} = -F^I_{\beta\ only}. \tag{4.10}$$

It is also clear that B_A is local on the boundary and, since F^I is conserved, B_A is gauge-invariant at least on-shell. The same calculation as in section 2 now shows that the deformation of the dual field theory action is the Legendre transform of B_A:

$$\langle \Delta S^{\mathrm{FT}} \rangle = B_A - \int_{\partial\mathcal{M}} \sqrt{\Omega} F^I_{\beta\ only} A_I. \tag{4.11}$$

Assuming that our boundary condition associates every $F^I_{\beta\ only}$ with some A_I, we may regard $\langle \Delta S^{\mathrm{FT}} \rangle$ as a function of $F^I_{\beta\ only}$. One would now like to functionally differentiate (4.11) with respect to $F^I_{\beta\ only}$. However, since we have worked on-shell, our expression

$\langle \Delta S^{\mathrm{FT}} \rangle$ is only defined for divergence-free vector fields $F^I_{\beta\ only}$. The result is therefore

$$\frac{1}{\sqrt{\Omega}} \frac{\delta \langle \Delta S^{\mathrm{FT}} \rangle}{\delta F^I_{\beta\ only}} = -A_I + \partial_I \Lambda. \tag{4.12}$$

Except for the term $\partial_I \Lambda$, this is the equation (3.6) satisfied by W_A. Thus we find $\Delta S^{\mathrm{FT}} = W_A + constant$ up to a term of the form $\int_{\partial \mathcal{M}} \sqrt{\Omega} F^I_{\beta\ only} \partial_I \Lambda$. Since $\partial_I F^I_{\beta\ only} = 0$ in the large N limit of the dual field theory, this amounts to the expected statement that $\Delta S^{\mathrm{FT}} = W_A + constant$ up to $1/N$ corrections (and perhaps a boundary term at $\Sigma_\pm$). The behavior at higher order in $1/N$ is determined by the structure of gauge anomalies in the bulk theory.

Similarly, for $d = 3$ one may regard a generic boundary condition as a deformation of the $F^I = constant$ theory via the action

$$S_{W_F} = S_{F=const} + B_F(F\big|_{\partial \mathcal{M}}), \tag{4.13}$$

defined by

$$\frac{1}{\sqrt{\Omega}} \frac{\delta B_F}{\delta F^I} = A_I + \partial_I \Lambda, \tag{4.14}$$

where Λ is arbitrary. Since the construction of the dual field theory deformation proceeds on-shell, this ambiguity in B_F leads at most to a boundary term at $\Sigma_\pm$. Again one finds that the $\langle \Delta S^{\mathrm{FT}} \rangle$ is the Legendre transform of B_F.

We wish to regard $\langle \Delta S^{\mathrm{FT}} \rangle$ as a functional of A_I. Because we now work on-shell, simply using the boundary condition to replace F^I by A_I would define $\langle \Delta S^{\mathrm{FT}} \rangle$ only for those A_I for which the boundary condition yields divergence-free F^I. Let us therefore consider only boundary conditions for which $every$ A_I differs from some $A_I^{div-free}$ only by a gauge transformation, where $A_I^{div-free}$ is a connection associated by the boundary condition to some divergence-free F^I. This is the natural analogue of the condition imposed above in discussing deformations of the $A_I = constant$ theory. Since ΔS^{FT} must be gauge-invariant up to boundary terms, our new assumption allows us to define $\langle \Delta S^{\mathrm{FT}} \rangle$ for all A_I. Taking a functional derivative then shows that for any $A_I^{div-free}$ we have $\Delta S^{\mathrm{FT}} = W_F$, up to an additive constant and the usual boundary terms at $\Sigma^\pm$. Thus, ΔS^{FT} is just the gauge-invariant version of W_F mentioned at the end of section 3.1.

Let us examine the particular case of linear boundary conditions in detail:

$$F^I_{\beta\ only} = \gamma^{IJ} A_J, \tag{4.15}$$

for some γ^{IJ} with inverse γ_{IJ}. (For $d = 4, 5$ we must have $\gamma_{IJ} \propto \delta_{It}\delta_{Jt}$ and γ^{IJ} does not exist.) Note that all solutions satisfying (4.15) will also will satisfy the gauge condition

$$\gamma^{IJ} \partial_I A_J = 0. \tag{4.16}$$

For $d = 3$ we have

$$W_F = \frac{1}{2} \int_{\partial \mathcal{M}} \sqrt{\Omega} A_I A_J \gamma^{IJ} = \frac{1}{2} \int_{\partial \mathcal{M}} \sqrt{\Omega} A_I (\gamma^{IJ} - \Box_\gamma^{-1} \gamma^{IK} \partial_K \gamma^{JL} \partial_L) A_J, \tag{4.17}$$

where $\Box_\gamma = \gamma^{IJ}\partial_I\partial_J$ and the inverse is defined using Dirichlet boundary conditions at $\Sigma_\pm$. In the last step, we have used the gauge condition (4.16). Note that this final form of W_F is invariant under gauge transformations which vanish on $\Sigma_\pm$.

The relevant (dim = 2) operator (4.17) will generate a renormalization-group flow away from the $F^I = 0$ CFT. The deformation is non-local when expressed in terms of gauge-invariant operators, but becomes local in Lorentz gauge. This is consistent with the fact that the bulk theory in this gauge satisfies local field equations and a local boundary condition. Although there is no $F^I = 0$ CFT for $d = 4, 5$, we will discuss a similar UV fixed point for $d = 5$ renormalization-group flows (and a logarithmic theory for $d = 4$) in section 4.4 below.

Of course, we can also describe a general boundary condition as a deformation of the $A_I = 0$ CFT by

$$W_A = \frac{1}{2}\int \sqrt{\Omega} F^I_{\beta\ only} F^J_{\beta\ only}\gamma_{IJ},\tag{4.18}$$

which is an irrelevant operator of dimension $2d - 2$. As in the case of scalar fields, it is thus natural to conjecture (for $d = 3$) that the renormalization-group flow from the $F^I = 0$ theory in the UV has an IR fixed point at the $A_I = 0$ CFT. See [16, 17] for further discussion of such flows from the CFT point of view.

4.4 Hybrid Boundary conditions and their deformations

As noted above, in $d = 4, 5$ the boundary conditions $F^I = 0$ are not allowed due to the failure of the vector modes associated with α_V to be normalizable. However, the scalar modes α_S *are* normalizable, and one may consider 'hybrid' boundary conditions of the form

$$A_i = J_{A_i}(x), \quad F^t_{\beta\ only} = J_{F^t}(x).\tag{4.19}$$

For $J_{A_i} = 0 = J_{F^t}$, these boundary conditions are again conformal for $d = 5$, though for $d = 4$ conformal invariance is broken by the logarithmic dependence on r. Furthermore, such boundary conditions may be deformed to yield any relationship of the form (3.31). These boundary conditions may also be used in $d = 3$, where other hybrid options also exist. For simplicity, we confine ourselves here to (4.19), but the other $d = 3$ hybrid boundary conditions can be handled similarly.

Consider the action

$$S_{\text{hybrid}} = S_{A=const} - \int_{\partial\mathcal{M}} \sqrt{-h} A_t F^t_{\beta\ only}.\tag{4.20}$$

Under a general variation which fixes boundary conditions at $\Sigma_\pm$, we find from (4.3) that

$$\delta S_{\text{hybrid}} = \int_{\partial\mathcal{M}} \sqrt{\Omega}\left(F^i_{\beta\ only}\delta A_i - A_t\delta F^t_{\beta\ only}\right),\tag{4.21}$$

where we have used the equations of motion for the background. Clearly, (4.21) vanishes when the variation preserves (4.19). The corresponding dual operators $\mathcal{O}^i_{A,}$, $\mathcal{O}_{F,t}$ satisfy

$$\langle\mathcal{O}_A,{}^i\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\text{hybrid}}}{\delta A_i} = F^i_{\beta\ only},$$

$$\langle\mathcal{O}_{F,t}\rangle = \frac{1}{\sqrt{\Omega}}\frac{\delta S_{\text{hybrid}}}{\delta F^t} = -A_t.\tag{4.22}$$

Here there are no restrictions on F^t, so that the functional derivative $\frac{\delta}{\delta F^t}$ is well-defined. The result is a set of local operators. For $d = 5$ these operators have conformal dimensions $\dim \mathcal{O}_{A,}{}^i = d - 1$ and $\dim \mathcal{O}_{F,t} = 1$.

Much as with the $d = 3$ theory with $F^I = 0$, for $d = 5$ we may regard the hybrid theory with $J_{A_i} = 0 = J_{F^t}$ as a UV fixed point which we can deform by relevant operators (such as $\int_{\partial\mathcal{M}} \sqrt{\Omega} A_t A_t$) to generate a renormalization-group flow. Again, we expect that this flow leads to an IR fixed point corresponding to the $A_I = 0$ theory. Although the hybrid theory breaks Lorentz invariance, we see that Lorentz invariance is restored at the IR fixed point.

Our hybrid theory also has an interesting class of marginal deformations. Given any anti-symmetric tensor ω_{IJ}, we may consider

$$W_\omega = \int_{\partial\mathcal{M}} \sqrt{\Omega} \omega_{it} \mathcal{O}_{F,t} \mathcal{O}_{A,}{}^i = -\int_{\partial\mathcal{M}} \sqrt{\Omega} \omega_{it} A_t F^i_{\beta only}, \tag{4.23}$$

which leads to boundary conditions related to (4.19) by a Lorentz transformation. Due to Lorentz symmetry in the bulk, this operator should be exactly marginal at all orders in $1/N$.

5. Discussion

In this work, we have studied field theories dual to AdS theories with deformed boundary conditions for vector fields. Our analysis used results from [14] concerning the asymptotics of vector gauge fields in AdS$_{d+1}$ to read off the general local boundary condition which leads to a well-defined phase space, and thus to a well-defined quantum theory. We then used the bulk action and the Schwinger variational principle to construct the associated multi-trace deformations of a dual CFT. The results are qualitatively similar to those obtained for general scalar field boundary conditions [7, 8, 23], which were also reviewed in detail.

The results are best summarized separately for each dimension d. The cases $d \leq 1$ are trivial as vector gauge fields have no propagating degrees of freedom.

For $d = 2$, there is a unique allowed class of local boundary conditions $F^I = constant$. In particular, the most familiar boundary condition $A_I = constant$ is *not* allowed, as it would fix all of the normalizable modes. This can be understood intuitively by considering the description in terms of a dual potential in the bulk, as in [15]. In three bulk dimensions, this is a massless scalar field defined by $*F = d\phi$, and $F^I = constant$ corresponds to the usual boundary condition for the scalar, fixing the slower falloff part.

In the original Maxwell field picture, one expects the dual operator to be another U(1) vector gauge field, and not the usual R-charge current. However, this vector gauge field is a dimension 1 operator (i.e., its anomalous dimension is 1 as well), and so has the same dimension as a conserved current. We also note that the typical AdS$_3$ gauge fields which arise in AdS$_3$/CFT$_2$ are not strict Maxwell fields, but have Chern-Simons terms which in $d = 2$ effectively provide a mixing between A_I and F^I. Clearly, these Chern-Simons terms should be taken into account in a complete analysis.

The most general boundary conditions arise for $d = 3$, and the results are similar to those for scalar fields near, but slightly above, the Breitenlohner-Freedman bound.

For $d = 3$, any local boundary condition relating A_I and F^I is allowed, so long as it is determined by a potential, see (3.6). We find Lorentz invariant CFTs associated with the boundary conditions $A_I = 0$ and $F^I = 0$, and any linear boundary condition is associated with a renormalization-group flow from the $F^I = 0$ theory (the UV fixed point) to the $A_I = 0$ theory (the IR fixed point).

As in the case of $d = 2$, the dual operator in the $F^I = 0$ theory is a vector gauge field with conformal dimension 1. Using the associated gauge freedom, the relevant operators that generate such renormalization-group flows can be expressed in two distinct ways. When expressed in a gauge-invariant form, the operator is non-local. However, with the gauge condition implied by the general boundary condition, the operator is completely local. This is consistent with the fact that the bulk theory in this gauge satisfies local field equations and a local boundary condition. In particular, the bulk advanced and retarded Green's functions $G^{\pm}(x, y)$ vanish unless x and y are connected by a causal curve. Since the supports of advanced and retarded Green's functions in the CFT are given by the boundary limits of those for the bulk Green's function, we see that the CFT satisfies the usual notion of causality in this gauge.

In the case $d = 4, 5$, one must fix the vector part of A_I, and there is no $F^I = 0$ theory. However, the scalar part still admits a variety of boundary conditions. For $d = 5$, this leads to a new 'hybrid' CFT defined by the boundary conditions $F^t = 0, A_i = 0$, which explicitly break Lorentz invariance. This CFT is a UV fixed point for renormalization-group flows that lead to the $A_I = 0$ CFT where Lorentz invariance is restored.[4] For $d = 4$ such boundary conditions lead to a logarithmic field theory. For $d \geq 6$, only the $A_I = 0$ theory is allowed.

Since we consider only gauge fields (which necessarily have vanishing mass), the dimension dependence above reflects the fact that, in the case of scalar fields, the freedom to choose non-trivial boundary conditions depends on the relation between the mass m and the dimension d. In that case one understands the allowed range (2.2) in terms of the unitarity bound $\Delta \geq (d-2)/2$ on the conformal dimension of scalar operators. If a CFT with 'conjugate' boundary conditions were allowed for scalars with mass above the upper boundary of (2.2), it would contain an operator violating this bound. Hence, it does not exist.[5] We see that the picture here is similar: any $F^I = 0$ CFT would contain a vector gauge field of conformal dimension 1. If such a theory were to exist for $d > 4$, the corresponding operator would have negative anomalous dimension. The case $d = 4$ is a marginal special case. It would be interesting to determine if the failure of the $A_I = 0$ theory for $d = 2$ and the failure of the hybrid theories for $d > 5$ can be understood in a similar way.

In the above, we considered a free Maxwell gauge field. It is interesting, however, to extrapolate our results to more complicated cases. For simplicity, we focus on the case $d = 3$. One immediate generalization is to the SO(8) non-abelian gauge fields of AdS$_4$

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[4]This hybrid CFT and others like it also exist for the case $d = 3$.

[5]The case where the upper bound of (2.2) is saturated and $\Delta = (d-2)/2$ is clearly marginal. In principle such a CFT is allowed, but the corresponding anomalous dimension would have to vanish. Since for this case normalizability fails in the bulk, one expect that there is no such AdS/CFT correspondence.

supergravity [45, 46]. As mentioned in [15], one expects that the asymptotics and thus the boundary conditions are governed by the linear theory, and that there is again a UV CFT dual to the boundary conditions $F^{IA} = 0$, where A is an adjoint SO(8) index. This CFT appears to contain an SO(8) gauge field in addition to the usual SU(N) gauge field. In some sense, the usual R-symmetry has been gauged.

Our results for vector gauge fields were based heavily on the analysis of Ishibashi and Wald [14], who also analyzed boundary conditions for rank 2 tensor fields in the bulk; i.e., for the linearized graviton. Again for this case, very general boundary conditions were allowed for $d = 3$. Extrapolating our results above, we therefore predict a new Lorentz-invariant AdS$_4$/CFT$_3$ correspondence where the graviton satisfies 'conjugate' boundary conditions in the bulk. With the usual boundary conditions, the graviton is dual to the CFT stress-energy tensor. However, for the conjugate boundary conditions the bulk graviton must be dual to a spin-2 operator with spin-2 gauge invariance; i.e., the CFT$_3$ is in fact a (conformal) quantum gravity theory! A similar observation was made in [16, 17] working from the CFT side. It is reassuring that quantum gravity in $d = 3$ is a finite theory [47 – 50] due to the lack of propagating degrees of freedom for the graviton [51, 52]. For $d = 4, 5$ we expect hybrid theories of what might still be called 'quantum gravity,' but which break (local) Lorentz invariance.

A further generalization would be the inclusion of supersymmetry. The theories discussed above, and those dual to deformations of bulk scalars, are not supersymmetric because they include no corresponding deformations of the Fermions. However, one expects the allowed boundary conditions for bulk spinor fields to be qualitatively similar to those for fields of integer spin, with appropriate combinations providing super-symmetric theories. We therefore conjecture that the 'conjugate' AdS$_4$/CFT$_3$ duality described above (with quantum gravity in the CFT) can be taken to be maximally supersymmetric. See [53] for details of $d = 3$ conformal supergravity.

Finally, one may ask about the stability of such exotic theories. Since such stability should be guaranteed by supersymmetry, stability itself may be taken as a test of the self-consistency of the above conjectures. At the linearized level for fields of spin 0,1,2, this question was fully analyzed for the dynamical modes by Ishibashi and Wald [14]. Interpreting their results in our language, the $F^I = 0$ and hybrid theories are indeed linearly stable.

After posting the first version of this paper on the hep-th arxiv, we became aware of a body of literature containing the main results presented here for the case $d = 3$. In particular, the fact that 'conjugate' boundary conditions in AdS$_4$ are dual to a CFT$_3$ with a dynamical gauge theory was described in [15] and discussed further in [16 – 19]. Our work extends this earlier work to other dimensions, and introduces the notion of hybrid boundary conditions.

The perspective we take is also rather different from that of this previous work. Whereas [15] started from the CFT description, and focused on the action of SL(2, Z), we have started from the bulk spacetime description, and considered all the possible boundary conditions such that the symplectic flux (2.6) vanishes through any region R on the boundary. That is, we start from a fixed notion of the bulk gauge potential and a fixed

form of the symplectic structure, and then consider all the allowed boundary conditions for this formulation of the bulk theory. In contrast, the approach of [15] was to consider the usual $A_I = 0$ boundary condition for the different notions of the bulk gauge potential related by $\mathrm{SL}(2, Z)$ and thus to derive boundary conditions on the original gauge potential. Note that the symplectic flux defined by the analogue of (2.6) for an $\mathrm{SL}(2, Z)$-transformed gauge potential $\tilde{A}_\mu$ will in general differ by a boundary term from the one we used here. As a result, some of the boundary conditions $\tilde{A}_I = 0$ will not preserve our choice of symplectic flux. However, the boundary term in the symplectic structure is just that associated with the addition to the action of a Chern-Simons boundary term, constructed in general from both the vector potential and the dual magnetic vector potential. Thus, so long as one is careful to include boundary terms in the action which provide an appropriate definition of symplectic flux, one can impose a general boundary condition in terms of any formulation of the bulk theory: for example, the general boundary condition $\epsilon^{IJK} \mathcal{D}_I A_J = \lambda_J^K F^J$ imposed by [15 – 19]. As one would expect, differences in perspective do not change the physics.

Acknowledgments

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