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CPT and other symmetries in string/M theory

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Abstract: We initiate a search for non-perturbative consistency conditions in M theory. Some non-perturbative conditions are already known in type-I theories; we review these and search for others. We focus principally on possible anomalies in discrete symmetries. It is generally believed that discrete symmetries in string theories are gauge symmetries, so anomalies would provide evidence for inconsistencies. Using the orbifold cosmic string construction, we give some evidence that the symmetries we study are gauged. We then search for anomalies in discrete symmetries in a variety of models, both with and without supersymmetry. In symmetric orbifold models we extend previous searches, and show in a variety of examples that all anomalies may be canceled by a Green-Schwarz mechanism. We explore some asymmetric orbifold constructions and again find that all anomalies may be canceled this way. Then we turn to type-IIB orientifold models where it is known that even perturbative anomalies are non-universal. In the examples we study, by combining geometric discrete symmetries with continuous gauge symmetries, one may define non-anomalous discrete symmetries already in perturbation theory; in other cases, the anomalies are universal. Finally, we turn to the question of CPT conservation in string/M theory. It is well known that CPT is conserved in all string perturbation expansions; here in a number of examples for which a non-perturbative formulation is available we provide evidence that it is conserved exactly.

Keywords: Superstrings and Heterotic Strings, M-Theory, M(atrix) Theories, Discrete and Finite Symmetries.

Contents

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1. Introduction

In field theory, the idea that perturbative and non-perturbative anomalies can render a theory inconsistent is familiar. In weakly coupled string theory, consistency conditions of various types are familiar. For closed strings, modular invariance is necessary for a unitary, Lorentz-invariant perturbation theory. Even classical solutions corresponding to smooth manifolds can violate this condition [1]. For open strings, one has tadpole conditions. Non-perturbatively, some examples of consistency conditions are known in type-I theories [2], and duality considerations suggest [3] (but do not firmly establish [4]) the existence of conditions in some closed string theories.

Lacking a non-perturbative formulation of string theory, a search for non-perturbative anomalies involves either study of topological objects (branes, etc.) or examination of features of the low energy theory. In this note, we adopt the latter approach, looking at string states which should have a conventional four dimensional effective field theory description. We focus, in particular, on the possibility of anomalies in discrete symmetries. It is generally believed that discrete symmetries in string theory are gauged. This follows from a general prejudice that global symmetries are implausible in a theory of gravity. It is also possible, in many cases, to show explicitly that these symmetries are remnants of, say, gauge and general coordinate invariances of a higher dimensional theory. The most convincing demonstration that any particular symmetry is a gauge symmetry comes from the study of cosmic string solutions [5]. We will discuss aspects of this problem further below.

To convincingly demonstrate an anomaly in such a symmetry in four dimensions, one must examine instantons in the low energy theory, and determine if they violate the symmetry. One must also check that one cannot cancel the anomalies by assigning transformation laws to moduli of the theory, i.e. by a Green-Schwarz mechanism. The possibility of such cancellations had been discussed [6] in the case of smooth Calabi-Yau compactifications, but it actually follows from an old result of Witten [7]. Witten showed the absence of global anomalies for any configuration which can be described by the effective ten-dimensional supergravity theory. Thus any such anomaly must arise through some inherently stringy effects. One might imagine that already some of the gauge symmetries of orbifold models are sufficiently "stringy" that Witten's result might not apply. However, while extensive searches among symmetric, supersymmetric orbifolds [8] found many examples of anomalies, all could be canceled by Green-Schwarz terms.

In the present note we extend this search in two ways. We consider non-supersymmetric, symmetric orbifolds. One might hope that, given that non-supersymmetric string theories seem to suffer from a variety of diseases (particularly tachyons and other instabilities), that perhaps they would often suffer from discrete anomalies as well. However, at least in the small sample of such orbifolds studied here, this does not occur.

We consider, also, asymmetric orbifolds. These one might imagine are more "stringy" than symmetric orbifolds. There is not, in general, an obvious procedure for blowing them up to obtain theories on smooth manifolds, so they may not fall within the class of theories considered by Witten. Here we construct a variety of asymmetric orbifolds of the heterotic string, both with and without space time supersymmetry, and do not find anomalies.

We then search for discrete anomalies in type-IIB orientifold constructions. Here we do find non-universal anomalies. In contrast to the symmetric and asymmetric orbifolds though, here there are gauge singlet massless states which have non-universal couplings to gauge fields, already at tree level. The non-universal discrete anomalies may in principle be canceled by assigning a non-linear transformation to these fields. The states though, are *twisted states*.

To convincingly demonstrate the *absence* of an anomaly, we would have to prove the existence of such a non-universal, non-linear transformation law. In the case of anomalies in continuous symmetries, there is a one-loop computation one can do to verify the existence of the Green-Schwarz term. There isn't something analogous here. Our approach is to assume that if the anomaly may be canceled by such a field, then it is. The fact that it is always possible to assume that the anomaly is canceled this way we take as evidence that this view is correct.

The remainder of the paper investigates a number of M-theory backgrounds in which it is possible to address non-perturbatively the question of CPT conservation. We study M-theory in eleven dimensions, on S_1/Z_2, and on T^3. At the classical level each of these theories has a CPT symmetry. One may ask whether this symmetry is preserved by quantum effects. To address this, we study matrix model descriptions for these backgrounds

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and identify a conserved CPT symmetry. Because the matrix models we study do not include all types of five-branes, we cannot argue that this is a proof of CPT conservation, but it is certainly strong evidence.

The paper is organized as follows. In the next section, we review some aspects of non-perturbative anomalies in type-I theories. We show that orbifold cosmic strings exist for the cases we study, and explain why this is evidence (but not proof) that these symmetries are gauged. In the section 4, we discuss some symmetric orbifolds without supersymmetry, and demonstrate the cancellation of anomalies. Section 5 contains our study of asymmetric orbifolds. We construct the required projectors, and give the spectra for several models. The gauge symmetries of the models are often very intricate, and perturbative anomaly cancellation provides a non-trivial check on the massless spectrum. We identify various discrete symmetries of the models, and compute the anomalies. The principal subtlety in this analysis lies in determining the transformation properties of the twisted ground states under the discrete symmetries. In section 6 we search for discrete anomalies in type-IIB string compactifications with D5 branes located at orbifold fixed points. The final section studies the CPT properties of a few M-theory backgrounds and their corresponding matrix model descriptions.

The appendices provide the details relevant for constructing the states in both the symmetric and asymmetric orbifolds and the type-IIB orientifolds, as well as their charges under the discrete symmetries of the low-energy theory.

2. Non-perturbative constraints in type-I theory and Hořava-Witten theory

It is known that there are non-perturbative consistency conditions for type-I theories. Ref. [2] provides a particularly simple example of the problem. When one mods out a weak coupling heterotic string by some symmetry, one of the conditions is that the action of the symmetry should be well defined. In the $O(32)$ heterotic string, this leads, among other constraints, to restrictions on the possible twists in order that the action of the symmetry on fermions and on spinor representations of $O(32)$ be sensible. In the type-I theory, however, the spinor representations of $O(32)$ only arise at the non-perturbative level. As a result, some vacua which seem perfectly sensible in perturbation theory are ill defined non-perturbatively. On the weak coupling heterotic side, the problem is obvious, but on the type-I side, an understanding of the non-perturbative structure of the theory is required.

This type of argument can be extended to many cases. For example, for weak coupling heterotic strings compactified on Calabi-Yau spaces, there are restrictions on the possible Wilson lines arising from modular invariance in twisted sectors; the condition is simply level matching. On the type-I side, these constraints can be understood by considering a $D1$ brane wrapped on a non-contractible loop. Now there is a consistency condition for the existence of this D brane: the states of the brane must level match. This can be understood by noting that level matching is just the statement that the state is invariant under shifts of σ by π, where now σ can be taken as a parameter describing the wrapped string. Since the $D1$ string is just the dual of the heterotic string, this level matching condition is identical to the level matching condition of the weak coupling theory.[1]

We can apply this sort of reasoning to the strong coupling limit of the $E_8 \times E_8$ theory, i.e. to Hořava-Witten theory [9]. Again, in the weak coupling limit, one has the constraints of level matching and consistency of spinor propagation. In the strong coupling limit one obtains these requirements by considering membranes stretched between the two walls and wrapped around non-contractible loops. It is interesting that if one does not impose the level matching condition, one does find discrete anomalies.

Because we understand in each of these cases the consistency condition on the weak coupling side, these conditions are not in some sense new. A new condition, however, arises in the case of Hořava-Witten theory when one breaks the supersymmetry by imposing a different chirality condition on each of the two walls. The condition is, again, level matching for the wrapped branes; the condition is new, because we do not know a weakly coupled dual.[2]

These examples illustrate that there are non-perturbative consistency conditions in M theory. For closed string theories, however, we do not have examples of non-perturbative consistency conditions. Indeed, with the exception of the non-supersymmetric Hořava-Witten example (and perhaps analogous examples in type-I theories), all of the known examples can be understood in terms of modular invariance in closed string duals. It would be interesting to find examples of inconsistencies, then, in weakly coupled closed string theories. Much of the rest of this note is devoted to such a search.

3. Discrete symmetries as gauge symmetries: cosmic strings

It is believed that discrete symmetries in string/M theory are gauged. Certainly this can be seen in many examples. The discrete symmetries of Calabi-Yau spaces are often discrete subgroups of the full ten-dimensional Lorentz group. Many duality symmetries are gauge symmetries, as is the symmetry which exchanges the two E_8's of the heterotic string. Perhaps the most conclusive demonstration that a discrete symmetry is gauged is provided by the presence cosmic strings. The point is illustrated by a U(1) theory with a Higgs field of charge N. A Higgs expectation value breaks $\mathrm{U}(1) \rightarrow Z_N$. Such a theory admits cosmic strings. If one brings a field of charge 1 around such a string, it picks up a phase $\alpha = e^{2\pi i/N}$. Similarly, the angular momenta of fields are fractional for such a field, $m - \frac{1}{N}$.

One way to test whether discrete symmetries in string theories are gauge symmetries, then, is to construct cosmic string solutions for which particles pick up a suitable phase when they circulate around the string. Orbifold cosmic strings [10] provide an example of this phenomenon. Consider a weakly coupled string orbifold with a Z_2, Z_3, Z_4 or Z_6 symmetry. For example, for the well-known Z_3 orbifold, there is, at suitable points in the moduli space, Z_6 symmetries which acts on each of the orbifold planes. One can consider compactification of this theory on an extremely large torus, with a Z_6 symmetry, and mod

[1] We thank Joe Polchinski for a discussion of these issues.

[2] We thank Tom Banks for this example.

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out by the product of one of the internal symmetries and this symmetry in what we will call the transverse space. In the language of orbifolds, one can think of modding out the torus, T_8, by a product of symmetries, g and h, where g is the usual Z_3 and h is the additional Z_6.

The fixed points of the symmetry h are naturally thought of as cosmic strings. Now consider the various sectors. First, there are the sectors *untwisted* by h. These sectors are distinguished by the existence of a set of momenta in the transverse directions. For a given momentum state, $|p\rangle$, one can construct states transforming as α^k, from

$$|p^{(k)}\rangle = \sum_{r=1}^{N} \alpha^{rk} h^r |p\rangle \qquad h|p^{(k)}\rangle = \alpha^k |p^{(k)}\rangle\,. \tag{3.1}$$

In general, states in this sector can be written as products,

$$|\psi\rangle = |\text{internal}\rangle \times |p^{(k)}\rangle\,. \tag{3.2}$$

Invariant states are states for which the transformation of the transverse momentum under h compensates the discrete transformation of the *internal* state. This is precisely the effect expected for a cosmic string.

Banks (unpublished) points to other examples. Consider the SL$(2,\mathbb{Z})$ symmetry of the type-IIB theory in ten dimensions. The analog of the cosmic string, in this case, is a seven brane. It is easy to see that $\tau \to \tau + 1$ is connected with the behavior of the dilaton-axion model as they encircle the seven brane.

4. A search for discrete gauge anomalies in weakly coupled heterotic strings: symmetric orbifolds with or without Wilson lines

The discussion above suggests that in many — and possibly all — cases discrete symmetries in string theory are gauge symmetries, and any violation of such symmetries would imply an inconsistency. So it is interesting to search for such violations. In the past, limited searches have been conducted among supersymmetric models, and have failed to find examples of such anomalies [8]. Here we enlarge the search, including models without supersymmetry. This case is particularly interesting, since such models have a number of troubling features. They typically contain tachyons, at least in regions of the moduli space, and they are subject to catastrophic decay processes [11, 12].

Before considering weakly coupled closed strings, it is useful to note that discrete anomalies are closely connected to the non-perturbative consistency conditions for type-I theory and Hořava-Witten theory which we mentioned earlier. If we consider compactification of these theories on a large radius Calabi-Yau, with Wilson lines, in general we find discrete gauge anomalies, i.e. we find that instantons in non-abelian groups in the low energy theory violate the symmetry. However, if one imposes the level-matching conditions, the anomalies cancel. This point is illustrated by the case of the quintic in CP^4, described at length in [13]. This theory has a set of Z_5 symmetries. Consider the $O(32)$ theory, for example, and includes a Wilson line $(1, 1, 0, \ldots, 0)$, where the notation means (in the fermionic formulation) group the fermions into sixteen complex pairs, and twist the first two by a Z_5 phase, while leaving the others alone. This particular choice of Wilson lines does not level match. In this case one finds that the discrete anomalies in the low energy groups are not the same. Similar examples arise in the $E_8 \times E_8$ theory.

4.1 Discrete symmetries

In this section, we focus on discrete symmetries of orbifold compactifications to four dimensions. We will limit our considerations to cases where the original torus is a product of three two-dimensional tori. In general, these models have a number of discrete symmetries. The orbifold group removes one linear combination of symmetries of the lattice, but others survive and are potentially anomalous. These are the quantum Z_N symmetry [14], and for toroidal compactifications, point group symmetries that are independent rotations of two of the three tori.

The quantum symmetry is the automorphism group of the orbifold group that for a Z_N orbifold is the Z_N cyclic permutation of its elements. Untwisted states are neutral, whereas states in the k-th twist sector have charge k/N. String interactions conserve this symmetry.

4.2 Models

Perhaps the simplest models to study in a search for such anomalies are symmetric orbifolds of the heterotic string. Z_6 orbifolds provide examples which are chiral and break supersymmetry. The Z_6 symmetry can be realized by a simple product of two dimensional lattices. At such points in moduli space, the models have 3 Z_6 symmetries (and possibly additional symmetries, such as permutations). One can mod out by one combination of symmetries; others survive, and are potentially anomalous. We have considered several variations on the Z_6 orbifold, with and without supersymmetry. In each case, we find that the anomalies cancel.

4.2.1 Supersymmetric Z_6

A supersymmetric Z_6 twist, satisfying level matching is

$$\phi = \frac{1}{6}(1, 1, -2) \tag{4.1}$$

Choosing the standard embedding for the gauge shift breaks SO(32) → SU(2) × U(1) × SO(26). The massless fermions are given in table 1. The table also lists the discrete charges of the states under the two Z_6 symmetries that remain after orbifolding. Here γ corresponds to rotations of the third plane, whereas η corresponds to simultaneous Z_6 rotations of the first two planes. All the symmetries are anomaly-free. One also finds that the quantum Z_6 symmetry is not anomalous (not shown).

4.2.2 Nonsupersymmetric Z_6 model with Wilson lines

A *non-supersymmetric* symmetric Z_6 orbifold with Wilson lines, satisfying the above level matching conditions is given by

$$\phi = \frac{1}{6}(0, 1, 1, 4)\,, \qquad \beta = \frac{1}{6}(1, 1, 4; 0^{13})\,, \qquad a_5 = -a_6 = \frac{1}{6}(0^3, 4^6, 0^7)\,. \tag{4.2}$$

sector	$G = \mathrm{SO}(26) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2$	Z_6	Z_6'
untwisted	$(\mathbf{325}, \mathbf{1}; 0, 0; \eta, \gamma^{1/2}) + (\mathbf{1}, \mathbf{3}; 0, 0, \eta, \gamma^{1/2})$ $(\mathbf{26}, \mathbf{1}; -1, -1; \eta^{-1}, \gamma^{\frac{1}{2}}) + 2(\mathbf{1}, \mathbf{2}; 2, 1; 1, \gamma^{-\frac{1}{2}}) + 2(\mathbf{26}, \mathbf{2}; -1, 0; 1, \gamma^{-\frac{1}{2}})$	(γ^3, γ^5)	(η^4, η^4)
$n = 1$	$6(\mathbf{1}, \mathbf{2}; 0, -\frac{2}{3}; \eta^{-\frac{5}{3}}, \gamma^{\frac{1}{6}}) + 3(\mathbf{26}, \mathbf{1}; -1, -\frac{2}{3}; \eta^{-\frac{2}{3}}, \gamma^{\frac{1}{6}})$	(γ, γ)	(η^2, η^2)
$n = 2$	$10(\mathbf{26}, \mathbf{1}; -1, -\frac{1}{3}; \eta^{-\frac{1}{3}}, \gamma^{-\frac{1}{6}}) + 5(\mathbf{26}, \mathbf{1}; -1, -\frac{1}{3}; \eta^{-\frac{1}{3}}, \gamma^{\frac{17}{6}})$ $+20(\mathbf{1}, \mathbf{2}; 0, -\frac{1}{3}; \eta^{-\frac{4}{3}}, \gamma^{-\frac{1}{6}}) + 10(\mathbf{1}, \mathbf{2}; 0, -\frac{1}{3}; \eta^{-\frac{4}{3}}, \gamma^{\frac{17}{6}})$ $+8(\mathbf{1}, \mathbf{2}; 0, -\frac{1}{3}; \eta^{\frac{8}{3}}, \gamma^{-\frac{7}{6}}) + 4(\mathbf{1}, \mathbf{2}; 0, -\frac{1}{3}; \eta^{\frac{8}{3}}, \gamma^{\frac{11}{6}})$	(γ, γ^5)	(η^2, η^4)
$n = 3$	$6(\mathbf{26}, \mathbf{1}; -1, 0; 1, \gamma^{-\frac{1}{2}}) + 5(\mathbf{26}, \mathbf{1}; 1, 0; \eta^4, \gamma^{-\frac{1}{2}})$ $+12(\mathbf{1}, \mathbf{2}; 0, 0; \eta^{-1}, \gamma^{-\frac{1}{2}}) + 10(\mathbf{1}, \mathbf{2}; 0, 0; \eta^5, \gamma^{-\frac{1}{2}})$	(γ, γ)	(η^4, η^2)
total		$(1, 1)$	$(1, 1)$

Table 1: Supersymmetric symmetric Z_6 orbifold with standard embedding. Here γ and η correspond to Z_6 rotations of the third torus and simultaneous Z_6' rotations of the first and second torus, with $\gamma^6 = \eta^6 = 1$. Z_6 and Z_6' discrete gauge anomalies are indicated.

The standard embedding for β combined with the choice of Wilson line breaks $\mathrm{SO}(32) \to \mathrm{SO}(14) \times \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3$. The states for this model are given in table 2. In the n-th twisted sector there will also be a Wilson line for each fixed point, and in general the states at each fixed point will be different. The third plane, which has the Wilson line, has 3 fixed points for $n = 1$ and 2, and no fixed points for $n = 3$. In table 2 the integer $m = 0,1$ and 2 refer to these fixed points. It also lists the discrete charges of the states under the two Z_6 symmetries that remain after orbifolding. Here γ corresponds to rotations of the third plane, whereas η corresponds to simultaneous Z_6 rotations of the first two planes.

The discrete anomalies from each sector are found in the last two columns of each table. One finds that the discrete anomalies are universal, and so may be canceled by a discrete Green-Schwarz mechanism. Note that the untwisted and twisted sectors each have non-universal anomalies and it is only their total that is universal. This feature is typical of the models we study. The quantum twist symmetry is also non-anomalous, although it too receives contributions from all twisted sectors (not shown).

5. A search for discrete gauge anomalies in weakly coupled heterotic strings: asymmetric orbifolds

Given Witten's results, one might have suspected that one would not find anomalies in symmetric orbifold models. In the supersymmetric case, these models can be blown up to smooth Calabi-Yau manifolds, and no anomalies can appear in this limit. In the non-supersymmetric cases, the situation is less clear, since one does not have an argument that one can find solutions of the classical equations corresponding to smooth manifolds, but one might still suspect that these models are not so different from the smooth cases, and that this is why we fail to find anomalies. Asymmetric orbifolds [15], one might hope, are more likely "stringy," and results obtained by considering smooth geometries might not hold. In this section, we search for (but do not find) discrete anomalies in asymmetric orbifold models, both with and without supersymmetry (in the weak coupling limit).

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(n,m)	$G = \mathrm{SO}(14)\times\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)^3$	Z_6	Z_6'
$(0,0)$	$2(\mathbf{14},\mathbf{1},\mathbf{1};0,-1,0;\gamma^{-\frac{1}{2}},1)+(\mathbf{14},\mathbf{1},\mathbf{2};-1,0,0;\gamma^{\frac{1}{2}},\eta^5)$ $+(\mathbf{1},\mathbf{1},\mathbf{2};\mp1,\pm1,0;\gamma^{\frac{1}{2}},\eta)+(\mathbf{1},\mathbf{1},\mathbf{2};1,1,0;\gamma^{\frac{1}{2}},\eta^5)$	$(1,1,\gamma^{5/2})$	$(\eta^2,1,\eta^5)$
$(2,0)$	$4(\mathbf{14},\mathbf{1},\mathbf{1};-\frac{2}{3},-\frac{1}{3},0;\gamma^{-\frac{1}{6}},\eta^{8/3})+4(\mathbf{1},\mathbf{6},\mathbf{1};-\frac{2}{3},-\frac{1}{3},1;\gamma^{-\frac{1}{6}},\eta^{8/3})$ $+4(\mathbf{1},\overline{\mathbf{6}},\mathbf{1};-\frac{2}{3},-\frac{1}{3},-1;\gamma^{-\frac{1}{6}},\eta^{8/3})+8(\mathbf{1},\mathbf{1},\mathbf{2};\frac{1}{3},-\frac{1}{3},0;\gamma^{-\frac{1}{6}},\eta^{5/3})$ $+5(\mathbf{1},\mathbf{1},\mathbf{2};\frac{1}{3},-\frac{1}{3},0;\gamma^{-\frac{7}{6}},\eta^{-1/3})$	$(\gamma^{14/3},\gamma^{14/3},\gamma^{29/6})$	$(\eta^{10/3},\eta^{10/3},\eta^{17/3})$
$(2,1)$	$5(\mathbf{1},\mathbf{1},\mathbf{2};\frac{1}{3},-\frac{1}{3},2;\gamma^{-\frac{1}{6}},\eta^{-\frac{1}{3}})+4(\mathbf{1},\overline{\mathbf{6}},\mathbf{1};-\frac{2}{3},-\frac{1}{3},1;\gamma^{-\frac{1}{6}},\eta^{8/3})$	$(1,\gamma^{-2/3},\gamma^{-5/6})$	$(1,\eta^{-4/3},\eta^{-5/3})$
$(2,2)$	$5(\mathbf{1},\mathbf{1},\mathbf{2};\frac{1}{3},-\frac{1}{3},-2;\gamma^{-\frac{1}{6}},\eta^{-\frac{1}{3}})+4(\mathbf{1},\mathbf{6},\mathbf{1};-\frac{2}{3},-\frac{1}{3},-1;\gamma^{-\frac{1}{6}},\eta^{8/3})$	$(1,\gamma^{-2/3},\gamma^{-5/6})$	$(1,\eta^{-4/3},\eta^{-5/3})$
$n=3$	$5(\mathbf{1},\mathbf{1},\mathbf{2};0,1,0;\gamma^{-\frac{1}{2}},\eta^2)+6(\mathbf{1},\mathbf{1},\mathbf{2};0,-1,0;\gamma^{-\frac{1}{2}},1)$ $+5(\mathbf{14},\mathbf{1},\mathbf{2};0,0,0;\gamma^{-\frac{1}{2}},\eta^4)$	$(\gamma^4,1,\gamma^{9/2})$	$(\eta^2,1,\eta^2)$
$(5,0)$	$2(\mathbf{14},\mathbf{1},\mathbf{1};\frac{1}{3},-\frac{1}{3},0;\gamma^{-\frac{1}{6}},\beta^{\frac{5}{3}})+2\times(\mathbf{1},\overline{\mathbf{6}},\mathbf{1};\frac{1}{3},-\frac{1}{3},-1;\gamma^{-\frac{1}{6}},\eta^{5/3})$ $2(\mathbf{1},\mathbf{6},\mathbf{1};\frac{1}{3},-\frac{1}{3},1;\gamma^{-\frac{1}{6}},\eta^{\frac{5}{3}})+3\times(\mathbf{1},\mathbf{1},\mathbf{2};-\frac{2}{3},-\frac{1}{3};0,\gamma^{-\frac{1}{6}},\eta^{8/3})$ $+(\mathbf{1},\mathbf{1},\mathbf{2};\frac{4}{3},-\frac{1}{3},0;\gamma^{-\frac{1}{6}},\eta^{\frac{2}{3}})+(\mathbf{1},\mathbf{1},\mathbf{2};-\frac{2}{3},-\frac{1}{3},0;\gamma^{-\frac{7}{6}},\eta^{\frac{2}{3}})$	$(\gamma^{16/3},\gamma^{16/3},\gamma^{25/6})$	$(\eta^{2/3},\eta^{2/3},\eta^{-8/3})$
$(5,1)$	$2(\mathbf{1},\overline{\mathbf{6}},\mathbf{1};\frac{1}{3},-\frac{1}{3},1;\gamma^{-\frac{1}{6}},\eta^{\frac{5}{3}})+(\mathbf{1},\mathbf{1},\mathbf{2};-\frac{2}{3},-\frac{1}{3},2;\gamma^{-\frac{1}{6}},\eta^{\frac{2}{3}})$	$(1,\gamma^{-1/3},\gamma^{-1/6})$	$(1,\eta^{10/3},\eta^{2/3})$
$(5,2)$	$2(\mathbf{1},\mathbf{6},\mathbf{1};\frac{1}{3},-\frac{1}{3},-1;\gamma^{-\frac{1}{6}},\eta^{\frac{5}{3}})+(\mathbf{1},\mathbf{1},\mathbf{2};-\frac{2}{3},-\frac{1}{3},-2;\gamma^{-\frac{1}{6}},\eta^{\frac{2}{3}})$	$(1,\gamma^{-1/3},\gamma^{-1/6})$	$(1,\eta^{10/3},\eta^{2/3})$
total		$(\gamma^2,\gamma^2,\gamma^2)$	(η^2,η^2,η^2)

Table 2: Nonsupersymmetric symmetric Z_6 model with the standard embedding and Wilson lines. Here γ and η correspond to Z_6 rotations of the third torus and simultaneous Z_6' rotations of the first and second torus, with $\gamma^6 = \eta^6 = 1$. Z_6 and Z_6' discrete gauge anomalies are indicated.

The asymmetric orbifold construction has been developed by a number of authors. We follow, particularly, the work of [16] and [17]. The procedure is straightforward. We start with a toroidal compactification of the heterotic string, on a lattice with a set of unbroken T-duality symmetries. We mod out this theory by a subgroup of this symmetry, and add twisted sectors so as to obtain a modular-invariant partition function. The result of this construction can be expressed in terms of a set of projectors for each twisted sector. We then construct the massless spectrum. Typically, perturbative anomaly cancellation provides a highly non-trivial check on the construction. Then we examine the discrete symmetries of the orbifold theory, and determine the transformation properties of the states under these symmetries. This is slightly non-trivial, since the twisted ground states themselves transform. With this information in hand, it is straightforward to ask whether instantons of the low energy theory violate the discrete symmetry.

5.1 Nonsupersymmetric examples

We first consider a set of non-supersymmetric orbifolds. As in the case of symmetric orbifolds above, we will not worry whether the states have tachyons (in the symmetric cases, anomaly cancellations occurred even with tachyons).

The fields in the theory are the 16 freely interacting left-moving (LM) real scalars H_L^a, three LM complex scalars X_L^i, three right-moving (RM) complex scalars X_R^i and their fermionic partners $\tilde{\psi}_i$. The scalars X cannot be interpreted as describing the coordinates of an internal manifold as the left and right movers are treated differently.

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For all of these orbifolds, we will take the underlying lattice to be

$$\Gamma_{(16)} \times \Gamma_{(4,4)}(D_4) \times \Gamma_{(2,2)}(A_2)\,. \tag{5.1}$$

The lattice $\Gamma_{(2,2)}(A_2)$ is formed from the SU(3) lie algebra in the manner described in appendix B. The weight lattice is generated by an element e_1 from the $\mathbf{3}$, and an element e_2 from the $\overline{\mathbf{3}}$. It is important to note that the weight lattice has both an obvious Z_3 and a less obvious Z_6 automorphism symmetry. The Z_3 is a Weyl symmetry of the weight lattice, corresponding to a discrete SU(3) rotation that permutes the elements of the $\mathbf{3}$. The lorentzian lattice $\Gamma_{(2,2)}(A_2)$ has a separate left and right Z_3 symmetry, since this automorphism preserves the condition $p_L - p_R \in \Lambda_R$. The Z_6 symmetry is also an automorphism of the SU(3) lattice, but it corresponds to a Z_6 rotation of $e_1 \to e_2$, $e_2 \to -e_1$, which is a mapping between the conjugacy classes. Consequently, this is a symmetry of $\Gamma_{(2,2)}(A_2)$ only if it acts simultaneously on the left and the right.

The $\Gamma_{(4,4)}(D_4)$ lattice is formed from the SO(8) lie algebra as described in appendix B. The weight lattice has four conjugacy classes, the root, the vector and the two spinorial classes. These can be generated by four elements; one element may be chosen from the spinorial class, and the other three can be taken from the vector class. The SO(8) lattice has a Z_6 Weyl symmetry [18]. Here we focus on the obvious Z_2 Weyl symmetry that sends a weight $w \to -w$. The lattice also has a S_3 permutation symmetry, which is just triality.

To construct an asymmetric orbifold we divide by a subset of these Weyl symmetries; in particular the Z_3 and Z_2. For the right movers, in each case, we will take the projector to be the Z_6 twist

$$X_R^i \to X_R^i e^{2\pi i \phi_R^i} \tag{5.2}$$

where

$$\phi_R = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right), \tag{5.3}$$

describing a Z_3 and Z_2 twist in the two and four dimensional lattices, respectively. This twist leaves no unbroken supersymmetries. The RM weights r, describing the positive helicity massless fermions for the different sectors are given in the table 3. (Both states with a '$\pm$' satisfy the projector and are included in the spectrum.) For the left movers we choose $\phi_L = 0$.

sector	Ramond state
untwisted	$r_1 = (\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ $r_2 = (\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2})$
$n=1$	none
$n=2$	$(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}) + (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$
$n=3$	$(\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2})$
$n=4$	$(\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}) + (\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{5}{2})$
$n=5$	$(\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{5}{2})$

Table 3: Non-supersymmetric Z_6: positive-helicity RM Ramond sector weights (r). The ground states are given by $r + n\phi_R$.

One may check that this twist satisfies the second and last level matching conditions (B.12) and (B.14). The details for satisfying (B.14) are provided in appendix B.

On the left, we will take the group action to be a shift by a vector β_L and, as already stated, $\phi_L = 0$. Our different models will be characterized by different choices of β_L that satisfy the level matching conditions (B.12) and (B.14).

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5.1.1 Projectors

It is straightforward to work out the projectors in the different sectors. In doing so, the character transformation formulae (see, for example, [17]), are useful. Performing so-called S and T transformations repeatedly, one obtains the partition functions. To obtain the projectors for the massless fermions, we note that the RM Ramond ground state is already massless, which has $p_R = 0$. In addition, for these models all the charged states do not contain any oscillators, so the oscillator contribution to the projectors are not included in the expressions below, although it is easy to add it back in. The projectors in the nth-twisted sectors for the massless states are then readily obtained (these results are special cases of the formulas in [17] and elsewhere):

$$
\begin{aligned}
n=0: \;& p_L\cdot\beta_L - r\cdot\phi_R = 0 \bmod 1\,,\\
n=1: \;& (p_L+\beta_L)\cdot\beta_L - \frac{1}{2}\beta_L^2 - r\cdot\phi_R - \sum_i \frac{\phi_R^i}{2} + \\
&+\left\{\frac{1}{3} \text{ if } p_{(2),L}\in Y_{i=1,2} \;\text{ or } 0 \text{ if } p_{(2),L}\in Y_0\right\} + \\
&+\left\{\frac{1}{2} \text{ if } p_{(4),L}\in Z_{i=1,2,3} \text{ or } 0 \text{ if } p_{(4),L}\in Z_0\right\} = 0 \bmod 1\,,\\
n=2: \;& (p_L+2\beta_L)\cdot\beta_L - \beta_L^2 - r\cdot\phi_R - \sum_i \frac{\phi_R^i}{2} + \\
&+\left\{\frac{2}{3} \text{ if } p_{(2),L}\in Y_{i=1,2} \text{ or } 0 \text{ if } p_{(2),L}\in Y_0\right\} = 0 \bmod 1\,,\; p_{(4),L}\in Z_0\,,\\
n=3: \;& (p_L+3\beta_L)\cdot\beta_L - \frac{3}{2}\beta_L^2 - r\cdot\phi_R - \sum_i \frac{\phi_R^i}{2} + \frac{1}{2} - \frac{1}{2}\phi_R^1 + \\
&+\left\{\frac{1}{2} \text{ if } p_{(4),L}\in Z_{i=1,2,3} \text{ or } 0 \text{ if } p_{(4),L}\in Z_0\right\} = 0 \bmod 1\,,\; p_{(2),L}\in Y_0\,.
\end{aligned}
\tag{5.4}
$$

Here Y_0 denotes the SU(3) root lattice, and $Y_{1,2}$ denote the lattices generated by the two fundamental weights. Similarly, Z_0 is the root lattice of SO(8), and $Z_{1,2,3}$ are the vector and two spinorial lattices respectively. The bracket notation means that there is an additional phase that depends on the weight momentum. For example, in the $n = 1$ twisted sector the value 1/2 is added to the projector if $p_{(4),L} \in Z_{i=1,2,3}$, whereas no factor (zero) is added if $p_{(4),L} \in Z_0$. In a symmetric orbifolds these phases do not appear as they cancel between the left and right movers. Finally, the states in the $n = 4$ and $n = 5$ sectors are obtained from the CPT conjugates of the states in the $n = 2$ and $n = 1$ sectors.

To complete the spectrum the bosonic and fermionic degeneracies for each sector must be specified, with the latter trivially obtained from table 3.

The bosonic degeneracy may be obtained either directly from the partition function or from the general formula presented in [15]. The partition function implies that in the $n = 1$, $n = 3$ and $n = 5$ sectors the bosonic degeneracy is $D = 2$, whereas in the $n = 2$ and $n = 4$ sectors it is $D = 1$. The general formula [15] for the bosonic degeneracy of the n-th twisted sector is

$$D_n = \frac{\prod_i 2\sin \pi n \phi_R^i}{\mathrm{vol}(I_n)}\,, \tag{5.5}$$

where $\mathrm{vol}(I_n)$ is the volume of the fundamental region of the lattice left invariant by the twist, and the product is over only non-vanishing twists. This gives $D_1 = 2$, $D_2 = 1$ and $D_3 = 2$, which agrees with the results obtained directly from the partition function.

5.2 Discrete symmetries and selection rules

Here we discuss some of the discrete symmetries that exist in the orbifolded theories. To compute the discrete anomalies, it is crucial to obtain the correct charges. These are rather straightforward to obtain for untwisted states and also for worldsheet fermions, untwisted or twisted, since there exists an explicit construction for twisted fermion vertex operators. This is discussed in subsection C.1.

The main subtlety is with the twisted bosons, and in particular the bosonic twist operator. In a symmetric orbifold this charge is readily obtained from the geometry. But for an asymmetric orbifold such geometric intuition is lacking, and we must resort to algebraic methods. Sections C.2 and C.3 compute this charge using two independent methods, which are found to agree.

To begin though, the asymmetric orbifolds considered here have the following abelian discrete symmetries:

- Quantum Z_N symmetry, where N is the order of the orbifold group [14]. Only twisted states are charged under this symmetry. Their charge is simply n/N, where n refers to the n-th twisted sector.

  In a symmetric orbifold the existence of the quantum symmetry is not hard to see. This is because a twisted string joined to a number of twisted strings may only form an untwisted string if the net twist is a multiple of N. For asymmetric orbifolds this argument is too naive, since here there is no geometric picture for the fixed points.

  To see the existence of a quantum Z_N symmetry in asymmetric orbifolds it is simplest to consider a correlation function involving a number of fermionic twist operators, τ_R (including excited states) and untwisted fermions,

$$\langle e^{-ir_1\cdot H_R(z_1)} \cdots e^{-ir_l\cdot H_R(z_k)} \tau_R^{(l_1)}(z_{l_1}) \cdots \tau_R^{(l_p)}(z_{l_p})\rangle\,. \tag{5.6}$$

  By explicitly evaluating this correlation (see for instance, [19]), one finds it is non-vanishing only if the sum of the momenta for all the operators, twisted and untwisted, vanishes. Now the i fermionic twist operator in the n_i-th twisted sector has momentum $r^a + n_i\phi_R^a$ where $\phi_R^a = k^a/N$ are the twists in the $n = 1$ sector, and at least one k^a and N are relatively coprime. Then the total momenta from the twisted sectors can only be canceled by untwisted states if $\sum_i n_i k^a/N$ is an integer. This is just the Z_N selection rule described above.

- Discrete Z_k symmetries of the lattice $\Gamma_{(d,d)}$ that remain after the orbifold projection. These are symmetries of the action and stress-tensor (or the Virasoro algebra) and are therefore symmetries of the perturbative string theory. These may act symmetrically or asymmetrically on p_L and p_R. For instance, the $\Gamma_{(2,2)}(A_2)$ lattice has the asymmetric Z_3 symmetry $(p_L, p_R) \to (p_L, \alpha p_R)$, $\alpha^3 = 1$, and the symmetric Z_6 symmetry $(p_L, p_R) \to (\gamma p_L, \gamma p_R)$ where $\gamma^6 = 1$. The $\Gamma_{(4,4)}(D_2)$ lattice has a Z_2 symmetry that acts asymmetrically.

All these symmetries provide selection rules for correlation functions. One can ask whether these symmetries are broken by instantons in the low-energy theory. More concretely, we compute the variation of the 't Hooft operator for the each of the gauge groups, and look to see if the variations are all universal. To do this, we need the discrete charges of the spacetime zero modes, and the number of zero modes for each representation in the one instanton background. The latter may be found in table 4. To compute the charges of the spacetime zero modes we need the charges of the worldsheet states. For worldsheet fermions this is described in subsection C.1 and for worldsheet bosons in subsections C.2 and C.3. Again, the main subtlety is finding the charge of the twisted bosonic ground state.

G	R	n
SU(N)	fund	1
	anti-sym	$N-2$
	adjoint	$2N$
SO($2N > 4$)	vector	2
	spinor	2^{N-3}
	adjoint	$4N-4$
SO(4)	spinor	1

Table 4: Instanton zero mode counting.

We find that these symmetries are often anomalous, but since the anomalies are always universal they may be canceled by a discrete Green-Schwarz mechanism. In subsections 5.3.1 and 5.4.3 we explain why a non-universal anomaly cannot be canceled, despite the existence of other massless scalars.

5.3 Models

Here we give in some detail the massless spectra of some models, and show that the anomalies are universal. These models are specified by a choice of a left-moving shift β_L that satisfies level matching. The massless fermions are then obtained from the projectors provided in the previous section. Their charge under the discrete Z_3 is obtained from the rules described in the appendix C.

In all of these models the perturbative, SU(2) Witten, quantum Z_N and discrete Z_3 anomalies either vanish, or may be canceled by a Green-Schwarz mechanism. It is interesting to note that the Z_3 anomalies are not universal in the untwisted sector, and are only universal after including the twisted sectors.

5.3.1 Other moduli?

We first ask: are there other moduli besides the dilaton which could have canceled non-universal anomalies? But in these examples, it is easy to see that the answer is no. The point is that there are simply no other massless scalars, apart from the dilaton, which are neutral under all symmetries. First, in the untwisted sector, the asymmetric twist projects out all of the geometric moduli. In the twisted sectors, because of the shifts all massless

sector	$\mathrm{SU}(4)\times\mathrm{SO}(22)\times\mathrm{SU}(3)'\times\mathrm{SO}(6)\times\mathrm{U}(1)^3$	Z_3 anomaly
untwisted	$2\times(\mathbf{4},\mathbf{22},\mathbf{1},\mathbf{1};-1,0,0;r_2)+2\times(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1};-2,0,0;r_1)$ $2\times(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{1};1,-1,0;r_2)+2\times(\mathbf{1},\mathbf{22},\mathbf{1},\mathbf{1};0,-1,0;r_1)$ $2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6};0,0,-1;r_1)$	$(1,1,1,\alpha)$
$n=1$	no states	
$n=2$	$2\times(\mathbf{4},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{1};\frac{1}{3},-\frac{1}{3},\frac{2}{3})+2\times(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{6};\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ $2\times(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{1};-\frac{5}{3},-\frac{1}{3},\frac{2}{3})$	$(1,1,\alpha,\alpha^2)$
$n=3$	$2\times(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1};0,1,0)+2\times(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{6};-1,0,0)$ $2\times(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1};1,0,-1)+2\times(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{4}_S\text{ and }\mathbf{4}_{S'};1,0,\frac{1}{2})$ $2\times(\mathbf{1},\mathbf{22},\mathbf{1},\mathbf{1};2,0,0)+2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};-2,-1,0)$	$(1,\alpha^2,1,\alpha^2)$
$n=4$	$2\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{1};-\frac{4}{3},-\frac{2}{3},-\frac{2}{3})+2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6};-\frac{4}{3},-\frac{2}{3},\frac{1}{3})$ $2\times(\mathbf{1},\mathbf{22},\mathbf{1},\mathbf{1};-\frac{4}{3},\frac{1}{3},-\frac{2}{3})+2\times(\overline{\mathbf{6}},\mathbf{1},\mathbf{1},\mathbf{1};\frac{2}{3},-\frac{2}{3},-\frac{2}{3})$	$(\alpha,\alpha,\alpha^2,\alpha)$
$n=5$	$2\times(\mathbf{1},\mathbf{22},\mathbf{1},\mathbf{1};-\frac{2}{3},\frac{2}{3},\frac{2}{3})+2\times(\overline{\mathbf{4}},\mathbf{22},\mathbf{1},\mathbf{1};\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ $2\times(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1};\frac{4}{3},-\frac{1}{3},\frac{2}{3})+2\times(\overline{\mathbf{4}},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{1};\frac{1}{3},\frac{2}{3},-\frac{1}{3})$ $2\times(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{4}_S\text{ and }\mathbf{4}_{S'};\frac{1}{3},\frac{2}{3},\frac{1}{6})+2\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{6};-\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$ $2\times(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{1};-\frac{5}{3},\frac{2}{3},-\frac{1}{3})+2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};-\frac{2}{3},-\frac{1}{3},-\frac{4}{3})$	$(1,\alpha,\alpha,\alpha)$
total		$(\alpha,\alpha,\alpha,\alpha)$

Table 5: Nonsupersymmetric asymmetric Z_6 model with $\beta_L=(\frac{1}{6}^4,\frac{1}{3},0^{11};0^2;\frac{1}{3},0^3)$, $\alpha^3=1$.

sector	$\mathrm{SU}(3)\times\mathrm{SO}(26)\times\mathrm{SU}(3)'\times\mathrm{SO}(8)\times\mathrm{U}(1)$	Z_3 anomaly
untwisted	$2\times(\mathbf{3},\mathbf{26},\mathbf{1},\mathbf{1};1;r_1)+2\times(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};-2;r_1)$	$(1,1,1,1)$
$n=1$	no states	
$n=2$	no states	
$n=3$	$2\times(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{8}_V\text{ and }\mathbf{8}_S\text{ and }\mathbf{8}_{S'};-1)$	$(1,1,1,1)$
$n=4$	$2\times(\mathbf{1},\mathbf{26},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{1};-1)+2\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{1};2)$ $2\times(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1};-2)+2\times(\overline{\mathbf{3}},\mathbf{26},\mathbf{1},\mathbf{1};1)$	$(1,1,1,1)$
$n=5$	$2\times(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{8}_V\text{ and }\mathbf{8}_S\text{ and }\mathbf{8}_{S'};-1)$ $2\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}},\mathbf{8}_V\text{ and }\mathbf{8}_S\text{ and }\mathbf{8}_{S'};1)$	$(1,1,1,1)$

Table 6: Nonsupersymmetric asymmetric Z_6 model with $\beta_L=(\frac{1}{3}^3,0^{13};0^2;0^4)$.

states are charged under the gauge symmetries. They are also charged under the quantum Z_N symmetry. Thus, even if there are massless bosons in the twisted sectors, and some of these are moduli, they cannot couple to $F\tilde{F}$, and thus could not have played a role in anomaly cancellation.

5.4 Supersymmetric examples

Let us turn, now, to supersymmetric theories.

5.4.1 Narain-Sarmadi-Vafa

It is instructive to first consider the Narain-Sarmadi-Vafa (NSV) supersymmetric asymmetric model [15]. The internal lattice is a product of the $E_8 \times E_8$ lattice with the lattices $(\Gamma_{(2,2)}(A_2))^3$. As discussed before the $\Gamma_{(2,2)}(A_2)$ lattice has independent left and right Z_3 symmetries, and a left-right symmetric Z_6 symmetry. The NSV model corresponds to performing an asymmetric Z_3 twist

$$\beta_L = \phi_R = \frac{1}{3}(-2, 1, 1)\,, \tag{5.7}$$

with no left-moving twist or right-moving shift ($\phi_L = 0$ and $\beta_R = 0$). This is modular invariant, and preserves $N = 1$ supersymmetry. The low-energy gauge group is $E_8 \times E_6 \times \mathrm{SU}(3)^4$. The gauge twist breaks one E_8 group and accounts for one of the SU(3) group factors. The other three SU(3) group factors correspond to left-moving lattice momentum in the three $\Gamma_{(2,2)}(A_2)$ lattices, which would otherwise be projected out in a symmetric orbifold construction of the same lattice.

The spectrum is easily worked out. The massless fermions in the untwisted sector are the gauginos, and from the matter $3 \times (\mathbf{3}, \mathbf{27}; \mathbf{1}, \mathbf{1}, \mathbf{1})$. In the twisted sector we have [15]

$$\begin{aligned}(\mathbf{3}, \overline{\mathbf{27}}; \mathbf{1}, \mathbf{1}, \mathbf{1}) + \big((\mathbf{1}, \mathbf{27}; \mathbf{3} + \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}) + \text{permutations in last three groups}\big) +\\ + \big((\overline{\mathbf{3}}, \mathbf{1}; \mathbf{3} + \overline{\mathbf{3}}, \mathbf{1}, \mathbf{3} + \overline{\mathbf{3}}) + \text{permutations in last 3 groups}\big)\,.\end{aligned} \tag{5.8}$$

Focusing on one of the $\Gamma_{(2,2)}(A_2)$ lattices, we can look for possible discrete anomalies in the Z_3 symmetry that acts only on the right-movers in this lattice. In the untwisted (RM) Ramond sector, the positive helicity massless states are $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2})$. Here as before we are using the bosonized formulation. In this case the gauginos are obtained from the state with all '+'s and have Z_3 charge $\gamma^{-1/2}$, whereas the matter states are obtained from the three remaining right-movers; two have charge $\gamma^{+1/2}$ and the other has charge $\gamma^{-1/2}$. The $(\mathrm{SU}(3), E_6, \mathrm{SU}(3)_{I=1,2,3})$ discrete anomalies in the untwisted sector are $(\gamma^{3/2}, 1, 1, 1, 1)$, where it is important to include the contribution from the gauginos. It is useful to recall that $t_2(\mathbf{27}) = 6$ and $t_2(\mathbf{78}) = 24$ in the normalization of table 4. We will find that this anomaly is canceled by a contribution from the twisted sector states.

In the twisted sector the discrete charges of the massless states receive two contributions. The first is from the twisted fermions. The Ramond RM ground state is unique and massless, and corresponds to the state $r + \phi$, with $r = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Using the fermionic twist operator, the world-sheet fermion contribution to the discrete charge of the twisted spacetime fermions is easily found to be $\gamma^{1/6}$. The twisted bosons provide a charge that is either $\gamma^{1/3}$ or $\gamma^{-2/3}$. Since for any twisted state the number of zero modes in a one instanton background is a multiple of 9, the bosonic charge does not contribute to an anomaly. In total the discrete anomalies from the twisted states are $(\gamma^{3/2}, 1, 1, 1)$, where the previous notation has been used. Although this is non-universal, a cancellation occurs between the twisted and untwisted sectors, leaving no Z_3 anomaly.

5.4.2 Z_6 models

Now a class of supersymmetric models with orbifold group Z_6 is constructed. We have already seen that the hexagonal lattice $\Gamma_{(2,2)}(A_2)$ has separate left and right Z_3 symmetries.

But the hexagonal lattice also has a Z_6 automorphism, which can be used to construct symmetric orbifold twists. Since this rotation exchanges the two SU(3) fundamental weights, it cannot be used to construct an asymmetric twist. But a symmetric Z_6 twist in this plane is allowed, since this is a symmetry of the lattice. (The symmetric Z_6 twist preserves the condition $p_L - p_R \in R$, whereas the asymmetric twist does not.)

The lattice we therefore consider the product of the internal SO(32) and the $(\Gamma_{(2,2)}(A_2))^3$ lattices. We orbifold by an asymmetric Z_3 twist on the first SU(3) lattice, and a symmetric Z_6 twist on the last two SU(3) lattices. In particular, consider

$$\phi_R = \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right) ; \qquad \phi_L = \left(0, \frac{1}{6}, \frac{1}{6}\right) . \tag{5.9}$$

This preserves $N = 1$ supersymmetry. It is straightforward to see that the last (non-trivial) level matching condition (B.14)is satisfied. The details may be found in appendix B. In addition, different models will be characterized by a shift β_L on the left-movers. These must satisfy the level matching conditions (B.12).

Proceeding as before, the projection operators for massless states in the nth-twisted sectors are:

$$n = 0: \; p_L \cdot \beta_L - r \cdot \phi_R + N_{\rm osc} = \; 0 \bmod 1 \tag{5.10}$$

$$\begin{aligned} n = 1: \; & p_L \cdot \beta_L - r \cdot \phi_R + \frac{1}{2}(\beta_L^2 - \phi_L^2) + \frac{1}{2}\phi_R^1 \\ & + \left\{ \frac{1}{3} \text{ if } p_{(1),L} \in Y^{(1)}_{i=1,2} \text{ or } 0 \text{ if } p_{(1),L} \in Y_0 \right\} + N_{\rm osc} \; = \; 0 \bmod 1 \end{aligned} \tag{5.11}$$

$$\begin{aligned} n = 2: \; & p_L \cdot \beta_L - r \cdot \phi_R + (\beta_L^2 - \phi_L^2) + \frac{1}{2}\phi_R^1 + \frac{1}{2} + \\ & + \left\{ \frac{2}{3} \text{ if } p_{(1),L} \in Y^{(1)}_{i=1,2} \text{ or } 0 \text{ if } p_{(1),L} \in Y_0 \right\} + N_{\rm osc} \\ & = \{0 \text{ mod1 or } \frac{1}{2} \bmod 1\} \end{aligned} \tag{5.12}$$

$$\begin{aligned} n = 3: \; & p_L \cdot \beta_L - r \cdot \phi_R + \frac{3}{2}(\beta_L^2 - \phi_R^2) + \phi_R^1 + \frac{1}{2} + N_{\rm osc} \\ & = \{0 \bmod 1 \text{ or } \frac{1}{3} \bmod 1 \text{ or } \; \frac{2}{3} \bmod 1\} , \quad p_{(1),L} \in Y_0 \, . \end{aligned} \tag{5.13}$$

Here $(p_{(1),L}, p_{(1),R})$ are momenta in the lattice with the asymmetric twist. In addition, the bosonic degeneracy for each choice of phase appearing on the right-side of the above equations is required. In the singly twisted sector the bosonic degeneracy is 1. For $n = 2$, the projector is

$$P = \frac{1}{6}(9 + \Delta + 9\Delta^2 + \Delta^3 + 9\Delta^4 + \Delta^5) , \tag{5.14}$$

where $\Delta = e^{2\pi i \Phi}$ and Φ is given by the expression appearing on the left side of (5.12). Thus the degeneracy of states with $\Phi = 0 \bmod 1$ is 5, whereas the degeneracy of states with $\Phi = 1/2 \bmod 1$ is 4. Similarly, in the $n = 3$ sector the projector is

$$P = \frac{1}{6}(16 + \Delta + \Delta^2 + 16\Delta^3 + \Delta^4 + \Delta^5). \tag{5.15}$$

sector	Ramond state
untwisted	$r_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \alpha^{-1/2})$, $r_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \alpha^{-1/2})$ $r_3^{\pm} = (\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}; \alpha^{1/2})$
$n = 1$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \alpha^{1/2}$ if $p_{(1),L} \in Y_0$ or $\alpha^{-1/2}$ if $p_{(1),L} \in Y_{1,2})$
$n = 2$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \alpha^{-1/2}$ if $p_{(1),L} \in Y_0$ or $\alpha^{1/2}$ if $p_{(1),L} \in Y_{1,2})$
$n = 3$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \alpha^{1/2})$

Table 7: Supersymmetric asymmetric Z_6 model: positive helicity RM supersymmetric Ramond sector weights (r) and the Ramond sector Z_3 charges ($\alpha^3 = 1$). The ground states are $r + n\phi_R$.

Again, Δ is defined as before with now Φ equal to the expression appearing on the left side of (5.13). The degeneracy of states with $\Phi = 0$ mod 1 ($\Delta = 1$) is 6, and those with $\Phi = 1/3$ mod 1 ($\Delta = \gamma^2$) or $2/3$ mod 1 ($\Delta = \gamma^4$) is 5.

There are a number of discrete anomalies that may be studied. There is a Z_3 symmetry of the $\Gamma_{(2,2)}(A_2)$ lattice with the asymmetric twist. This is the same asymmetric Z_3 symmetry studied in the non-supersymmetric models. The discrete charges are computed as in the previous section and listed in table 7.

There is also a symmetric Z_6 symmetry acting on either of the $\Gamma_{(2,2)}(A_2)$ lattices with the symmetric twist. Here now the Z_6 charges in the twisted sector also depend on the fixed point. This too has been described in the section on symmetric orbifolds.

In addition there is another symmetric Z_6 symmetry acting on the $\Gamma_{(2,2)}(A_2)$ lattice that has the asymmetric twist. But since this lattice has an enhanced SU(3) gauge symmetry, the Z_6 symmetry permutes, for example, the $\mathbf{3}$ and the $\overline{\mathbf{3}}$ of SU(3). It is unclear how to study the instanton anomalies of this symmetry. Finally there is the quantum Z_N symmetry.

The massless spectra for three models is given in tables 8, 9 and 10. Again, perturbative anomalies cancel, with a Green-Schwarz mechanism required for models II and III. This provides a non-trivial consistency check on the models. Discrete anomalies are computed as in the previous section. Of course, one must now take account of the gaugino contribution from the untwisted sector. The quantum Z_6 anomalies cancel as well, although a discrete "Green-Schwarz" mechanism is required for models II and III. The symmetric Z_6 symmetry is also found to be non-anomalous, again, with a discrete "Green-Schwarz mechanism" required for models II and III. The asymmetric Z_3 anomalies also cancel, but with a discrete "Green-Schwarz mechanism" required for model III. It is interesting to note that in all of these models the discrete anomalies are non-universal in the untwisted sector, and only become universal after adding in the contributions from the twisted sectors.

5.4.3 Universal couplings of moduli

One must again ask whether other moduli might cancel the anomalies. In this case, the question is not quite as simple as in the non-supersymmetric examples of the previous subsection. There are now moduli, neutral under all symmetries, in the untwisted sector. However, using the methods of [20], one can readily show that these moduli couple univer-

sector	$SU(2) \times SO(28) \times SU(3) \times U(1)$	Z_3 anomaly	Z_6 anomaly
untwisted	$(\mathbf{3},\mathbf{1},\mathbf{1};0;r_0)+(\mathbf{1},\mathbf{373},\mathbf{1};0;r_0)+(\mathbf{1},\mathbf{1},\mathbf{8};0;r_0)$ $(\mathbf{2},\mathbf{28},\mathbf{1};1;r_3^{\pm})$	$(\alpha^2,\alpha^2,1)$	$(\gamma^2,\gamma^2,1)$
$n=1$	$(\mathbf{2},\mathbf{28},\mathbf{1};-\frac{2}{3})+4\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}};\frac{1}{3})$	$(\alpha^2,\alpha^2,\alpha^2)$	$(\gamma^{2/3},\gamma^{8/3},\gamma^{-2/3})$
$n=2$	$4\times(\mathbf{2},\mathbf{28},\mathbf{1};-\frac{1}{3})+5\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}};-\frac{4}{3})$ $8\times(\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}};\frac{2}{3})$	(α,α,α)	$(\gamma^{4/3},\gamma^{-2/3},\gamma^{2/3})$
$n=3$	$5\times(\mathbf{2},\mathbf{28},\mathbf{1};0)$	$(\alpha,\alpha,1)$	$(\gamma^2,\gamma^2,1)$
total		$(1,1,1)$	$(1,1,1)$

Table 8: Supersymmetric asymmetric Z_6 model I: $\beta_L=(\frac{1}{6}^2,0^{14};0^2)$, $\alpha^3=1$ and $\gamma^6=1$. Some states have oscillators.

sector	$SU(2)\times SU(6)\times SO(16)\times SU(3)\times U(1)^2$	Z_3 anomaly	Z_6 anomaly
untwisted	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1};0,0;r_0)\ +\ (\mathbf{1},\mathbf{35},\mathbf{1},\mathbf{1};0,0;r_0)$ $(\mathbf{1},\mathbf{1},\mathbf{120},\mathbf{1};0,0;r_0)\ +\ (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8};0,0;r_0)$ $(\mathbf{2},\mathbf{1},\mathbf{16},\mathbf{1};1,0;r_3^{\pm})\ +\ (\mathbf{1},\mathbf{6},\mathbf{16},\mathbf{1};0,-1;r_2)$ $(\mathbf{2},\overline{\mathbf{6}},\mathbf{1},\mathbf{1};-1,1;r_3^{\pm})\ +\ (\mathbf{1},\overline{\mathbf{15}},\mathbf{1},\mathbf{1};0,2;r_2)$	$(\alpha^2,\alpha,\alpha^2,1)$	$(\gamma^2,\gamma^2,\gamma^2,1)$
$n=1$	$(\mathbf{2},\mathbf{6},\mathbf{1},\mathbf{1};-\frac{2}{3},1)\ +\ 2\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}};\frac{1}{3},2)$ $2\times(\mathbf{1},\mathbf{15},\mathbf{1},\mathbf{1};\frac{1}{3},0)$	$(1,\alpha^2,1,\alpha)$	$(\gamma^4,\gamma^{8/3},1,\gamma^{2/3})$
$n=2$	$5\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3}\text{ and }\overline{\mathbf{3}};\frac{2}{3},-2)\ +\ 4\times(\mathbf{2},\overline{\mathbf{6}},\mathbf{1},\mathbf{1};-\frac{1}{3},-1)$ $5\times(\mathbf{1},\overline{\mathbf{15}},\mathbf{1},\mathbf{1};\frac{2}{3},0)$	$(1,\alpha^{-2},1,\alpha^2)$	$(\gamma^2,\gamma^{-8/3},1,\gamma^{10/3})$
$n=3$	$5\times(\mathbf{2},\mathbf{1},\mathbf{16},\mathbf{1};0,0)\ +\ 5\times(\mathbf{2},\overline{\mathbf{6}},\mathbf{1},\mathbf{1};0,1)$ $6\times(\mathbf{2},\mathbf{6},\mathbf{1},\mathbf{1};0,-1)$	$(\alpha,\alpha^2,\alpha,1)$	$(\gamma^2,\gamma^2,\gamma^2,1)$
$n=4$	none		
$n=5$	none		
total		$(1,1,1,1)$	$(\gamma^4,\gamma^4,\gamma^4,\gamma^4)$

Table 9: Supersymmetric asymmetric model II: $\beta_L=(\frac{1}{6}^2,\frac{1}{3}^6,0^8;0^2;0^4)$, $\alpha^3=1$ and $\gamma^6=1$.

sally to all gauge groups, in a fashion similar to the weakly coupled string dilaton. As a result, they could not have helped with anomaly cancellation.

6. A search for discrete gauge anomalies in type-IIB orientifolds

This section describes models obtained by the orientifold compactification of type-IIB string theory on T^6. At low-energies they describe four-dimensional, $\mathcal{N}=1$ supersymmetric theories. In contrast to the heterotic models discussed above, the type-IIB orientifolds models discussed here *do* have non-universal discrete anomalies. If these are to be canceled there must be massless states with non-universal couplings to the gauge bosons. But how is this possible? For the only untwisted scalars that are not charged under the Z_M discrete symmetry are the dilaton multiplet and the T moduli. In the first instance the coupling to D9 brane matter is universal, and in the second the T moduli do not couple to D9 matter at tree-level.

sector	$\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{SO}(14) \times \mathrm{SU}(3) \times \mathrm{U}(1)^3$	Z_3 anomaly	Z_6 anomaly
untwisted	$(\mathbf{24}, \mathbf{1}, \mathbf{1}, \mathbf{1}; 0, 0, 0; r_0) + (\mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1}; 0, 0, 0; r_0)$ $(\mathbf{1}, \mathbf{1}, \mathbf{91}, \mathbf{1}; 0, 0, 0; r_0) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8}; 0, 0, 0; r_0)$ $(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{14}, \mathbf{1}; 0, -1, 0; r_2) + (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; 0, 2, 0; r_2)$ $(\overline{\mathbf{10}}, \mathbf{1}, \mathbf{1}, \mathbf{1}; -2, 0, 0; r_2) + (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{1}; 1, 0, \pm 1; r_2)$ $(\mathbf{5}, \mathbf{1}, \mathbf{14}, \mathbf{1}; 1, 0, 0; r_3^{\pm}) + (\overline{\mathbf{5}}, \mathbf{3}, \mathbf{1}, \mathbf{1}; -1, 1, 0; r_3^{\pm})$ $(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; 0, -1, \pm 1; r_3^{\pm})$	$(\alpha^{\frac{1}{2}}, \alpha^{\frac{5}{2}}, \alpha, 1)$	$(\gamma, \gamma^3, 1, 1)$
$n = 1$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; -\frac{1}{6}, 0, \frac{1}{2}) + (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{3} \text{ and } \overline{\mathbf{3}}; \frac{5}{6}, 0, -\frac{1}{2})$ $(\mathbf{1}, \mathbf{1}, \mathbf{14}, \mathbf{1}; \frac{5}{6}, 1, -\frac{1}{2}) + 2 \times (\overline{\mathbf{5}}, \mathbf{1}, \mathbf{1}, \mathbf{1}; -\frac{1}{6}, 1, -\frac{1}{2})$ $2 \times (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}; \frac{5}{6}, -1, \frac{1}{2}) + 2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3} \text{ and } \overline{\mathbf{3}}; \frac{5}{6}, 1, \frac{1}{2})$ $3 \times (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; \frac{5}{6}, 0, -\frac{1}{2})$	$(\alpha^{\frac{5}{2}}, \alpha^2, \alpha, \alpha)$	$(\gamma^{-\frac{2}{3}}, \gamma^{\frac{2}{3}}.\gamma^{\frac{4}{3}}, \gamma^{-\frac{4}{3}})$
$n = 2$	$4 \times (\overline{\mathbf{10}}, \mathbf{1}, \mathbf{1}, \mathbf{1}; -\frac{1}{3}, -1, 0) + 4 \times (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}; \frac{5}{3}, 1, 0)$ $4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3} \text{ and } \overline{\mathbf{3}}; \frac{5}{3}, -1, 0) + 5 \times (\overline{\mathbf{5}}, \mathbf{3}, \mathbf{1}, \mathbf{1}; \frac{2}{3}, 0, 0)$	$(\alpha^{\frac{3}{2}}, \alpha^{\frac{1}{2}}, 1, \alpha)$	$(\gamma^3, \gamma^{-\frac{7}{3}}, 1, \gamma^{\frac{8}{3}})$
$n = 3$	$5 \times (\mathbf{5}, \mathbf{1}, \mathbf{1}, \mathbf{1}; -\frac{3}{2}, 0, \frac{1}{2}) + 5 \times (\overline{\mathbf{5}}, \mathbf{1}, \mathbf{1}, \mathbf{1}; \frac{3}{2}, 0, -\frac{1}{2})$ $5 \times (\overline{\mathbf{10}}, \mathbf{1}, \mathbf{1}, \mathbf{1}; \frac{1}{2}, 0, \frac{1}{2}) + 6 \times (\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{1}; -\frac{1}{2}, 0, -\frac{1}{2})$	$(\alpha^{\frac{1}{2}}, 1, 1, 1)$	$(\gamma^4, 1, 1, 1)$
total		$(\alpha^2, \alpha^2, \alpha^2, \alpha^2)$	$(\gamma^{\frac{4}{3}}, \gamma^{\frac{4}{3}}, \gamma^{\frac{4}{3}}, \gamma^{\frac{4}{3}})$

Table 10: Supersymmetric asymmetric model III: $\beta_L = (\frac{1}{6}^5, \frac{1}{3}^3, \frac{1}{2}, 0^7; 0^2)$, $\alpha^3 = 1$ and $\gamma^6 = 1$.

If this were the end of the story then there would be a puzzle. But there is an important distinction between orientifolds and orbifolds. In orientifold models the world-sheet parity projection breaks the quantum Z_N symmetry. Consequently, a linear coupling of a massless R-R state in the twisted sector to gauge bosons is not forbidden. Non-universal discrete anomalies can then be canceled by assigning a shift to these moduli. Indeed this is not surprising, for such couplings are needed to cancel the multiple U(1) anomalies typically found in these class of models [21, 22].

6.1 Discrete charges

The symmetries we study are the discrete isometries of the internal manifold. Initially there are three such isometries, but the orientifold projection removes one. We initially focus on the Z_M symmetry of the third torus, which in our notation corresponds to the last entry in r. This symmetry acts only on the world sheet variables and not the Chan-Paton factors. The charges of the states can be directly read off from (D.1), (D.2) and (D.3):

- The Z_M charge of a 99 or 55 state (D.1) with $r = (\cdots \pm)$ is simply $\gamma^{\pm/2}$, where $\gamma^M = 1$. Clearly this is a discrete R-symmetry.
- In the 95 sector the spacetime bosons are neutral under this Z_M, whereas spacetime fermions with negative helicity ($s_0 = -1/2$) have charge $\gamma^{1/2}$.

In models where the orbifold group has a single Z_N factor one may also study the remaining discrete rotation, which is the product of a discrete Z_6 rotation in the first torus with an anti-rotation in the third torus. The gauginos are neutral under this linear combination, so this is a non-R discrete symmetry. The charges of the other states are easily obtained from the formulae given above.

6.2 Models

The three models we study have been constructed and presented in the literature [21, 22]. The new ingredient here is the computation of the discrete anomalies.

6.2.1 Z_3

Here the twist is generated by $\phi = (1,1,-2)/3$. This model has only D9 branes. The embedding of the twist into the gauge group is described by the shift vector $V = (1^{12}, 0^4)/3$. The gauge group is $\mathrm{U}(12) \times \mathrm{SO}(8)$. The massless charged fermions and gauginos are [21] (for $s_0 = -1/2$)

$$\begin{aligned} 99: \ & 2(\mathbf{12},\mathbf{8};1;\gamma^{1/2}) + (\mathbf{12},\mathbf{8};1;\gamma^{-1/2}) + 2(\overline{\mathbf{66}},\mathbf{1};-2,\gamma^{1/2}) + \\ & +(\overline{\mathbf{66}},\mathbf{1};-2,\gamma^{-1/2}) + (\mathbf{143},\mathbf{1};0,\gamma^{-1/2}) + (\mathbf{1},\mathbf{28};0,\gamma^{-1/2}) \end{aligned} \tag{6.1}$$

with the second-to-last entry indicating the U(1) charge and the last entry indicating the Z_6 charge. The U(1) in this model is anomalous and non-universal. In particular, the mixed U(1) and non-abelian gauge anomalies are: $(\mathrm{U}(1)\,\mathrm{SU}(12)^2, \mathrm{U}(1)\,\mathrm{SO}(8)^2) \propto (1,-2)$[22]. This can be canceled by a shift in one of the neutral twisted R-R scalars [22].

Now we compute the anomalies for the Z_6 symmetry of the third torus, say. The Z_6 charges of the matter and gauginos are easily computed using the rule given in the previous section. One finds non-universal discrete anomalies

$$(Z_6\,\mathrm{SU}(12)^2, Z_6\,\mathrm{SO}(8)^2) = (\gamma^3, 1)\,. \tag{6.2}$$

Note that this is non-universal. But since the discrete anomalies are in the same ratio as the U(1) anomalies, they may be canceled by assigning a shift to the R-R scalar $\mathcal{M}$ used to cancel the U(1) anomalies. *In fact, one can define discrete symmetries which are unbroken and free of anomalies by combining the discrete transformations with suitable* U(1) *transformations.*

One can also consider the non-R Z_6 symmetry which is a combination of a rotation in the third torus with an anti-rotation in, say, the first torus. One finds that this symmetry is not anomalous.

6.2.2 $Z_3 \times Z_3$

Here the first Z_3 is generated by a twist with $\phi_1 = (1,-1,0)/3$ and the second by a twist with $\phi_2 = (0,1,-1)/3$. The shifts associated with these twists are $V_1 = (1^4, -1^4, 0^8)/3$ and $V_2 = (0^4, 1^4, -1^4, 0^4)/3$. The gauge group is then $\mathrm{U}(4)^3 \times \mathrm{SO}(8)$. The massless charged fermions and gauginos are [21] (for $s_0 = 1/2$)

$$\begin{aligned} 99: \ & (\mathbf{4},\mathbf{4},\mathbf{1},\mathbf{1};1,1,0;\gamma^{1/2}) + (\mathbf{4},\mathbf{1},\mathbf{4},\mathbf{1};1,0,1;\gamma^{-1/2}) + (\mathbf{1},\mathbf{4},\mathbf{4},\mathbf{1};0,1,1;\gamma^{-1/2}) + \\ & +(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1};2,0,0;\gamma^{-1/2}) + (\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{1};0,2,0;\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\mathbf{6},\mathbf{1};0,0,2;\gamma^{1/2}) + \\ & +(\overline{\mathbf{4}},\mathbf{1},\mathbf{1};\mathbf{8};-1,0,0;\gamma^{-1/2}) + (\mathbf{1},\overline{\mathbf{4}},\mathbf{1};\mathbf{8};0,-1,0;\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\overline{\mathbf{4}};\mathbf{8};0,0,-1;\gamma^{1/2}) + \\ & +(\mathbf{15},\mathbf{1},\mathbf{1},\mathbf{1};0,0,0;\gamma^{1/2}) + (\mathbf{1},\mathbf{15},\mathbf{1},\mathbf{1};0,0,0;\gamma^{1/2}) + (\mathbf{1},\mathbf{1},\mathbf{15},\mathbf{1};0,0,0;\gamma^{1/2}) + \\ & +(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{28};0,0,0;\gamma^{1/2}) \end{aligned} \tag{6.3}$$

The U(1)'s in this model are anomalous and given by [22] $(\mathrm{SU}(4)_i^2\,\mathrm{U}(1)_j; \mathrm{SO}(8)^2\,\mathrm{U}(1)_j) \propto$

$(1,1,1;-2)$. The Z_6 charges of the matter and gauginos are easily computed using the rule given in the previous section and are given by the last entry of each item in (6.3). One finds the discrete anomalies

$$(Z_6\,\mathrm{SU}(4)_i^2; Z_6\,\mathrm{SO}(8)^2) = (\gamma^{-1}, \gamma^{-1}, \gamma^{-1}; \gamma^2)\,, \tag{6.4}$$

where $\gamma^6 = 1$. As in the previous model these are non-universal, but in the same ratio as the U(1) anomalies. Thus the twisted R-R scalar that is responsible for canceling the U(1) anomalies may also be used to cancel the discrete anomalies. Alternatively, we may again define a combination of the original discrete symmetry transformation and discrete U(1) transformations to define unbroken anomaly free discrete symmetries.

6.2.3 Z_6

This orbifold is generated by the Z_6 twist $\phi = (1,1,-2)/6$. Since this group contains an element of order 2, this model has both D5 and D9 branes. For simplicity the D5 branes are placed at the origin. Then with the choice of shifts $V_{99} = V_{55} = (1^6, 5^6, 3^4)/12$ that satisfy the tadpole cancellation requirement, the gauge group is $\mathrm{U}(6) \times \mathrm{U}(6) \times \mathrm{U}(4)$ for the 99 matter and a different $\mathrm{U}(6) \times \mathrm{U}(6) \times \mathrm{U}(4)$ for the 55 matter. The massless charged fermions and gauginos are [21] (for $s_0 = -1/2$)

$$\begin{aligned}
99:\ & 2(\mathbf{15},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{1/2}) + (\mathbf{6},\overline{\mathbf{6}},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + (\overline{\mathbf{6}},\mathbf{1},\overline{\mathbf{4}};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + \\
& +2(\overline{\mathbf{6}},\mathbf{1},\mathbf{4};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{1/2}) + (\mathbf{1},\mathbf{6},\mathbf{4};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + 2(\mathbf{1},\mathbf{6},\overline{\mathbf{4}};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{1/2}) + \\
& +2(\mathbf{1},\overline{\mathbf{15}},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{1/2}) + \\
& +(\mathbf{35},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + (\mathbf{1},\mathbf{35},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\mathbf{15};\mathbf{1},\mathbf{1},\mathbf{1};\gamma^{-1/2}) \\
55:\ & 2(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{15},\mathbf{1},\mathbf{1};\gamma^{1/2}) + (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{6},\overline{\mathbf{6}},\mathbf{1};\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{6}},\mathbf{1},\overline{\mathbf{4}};\gamma^{-1/2}) + \\
& +2(\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{6}},\mathbf{1},\mathbf{4};\gamma^{1/2}) + (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{6},\mathbf{4};\gamma^{-1/2}) + 2(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{6},\overline{\mathbf{4}};\gamma^{1/2}) + \\
& +2(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{15}},\mathbf{1};\gamma^{1/2}) + \\
& +(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{35},\mathbf{1},\mathbf{1};\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{35},\mathbf{1};\gamma^{-1/2}) + (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{15};\gamma^{-1/2}) \\
95+59:\ & (\mathbf{6},\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1},\mathbf{1};\gamma^{1/2}) + (\overline{\mathbf{6}},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{4};\gamma^{1/2}) + (\mathbf{1},\overline{\mathbf{6}},\mathbf{1};\mathbf{1},\overline{\mathbf{6}},\mathbf{1};\gamma^{1/2}) + \\
& +(\mathbf{1},\mathbf{6},\mathbf{1};\mathbf{1},\mathbf{1},\overline{\mathbf{4}};\gamma^{1/2}) + (\mathbf{1},\mathbf{1},\mathbf{4};\overline{\mathbf{6}},\mathbf{1},\mathbf{1};\gamma^{1/2}) + (\mathbf{1},\mathbf{1},\overline{\mathbf{4}};\mathbf{1},\mathbf{6},\mathbf{1};\gamma^{1/2})\,.
\end{aligned} \tag{6.5}$$

The U(1) charges are suppressed but easily obtained. One finds that the mixed U(1) anomalies are not universal.

The Z_6 discrete anomalies can be computed as in the previous models. The charges of the states under a Z_6 rotation of the third torus are indicated in (6.5). Here the anomalies are universal. In particular, $(Z_6\,\mathrm{SU}(6)_i^2, Z_6\,\mathrm{SU}(4)_j^2) = \gamma^2 \times (1,1)$. These may be canceled by assigning a shift to both the dilaton axion which couples to the D9 brane gauge bosons and to the axion of the T_3 modulus which couples to the D5 brane gauge bosons.

As in the previous examples, one can also consider the non-R Z_6 symmetry. One finds that this symmetry has a non-universal anomaly

$$(Z_6\,\mathrm{SU}(6)_i^2, Z_6\,\mathrm{SU}(4)_j^2) = (\delta^3, 1)\,. \tag{6.6}$$

Again, one may redefine this discrete symmetry to include a U(1) factor such that the above anomaly is universal.

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7. CPT

In this section we study the C, P and T symmetries of M-theory in a number of backgrounds. We first consider the low-energy limits given by a supergravity theory in various backgrounds. Because these theories are local and polynomial in fields, they respect a CPT symmetry [23], which we identify, along with other discrete symmetries.

We wish to ask whether these symmetries of the low-energy theory are indeed exact. For backgrounds which are believed to be described non-perturbatively by matrix models, this is a straightforward exercise. The theories we examine in detail are M-theory on flat eleven-dimensional space (section 7.2), M-theory on T^3 (section 7.3), and M-theory on S_1/Z_2 (section 7.4), along with their corresponding matrix theory descriptions. In section 7.5 we comment on M-theory on T^5. We only discuss matrix models that describe the supergravity and membrane dynamics. Whether CPT still exists when the M5 branes are included is beyond the scope of this paper, and is left for future work.

In all the examples that we examine, we find that the discrete symmetries of the classical supergravity theory can be found in the corresponding matrix models. This includes CPT. Typically it is either a T or PT symmetry of the matrix model that corresponds to the CPT symmetry of the space-time theory.

We also emphasize that we focus on theories that are Lorentz invariant. A violation of CPT in a Lorentz invariant theory is much more non-trivial, and interesting.

7.1 Chern-Simons theory in five dimensions

It is instructive to start by discussing the discrete symmetries of a five-dimensional Chern-Simons theory coupled to a current. This theory has some similarities to the eleven-dimensional theory considered in the next section, but since it is simpler we begin here first.

Electromagnetism in five dimensions coupled to a current preserves three symmetries: C, T, and P. Since here there are an even number of spatial dimensions, the parity symmetry corresponds to a reflection about an odd number of spatial directions. The addition of a Chern-Simons coupling violates both the charge conjugation and parity symmetries. But T and the combination CP are preserved.

7.2 M-theory in eleven-dimensions

7.2.1 Discrete symmetries of the classical theory

The low-energy limit of a supersymmetric quantum theory in eleven-dimensions is described by eleven-dimensional supergravity. It contains a graviton, gravitino and also a three-form potential C_3 that has a Chern-Simons coupling. The lagrangian is, schematically [24],

$$\begin{aligned}\mathcal{L} = &-eR - eG^2 + ie\overline{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho + \\ &+ C \wedge G \wedge G - ie\left(\overline{\psi}_\lambda \Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi_\tau + \overline{\psi}^\mu \Gamma^{\nu\rho}\psi^\sigma\right)(G+\hat{G})_{\mu\nu\rho\sigma}\,. \end{aligned} \tag{7.1}$$

Here $G = dC_3$ is the four-form field strength, ψ_μ is the gravitino, D_μ is the covariant derivative including the spin connection and e is the determinant of the vielbein. $\hat{G}$ is a combination of the four-form field strength and a term involving two gravitinos. Under C,P and T it transforms like G. The Γ matrices will be chosen to be real, with Γ^0 antisymmetric and all the others symmetric. Γ's with n Lorentz indices are given by the antisymmetric product of n Γ matrices and are all real. Finally, $\overline{\psi} \equiv \psi^\dagger \Gamma^0$.

From the previous example we expect the eleven-dimensional theory to preserve only T and CP. Under T the three-form potential and field-strength transform as pseudotensors. Under CP the three-form potential transforms as a tensor under the reflection, and acquires an overall $(-)$ sign from the charge conjugation. It is straightforward to confirm that the bosonic part of the action is invariant under either of these symmetries.

Verifying that the supersymmetric theory is CP and T invariant requires a bit more work. Under CP (a reflection in x^i say), a Majorana SO(10, 1) spinor transforms

$$\theta(x, x^i) \to \Gamma^i \theta(x, -x^i) \,. \tag{7.2}$$

Then

$$\overline{\theta}\Gamma_{\mu_1 \cdots \mu_n}\theta \tag{7.3}$$

transforms as a tensor when n is odd and as a pseudo-tensor when n is even. This pseudotensor property is crucial to make the action CP invariant.

Under T, a spinor transforms as

$$\theta(x^0, x^i) \to \Gamma^0 \theta(-x^0, x^i) \,, \tag{7.4}$$

with the gravitino transforming as a vector-spinor. One may check that all the interactions are T invariant. This is because under T

$$i\overline{\theta}\Gamma^{\mu_1 \cdots \mu_n}\theta \tag{7.5}$$

transforms as a tensor when n is odd and as a pseudo-tensor when n is even.

The classical theory also has membrane solutions. The lagrangian for the membrane in flat space is [25] (dropping numerical factors)

$$\mathcal{L} = \sqrt{g(x,\theta)} + i\epsilon^{ijk} \left(\partial_i X^\mu (\partial_j X^\nu + i\overline{\theta}\Gamma^\nu \partial_j \theta) + \overline{\theta}\Gamma^\mu \theta \overline{\theta}\Gamma^\nu \theta\right) \overline{\theta}\Gamma_{\mu\nu}\partial_k \theta \,. \tag{7.6}$$

Here $X^\mu(\tau, \sigma^i)$ parameterize the position of the membrane, and $\theta(\tau, \sigma^i)$ is a Majorana spinor which has 32 real components.

This action has both a CP and a T invariance if we extend the symmetries to include a world-sheet parity and world-sheet time-reversal.

First consider CP. Since $\overline{\psi}\Gamma_{(2)}\psi$ transforms as a pseudo-tensor whereas $\overline{\psi}\Gamma_{(1)}\psi$ transforms as a vector, none of the fermion terms are invariant. Also note that the fermion terms have an odd number of world-sheet derivatives, so the lagrangian is not invariant under a world-sheet parity transformation either. These two transformations can be combined so that the action is invariant under the spacetime CP and a world-sheet parity transformation.

Next consider T. The membrane lagrangian is invariant under T if the world-sheet time τ is also flipped. This too is reasonable. For the above lagrangian, although covariant, has a large gauge symmetry which includes (proper) reparameterization invariance. This redundancy is fixed in the light-cone frame, where $X^+ = \tau$. A time-reversal in X^0 implies a time-reversal in the world-sheet time.

Finally the coupling of the membrane to the three-form potential is given by

$$\int_{M_2} C_3 = \int_{M_2} dt d^2\sigma C_{\mu\nu\gamma} \frac{dx^\mu}{dt} \{X^\nu, X^\gamma\}_{PB} \,. \tag{7.7}$$

Using the information provided above one finds that these too are invariant under T and CP. For under CP or T the three-form potential transforms as a pseudo-tensor. For CP we must also include a world-sheet parity reflection which causes the Poisson-bracket to transform as pseudo-tensor. The current-coupling is then invariant. For T we must also flip the world-sheet time. The velocity $\dot{X}^\mu$ transforms as a pseudo-vector (that is, as a momentum), whereas the Poisson bracket now transforms as a tensor. The membrane current-coupling is then T invariant.

In summary, the eleven-dimensional supergravity theory, including the membrane dynamics, is invariant under two discrete symmetries: T and CP. We leave the issue of whether these symmetries are preserved when M5 branes are included to future research, but we expect that the answer is yes. It is natural to then ask whether these symmetries are preserved in the exact quantum theory. This leads us to matrix theory, which is conjectured to be an exact quantum gravity theory whose low-energy limit is eleven-dimensional supergravity, including the membrane excitations.

7.2.2 Matrix model

The matrix model is given by the truncation of $N = 1$ $d = 10$ Super-Yang-Mills U(N) theory down to zero dimensions. The large-N limit of this quantum mechanical system is conjectured [26] to describe M-theory in eleven-dimensional Minkowski space, while for finite N it is believed to give the discrete light cone quantization of the theory.

We shall see that the matrix model has two discrete symmetries, $\mathrm{C_M}$ and $\mathrm{T_M}$. These correspond to the charge and parity-time reversal symmetries of the ten-dimensional minimal super–Yang-Mills theory. As we shall see, the correspondence we will establish is:

$$\begin{array}{ccc} \text{11 dimensions} & & \text{Matrix theory} \\ CR_{10}T & \longleftrightarrow & \mathrm{T_M} \\ CR_1 \cdots R_9 & \longleftrightarrow & \mathrm{C_M} \end{array} \tag{7.8}$$

where R_i reverses the sign of the i'th coordinate. *Note that CPT in the eleven-dimensional theory is equivalent to a time-reversal symmetry of the matrix model.* To minimize confusion, discrete symmetries of the eleven-dimensional or space-time theory will be italicized, while those of the matrix model will be given in Roman type.

The lagrangian is

$$L = \frac{1}{2} \mathrm{tr} \left(D_t \mathbf{X} D_t \mathbf{X} + [X^k, X^l]^2 + i\theta D_t \theta - \theta \gamma^k [X_k, \theta] \right) . \tag{7.9}$$

Here X^k and θ are $N \times N$ hermitian matrices, with $k = 1 \ldots 9$. θ is an anti-commuting SO(9) spinor with 16 real components. With the hermitian conjugation of two anti-commuting variables defined to be $(\lambda\eta)^\dagger = \eta^*\lambda^*$, the two fermionic terms in the lagrangian are hermitian. The γ^i matrices are 16×16 matrices satisfying the SO(9) Clifford algebra $\{\gamma_i,\gamma_j\} = 2\delta_{ij}$ and are all real and symmetric.

We recognize the first two terms as the ten-dimensional gauge field strength, and the last two as the gaugino kinetic term and current coupling.

This theory has two real sixteen component supercharges, which are [27]:

$$\begin{aligned} q_\alpha &= \mathrm{tr}\theta_\alpha \\ Q_\beta &= \mathrm{tr}\left(P^i\gamma^i_{\beta\alpha}\theta_\alpha + i\gamma^{ij}_{\beta\alpha}[X^i,X^j]\theta_\alpha\right) . \end{aligned} \tag{7.10}$$

Here $\gamma^{ij}\sim[\gamma^i,\gamma^j]$ is purely real. For future reference,

$$\{q_\alpha, Q_\beta\} \sim \gamma^i_{\alpha\beta}\mathrm{tr}P_i + Z_{ij}\gamma^{ij}_{\alpha\beta}\,, \tag{7.11}$$

where

$$Z_{kl} = i\mathrm{tr}[X^k,X^l]\,. \tag{7.12}$$

is the charge of a membrane stretched in the i-th and j-th directions [28].

We next discuss the $\mathrm{C_M}$ and $\mathrm{T_M}$ symmetries of this quantum mechanical model.

$\mathbf{C}_\mathrm{M}$: as asserted above, the matrix model has a charge conjugation symmetry. In fact there are nine [29]. These descend from the charge conjugation symmetry existing in ten-dimensions and the SO(9) rotation group. Although the ten-dimensional theory is chiral, a charge conjugation symmetry can still be imposed. This is possible in $d = 4k+2$ dimensions, but not in $d = 4k$ (which includes the familiar four dimensions) [19]. A Majorana condition can also be imposed if the representation is real. In this case it is, since the fermions are in the adjoint representation. For our purposes it is sufficient to study the charge conjugation symmetry which is:

$$X_i(t)\to -X_i^T(t)\,,\theta(t)\to\theta^T(t) \tag{7.13}$$

with the transpose acting on the U(N) indices. Normally the spinors are conjugated, but this is trivial since the spinors are real.

Since commuting diagonal matrices X^i_{aa} describe the transverse center-of-mass coordinate of a particle in the light-cone, this charge conjugation describes the reflection (not connected to the identity) $R_1\cdots R_9$ in eleven-dimensions. The momenta also transform correctly, appropriate for a reflection. This transformation also induces C in the space-time theory. The matrix model expression for the charge of an infinite M2 brane at rest is given by (7.12). Since under C_M the matrices are transposed, the charge of the membrane is flipped. This is also confirmed by inspecting the supersymmetry algebra (7.10) and (7.11). For under C_M, $Q\to -Q$, and consistency of the algebra requires that $Z\to -Z$.

$\mathbf{T}_\mathrm{M}$: this is just the PT transformation of a zero mode gauge field in ten-dimensions. In general, this is described by antiunitary operator K that sends t to $-t$ and has also acts on the variables of the model - in this case, $N\times N$ matrices. For a non-relativistic electron, for example, $K = \sigma_y K_0$ where K_0 complex conjugates the electron wavefunction. Here we *define* time reversal by the antiunitary operator K that acts on the $N\times N$ matrices by:

$$KX_i(t)K^{-1} = X_i(-t)\,,\qquad K\theta(t)K^{-1} = \theta(-t)\,,\qquad KcK^{-1} = c^*\,. \tag{7.14}$$

Here c is any complex number that is *not* in the $N \times N$ matrices X or θ. We later demonstrate that K defined in this way acts on the supergraviton states in a way that agrees with the $CP_{10}T$ transformation in the eleven-dimensional theory. To begin though, it is straightforward to verify that the matrix model lagrangian is invariant under this transformation, in that $KL(t)K^{-1} = L(-t)$.

Next we describe the eleven-dimensional interpretation of this symmetry. First note that the transverse coordinates transform rather trivially. The transverse momenta, however, do change sign. Both of these are what we expect of an eleven-dimensional time-reversal. *But the eleven-dimensional interpretation requires an inversion of both of the light-cone coordinates.* This is because the quantum mechanical mass, which is the longitudinal momentum, is invariant. This means that the matrix model $\mathrm{T_M}$ symmetry reverses both light-cone directions, corresponding at least to a $P_{10}T$ symmetry in eleven dimensions. Finally, this symmetry also implies a charge conjugation in eleven-dimensions. For under K the charge (7.12) of a membrane transforms as $Z_{ij} \to -Z_{ij}$. To keep the current coupling invariant this requires a charge conjugation of the three-form.

We can also inspect the expression for the supercharges to find that under $\mathrm{T_M}$ the hamiltonian is invariant. This is sufficient to establish that $\mathrm{T_M}$ is a symmetry of the quantum mechanical model. Further, since it is anti-unitary, it will exchange ingoing and outgoing states. From the preceding discussions, $\mathrm{T_M}$ corresponds to the $CR_{10}T$ symmetry of the eleven-dimensional theory, which since eleven-dimensions has an even number of spatial directions, *is* the CPT symmetry. Assuming eleven-dimensional (proper) Lorentz invariance, this symmetry is unique.

So far we have demonstrated that the matrix model $\mathrm{T_M}$ symmetry has the correct interpretation as a CPT symmetry in eleven dimensions, and is a symmetry of the matrix model hamiltonian. It remains to establish that it acts appropriately on the states of the theory.

We have already seen that K reverses the transverse momenta, preserves the longitudinal momentum, and also charge conjugates. But a $CP_{10}T$ transformation in eleven dimensions should also change the helicity of the supergraviton states.

To see this, we focus on states that have zero transverse momenta. (But in eleven dimensions they have non-zero longitudinal momentum.) The helicity of these states is then given by their charge under U(1) subgroups of the SO(9) little group.

In addition, we study the asymptotic properties of the supergraviton states in an $N = 2$, U(2) model. Although the Hilbert space for this model has only two supergravitons, it is large enough for our purposes. Plefka and Waldron [30] have discussed the construction and computation of scattering amplitudes in this model. We follow their construction and notation.

For $N = 2$ we expand

$$X_i = X_i^0 \mathbf{1} + \vec{X}_i \cdot \vec{\mathbf{T}} \tag{7.15}$$

$$\theta_\alpha = \theta_\alpha^0 \mathbf{1} + \vec{\theta}_\alpha \cdot \vec{\mathbf{T}} \tag{7.16}$$

where X_i^0 describes the center-of-mass position of the supergraviton with fermionic zero modes θ^0, and $\vec{X}_i$ and its superpartner describe the relative separation. Vectored quantities denote values in the SU(2) algebra. The U(1) part corresponds to the center-of-mass motion, and variables in the SU(2) part describe the relative separation of two supergravitons.

In terms of these variables the hamiltonian has a simple form,

$$H = P_i^0 P_i^0 + \vec{P}_i \cdot \vec{P}_i + (\vec{X}_i \times \vec{X}_j)^2 + i\vec{X}_j \cdot (\vec{\theta} \times \gamma_j \vec{\theta})\,, \tag{7.17}$$

and the action of K is given by

$$KZ^I K^{-1} = Z^I\,, \quad I = 0,1,3\,, \qquad KZ^I K^{-1} = -Z^I\,, \quad I = 2\,, \qquad KcK^{-1} = c^*\,, \tag{7.18}$$

where the label $I = 0,1,2,3$ refers to the generators of SU(2), and $Z = X$ or θ. It can be verified that the hamiltonian is invariant.

To discuss the construction of the states, it is more convenient to only keep invariance under the SO(7)$\times$U(1) subgroup of the SO(9) little group manifest, as described in [25, 30]. The U(1) generator J_{89} gives the helicity of a state in the 89 plane. Under SO(7)$\times$U(1), the 16 dimensional spinor θ^0 decomposes into two 8 dimensional SO(7) spinors θ^0_+ and θ^0_- that differ by their SO(8) chirality. They can be organized into $\lambda = \theta^0_+ + i\theta^0_-$ and its conjugate $\lambda^\dagger$. These obey the algebra $\{\lambda_\alpha, \lambda^\dagger_\beta\} = \delta_{\alpha\beta}$. The supergraviton multiplet is constructed by applying the raising operators $\lambda^\dagger$ to the ground state $|-\rangle$ which is annihilated by all the lowering operators. This state has U(1) charge -1. A transverse graviton state is, for example [30],

$$e^{ik_i X_i^0} h_{ij} (\lambda^\dagger \gamma_i \lambda^\dagger)(\lambda^\dagger \gamma_j \lambda^\dagger)|-\rangle \tag{7.19}$$

with the γ_i matrices real antisymmetric SO(7) Dirac matrices. A transverse 3-form state is [30]

$$e^{ik_i X_i^0} c_{ijk} (\lambda^\dagger \gamma_{[ij} \lambda^\dagger)(\lambda^\dagger \gamma_{k]} \lambda^\dagger)|-\rangle \tag{7.20}$$

and so on. As an aside, we note that given this explicit expression for the three-form state, it can be verified that under C_{YM} this state transforms as $c_{ijk} \to c_{ijk}$ with the transverse momentum reversed. This is indeed the correct $CR_1 \cdots R_9$ transformation of the three-form.

To discuss the action of $\mathrm{T_M}$ it is convenient to expand our state into modes with definite helicity. The basis appearing in (7.19) is not well suited for this purpose, since it corresponds to a real basis. But it is straightforward to construct the helicity basis corresponding to states with definite charge. For instance, focus on an SO(3) subgroup of SO(7). From a vector of SO(3), we can construct states with charges $m_{12} = 0, \pm 1$ under J_{12}. In the matrix model this corresponds to the states

$$\begin{aligned}
|B_\pm\rangle &= \lambda^\dagger (\gamma_1 \pm i\gamma_2)\lambda^\dagger |-\rangle \\
|\tilde{B}_\pm\rangle &= \lambda (\gamma_1 \pm i\gamma_2)\lambda |+\rangle \\
|B_0\rangle &= \lambda^\dagger \gamma_3 \lambda^\dagger |-\rangle \\
|\tilde{B}_0\rangle &= \lambda \gamma_3 \lambda |+\rangle
\end{aligned} \tag{7.21}$$

with $\lambda^\dagger|+\rangle = 0$. In addition, the states with $|-\rangle$ ($|+\rangle$) have charge $m_{89} = +1$ (-1) under

J_{89}. Since K acts as

$$K|-\rangle = |+\rangle\,, \qquad K\lambda K^{-1} = \lambda^\dagger \tag{7.22}$$

we see that

$$K|B_\pm\rangle = |\tilde{B}_\mp\rangle\,, \qquad K|B_0\rangle = |\tilde{B}_0\rangle\,. \tag{7.23}$$

That is, K changes a state (m_{12}, m_{89}) into another having the opposite helicities. It is clear that we can use this method to construct, for example, a **5** which will have well-defined helicity $m_{12} = -2, \ldots 2$. These describe a graviton with polarizations in the transverse $1, 2, 3$ directions. It is also clear that as before, K will change a state with helicity m_{12} into one with $-m_{12}$. Thus K acts on the transverse supergraviton states in a way that is appropriate for a CPT transformation in eleven-dimensions.

This section concludes with some remarks summarizing what has been established. Matrix theory has two discrete symmetries, which in the eleven-dimensional interpretation correspond to the discrete symmetries of the low-energy theory, CPT and CP. It is curious that the CPT symmetry of the eleven-dimensional theory does not correspond to the CPT symmetry of matrix model, inherited from ten dimensions. Rather, it is PT. We do not understand why it turned out this way.

But which dynamical processes, viewed from eleven-dimensions, will preserve these symmetries? It is conjectured that matrix theory is complete in the following sense [26]. It should describe any configuration or dynamics that can be produced by finite energy multigraviton scattering. This includes supergraviton scattering at low-energies, but also more exotic (and interesting) processes. For instance, since membranes or fivebranes of finite size (and energy) carry no charge, they can be produced by multigraviton scattering. These are not static configurations, since they eventually collapse to form non-extremal black holes. The process leading from (super)gravitons in the initial state to the (super)gravitons produced by the decaying black holes will preserve CPT and CP.

But there are other notable degrees of freedom — membranes and fivebranes of infinite size. These cannot be produced by supergraviton scattering, so a separate argument is needed to argue that their dynamics preserves CPT and CP. Matrix theory descriptions exist for the infinite membrane and antimembrane [26], so processes in these backgrounds will preserve CPT and CP.

When we come to fivebranes of infinite size our arguments are incomplete. This is because matrix theory does not describe all fivebranes. There do exist matrix configurations for fivebranes carrying a fivebrane and membrane charge [28], and these will obey CPT and CP. But there are other fivebranes that require a different matrix theory. For instance, k longitudinal fivebranes are described [31] by the dimensional truncation to zero dimensions of $N = 2$, $d = 4$ super-Yang-Mills with k fundamental hypermultiplets.

We also lack arguments for transverse fivebranes (of infinite size). By Seiberg's argument [32] they are described by the six dimensional $(2, 0)$ little string theory. This is not a local quantum field theory, so there is no argument for the existence of a CPT symmetry. It would be interesting to investigate this question further in the recent matrix model proposal for transverse five-branes [33].

7.3 M-theory on $\mathbf{T}^3$

7.3.1 Discrete symmetries of the classical theory

At the classical level the eight-dimensional model we consider is given by the compactification of 11-dimensional supergravity and membranes on a three-torus. (As in our discussion of the eleven-dimensional theory, here we focus on backgrounds that do not include M5 branes.) The discrete symmetries we focus on are those inherited from the eleven-dimensional theory. Recall that the eleven-dimensional theory has two discrete symmetries: T and CP. For CP, we may choose the reflection to occur in one of the seven non-compact directions. For instance, this could be the longitudinal direction (R_{10}) of the light-cone gauge. But for suitable choice of torus there are also three more parity symmetries P_i, which combine a reflection in one of the internal directions (R_7, R_8 or R_9, say), with a reflection in all of the non-compact directions. They correspond to unbroken discrete elements of the SO(10) rotation group in eleven-dimensions. As the product of any two of these is a discrete rotation in the internal three-torus, the three parity symmetries, together with the CP symmetry, can be combined to give a C and P symmetry, and two internal symmetries. In total, the 7+1 dimensional theory has five independent discrete symmetries: P, C, T and two internal discrete symmetries. The product of CP and T is the CPT_8 symmetry of the eight-dimensional theory.

These symmetries may be spontaneously broken by the vev of a field. An interesting example is if C_{789} is non-zero. This breaks the eleven-dimensional CPT, but still preserves the eight-dimensional CPT and Lorentz invariance.

In the next section we examine the matrix model for M theory in this background, with and without $C_{789} = 0$, and ask if there is an unbroken symmetry which corresponds to the eight-dimensional CPT. In both cases the answer is yes.

7.3.2 Matrix theory description

By Seiberg's argument [32], the matrix theory description of M-theory on T^3 in the light-cone is given by the large-N limit of the $d = 3 + 1$, $\mathcal{N} = 4$ supersymmetric U(N) theory compactified on the dual three-torus $\widetilde{T}^3$. As we now show, the discrete symmetries of this theory correspond to those found in the classical 7+1 dimensional theory.

The couplings of the four-dimensional theory are the gauge coupling and the theta parameter. We begin by first setting the theta parameter to zero.

Then the gauge theory has separate $\mathrm{CP_{YM}}$, $\mathrm{C_{YM}}$ and $\mathrm{T_{YM}}$ symmetries. As in the original matrix model example, the interpretation of these symmetries in the eight-dimensional theory is non-trivial.

Thinking about this theory as arising from the dimensional truncation of a ten-dimensional Yang-Mills theory, one might have expected only two discrete symmetries. The four-dimensional $\mathrm{C_{YM}}$ and $\mathrm{PT_{YM}}$ are inherited directly from the ten-dimensional $\mathrm{C_{YM}}$ and $\mathrm{PT_{YM}}$ symmetries. But there is an additional discrete symmetry in the four-dimensional theory, which was part of the Lorentz group in ten dimensions. Four-dimensional $\mathrm{P_{YM}}$ is given by a reflection of the three non-compact directions together with one of the internal directions.

On the $\tilde{T}^3$, the rotation group is broken to the discrete subgroup of rotations preserved by the three-torus. We may combine $\mathrm{P_{YM}}$ with these discrete symmetries to obtain three parity symmetries $\mathrm{P}^i{}_{\mathrm{YM}}$, each corresponding to a reflection in one of the directions of the dual three-torus. The Yang-Mills theory also has a SU(4) symmetry, which in the eight-dimensional theory corresponds to the manifest rotation symmetry in light-cone gauge.

To obtain the mapping between the symmetries of the Yang-Mills theory and the supergravity theory, we need to discuss the BPS states of the two theories [34]. In the Yang-Mills theory we can have field configurations with an electric field $\vec{E}$, magnetic field $\vec{B}$, and/or a non-zero vev for the six scalar adjoints X^i. On the moduli space the six scalars parameterize the position of the (super)graviton in the (non-compact) transverse six dimensions. Field configurations with quantized electric flux correspond to the KK momentum of supergravity states. Field configurations with magnetic flux correspond to transverse membranes (i.e., not along the light-cone direction R_{10}) that are wrapped around two one-cycles of the three-torus. The winding number of the membrane corresponds to the quantized magnetic flux.

We discuss each of these in turn.

C$_{\mathrm{YM}}$: this flips all of the transverse scalars. Since there are six of these, this is just an element of SO(6). But since the electric field also changes, in the supergravity theory this must correspond to flipping all the components of the KK momentum. Thus $\mathrm{C_{YM}}$ includes a reflection in 7,8 and 9. But under $\mathrm{C_{YM}}$ the magnetic field changes as well. In the supergravity theory this corresponds to flipping the winding number of the membrane. But the winding number is just the charge of the membrane. So $\mathrm{C_{YM}}$ includes a charge conjugation in the supergravity theory. Therefore $\mathrm{C_{YM}} \leftrightarrow CR_7R_8R_9$.

To see this more formally we have to discuss how the charge of a membrane transforms under CP. The charge or winding number of a membrane wrapped around cycles of the torus in the X^i and X^j directions, with i, j one of 7,8,9, is given by

$$Z_{ij} \sim \int d^2\sigma \{X^i, X^j\}_{PB} \,. \tag{7.24}$$

Under a reflection in 7,8 and 9 the charge is invariant. But under a CP transformation ($CR_7R_8R_9$) the orientation of the membrane is also changed. This CP changes the charge, which agrees with the corresponding change in the Yang-Mills magnetic field. So we have learned that $\mathrm{C_{YM}} \leftrightarrow CR_7R_8R_9$.

PT$_{\mathrm{YM}}$: this reverses the Yang-Mills time coordinate, which is the light-cone time coordinate of the M-theory. So $\mathrm{PT_{YM}}$ implies TR_{10}. Now under $\mathrm{PT_{YM}}$ both the electric and magnetic fields change. A sign change in the electric fields means that in the M-theory all the KK momenta change. The action of TR_{10} alone does this, so no additional reflection in the internal space is required. That the magnetic field changes means that in the M-theory the membrane charge changes. But (7.24) implies that the membrane charge does not change sign under TR_{10} alone. So to obtain the correct winding number transformation, $\mathrm{PT_{YM}}$ must also imply a charge conjugation in the M-theory, which acts to reverse the membrane orientation.

Note that again it is $\mathrm{PT_{YM}}$, and not $\mathrm{CPT_{YM}}$, that corresponds to the eight-dimensional CPT symmetry. This can be better understood if we try, given the symmetries of the supergravity theory, to see what it could have been. Since a time reversal is involved, the Yang-Mills parity-time reversal symmetry could *a priori* only correspond to T, CT, $TR_{10}R_7R_8R_9$, $R_{10}T$, or $CR_{10}T$ ($=CPT$) of the eight-dimensional theory. The arguments of the preceding paragraph exclude all but the last possibility.

$\mathbf{P}^i_{\mathrm{YM}}$: in the ten-dimensional theory this was an element of the SO(9) rotation group. Therefore one and only one of the adjoint scalars transforms. A change in one of the adjoint scalars corresponds in the supergravity theory to a parity reflection in one of the transverse non-compact directions. Therefore $\mathrm{P}^i_{\mathrm{YM}}$ implies P_i of the M-theory. Returning to the Yang-Mills theory, $\mathrm{P}^i_{\mathrm{YM}}$ changes the electric field only in the $i-$th direction. By the correspondence, the graviton momentum along the $i-$th cycle should be modified. The most natural choice consistent with these two observations is that this symmetry of the Yang-Mills theory is P_i of the M-theory. This guess also preserves the correspondence between the magnetic flux and membrane charge. For under $\mathrm{P}^i_{\mathrm{YM}}$, F_{kl} does not change for k and l not equal to i, whereas F_{ik} does change. But this is precisely the transformation of the winding number under P_i: flipping only R_i changes Z_{ij} and leaves the other components invariant. From this we find that $\mathrm{P}^i_{\mathrm{YM}} \leftrightarrow P^i$.

To summarize:

$$\begin{array}{ccc} \text{7+1 dimensions} & & \text{Matrix theory} \\ CR_{10}T & \longleftrightarrow & \mathrm{PT_{YM}} \\ CR_7R_8R_9 & \longleftrightarrow & \mathrm{C_{YM}} \\ P_i & \longleftrightarrow & \mathrm{P}^i{}_{\mathrm{YM}} \end{array} \tag{7.25}$$

In particular, the eight dimensional CPT symmetry is $\mathrm{PT_{YM}}$ of the Yang-Mills theory. So far we have assumed that the theta parameter of the Yang-Mills theory vanishes. If instead it is non-zero, then the time and parity symmetries of the Yang-Mills theory are broken, but the combinations $\mathrm{PT_{YM}}$ and $\mathrm{C_{YM}}$ are preserved. In addition, each of the $\mathrm{P}^i_{\mathrm{YM}}$ symmetries are broken, but the product of any two is not. In total, the theta parameter violates only $\mathrm{P_{YM}}$ and T_{YM} but preserves four discrete symmetries. In particular, $\mathrm{PT_{YM}}$ is preserved. This implies that M-theory in this background should have an unbroken eight-dimensional CPT.

A non-zero theta term corresponds in the supergravity theory to $C_{789} \neq 0$[34]. This is satisfying, for this preserves the eight-dimensional CPT. In fact, one finds that the symmetries preserved in the Yang-Mills theory with a non-zero theta parameter are indeed, by the correspondence given above, the same symmetries preserved by C_{789}.

To summarize: we have found that the matrix model has an unbroken PT symmetry, and that it corresponds to the CPT symmetry of the eight-dimensional M-theory. Just as before, we have not studied backgrounds including M5 branes.

7.4 M-theory on S_1/Z_2

In this section, we consider M-theory on S_1/Z_2. The large radius limit of this theory is the Hořava-Witten model, and the small radius limit is the $E_8 \times E_8$ weakly coupled heterotic string. We begin by recalling in section 7.4.1 the CPT symmetry of the weakly coupled heterotic theory, and then in section 7.4.2 identify this symmetry in the matrix model description.

7.4.1 Discrete symmetries of the heterotic string

The heterotic string has one discrete symmetry θ which is CPT in ten dimensions [19]. In the covariant formulation θ acts as $X^{0,1}(\sigma,\tau) \to -X^{0,1}(\sigma,\tau)$, where $X^{0,1}$ are the two light-cone directions. In the (RNS) formulation where the world-sheet fermions ψ^ν transform as space-time vectors, θ acts to also change the signs of $\psi^{0,1}$. All other fields transform trivially, including the 32 left-moving λ's.

In light cone gauge, the Weyl and diffeomorphism invariance of the world-sheet may be used to completely fixed these coordinates and their fermionic partners to be $\psi^- = 0$, $\partial_+ X^+ = p^+/2$ and [13]

$$\begin{aligned} \partial_+ X^- &= \frac{(\partial_+ X^i \partial_+ X^i + \frac{i}{2}\psi^i \partial_+ \psi^i)}{p^+}\,, \\ \psi^- &= \frac{2\psi^i \partial_+ X^i}{p^+}\,, \end{aligned} \tag{7.26}$$

along with setting the world-sheet metric to be canonical. The only residual reparameterization transformations are constant shifts of the origin of the world-sheet spatial coordinate, which lead to the level matching condition. In the light-cone gauge θ acts as $(\sigma,\tau) \to (-\sigma,-\tau)$ and leaves the transverse bosonic and fermionic variables invariant:

$$X^i(x^+) \to X^i(-x^+)\,, \qquad \psi^i(x^+) \to \psi^i(-x^+)\,. \tag{7.27}$$

Because of the gauge fixing, this is not a world sheet diffeomorphism. In light-cone gauge $\tau \to \tau'(\tau) = -\tau$ is a combination of time-parity reversal in the target space with a reflection of the world-sheet coordinates. This means that X_+ should transform as

$$X_+(\tau,\sigma) \to X'_+(-\tau,-\sigma) = -X_+(\tau,\sigma) = -\tau \tag{7.28}$$

with the last equality following from the light-cone gauge condition. But in the lightcone gauge we must have $X'_+(-\tau) = -\tau$ which is now consistent with the r.h.s., due to the time-parity reversal in the spacetime.

We can check that X_- and ψ^- transform properly. Inspecting (7.26) indicates that reflecting the worldsheet coordinates implies that $\partial_+ X^-$ is invariant and ψ^- flips sign. Combining these with the invariance of $\partial_+ X^+$ and $\psi^+ = 0$, one finds that a reversal of the worldsheet coordinates, while *staying within the light-cone gauge*, requires $X^{0,1}$ and $\psi^{0,1}$ to also change sign. Assuming diffeomorphism invariance of the quantized string, this action is precisely θ.

In ten dimensions one finds the effective action at two derivatives has a PT symmetry. What about C? C is trivial, since the E_8 gauge group has no non-trivial outer automorphisms. So the PT symmetry is the same as CPT.

7.4.2 Discrete symmetries of the matrix model

The matrix model describing the $E_8 \times E_8$ heterotic string or Hořava-Witten theory has been developed by a number of authors [35, 29, 36, 37]. It is given by 1+1 dimensional super-Yang-Mills $O(N)$ theory with $(0,8)$ supersymmetry. The gauge supermultiplet (A_μ, λ_-) includes eight left-moving gauginos. In addition there are eight bosons X^i and eight right-moving fermions ψ^i each in the symmetric representation, and 32 left-moving fermions χ^r in the vector representation. These fermions must be added to cancel anomalies [36]. This theory has a spin(8) R-symmetry. Under it, λ and ψ transform as $\mathbf{8_s}$ spinors of opposite chirality, χ is neutral and X^i transforms as an $\mathbf{8_v}$ vector.

The relation of this theory to the weakly and strongly coupled limits of the heterotic theory is described by Banks and Motl [37]. The gauge coupling of the heterotic matrix theory is proportional to the volume of the dual circle that the gauge theory is compactified on. The weakly coupled heterotic string limit corresponds to small radius in M-theory which translates to large radius in the Yang-Mills description. In two dimensions the gauge coupling is a relevant parameter, so this is a strong coupling limit. On the moduli space the lowest energy excitations are diagonal matrices, which gives eight bosons, eight right-moving fermions and 32 left-moving fermions. The $O(N)$ gauge theory is completely broken to a number of Z_2 subgroups (which act trivially on the X's). The truncation of the matrix theory to these states reproduces the free heterotic string theory.

The lagrangian is given by [36]

$$\begin{aligned} L = \operatorname{tr}\Big(& i\lambda D_+ \lambda + \operatorname{Tr} F_{\mu\nu}^2 + D^\mu X^i D_\mu X^i + i\psi^i D_- \psi^i + i\chi^r D \chi^r - \\ & - (X^i T^a X^j)^2 + i\psi \tilde{\gamma}^i \lambda X^i \Big) . \end{aligned} \tag{7.29}$$

Here $\tilde{\gamma}^i$ are eight dimensional matrices and are all real (but not all symmetric). They are defined such that $\gamma^i \equiv \tilde{\gamma}^i \otimes \sigma_1$ satisfy the spin(8) Clifford algebra (with positive signature).

There is one symmetry θ_M which is not part of the spin(8) R-symmetry. It includes a reversal of the 1+1 dimensional coordinates and is given by

$$\begin{aligned} A^a_\mu(z) &\to -A^a_\mu(-z)\,, \qquad & \chi_a(z) &\to \chi_a(-z) \\ X^i_{ab}(z,\bar{z}) &\to X^i_{ab}(-z,-\bar{z}) & & \\ \lambda_{ab}(z) &\to -\lambda_{ab}(-z)\,, \qquad & \psi_{ab}(\bar{z}) &\to \psi_{ab}(-\bar{z}) \end{aligned} \tag{7.30}$$

where a, b are $O(N)$ color indices, and the spin(8) index has been suppressed on the spinors. Note that there is a crucial $(-)$ sign appearing in the transformation of the gauginos λ. This is needed to make the Yukawa coupling invariant. (And in our notation $T^{-1} c\theta\eta T = c^* T^{-1} \theta T T^{-1} \eta T$.)

The transformation θ_M acts to send $L(z) \to L(-z)$ which is sufficient for it to be a symmetry of the theory. To verify this, it is important to remember to complex conjugate all constants appearing in the lagrangian. This includes the generators appearing in both the covariant derivatives and explicitly in the lagrangian. For all the $O(N)$ representations appearing here the generators are purely imaginary. (So that, for example, $A^a T^a$ is invariant under θ_M.)

On the moduli space all the gauginos and gauge bosons are massive. Restricting attention to just the zero modes, the action becomes the heterotic string theory action in the light-cone gauge. Since the 1+1 dimensional spinors are in the spinor representation of spin(8), the theory we arrive at is the Green-Schwarz formulation of the heterotic theory. The action of the θ_M symmetry on the diagonal zero modes X^j, ψ^j and the 32 χ^r, is rather trivial, for it only reverses the worldsheet coordinates, without any additional transformation on the fields. Transforming to the RNS formulation, this symmetry still reverses only the world-sheet coordinates. As described in the previous section, this is just θ in the light-cone gauge.

Thus CPT of the weakly coupled heterotic theory is also a symmetry of its non-perturbative formulation. This also establishes that CPT is a symmetry of the Hořava-Witten theory. Again, as before our arguments do not apply to backgrounds with M5 (NS5) branes.

7.5 M-theory on T^5

By Seiberg's argument this is equivalent to the $(2,0)$ little string theory on the dual five-torus. The little string theory is not a local quantum field theory. So we cannot use this non-perturbative formulation to argue that this theory has a CPT symmetry. Discrete Light Cone Quantizations of this theory exist and are given by 1+1 dimensional conformal field theories [38]. We leave it to future work to decide whether these theories preserve CPT. Given that so little is known about little string theories, we can not make a reliable statement here.

8. Conclusions

We have conducted a search for anomalies in discrete symmetries in a variety of models. In both asymmetric and symmetric orbifold models, with and without supersymmetry, we have not found discrete anomalies. This perhaps might be surprising, given that asymmetric orbifolds are inherently more "stringy" than symmetric orbifolds.

We have also seen that the *untwisted* sector often has non-universal anomalies. But in string theory the states in the *twisted* sector are also charged under these symmetries. This is not surprising from string theory, but it is from effective field theory. The discrete symmetries we are studying are, after all, discrete isometries of the internal manifold (in this case, an orbifold). It is only after adding the contributions from the untwisted and twisted sectors are the anomalies universal and may be canceled by a shift of the universal axion.

In asymmetric orbifolds a similar pattern was found, but there is an additional subtlety. Here the charge assignments in the twisted sector involved a delicate correlation between the gauge quantum numbers of the left-movers and the bosonic twist operators of the right-movers. Only after properly including this correlation are universal anomalies obtained.

We also looked for discrete anomalies in $N = 1$ type-IIB orientifold models. Here non-universal anomalies were found, but since the world-sheet parity projection breaks the quantum symmetry of the orbifold, states in the twisted sector can now couple to $F\tilde{F}$. As a result, a non-universal discrete anomaly may be canceled by assigning a shift to a twisted scalar. In fact, in all of the examples we studied, we found that by including in the discrete symmetry a discrete gauge transformation, unbroken, non-anomalous discrete symmetries could be identified.

We also examined several matrix model descriptions of M-theory compactifications and found evidence for CPT conservation. This would be non-perturbative evidence that the CPT symmetry of the classical theory is preserved by quantum gravity. Our arguments are not complete, however, since our backgrounds did not include all types of five-branes. We did not consider the AdS-CFT correspondence. Clearly in this case, the CPT of the boundary field theory establishes a corresponding symmetry of boundary correlators. Again, this is suggestive that CPT is always a non-perturbative symmetry of string/M theory.

In undertaking this work, we were hoping that one might find anomalies in the case of non-supersymmetric strings, but no anomalies for supersymmetric ones. This might have provided evidence that string theory "prefers" supersymmetry. But from this perspective, there seems to be no difference between supersymmetry and its absence.

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A. Review of symmetric orbifolds with Wilson lines

A.1 Construction of models

We begin with a brief discussion of the construction of symmetric orbifolds, with or without Wilson lines [39]. See for instance, [16] for a more detailed discussion.

We consider compactification of heterotic strings on a six-dimensional lattice with periodic boundary conditions. At special points in the moduli space the lattice may have a point group of symmetries. In addition to the point group the torus has a larger space group, which includes invariance under translations by lattice vectors e^i_α:

$$X^i \to X^i + m^\alpha e^i_\alpha \,. \tag{A.1}$$

The construction of the orbifold proceeds by first selecting an element of the point group

$$\theta = e^{2\pi i \vec{\phi}\cdot\vec{R}} \tag{A.2}$$

which acts on the spacetime variables X^i. This is identified with an element

$$e^{2\pi i \vec{\beta}\cdot\vec{T}} \tag{A.3}$$

of the SO(32) or $E_8 \times E_8$ gauge group.

To add Wilson lines one embeds the space group into the gauge group. This is done by associating a translation element e_α with an action by the gauge group:

$$\lambda \to \gamma(e_\alpha)\lambda\,. \tag{A.4}$$

We denote

$$\gamma(e_\alpha) = e^{2\pi i \vec{a}_\alpha \cdot \vec{T}} \tag{A.5}$$

and the $\vec{a}_\alpha$ are referred to as Wilson lines. The orbifold group g then consists of the elements $(\theta, 0; \beta(\theta))$ and $(1, e_\alpha, \gamma(e_\alpha))$. Loosely speaking, the states of the theory are those that are invariant under this group.

There are constraints on the allowed values of the Wilson lines. In addition to the requirement that $N\vec{\phi} = N\vec{\beta} = N\vec{a}_\alpha = 0 \bmod 1$ which follows from the requirement that $g^N = 1$ (N is the order of the group), Wilson lines also satisfy [16]

$$\gamma(\theta e_\alpha) = \beta^{-1}(\theta)\gamma(e_\alpha)\beta(\theta)\,. \tag{A.6}$$

This follows from requiring that the multiplication law for the gauge group elements is homomorphic to the multiplication law for the space group. For abelian embeddings it reduces to $\vec{a}_{\theta e_\alpha} = \vec{a}_\alpha + \vec{n}$. For the Z_6 twists that we study this constraint is enough to forbid any Wilson lines on the Z_6 twisted planes. A Wilson line in a Z_3 twisted plane is allowed provided that $3a_i = 0 \bmod 1$.

The level matching conditions, necessary for modular invariance, with a Wilson line are

$$(\beta + m^\alpha a_\alpha)^2 - \phi^2 = 0 \bmod 2N\,; \qquad N\mathrm{tr}a_\alpha = 0 \bmod 2\,, \qquad N\mathrm{tr}\phi = 0 \bmod 2\,. \tag{A.7}$$

where N is even.

States in the untwisted sector are obtained by projecting onto states that are invariant under $(\theta, 0; \beta(\theta))$. If in addition Wilson lines are present, untwisted states must also be invariant under $(1, e_\alpha, \gamma(e_\alpha))$.

States in the twisted sectors are obtained by projecting onto states invariant under the point group, consistent with the GSO projection. When Wilson lines are present there is one additional rule - one only keeps states invariant under those space group elements that commute with the twist for that sector.

Here though there a few subtleties in obtaining the orbifold group transformation property of the twisted (worldsheet) bosonic and fermionic ground states. In particular, the fixed points will in general not be invariant, but will transform into one another. Fixed points with well-defined charge are obtained simply by diagonalization.

The other subtlety is to correctly obtain the orbifold transformation of the world-sheet fermion ground states. This is straightforward in the bosonic formulation since, for instance, the Ramond right-mover (RM) ground state corresponds to the twist operator

$$\tau_R = e^{-i([n\phi_R]-1/2)\cdot H_R}\,, \tag{A.8}$$

where $0 \leq [\xi] < 1$. Under g

$$H_R \to H_R + 2\pi\phi \tag{A.9}$$

from which the transformation of the twist operator under g is found. The charge of the left-mover (LM) fermions is obtained in a similar way.

A.2 Discrete charges

In general the charges of the worldsheet fermions are more easily obtained by bosonization. In the untwisted sector the RM Ramond ground state is described by the vertex operator

$$e^{-ir\cdot H_R} \tag{A.10}$$

where r is an SO(8) spinor $(\pm,\pm,\pm,\pm)/2$ and the GSO projection requires an even number of $+$'s. The charge of this state under a Z_k rotation of the third plane $H_R^3 \to H_R^3 + 2\pi q/k$, for instance, is then obtained directly from (A.10). The contribution from any oscillator are obtained rather trivially. The total charge of a physical state built from oscillators and the R ground state is then easily obtained.

In the twisted sectors both the fermionic ground state and the bosonic fixed points contribute a charge. The former is obtained from the bosonized Ramond sector twist operator (A.8). Under a Z_k rotation where the third torus transforms as above the discrete charge of this operator is obtained rather straightforwardly.

The other non-trivial contribution involves the bosonic ground states, which are also charged under these discrete symmetries. This is understandable, since the fixed points themselves transform. For example, consider the Z_6 orbifold with twist $(1,1,-2)/6$. In the $n=1$ sector there are 3 fixed points coming from the third torus. They are invariant under simultaneous Z_6 rotations η of the first two tori. But under a Z_6 rotation γ of the third torus they transform as $3 = 2+\gamma^3$. In the doubly twisted sector the 27-fold degeneracy transforms as

$$27 \to 10(1;1) + 5(1;\gamma^3) + 8(\alpha^3;\eta^3) + 4(\alpha^3;\gamma^3\eta^3) \tag{A.11}$$

where the phase α ($\alpha^6 = \eta^6 = 1$) under the orbifold transformation (g) has also been included. In the triply twisted sector the 16-fold degeneracy transform as

$$16 \to 6(1;1) + 5(\alpha^2;\eta^2) + 5(\alpha^4;\eta^4)\,. \tag{A.12}$$

Putting these elements together gives the total discrete charge of a state.

B. Review of asymmetric orbifolds

Asymmetric orbifolds are described by a set of fields valued in an internal lattice (described below) together with a specification of the orbifold group action, consistent with modular invariance. The fields in the theory are the 16 freely interacting left-moving (LM) real scalars H_L^a, three LM complex scalars X_L^i, three right-moving (RM) complex scalars X_R^i and their fermionic partners $\tilde{\psi}_i$. The scalars X cannot be interpreted as describing the coordinates of an internal manifold as the left and right movers are treated differently.

To construct an asymmetric orbifold one first begins with a lorentzian, even, self-dual lattice $\Lambda_L \equiv \Gamma_{(22,6)}$. These conditions are required by modular invariance and consistency of operator products. The lattices we consider are of the form $\Gamma_{(22,6)} = \Gamma_{(16)} \times \Gamma_{(6,6)}$. Both the SO(32) and $E_8 \times E_8$ lattices are self-dual and even. For the non-supersymmetric models we consider the SO(32) lattice and for the supersymmetric models we consider both.

A simple construction [15] of a lattice $\Gamma_{(6,6)}$ that satisfies these properties is to begin with a Lie algebra $\mathcal{G}$ and to consider momenta in the weight lattice of $\mathcal{G}$,

$$p_L, p_R \in \Lambda_W \,. \tag{B.1}$$

The weight lattice is the integer sum of all weights of $\mathcal{G}$. The weight lattice can be decomposed into cosets

$$\Lambda_W = \uplus_a (\Lambda_R + w_a) \,, \tag{B.2}$$

with one w_a from each conjugacy class. A lorentzian lattice $\Lambda_L(\mathcal{G})$ is formed by the elements (p_L, p_R), where the inner product is chosen with the appropriate signature. With the difference of the left and right momenta restricted to lie in the root lattice,

$$p_L - p_R \in \Lambda_R \,, \tag{B.3}$$

it follows that the lorentzian lattice is both integral, $l \circ l' = p_L \cdot p'_L - p_R \cdot p'_R = 0 \bmod 1$, and even, $l \circ l = 0 \bmod 2$. If we further choose both p_L and p_R from the same conjugacy class, then the lattice is also self-dual [40].

These models still have 16 supercharges. To obtain nonsupersymmetric and $N = 1$ supersymmetric models we need to orbifold. This is done, as before, by dividing out by an abelian point group symmetry of the lattice. One set of symmetries of the lattice consists of the Weyl groups g of $\mathcal{G}$. From the construction of the lattice we see that there is an independent Weyl symmetry for both the left and right movers. In an asymmetric orbifold one may choose to twist only the left or the right by these Weyl symmetries.

The full lattice may have additional discrete symmetries that are not Weyl symmetries. There are, for instance, symmetries which interchange the conjugacy classes.

In the remainder of this appendix we provide the orbifold group action, the mass formulae of states, and the constraints provided by level matching.

The scalars H^a_L have momenta on the SO(32) lattice $\Gamma_{(16)}$. The orbifold action on these fields may include a shift

$$H^a_L \sim H^a_L + 2\pi\beta^a_L \,. \tag{B.4}$$

The scalars (X^i_L, X^i_R) are valued in the torus $R^{6,6}/\Gamma_{(6,6)}$. A twist or shift is given by

$$X^i_{L(R)} \to e^{2\pi i \phi^i_{L(R)}} X^i_{L(R)} + \beta^i_{L(R)} \,. \tag{B.5}$$

By worldsheet supersymmetry there is an action on the right-moving fermions:

$$\tilde{\psi}_i \to e^{-2\pi i \phi^i_R} \tilde{\psi}_i \,. \tag{B.6}$$

To describe the right-moving fermions $\tilde{\psi}^a$, $a = 1, \ldots, 8$ and obtain the projectors it is more convenient to bosonize the right-moving fermions to four real scalars H^a_R, $a = 1, \ldots, 4$. The relation between the fermions and bosons is given by $\tilde{\psi}^a = e^{-iH^a_R}$ for the Neveu-Schwarz sector, and $\tilde{\psi}^a = e^{\pm i \frac{1}{2} H^a_R}$ for the Ramond sector. The GSO projection requires the momentum r^a of H_R^a to be from either the vector or one of the spinorial weight lattices of SO(8), for the Neveu-Schwarz or Ramond sectors respectively. The group action of the orbifold is given by

$$H_R \to H_R + 2\pi\phi_R\,. \tag{B.7}$$

The momentum in the twisted sectors is then given by $r + n\phi_R$. Twisted states without oscillators correspond to the fermion twist operator

$$\tau_R \sim e^{-i(r+n\phi_R)\cdot H_R}\,. \tag{B.8}$$

States with oscillators correspond to multiplying this operator by the appropriate factors of $\overline{\partial}^n H_R$.

Finally, the mass formulae for the right and left movers in the n-th twisted sector are respectively

$$E_R = \overline{N}_{\rm osc} + \frac{1}{2}(r+n\phi_R)^2 + \frac{1}{2}(p_R+n\beta_R)^2 + h_R - \frac{1}{2}\,, \tag{B.9}$$

$$E_L = N_{\rm osc} + \frac{1}{2}(p_L+n\beta_L)^2 + h_L - 1\,. \tag{B.10}$$

Here $N_{\rm osc}$ and $\overline{N}_{\rm osc}$ are the number operators for the left and right moving oscillators. Also,

$$h_{L(R)} = \frac{1}{2}\frac{k_{L(R)}}{N}\left(1 - \frac{k_{L(R)}}{N}\right) \tag{B.11}$$

where $0 \le k/N < 1$, are the shifts in the zero point energies due to the twisted oscillators and contains an implicit sum over all the twisted bosons.

Level matching provides a constraint on the allowable left-mover shift β_L and twists (ϕ_L, ϕ_R), and is necessary for maintaining one-loop modular invariance. For N even the constraints are [15]

$$N(\beta_L\cdot\beta_L - \phi_L\cdot\phi_L - \beta_R\cdot\beta_R) = 0 \bmod 2 \tag{B.12}$$

$$N\sum_i \phi_R^i = 0 \bmod 2\,, \qquad N\beta_L^a \in \Lambda_R\,, \qquad N\beta_L \in \Lambda_R\,, \tag{B.13}$$

$$p_L\cdot\theta_L^{N/2}p_L - p_R\cdot\theta_R^{N/2}p_R = 0 \bmod 2\,. \tag{B.14}$$

For N odd (B.13) is still required but (B.14) is not, and (B.12) must be satisfied mod 1. In [41] Freed and Vafa prove that for abelian symmetric orbifolds, these level matching conditions are necessary and sufficient to guarantee higher loop modular invariance. It is implied that a similar statement is also true for abelian asymmetric orbifolds (see footnote 2 of [41].)

Modular invariance requires the states to satisfy certain projection rules. These may be obtained from the twisted partition functions, which are found by applying a sequence of modular transformations to the untwisted partition function. We checked that the partition functions obtained this way satisfied the following important self-consistency condition. Starting from a partition function $Z_{(m,n)}$ twisted some number of times in the two directions of the torus, perform a sequence of modular transformations that close to the identity. Then one should find that $Z_{(m,n)}$ is invariant under this action, which is indeed confirmed by explicit computations.

B.1 Level matching condition in asymmetric Z_6 models

For the non-supersymmetric models we considered the twist

$$\phi_R = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right). \tag{B.15}$$

Here we show that this twist satisfies the last level matching condition (B.14). To see this, write $p = p_{(2)} + p_{(4)}$ with $p_{(2)} \in \Lambda_W(\mathrm{SU}(3))$ and $p_{(4)} \in \Lambda_W(\mathrm{SO}(8))$. For then

$$\begin{aligned} p_L\theta^3 p_L - p_R\theta^3 p_R &= p_{(2),L}^2 - p_{(2),R}^2 + p_{(4),L}^2 + p_{(4),R}^2 \\ &= p_L^2 - p_R^2 + 2p_{(4),R}^2\,. \end{aligned} \tag{B.16}$$

Now the first two terms are just $l \circ l$ which is even (since the lattice is even). The last term is also even, since $p_{(4),R}^2 = 0 \bmod 1$ for any weight of the SO(8) lattice. It follows that the right side of the above equation is even as required.

For the supersymmetric models we considered

$$\phi_R = \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right); \qquad \phi_L = \left(0, \frac{1}{6}, \frac{1}{6}\right). \tag{B.17}$$

We now see that the last level matching condition is satisfied. With $(p_{(i),L}, p_{(i),R}) \in \Gamma^i_{(2,2)}(A_2)$,

$$\begin{aligned} p_L\theta^3 p_L - p_R\theta^3 p_R &= p_{(1),L}^2 - p_{(1),R}^2 - \left(p_{(2),L}^2 - p_{(2),R}^2 + p_{(3),L}^2 - p_{(3),R}^2\right) \\ &= -p \circ p + 2(p_{(1),L}^2 - p_{(1),R}^2) \\ &= -p \circ p + 2r \cdot (p_{(1),L} + p_{(1),R}) \end{aligned} \tag{B.18}$$

where r is the root vector $p_{(1),L} - p_{(1),R}$. Both terms in the last line above are manifestly even.

C. Discrete charges of bosonic and fermionic twist operators in asymmetric orbifolds

C.1 Discrete charge of world-sheet fermions

It is easiest to begin with the charges of the worldsheet fermions. We focus on a Z_M symmetry that acts on the right-moving bosonized NS fermions as

$$H_R^a \to H_R^a + \frac{k_a}{M}\,. \tag{C.1}$$

In the Ramond sector the fermions are half-integer moded, in which case the Z_M symmetry is realized as a Z_{2M} symmetry. This is obvious from the bosonized expression for these states,

$$\tilde{\psi}_R \sim e^{-ir\cdot H_R}\,, \tag{C.2}$$

since r is half-integer valued.

A generic single particle state from a twisted sector involves oscillators and/or lattice momenta acting on the ground state of that sector. The charge of the twisted ground state is obtained by inspecting the fermion twist operator of the n-th twisted sector, which is

$$\tau_R \sim e^{-i(r+n\phi_R)\cdot H_R}\,. \tag{C.3}$$

Here r is the momentum of the R or NS ground state in the n-th twisted sector. Given this explicit construction it is straightforward to determine the Z_M charge of the worldsheet fermions in the twisted sectors using the transformation (C.1). In particular, the charge is

$$e^{-2\pi i(r+n\phi_R)\cdot\frac{k}{M}}\,, \tag{C.4}$$

which is not a multiple of $1/M$. Actually, in the twisted sectors the Z_M symmetry is instead Z_{2NM}!

It is not unreasonable that the worldsheet fermions have charges that are Z_{2NM}: this is due to the quantum Z_N symmetry. For example, the operator product of N singly twisted fermion operators does not contain the identity element, but rather contains an untwisted fermion. This is because the product conserves the quantum Z_N charge, and so describes in general an excited state r' of the untwisted Hilbert space. Schematically,

$$(\tau_F)^N \sim e^{-ir'\cdot H_R}\,. \tag{C.5}$$

Since an untwisted worldsheet fermion has a charge q/M, the fermion twist operator must have a charge that is a multiple of $1/(2NM)$. By computing a sufficient number of correlation functions the Z_M charge is uniquely determined up to an additive shift proportional to the quantum Z_N charge.

Given these results, one then also expects the bosonic twist operators to be charged.

C.2 Bosonic twist operator: discrete charges from branch cuts

In order to compute the anomalies it is necessary to compute the transformation properties of the different states. In twisted sectors, one must be a bit careful, since the ground states transform. For the fermionic part of the state, it is useful to bosonize the right moving fermions, for then the twist acts as a shift and it is easy to read off (as in the symmetric orbifold case) the charge of the twist operator. For the bosonic coordinates, we will resort to a somewhat more indirect argument. The basic idea is simply to note that the bosonic twist and antitwist operators contain, in their operator product expansion, the unit operator. The coefficient function is determined in terms of the dimension of these operators and is in general not single-valued. In correlation functions, the effect of the twist operator is to introduce a branch cut, corresponding to the Z_3 charge of the operator. From the OPE, then, one can read off the transformation of τ_R from the value of the branch cut.

Thus if the orbifold acts as $X^i_R \to e^{2\pi i/N} X^i_R$, a bosonic twist operator from the singly twisted sector, $\tau_B(0)$, located at the origin introduces a branch cut:

$$\overline{\partial} X^i_R(\bar z)\tau_B(0) \sim \bar z^{-1/N}\tau'_B(0)\,. \tag{C.6}$$

But note that the branch cut is the same as the Z_M (here, Z_3) charge of X^i_R. This is because in this example, the orbifold group contains in its product the discrete symmetry of this torus.

Before considering the twisted bosons, we note that the RM Z_3 charge of a world-sheet fermion can be inferred from the branch cut in this way. For illustrative purposes we consider an untwisted NS fermion and a twisted R fermion, although the charge for untwisted R fermions or twisted NS fermions can also be obtained this way.

- Untwisted NS fermion. The leading term in the OPE for the untwisted NS fermion and the NS fermion twist operator (for which $r_{NS} = (1, 0, 0, 0)$ and $r_{NS} \cdot \phi = 0$) is:

  $$: e^{\pm i H_I(z)} :: e^{i(r_{NS}+\phi)\cdot H(0)} :\sim z^{\pm\phi_I} : e^{i(r^I_{NS}+\phi^I\pm 1)H_I(0)} e^{i(r_{NS}+\phi)\hat{\cdot} H(0)} :, \tag{C.7}$$

  with $\hat{\cdot}$ denoting the inner product omitting H^I. Note that the power of z follows from dimensional analysis. In general it is the difference in the dimensions of the operators appearing on the right side with those on the left side. The branch cut is

  $$e^{\pm 2\pi i \phi_I} \tag{C.8}$$

  which is the same as the Z_M charge of the untwisted NS fermion.

- Twisted Ramond fermions. The leading term in the OPE between the Ramond and NS twisted fermionic operators is

  $$: e^{i(-\frac{1}{2}+\phi)\cdot H(z)} e^{i(r_{NS}+\phi)\cdot H(0)} : \ \sim z^{(\phi-1/2)\cdot\phi-1/2} : e^{i(r_{NS}+\phi+\phi-\frac{1}{2})\cdot H(0)} : . \tag{C.9}$$

  (The last factor of $-1/2$ in the power of z is from the untwisted field H^0). Using factorization the branch cut gives the correct charge,

  $$e^{2\pi i \phi_I(\phi_I - 1/2)} . \tag{C.10}$$

Now consider the twisted bosons and focus on the Z_3 symmetry of the lattice $\Gamma^{(2)}(A_2)$ of the third torus. It is readily confirmed that for untwisted bosons this procedure gives the correct answer. As stated before, this is because of factorization and because the orbifold group contains in its product the discrete symmetry we are studying.

To obtain the charge of the twist operator itself, we note that the OPE of this operator with its inverse contains the identity:

$$V_B(z,\bar{z}) V_B^{-1}(0) \sim z^{2\Delta_L} \bar{z}^{2\Delta_R} \times \mathbf{1} . \tag{C.11}$$

The scaling on the right-side is determined by conformal invariance to be the difference of the conformal weights of the operators appearing on both sides of the equation. The factor of 2 appears, since the twist operator appears twice on the left side. Here we note that the twist operator is *not* the full vertex operator, but only that part of the twisted vertex operator constructed from the fields valued in the sublattice $\Gamma_{(2,2)}(A_2)$. In particular, this includes a contribution from the LM and RM momenta of the ground state, so that

$$V_B \sim e^{i p_L \cdot X_L(z)} e^{-i p_R \cdot X_R(\bar{z})} \tau_B(z,\bar{z}) \tag{C.12}$$

with $(p_L, p_R) \in \Gamma_{(2,2)}(A_2)$. (In our models $\beta_L = \beta_R = 0$ when restricted to $\Gamma_{(2,2)}(A_2)$.) Also the energy shift from fields valued in this sublattice is

$$\Delta_L = \frac{1}{2} p_L^2 + h_L \tag{C.13}$$

and with a similar expression for Δ_R. Again, h_L is not the full shift in energy due to all the twisted bosons in $\Gamma_{(6,6)}$, but only that part from the twisted bosons in $\Gamma_{(2,2)}(A_2)$. The net branch cut is then:

$$e^{2\pi i(2h_L - 2h_R + p_L^2 - p_R^2)} . \tag{C.14}$$

Note that this branch cut vanishes for a symmetric twist: the contribution from the zero point energy cancels between left and right, and $p_L = p_R = 0$ since both X_L and X_R are twisted. As advertised earlier, in a symmetric orbifold the bosonic twist operator is neutral under the discrete symmetry.

Now focus on our example: the RM Z_3 twist in the $\Gamma_{(2,2)}(A_2)$ lattice. Then $p_R = 0$ for the massless states, $h_L = 0$ and $h_R = 1/9$. Then since $p_L = 0$ or $p_L \in Y_{1,2}$ (for which $p_L^2 = 2/3$), the branch cuts are

$$\begin{aligned} &e^{2\pi i(-2/9)} && \text{if } p_L = 0 \\ &e^{2\pi i(4/3-2/9)} = e^{2\pi i(1/9)} && \text{if } p_L^2 = 2/3 . \end{aligned} \tag{C.15}$$

From this we infer that the Z_3 charges of the bosonic ground states are:

$$\begin{aligned} &\alpha^{-2/3} , && p_L \in Y_0 \\ &\alpha^{1/3} , && p_L \in Y_{1,2} . \end{aligned} \tag{C.16}$$

Notice that the charge depends on the choice of LM momentum p_L. This correlation is rather surprising, since the LM are not charged under the discrete Z_3! The naive expectation that all the right-moving ground state have the same charge is not correct. We note that these charge assignments (C.16) are crucial to obtain universal anomalies.

Independent confirmation of these charges is provided in the next subsection, where they are obtained from a completely different method. Fortunately, the two methods agree.

C.3 Bosonic twist operator: discrete charges from an explicit construction

Recall that in the case of the twisted fermions the charge assignments were easily found by bosonizing the fermions. To confirm the charges in (C.16) it would be nice to do something similar and obtain an explicit form for the bosonic twist operator. That is, to express the right-moving scalars as the exponential of another set of scalars. Then the linear action of the twist on the original variables would act as a non-linear shift on the new set of scalars. For the $\Gamma_{(2,2)}(A_2)$ lattice with the Z_3 twist, it turns out that an explicit rebosonization can be found and is given by [42]

$$\bar{\partial} X_R^{(3)} = \frac{1}{\sqrt{3}} (e^{ie_1 \cdot B} + e^{ie_2 \cdot B} + e^{ie_3 \cdot B}) . \tag{C.17}$$

The expression for $\overline{\partial}\,\overline{X}_R^{(3)}$ is given by the hermitian conjugate. Here $e_{i=1,2,3}$ are three SU(3) roots such that $e_i \cdot e_j = -1$ for $i \neq j$. B_I is a two-component RM freely interacting real scalar.

One may confirm that this rebosonization is consistent with the conformal weights, OPE's, and world-sheet statistics expected for either of these bosons [42].

Under the orbifold action the complex scalar ϕ transforms as

$$B \to B + 2\pi \frac{e_4}{3} \tag{C.18}$$

where e_4 is the SU(3) root for which $e_4 \cdot e_1 = e_4 \cdot e_2 = 1$ and $e_4 \cdot e_4 = 2$. With these definitions and transformation law for ϕ one indeed finds that $\overline{\partial} X_R^{(3)} \to \gamma \overline{\partial} X_R^{(3)}$ and $\overline{\partial X}_R^{(3)} \to \gamma^2 \overline{\partial X}_R^{(3)}$.

Using this equivalence it is now straightforward to explicitly determine the bosonic twist operators and their Z_3 charges. They are

$$\tau_B \sim e^{-i(p_\phi + n\frac{e_4}{3})\cdot B}\,, \tag{C.19}$$

with $p_\phi \equiv p_{(2),R}$.

To find the Z_3 charge of the bosonic ground state we need to determine the momentum of the ground state appearing in (C.19). In the untwisted sector the B momentum p_B is in the SU(3) weight lattice and for the twisted sectors in the shifted weight lattice. Since this scalar is not twisted it does not modify the zero point energy in the twisted sectors. The scalars in the other sublattice $\Gamma_{(2,2)}(D_4)$ are still twisted though. The expression (B.9) for the RM Ramond energy in the singly twisted sector is then

$$E_R = \overline{N}_{\rm osc} + \frac{1}{2}(r + \phi_R)^2 + \frac{1}{2}\left(p_B + \frac{e_4}{3}\right)^2 - \frac{1}{4}\,. \tag{C.20}$$

This has three ground state solutions, one from each conjugacy class: $p_B = 0$, $p_B = w_1$ and $p_B = w_2$, where w_1 and w_2 are the two weights from the fundamental and anti-fundamental conjugacy classes such that $w_1 + w_2 = -e_4$. The three twist operators corresponding to these three solutions are

$$e^{-ie_4\cdot B/3}\,,\ p_B = 0\,; \qquad e^{ie_1\cdot B/3}\,,\ p_B = w_1\,; \qquad e^{ie_2\cdot B/3}\,,\ p_B = w_2\,. \tag{C.21}$$

The conformal weight of these operators is 1/9 which is correct, since in the other twisted formulation the weight is equal to the shift in the zero point energy which in that case is also 1/9.

The Z_3 charges of the twist operators are easily obtained using the explicit transformation (C.18). They are: $\alpha^{-2/3}$, $\alpha^{1/3}$, $\alpha^{1/3}$, respectively. Note that the charge assignments are not identical but depend on the choice of RM bosonic ground state. This surprising result was also found in the method of the previous subsection. More importantly, the charges inferred from either method agree.

Since there are three RM ground states it appears that the bosonic degeneracy in the single twist sector is 3. Naively this disagrees with the construction of this model in the previous (twisted) formulation, since there the degeneracy was one. There is no contradiction though, since the three RM ground states are not all paired with each LM

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sector	Ramond state and Z_3 charge
untwisted	$(r_1 : \alpha^{-\frac{1}{2}}) \oplus (r_2 : \alpha^{\frac{1}{2}})$
$n=1$	no states
$n=2$	$(p_{(2),L} \in Y_0 : \alpha^{\frac{1}{2}}) \oplus (p_{(2),L} \in Y_{1,2} : \alpha^{-\frac{1}{2}})$
$n=3$	$\alpha^{\frac{1}{2}}$
$n=4$	$(p_{(2),L} \in Y_0 : \alpha^{-\frac{1}{2}}) \oplus (p_{(2),L} \in Y_{1,2} : \alpha^{\frac{1}{2}})$
$n=5$	$(p_{(2),L} \in Y_0 : \alpha^{\frac{1}{2}}) \oplus (p_{(2),L} \in Y_{1,2} : \alpha^{-\frac{1}{2}})$

Table 11: Z_3 charges for positive helicity massless fermions.

ground state. In fact, only one RM state is selected according to the correlation $p_{(2),L} - p_B \in R$ and the choice of $p_{(2),L}$. That is, $p_{(2),L} \in Y_0 \leftrightarrow p_B = 0$ and $p_{(2),L} \in Y_{1,2} \leftrightarrow p_B = w_{1,2}$. Thus the bosonic degeneracy remains equal to one, consistent with the degeneracy found in the twisted formulation.

To verify that the states are the same in either formulation and that the degeneracies match, we need to look at the projectors. One finds that each LM state is always paired up with only one of the three RM bosonic ground states. And as they should be, one finds that the projectors are the same in either formulation.

Using the explicit expression for the twist operator, the total Z_3 charge, bosonic plus fermionic, in the singly-twisted sector is

$$\alpha^{-1/2} \ : \ p_L \in Y_0 \, ; \qquad \alpha^{1/2} \ : \ p_L \in Y_{1,2} \, . \tag{C.22}$$

Thus the discrete charge of the twisted state depends on the left-moving quantum numbers and their values agree with those inferred from the branch cut. By factorization and CPT the charges in the other twist sectors are uniquely determined. They may be found in table 11.

D. Open string states in type-IIB orientifolds

A general discussion for constructing the states in a $\mathcal{N}=1$, D=4 type-IIB orientifold can be found in [21, 22]. We briefly summarize their main results that are relevant here.

The orientifolds studied here are obtained by first compactifying type IIB on $T^2 \times T^2 \times T^2$ where each torus has a Z_6 isometry. The orientifold group is generated by the group elements $G_1 = g$ and $G_2 = g \times \Omega$ with $g^N = 1$. Here Ω is the world-sheet parity transformation, and g generates a discrete isometry of the internal manifold and also a discrete U(16) transformation of the open string endpoints. The spacetime embedding of g is chosen to be an element of SU(3) so that at low energies these models describe $\mathcal{N}=1$ supersymmetric theories in four dimensions.

Since $G_2 = G_1 \times \Omega$, states in the closed string sector are obtained by first projecting onto g-invariant states and then onto Ω-invariant states. In the massless untwisted bosonic sector the NS-NS antisymmetric two-form, R-R zero form and R-R four form states are discarded, but the dilaton, graviton and R-R two form states are kept. The twisted sector initially contains states obtained by the standard orbifold construction. But the world sheet parity transformation exchanges states in the n-th twisted sector with those in the $(N-n)$-th twisted sector. Only the symmetric combination survives the world-sheet parity projection. As a result these models do not have a quantum Z_N symmetry.

As is well-known, in these models the charged matter is from the open string sector which are all untwisted. All the models here have D9 branes. They will also have D5 branes when the orbifold group contains an element of order 2. Open strings consist of sectors that are distinguished by whether their endpoints have Dirichlet or Neumann boundary conditions. So there will in general be 99,55 and 95 open strings.

- Massless fermions in the 99 sector are of the form

$$\lambda^i_{ab}|ab\rangle \times |r^i\rangle \,. \tag{D.1}$$

  Here λ is a Chan-Paton matrix for U(16) and $r = (\pm\pm\pm\pm)$ is an SO(8) spinor. The GSO projection requires an even number of $-$ signs. The orbifold action on the spinor r is given by an element of the SU(4) subgroup. The embedding of the spacetime twist into the gauge group is, for abelian orbifolds, described by a shift vector v which is a Cartan element of U(16). Conditions on the Chan-Paton matrices and the shift vectors to ensure tadpole cancellation are derived in [21]. The orientifold projection keeps all states that are separately invariant under the orbifold action g and the world-sheet parity projection.

- D5 branes are present in the Z_6 model discussed in the main text, but not in the Z_3 models. The 5 branes are assumed to be wrapped around the third torus, and are referred to as 5_3 branes. The shift vector for the D5 branes could in principle be different from the D9 branes, although here they are chosen to be identical. The matter content of the 55 strings are of the form (D.1).

- The new ingredient for the 59 strings is that the bosonic modes perpendicular to the 5-brane direction have half-integer moding. This is due to the mixed Dirichlet and Neumann boundary conditions. In the NS sector supersymmetry implies that the superconformal partners ψ^i, with i not one of the 5_3 brane directions, have integer moding. These worldsheet fermions have zero modes. Consequently, spacetime scalars are of the form

$$\lambda_{ab}|ab\rangle|s_1 s_2\rangle \,. \tag{D.2}$$

  The GSO projection is $s_1 = s_2$, i.e. an even number of $-$ signs. In the R sector the ψ^i's, with i orthogonal to the 5_3 brane, have half-integer moding and do not contribute any zero modes. In this sector the only zero modes are from worldsheet fermions with indices parallel to the 5_3 brane. This leads to space-time fermions of the form

$$\lambda_{ab}|ab\rangle|s_0 s_3\rangle \,. \tag{D.3}$$

  The GSO projection requires an odd number of $-$ signs. The orientifold projection keeps the symmetric combination of 95 and 59 strings.

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