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On the Superconformal Flatness of AdS Superspaces

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On the superconformal flatness of AdS superspaces

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Abstract: The superconformal structure of coset superspaces with $AdS_m \times S^n$ geometry of bosonic subspaces is studied. It is shown, in particular, that the conventional superspace extensions of the coset manifolds $AdS_2 \times S^2$, $AdS_3 \times S^3$ and $AdS_5 \times S^5$, which arise as solutions of corresponding $D = 4, 6, 10$ supergravities and have been extensively studied in connection with AdS/CFT correspondence, are not superconformally flat, though their bosonic submanifolds are conformally flat. We give a group-theoretical reasoning for this fact. We find that in the $AdS_2 \times S^2$ and $AdS_3 \times S^3$ cases there exist different supercosets based on the supergroup OSp$(4^*|2)$ which are superconformally flat. We also argue that in $D = 2, 3, 4$ and 5 there exist superconformally flat "pure" AdS_D supercosets. Two methods of checking the superconformal flatness are proposed. One of them consists in solving the Maurer-Cartan structure equations and the other is based on embedding the isometry supergroup of the $AdS_m \times S^n$ superspace into a superconformal group in $(m+n)$-dimensional Minkowski space. Finally, we discuss some applications of the above results to the description of supersymmetric dynamical systems.

Keywords: M-Theory, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence, Supergravity Models.

Contents

1. Introduction

Space-times of anti-de-Sitter geometry have attracted great deal of attention because they appear in various physical problems, e.g. cosmology, black holes, supergravity and compactification, AdS/CFT correspondence, the theory of higher spins, etc. The basic geometrical feature of the D-dimensional AdS spaces is that their isometry group $\mathrm{SO}(2, D-1)$ acts as a group of conformal transformations of the AdS boundary which may be identified with a $D-1$ dimensional Minkowski space. On the other hand, the AdS metric is invariant, up to a dilaton factor, under a higher group of conformal transformations, namely $\mathrm{SO}(2, D)$. A consequence of this fact is that (locally) there exists a set of AdS coordinates x^m in which the AdS metric is conformally flat[1]

$$ds^2 = e^{4\phi(x)} dx^m dx^n \eta_{mn}\,, \quad \eta_{mn} = \mathrm{diag}\,(+1, -1\ldots, -1)\,, \quad m, n = 0, 1, \ldots, D-1\,, \quad (1.1)$$

where $\phi(x)$ is a conformal factor.

[1] One can mention another peculiar property of the AdS metric to have (in a certain set of coordinates) a "Kahler-like" potential structure $g_{mn} = \partial_m \partial_n V(x)$ [1].

Among compactifications of $D = 10$ and $D = 11$ supergravities (for reviews see [2]) there appear vacuum configurations having the geometry of the direct product of an AdS space and a sphere, i.e. $AdS_p \times S^q$ $(p+q = D)$. These configurations provide a geometrical ground for the AdS/CFT correspondence in string theory and M-theory [3], and also are relevant (for $p = q = 2$) to the superconformal quantum mechanics [4] of particles in the background of Reissner-Nordström black holes [5]. The isometry group of such spaces is $\mathrm{SO}(2, p-1) \times \mathrm{SO}(q+1)$ which is a bosonic subgroup of the appropriate superconformal symmetry of the given configuration of string or M-theory on the boundary of AdS.

It is known that the $AdS_5 \times S^5$ space of the compactified type-IIB $D = 10$ supergravity (or string theory) and the $AdS_2 \times S^2$ space of Reissner-Nordström black hole (see e.g. [5, 6]) are conformally flat since the radii of the AdS spaces and spheres are adjusted to be equal. This is also the case for superstrings and superparticles in $AdS_2 \times S^2$ and $AdS_3 \times S^3$ [8, 9, 10], while the compactifications of $D = 11$ supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ are not conformally flat. Conformal flatness of classical solutions has been considered as a guarantee of their exactness, i.e. the absence of quantum corrections to these solutions due to higher derivative terms in the supergravity effective actions [11]. As has been discussed in [12], since the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ solutions of $D = 11$ supergravity are not conformally flat, the exactness argument for them should be based on another reasoning, such as unbroken supersymmetry. In this connection one can also ask whether the conformal flatness is compatible with the supersymmetry properties of the corresponding configurations.

So a natural question arises whether the *superspaces* with a conformally flat bosonic body $AdS_p \times S^q$ are also *superconformally* flat. A naive expectation might be that the answer to this question is always positive. However, we shall see that, among physically interesting examples, indeed the AdS_4 coset superspace $\mathrm{OSp}(N|4; R)/(\mathrm{SO}(1,3) \times \mathrm{SO}(N))$ (for a generic N) is superconformally flat, while the $AdS_2 \times S^2$ coset superspace $\mathrm{SU}(1,1|2)/(\mathrm{SO}(1,1) \times \mathrm{SO}(2))$, the $AdS_3 \times S^3$ coset superspace $(\mathrm{SU}(1,1|2) \times \mathrm{SU}(1,1|2))/(\mathrm{SO}(1,2) \times \mathrm{SO}(3))$ and the $AdS_5 \times S^5$ coset superspace $\mathrm{SU}(2,2|4)/(\mathrm{SO}(1,4) \times \mathrm{SO}(5))$ are not, though their bosonic subspaces are conformally flat. The reason for this somewhat surprising conclusion is that the isometry supergroups $\mathrm{OSp}(N|4; R)$ of the corresponding four-dimensional AdS superspaces are *subgroups* of the N-extended $D = 4$ superconformal group $\mathrm{SU}(2,2|N)$,[2] like the super Poincaré group of flat N-extended $D = 4$ superspaces. On the contrary, the isometry supergroups $\mathrm{SU}(1,1|2)$, $\mathrm{SU}(1,1|2) \times \mathrm{SU}(1,1|2)$ and $\mathrm{SU}(2,2|4)$ of the $AdS_{D/2} \times S^{D/2}$ $(D = 4, 6, 10)$ superspaces are not *appropriate* subgroups of corresponding $N = 2$, $D = 4, 6, 10$ superconformal groups. By "appropriate" we mean that the bosonic subgroups of these AdS supergroups must be subgroups of the bosonic conformal subgroups of the corresponding superconformal groups. This is the group-theoretical argument why the above mentioned coset superspaces are not superconformal, though their bosonic $AdS_{D/2} \times S^{D/2}$ subspaces are conformal.[3]

[2]The Cartan forms of the N extended superconformal group $\mathrm{SU}(2,2|N)$ which are suitable for the construction of AdS_5 coset superspaces, were computed in [7]. We also note that the bosonic subgroup of $\mathrm{SU}(n,n|2n)$ $(n = 1,2)$ is $\mathrm{SU}(n,n) \times \mathrm{SU}(2n)$, while for a generic $\mathrm{SU}(n,n|N)$ it is $\mathrm{SU}(n,n) \times \mathrm{U}(N)$.

[3]Let us note that (super)conformally flat (super)spaces are (super)conformal in the sense that, with a

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For instance, the isometry supergroup SU$(2,2|4)$ of the $AdS_5\times S^5$ solution of type-IIB $D=10$ supergravity and the type-IIB $D=10$ super Poincaré group are *not subgroups* of a generalized extended ("$N=2$") superconformal group in ten dimensions (usually chosen to be either OSp$(2|32;R)$ or OSp$(1|64;R)$). At the same time, the bosonic $D=10$ Poincaré group and the isometry group SO$(2,4)\times$SO(6) of $AdS_5\times S^5$ are subgroups of the $D=10$ conformal group SO$(2,10)$.

As was already discussed in the literature [13, 15, 16], only "central" extensions of the type-IIB $D=10$ super Poincaré group by tensorial generators are subgroups of OSp$(2|32;R)$ and/or OSp$(1|64;R)$,[4] while SU$(2,2|4)$ is *not* a subgroup of these superconformal groups (we give a simple reasoning for this in subsection 2.4). As a result, the $D=10$ super Poincaré group and the SU$(2,2|4)$ supergroup and, respectively, the flat $D=10$ superspace and the $AdS_5\times S^5$ superspace, cannot be related to each other by a super Weyl transformation (in the sense explained in sections 3 and 4). Hence the $AdS_5\times S^5$ superspace is not superconformal.

In the cases of $AdS_2\times S^2$ and $AdS_3\times S^3$, however, there exist different coset superspaces with the same bosonic body which are superconformal. These are OSp$(4^*|2)/($SO$(1,1)\times$ SO$(2)\times SU(2))$ and OSp$(4^*|2)\times$ OSp$(4^*|2)/($SO$(1,2)\times$SO$(3)\times$SU$(2)\times$SU$(2))$. These superspaces contain $AdS_2\times S^2$ and $AdS_3\times S^3$ as bosonic subspaces and have the same number of fermionic coordinates as the supercosets SU$(1,1|2)/($SO$(1,1)\times$SO$(2))$ and $($SU$(1,1|2)\times$SU$(1,1|2))/($SO$(1,2)\times$SO$(3))$, respectively. Their crucial difference from the latter ones is that the supergroups OSp$(4^*|2)$ and OSp$(4^*|2)\times$ OSp$(4^*|2)$ are subgroups of the corresponding superconformal groups SU$(2,2|2)$ and OSp$(8^*|4)$. These coset superspaces are related to the "conventional" ones by an "α-deformation" of supercosets based on the exceptional supergroup $D(2,1;\alpha)$ (see sections 2 and 4).

A generic form of the supervielbeins of supermanifolds with superconformally flat geometry is [17]

$$\begin{aligned} E^a &= e^{2\Phi(x,\theta)}(dx^a - i\bar\theta\gamma^a d\theta) \equiv e^{2\Phi(x,\theta)}\Pi^a\,, \\ E^\alpha &= e^{\Phi(x,\theta)}\Lambda^\alpha_\beta\left(d\theta^\beta + iD_\gamma\Phi\gamma_a^{\gamma\beta}\Pi^a\right)\,, \end{aligned} \tag{1.2}$$

where x^m and θ^α are bosonic and fermionic coordinates of the supermanifold, $\Phi(x,\theta)$ is a dilaton factor, $\Lambda^\alpha{}_\beta(x,\theta)$ is a matrix (in some cases it can be equal to unity), D_α is a flat superspace covariant derivative, and Π^a and $d\theta^\alpha$ are covariant flat supervielbeins.

In this paper we shall derive the superconformally flat form of the supervielbeins and connections of the AdS_4 coset superspaces OSp$(N|4;R)/($SO$(1,3)\times$SO$(N))$ $(N=1,2)$ and of an $AdS_2\times S^2$ coset superspace OSp$(4^*|2)/($SO$(1,1)\times$SO$(2)\times$SU$(2))$. Actually, the first type of superspaces will be shown to be superconformally flat for generic N. In section 2 we show that the superconformally flat ansatz is compatible with the Maurer-Cartan structure equations of the corresponding coset superspaces (while it is not compatible with

proper choice of coordinates, the whole (super)conformal group is realized in these (super)spaces in the same way as in the corresponding flat (super)spaces. So below we shall often use "(super)conformal" as a synonym of "(super)conformally flat".

[4]These central extensions non-trivially transform under the superconformal transformations.

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the $AdS \times S$ solutions of $N = 2$, $D = 4, 6$ supergravities), and in sections 3 and 4, using a "bottom-up" approach, we construct explicitly the superconformal factors of the AdS supervielbeins. In subsection 2.4 we demonstrate that the supercoset extension of $AdS_5 \times S^5$ is not superconformally flat. In Conclusion we also discuss the issue of the supeconformal flatness of "pure" AdS_D supercosets, i.e. those without "S-factors". In particular, we argue that there exists only one superconformlly flat "pure" AdS_5 supercoset, with $\mathrm{SU}(2,2|1)$ as the superisometry group.

The results obtained can be useful for the description of supersymmetric black holes, superparticles and superstrings in AdS superbackgrounds, and for studying issues of their quantization and the influence of higher order quantum corrections. As an example, in section 5 we demonstrate that the classical dynamics of massless superparticles propagating on the AdS_4 coset superspace $\mathrm{OSp}(2|4;R)/(\mathrm{SO}(1,3) \times \mathrm{SO}(2))$, on the $AdS_2 \times S^2$ coset superspace $\mathrm{OSp}(4^*|2)/(\mathrm{SO}(1,1) \times \mathrm{SO}(2) \times \mathrm{SU}(2))$ and in flat $N = 2$, $D = 4$ superspace are equivalent, because these superspaces are superconformal, with the same superconformal group $\mathrm{SU}(2,2|2)$ acting in all three cases. But their quantized dynamics differ because of different geometrical and symmetry properties of the superbackgrounds which the quantization procedure should respect.

2. Solving the Maurer-Cartan equations of the AdS cosets

In this section we use two-component Weyl spinors θ^α, $\bar\theta^{\dot\alpha}$ to define Grassmann coordinates and the matrix representation $x^{\alpha\dot\alpha} = \sigma_a^{\alpha\dot\alpha} x^a$ for the vector coordinates of $D = 4$ superspaces. The metric of the flat space-time is chosen to be almost negative, $\eta_{ab} = \mathrm{diag}(+,-,-,-)$.

2.1 AdS_4 superspaces

The structure equations of the AdS_4 coset superspace $\mathrm{OSp}(1|4;R)/\,\mathrm{SO}(1,3)$ are (see e.g. [37, 38])

$$\mathcal{D}E^{\alpha\dot\alpha} = dE^{\alpha\dot\alpha} - E^{\beta\dot\alpha} \wedge w_\beta{}^\alpha - E^{\alpha\dot\beta} \wedge w_{\dot\beta}{}^{\dot\alpha} = -2iE^\alpha \wedge \bar E^{\dot\alpha}\,, \tag{2.1}$$

$$\mathcal{D}E^\alpha = dE^\alpha - E^\beta \wedge w_\beta{}^\alpha = -\frac{1}{R}\bar E_{\dot\beta} \wedge E^{\alpha\dot\beta}\,, \tag{2.2}$$

$$R^{\alpha\beta} = dw^{\alpha\beta} - w^{\alpha\gamma} \wedge w_\gamma{}^\beta = -\frac{2i}{R}E^\alpha \wedge E^\beta + \frac{1}{R^2}E^{\alpha\dot\gamma} \wedge E^\beta{}_{\dot\gamma}\,, \tag{2.3}$$

where $w^{\alpha\beta}$ and $R^{\alpha\beta}$ are the spin connection and the curvature form, and R is the AdS radius. The eqs. (2.1)–(2.3) should be supplemented with their complex conjugate.

Let us assume that the $N = 1$ AdS_4 superspace is superconformally flat. In this case eqs. (2.1)–(2.3) should have a non-trivial solution of the form

$$E^{\alpha\dot\alpha} = e^{2\Phi(x,\theta,\bar\theta)}\Pi^{\alpha\dot\alpha}\,, \tag{2.4}$$

where

$$\Pi^{\alpha\dot\alpha} = dx^{\alpha\dot\alpha} - id\theta^\alpha\bar\theta^{\dot\alpha} + i\theta^\alpha d\bar\theta^{\dot\alpha}\,. \tag{2.5}$$

Substituting (2.4) into the structure eq. (2.1), one finds that it fixes the form of the fermionic supervielbeins E^α, $\bar{E}^{\dot\alpha}$ up to a phase factor $e^{iW(x,\theta,\bar\theta)}$, where W is a real superfield:

$$E^\alpha = e^{\Phi(x,\theta,\bar\theta)+iW(x,\theta,\bar\theta)}\left(d\theta^\alpha + 2i\Pi^{\alpha\dot\alpha}\bar{D}_{\dot\alpha}\Phi\right), \tag{2.6}$$

$$E^{\dot\alpha} = e^{\Phi(x,\theta,\bar\theta)-iW(x,\theta,\bar\theta)}\left(d\bar\theta^{\dot\alpha} - 2i\Pi^{\alpha\dot\alpha}D_\alpha\Phi\right). \tag{2.7}$$

From eq. (2.1) one also finds the Grassmann components of the connection 1-form

$$w_\beta{}^\alpha = -2d\theta^\alpha D_\beta\Phi - 2d\theta_\beta D^\alpha\Phi + \Pi^{\gamma\dot\gamma}w^\alpha_{\gamma\dot\gamma\beta}\,, \quad w_{\dot\beta}{}^{\dot\alpha} = (w_\beta{}^\alpha)^*\,. \tag{2.8}$$

In eqs. (2.6)–(2.8)

$$D_\alpha = \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \qquad \bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} \tag{2.9}$$

are the flat superspace covariant spinor derivatives.

In order to obtain eqs. (2.6)–(2.8) one takes the covariant differential of the bosonic supervielbein form (2.4)

$$\mathcal{D}E^{\alpha\dot\alpha} = -2ie^{2\Phi}d\theta^\alpha\wedge d\bar\theta^{\dot\alpha} + 2e^{2\Phi}\Pi^{\alpha\dot\alpha}\wedge d\Phi - e^{2\Phi}\Pi^{\alpha\dot\beta}\wedge\bar\omega_{\dot\beta}{}^{\dot\alpha} - e^{2\Phi}\Pi^{\beta\dot\alpha}\wedge\omega_\beta{}^\alpha$$

and compares the result with the right hand side of (2.1).

Let us substitute into the right hand side of eq. (2.1) the most general expression for the fermionic supervielbeins in terms of the independent covariant 1-superforms $d\theta$, $d\bar\theta$ and Π

$$E^\alpha = d\theta^\beta\mathcal{E}_\beta{}^\alpha + d\bar\theta^{\dot\beta}\bar{\mathcal{E}}_{\dot\beta}{}^\alpha + \Pi^{\beta\dot\beta}\tilde\psi_{\beta\dot\beta}{}^\alpha\,,$$

$$\bar{E}^{\dot\alpha} = d\bar\theta^{\dot\beta}\bar{\mathcal{E}}_{\dot\beta}{}^{\dot\alpha} + d\theta^\beta\mathcal{E}_\beta{}^{\dot\alpha} + \Pi^{\beta\dot\beta}\bar{\tilde\psi}_{\beta\dot\beta}{}^{\dot\alpha}\,.$$

Then, equating the components of the terms containing the basic form $d\theta^\beta\wedge d\bar\theta^{\dot\beta}$ on the left and right hand sides of eq. (2.1), we have

$$\mathcal{E}_\beta{}^\alpha\bar{\mathcal{E}}_{\dot\beta}{}^{\dot\alpha} + \bar{\mathcal{E}}_{\dot\beta}{}^\alpha\mathcal{E}_\beta{}^{\dot\alpha} = \delta_\beta{}^\alpha\delta_{\dot\beta}{}^{\dot\alpha}e^{2\Phi}\,,$$

while the components of the basic form $d\theta^\beta\wedge d\theta^\gamma$ give rise to the relations

$$2\mathcal{E}_{(\beta}{}^\alpha\mathcal{E}_{\gamma)}{}^{\dot\alpha} \equiv \mathcal{E}_\beta{}^\alpha\mathcal{E}_\gamma{}^{\dot\alpha} + \mathcal{E}_\gamma{}^\alpha\mathcal{E}_\beta{}^{\dot\alpha} = 0\,.$$

Under the assumption that $\mathcal{E}_\beta{}^\alpha$ is invertible, the solution of the second equation is

$$\mathcal{E}_\beta{}^{\dot\alpha} = (\mathcal{E}_\beta{}^{\dot\alpha})^* = 0\,,$$

then, from the first equation one gets

$$\mathcal{E}_\beta{}^\alpha\bar{\mathcal{E}}_{\dot\beta}{}^{\dot\alpha} = \delta_\beta{}^\alpha\delta_{\dot\beta}{}^{\dot\alpha}e^{2\Phi} \Longrightarrow \mathcal{E}_\beta{}^\alpha = \delta_\beta{}^\alpha e^{\Phi+iW}\,, \quad \bar{\mathcal{E}}_{\dot\beta}{}^{\dot\alpha} = \delta_{\dot\beta}{}^{\dot\alpha}e^{\Phi-iW}\,.$$

Taking the above relations into account, one finds that the component of the basic form $\Pi^{\beta\dot\beta}\wedge d\bar\theta^{\dot\gamma}$ in eq. (2.1) can be written in the form

$$\epsilon_{\dot\alpha\dot\beta}\omega_{\dot\gamma\beta\alpha} + \epsilon_{\alpha\beta}\bar\omega_{\dot\gamma\dot\beta\dot\alpha} = -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar{D}_{\dot\gamma}\Phi - 2i\epsilon_{\dot\alpha\dot\gamma}\psi_{\beta\dot\beta\ \alpha}\,, \tag{2.10}$$

where $\tilde{\psi}_{\beta\dot{\beta}\ \alpha} = e^{\Phi+iW}\psi_{\beta\dot{\beta}\ \alpha}$ and the indices are "lowered" by the unit antisymmetric matrix $\epsilon_{\alpha\beta}$, ($\epsilon_{12} = -\epsilon_{21} = 1$), e.g. $V_{\alpha\dot{\alpha}} = \epsilon_{\alpha\alpha'}\epsilon_{\dot{\alpha}\dot{\alpha}'}V^{\alpha'\dot{\alpha}'}$. Now we should decompose the above equation into the irreducible parts, using the fact that the spin connections are symmetric

$$\bar{\omega}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = \bar{\omega}_{\dot{\gamma}\dot{\alpha}\dot{\beta}}\,, \qquad \omega_{\dot{\gamma}\beta\alpha} = \omega_{\dot{\gamma}\alpha\beta}\,,$$

and decomposing the spin–ensor $\psi_{\beta\dot{\beta}\ \alpha}$ as follows:

$$\psi_{\beta\dot{\beta}\ \alpha} = \bar{\psi}_{0\dot{\beta}}\epsilon_{\alpha\beta} + \chi_{\dot{\beta}\ \beta\alpha}\,, \qquad \chi_{\dot{\beta}\ \beta\alpha} = \chi_{\dot{\beta}\ \alpha\beta}\,.$$

The irreducible part of eq. (2.10) symmetric in $\dot{\alpha}$, $\dot{\beta}$ and α, β results in

$$\epsilon_{\dot{\gamma}(\dot{\alpha}}\chi_{\dot{\beta})\ \beta\alpha} = 0 \Longrightarrow \chi_{\dot{\beta}\ \beta\alpha} = 0\,,$$

while the part antisymmetric in $\dot{\alpha}\dot{\beta}$ and α,β determines $\bar{\psi}_{0\dot{\beta}}$ as the spinor derivative of the superfield Φ

$$\bar{\psi}_{0\dot{\beta}} = 2i\bar{D}_{\dot{\beta}}\Phi\,. \qquad (2.11)$$

Thus we obtain eqs. (2.6) and (2.7).

Substituting the results back into the above equation and analyzing the irreducible parts which remain we obtain eq. (2.8).[5]

To check the coefficient in the right hand side of (2.11) let us substitute the expressions obtained (except for eq. (2.11)) into eq. (2.10). We thus have

$$\epsilon_{\alpha\beta}\bar{\omega}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\gamma}}\Phi - 2i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\psi}_{0\dot{\beta}} \qquad (2.12)$$

or, omitting $\epsilon_{\alpha\beta}$,

$$\bar{\omega}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\gamma}}\Phi - 2i\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\psi}_{0\dot{\beta}}\,. \qquad (2.13)$$

Recall that $\bar{\omega}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = \bar{\omega}_{\dot{\gamma}\dot{\alpha}\dot{\beta}}$ and hence $\bar{\omega}_{\dot{\gamma}\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} = 0$. Thus, contracting (2.13) with $\epsilon^{\dot{\alpha}\dot{\beta}}$ one derives eq. (2.11).

Now, substituting (2.6), (2.7), (2.8) and (2.4) into the fermionic structure eq. (2.2), one finds from the analysis of the $d\theta^{\beta} \wedge d\theta^{\gamma}$ component that

$$D_{\beta}(3\Phi + iW) = 0\,,$$

or

$$3\Phi + iW = 6\bar{\phi}\,, \qquad D_{\alpha}\bar{\phi} = 0\,. \qquad (2.14)$$

The complex conjugate equation

$$\bar{D}_{\dot{\beta}}(3\Phi - iW) = 0 \Longrightarrow 3\Phi - iW = 6\phi\,, \qquad \bar{D}_{\dot{\alpha}}\phi = 0\,, \qquad (2.15)$$

follows from the $\propto d\theta^{\beta} \wedge d\bar{\theta}^{\dot{\gamma}}$ component of (2.2).

Thus both Φ and W are expressed through the chiral and antichiral superfields ϕ ($\bar{D}_{\dot{\alpha}}\phi = 0$) and $\bar{\phi}$ ($D_{\alpha}\bar{\phi} = 0$)

$$\Phi = \phi + \bar{\phi}\,, \qquad iW = 3(\bar{\phi} - \phi)\,. \qquad (2.16)$$

[5]The part with $[\dot{\alpha}\dot{\beta}]$, $(\alpha\beta)$ requires $\omega_{\dot{\gamma}\alpha\beta} = 0$, then the part with $(\dot{\alpha}\dot{\beta})$, $[\alpha\beta]$ gives the expression for $\bar{\omega}_{\dot{\gamma}\dot{\beta}\dot{\alpha}}$.

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The component of the basic form $\Pi^{\gamma\dot\gamma}\wedge d\bar\theta^{\dot\beta}$ of eq. (2.2) produces the equation for the chiral and antichiral superfields

$$\bar D_{\dot\alpha}\bar D^{\dot\alpha}e^{4\bar\phi}=\frac{4i}{R}e^{8\phi}\,. \tag{2.17}$$

The left hand side of this equation contains only the antichiral superfield while its right hand side contains only the chiral one.

The $\Pi^{\beta\dot\beta}\wedge d\theta^{\gamma}$ component of eq. (2.2) specifies the form of the vector component $w_{\beta\dot\beta\gamma}{}^{\alpha}$ of the spin connection

$$w_{\beta\dot\beta\gamma}{}^{\alpha}=4\partial_{\beta\dot\beta}\bar\phi\delta_{\gamma}{}^{\alpha}-4\partial_{\gamma\dot\beta}\bar\phi\delta_{\beta}{}^{\alpha}-2\partial_{\beta\dot\beta}\phi\delta_{\gamma}{}^{\alpha}-8iD_{\beta}\phi\bar D_{\dot\beta}\bar\phi\delta_{\gamma}{}^{\alpha}+8iD_{\gamma}\phi\bar D_{\dot\beta}\bar\phi\delta_{\beta}{}^{\alpha}\,. \tag{2.18}$$

Since by definition the spin connection is traceless, $w_{\alpha}{}^{\alpha}\equiv 0$, one finds from (2.18) that

$$\partial_{\beta\dot\beta}(\phi-\bar\phi)=-2iD_{\beta}\phi\bar D_{\dot\beta}\bar\phi\,, \tag{2.19}$$

i.e., $\partial_{\beta\dot\beta}W=6D_{\beta}\phi\bar D_{\dot\beta}\bar\phi$. Equation (2.19) can be used to simplify the expression (2.18) for $w_{\beta\dot\beta\gamma}{}^{\alpha}$

$$w_{\beta\dot\beta\gamma}{}^{\alpha}=2\partial_{\beta\dot\beta}\phi\delta_{\gamma}{}^{\alpha}-4\partial_{\gamma\dot\beta}\phi\delta_{\beta}{}^{\alpha}\,. \tag{2.20}$$

This expression can be also obtained directly from the $d\theta\wedge d\bar\theta$ component of eq. (2.3). As a consequence of eqs. (2.20), (2.8) and (2.16), the "left chiral" spin connection is expressed through the derivatives of the chiral field ϕ only

$$w_{\beta}{}^{\alpha}=-2d\theta^{\alpha}D_{\beta}\phi-2d\theta_{\beta}D^{\alpha}\phi-2\Pi^{\alpha\dot\gamma}\partial_{\beta\dot\gamma}\phi+2\Pi_{\beta\dot\gamma}\partial^{\alpha\dot\gamma}\phi\,. \tag{2.21}$$

To check the compatibility of eqs. (2.17) and (2.19) one should verify that applying the derivative $\bar D_{\dot\beta}$ to eq. (2.19) produces the equation which coincides with the one obtained by applying D_{β} to eq. (2.17). This fermionic integrability condition can be written in the form

$$\partial_{\beta\dot\beta}\bar D^{\dot\beta}\bar\phi=-\frac{2}{R}D_{\beta}\phi\, e^{8\phi-4\bar\phi}-4\partial_{\beta\dot\beta}\bar\phi\bar D^{\dot\beta}\bar\phi\,. \tag{2.22}$$

However, the first fermionic component in the θ-expansion of eq. (2.19) is more restrictive than the same component of eq. (2.17). Indeed, acting on eq. (2.19) by $\bar D_{\dot\alpha}$ (no contraction of the dotted indices is made in contrast to (2.22)), one finds

$$\partial_{\beta\dot\beta}\bar D_{\dot\alpha}\bar\phi=i\epsilon_{\dot\alpha\dot\beta}\bar D\bar D\bar\phi\, D_{\beta}\phi+\partial_{\beta\dot\alpha}\phi\,\bar D_{\dot\beta}\bar\phi\,. \tag{2.23}$$

Equation (2.23) also implies

$$\partial_{\beta\dot\beta}\bar D\bar D\bar\phi=4\bar D\bar D\bar\phi\partial_{\beta\dot\beta}\phi\,. \tag{2.24}$$

At this step the check of consistency can be carried out by verifying that the equation obtained by taking the derivative $\partial_{\beta\dot\beta}$ of eq. (2.17) is satisfied identically when eqs. (2.24) and (2.23) are taken into account.

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The $SO(1,3)$ invariant solutions of eqs. (2.17) and (2.19) specify the form of the chiral superfield ϕ for the AdS_4 coset superspace $OSp(1|4;R)/SO(1,3)$ and thus define the superconformally flat parametrization of the latter.

In a similar way one can demonstrate the superconformal flatness of the coset superspace $OSp(N|4;R)/(SO(1,3) \times SO(N))$ for any N.

In section 3 we shall describe a method which allows one to derive the explicit form of the conformal factor $\Phi(x,\theta)$ and the phase factor $W(x,\theta)$ for these supermanifolds without solving the Maurer-Cartan equations.

2.2 $AdS_2 \times S^2$ superspaces

Let us consider now a coset superspace $SU(1,1|2)/(SO(1,1) \times SO(2))$ whose bosonic body is the space $AdS_2 \times S^2$. When the radii R of AdS_2 and S^2 are equal, the metric of the bosonic space is conformally flat.

We shall now show that the coset superspace $SU(1,1|2)/(SO(1,1) \times SO(2))$ is however **not** *super* conformally flat and that another supercoset with the bosonic subspace $AdS_2 \times S^2$, namely $OSp(4^*|2)/(SO(1,1)\times SO(2)\times SU(2))$, is superconformally flat. The supergroup $OSp(4^*|2)$ has eight fermionic generators, like $SU(1,1|2)$, but a larger bosonic subgroup $SO(2,1) \times SO(3) \times SU(2)$ (see subsection 4.2 for details).[6]

The $AdS_2 \times S^2$ space is the Reissner-Nordström extreme black hole solution of $N = 2$, $D = 4$ supergravity (as discussed, e.g. in [5, 22]). In [22] it was demonstrated that the corresponding coset superspace $SU(1,1|2)/(SO(1,1) \times SO(2))$ is a solution of $N = 2$, $D = 4$ superfield supergravity constraints. On this solution the $N = 2$, $D = 4$ superspace coordinates $x^a = (x^{a'}, x^{\hat a})$ $(a' = 0,3;\ \hat a = 1,2)$, θ^α_i and $\bar\theta^{\dot\alpha\, i}$ $(i = 1,2)$ split into $AdS_2 \times S^2$ vectors and spinors, and the supergravity constraints reduce to the Maurer-Cartan structure equations of $SU(1,1|2)/(SO(1,1) \times SO(2))$.

Instead of writing the Maurer-Cartan equations of $SU(1,1|2)/(SO(1,1) \times SO(2))$ we shall consider Maurer-Cartan equations of the one-parameter set of supercosets $D(2,1;\alpha)/(SO(1,1) \times SO(2) \times SU(2))$, where $D(2,1;\alpha)$ is an exceptional supergroup with eight fermionic generators and the bosonic subgroup $SO(2,1) \times SO(3) \times SU(2)$, α being a numerical parameter. When $\alpha = 1$, $D(2,1;\alpha)$ becomes isomorphic to $OSp(4^*|2)$ and, when $\alpha = -1$, it reduces to the semi-direct product of $SU(1,1|2)$ and the outer automorphism group $SU(2)$ (see [19] and subsection 4.2 for details). Thus all the coset superspaces $D(2,1;\alpha)/(SO(1,1) \times SO(2) \times SU(2))$ contain $AdS_2 \times S^2$ as the bosonic body and possess the same number of fermionic dimensions. The supercosets $OSp(4^*|2)/(SO(1,1) \times SO(2) \times SU(2))$ and $SU(1,1|2)/(SO(1,1) \times SO(2))$ are recovered at $\alpha = 1$ and $\alpha = -1$, respectively.

The torsion and the curvature constraints have the form

$$T^{a'} \equiv dE^{a'} - E^{b'} \wedge w_{b'}{}^{a'} = -2i\, E^\alpha_i \wedge E^{\dot\beta i}\, \sigma^{a'}_{\alpha\dot\beta}\,, \tag{2.25}$$

$$T^{\hat a} \equiv dE^{\hat a} - E^{\hat b} \wedge w_{\hat b}{}^{\hat a} = 2i\alpha\, E^\alpha_i \wedge E^{\dot\beta i}\, \sigma^{\hat a}_{\alpha\dot\beta}\,, \tag{2.26}$$

$$T^\alpha_i \equiv dE^\alpha_i - E^\beta_i \wedge w_\beta{}^\alpha - E^\alpha_j \wedge \omega_j{}^i = \frac{2(1-i)}{R} E^a \wedge \bar E^{\dot\beta}_j f^{\alpha\beta} \sigma_{a\beta\dot\beta}\,, \tag{2.27}$$

[6] Note that the supergroup $SU(1,1|2)$ has a group of outer automorphisms $SU(2)$ whose generators do not appear in the anticommutator of the supercharges.

$$
\begin{aligned}
T^{\dot\alpha i} &\equiv dE^{\dot\alpha i} - E^{\dot\beta i}\wedge w_{\dot\beta}{}^{\dot\alpha} - \bar E_j^{\dot\alpha}\wedge \omega_j{}^i = -\frac{2(1+i)}{R}E^a\wedge E_j^\beta \epsilon^{ij}\sigma_{a\beta\dot\beta}\bar f^{\dot\alpha\dot\beta}\,, && (2.28)\\
R^{a'b'} &\equiv dw^{a'b'} - w^{a'c}\wedge w_c{}^{b'}\\
&= -\frac{i}{R}\left(E^{\alpha i}\wedge E_{\alpha i} + \bar E_{\dot\alpha i}\wedge \bar E^{\dot\alpha i}\right)\epsilon^{a'b'} - \frac{1}{4R^2}E^{d'}\wedge E^{c'}\epsilon_{c'd'}\epsilon^{a'b'}, && (2.29)\\
R^{\hat a\hat b} &\equiv dw^{\hat a\hat b} - w^{\hat a c}\wedge w_c{}^{\hat b}\\
&= \frac{\alpha}{R}\left(E^{\alpha i}\wedge E_{\alpha i} - \bar E_{\dot\alpha i}\wedge \bar E^{\dot\alpha i}\right)\epsilon^{\hat a\hat b} + \frac{1}{4R^2}E^{\hat d}\wedge E^{\hat c}\epsilon_{\hat c\hat d}\epsilon^{\hat a\hat b}\,, && (2.30)\\
R^{ij} &= d\omega^{ij} - \omega^{ik}\wedge\omega_k^{\;j}\\
&= \frac{1+\alpha}{2R}\Big((1-i)E^{i\alpha}\wedge E^{j\beta}f_{\alpha\beta} - (1+i)\bar E^{i\dot\alpha}\wedge\bar E^{j\dot\beta}\bar f_{\dot\alpha\dot\beta}\Big)\,. && (2.31)
\end{aligned}
$$

Here

$$
E^{\underline{\alpha}} \equiv (E_i^\alpha\,, \bar E^{\dot\alpha i})\,, \qquad \bar E^{\dot\alpha i} = (E_i^\alpha)^*\,, \tag{2.32}
$$

w^{ab}, $w_\beta{}^\alpha = (1/4)w^{ab}(\sigma_a\tilde\sigma_b)_\gamma{}^\alpha$ and $w_{\dot\beta}{}^{\dot\alpha} = (w_\beta{}^\alpha)^*$ are $AdS_2\times S^2$ spin connections with the curvature R^{ab}, $\omega_j{}^i$ is the SU(2) connection and $R^{ij} = R^{ji}$ is the corresponding curvature. Note that in the case of $\mathrm{SU}(1,1|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2))$ when $\alpha=-1$, the SU(2) curvature R^{ij} (2.31) is zero and the SU(2) connection can be gauged away by an appropriate local SU(2) transformation of $E^{\alpha i}$ and $\bar E_i^{\dot\alpha}$.

The symmetric spin tensors $f^{\alpha\beta} = f^{\beta\alpha}$, $\bar f^{\dot\alpha\dot\beta} = \bar f^{\dot\beta\dot\alpha} = (f^{\alpha\beta})^*$ are related by

$$
f^{\alpha\beta} = \frac14\epsilon^{\alpha\gamma}(\sigma^a\tilde\sigma^b)_\gamma{}^\alpha f_{ab}\,, \qquad \bar f^{\dot\alpha\dot\beta} = \frac14\epsilon^{\dot\beta\dot\gamma}(\tilde\sigma^a\sigma^b)^{\dot\alpha}{}_{\dot\gamma}f_{ab} \tag{2.33}
$$

to the $\mathrm{SO}(1,1)\times\mathrm{SO}(2)$ invariant antisymmetric tensor

$$
f_{ab} = \begin{pmatrix}\epsilon_{a'b'} & 0\\ 0 & \epsilon_{\hat a\hat b}\end{pmatrix}, \quad a',b' = 0,3\,, \quad \hat a,\hat b = 1,2\,, \tag{2.34}
$$

which in the case of the supercoset $\mathrm{SU}(1,1|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2))$ (i.e., when $\alpha=-1$) can be associated with the "vacuum" value of the field strength F_{ab} of the abelian gauge field of the $N=2$, $D=4$ supergravity multiplet. In the superspace the corresponding constant closed form $F=dA$ is

$$
F\epsilon^{ij} = \frac12 E^{\alpha i}\wedge E^{\beta j}\epsilon_{\alpha\beta} + \frac12\bar E^{\dot\alpha i}\wedge\bar E^{\dot\beta j}\epsilon_{\dot\alpha\dot\beta} + \frac12 E^a\wedge E^b f_{ab}\epsilon^{ij}\,, \qquad dF\equiv 0\,. \tag{2.35}
$$

One can verify that the constraints (2.26)–(2.32) admit a superconformally flat solution only for $\alpha=1$, i.e. in the case of the supercoset $\mathrm{OSp}(4^*|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2)\times\mathrm{SU}(2))$. The superconformally flat form of the supervielbeins is

$$
\begin{aligned}
E^a &= e^{\Phi+\bar\Phi}\Pi^b L_b{}^a(x,\theta)\,, \qquad \Pi^a = dx^a - i\left(d\theta_i^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta i} - \theta_i^\alpha\sigma^a_{\alpha\dot\beta}d\bar\theta^{\dot\beta i}\right), && (2.36)\\
E_i^\alpha &= (1-i)\,e^{\bar\Phi}\left(d\theta_i^\beta + \frac{i}{2}\Pi^a\tilde\sigma_a^{\dot\beta\beta}\bar D_{\dot\beta i}\bar\Phi\right)L_\beta{}^\gamma f_\gamma{}^\alpha\,, && (2.37)\\
\bar E^{\dot\alpha i} &= (1+i)\,e^{\Phi}\left(d\bar\theta^{\dot\beta i} - \frac{i}{2}\Pi^a\tilde\sigma_a^{\dot\beta\beta}D_\beta^i\Phi\right)\bar L_{\dot\beta}{}^{\dot\gamma}\bar f_{\dot\gamma}{}^{\dot\alpha}\,, && (2.38)
\end{aligned}
$$

where

$$\bar{D}_{\dot{\beta}j}\Phi = 0\,, \qquad D^j_\beta\bar{\Phi} = 0\,, \qquad \bar{\Phi} = (\Phi)^*\,, \tag{2.39}$$

and $L_b{}^a(x,\theta)$, $L_\beta{}^\alpha(x,\theta)$ and $\bar{L}_{\dot{\beta}}{}^{\dot{\alpha}}(x,\theta)$ are some SO(1,3) matrices in vector and spinor representations, respectively.

The crucial point where the superconformally flat ansatz fails for the supercoset $\mathrm{SU}(1,1|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2))$ is the spinor torsion constraints (2.28) and (2.29), while these are compatible with the superconformally flat ansatz (2.37)–(2.39) for $\mathrm{OSp}(4^*|2)/(\mathrm{SO}(1,1)$ $\times$ $\mathrm{SO}(2)$ $\times$ $\mathrm{SU}(2))$ and produce differential equations for the Weyl and U(1) factor, such as

$$\bar{D}_{\dot{\gamma}i}\bar{D}^{\dot{\gamma}}_j\bar{\Phi} + 2\bar{D}_{\dot{\gamma}i}\bar{\Phi}\;\bar{D}^{\dot{\gamma}}_j\bar{\Phi} = 0\,, \tag{2.40}$$

$$\bar{D}_{\dot{\alpha}i}\bar{D}^i_{\dot{\beta}}\bar{\Phi} - 2\bar{D}_{\dot{\alpha}i}\bar{\Phi}\;\bar{D}^i_{\dot{\beta}}\bar{\Phi} = -\frac{8i}{R}e^{2\Phi}(\bar{L}\bar{f}\bar{L}^{-1})_{\dot{\alpha}\dot{\beta}}\,. \tag{2.41}$$

Thus the $AdS_2\times S^2$ superspace which appears as a maximally supersymmetric solution of $N=2$, $D=4$ supergravity is not superconformal. An intrinsic nature of this somewhat surprising feature, as well as the reason why an alternative coset $\mathrm{OSp}(4^*|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2)\times\mathrm{SU}(2))$ is superconformal are explained in detail in subsection 4.2. They are traced to the fact that $\mathrm{OSp}(4^*|2)$ can be embedded as an appropriate subgroup into the $N=2$, $D=4$ superconformal group, while $\mathrm{SU}(1,1|2)$ cannot. In subsection 4.2 we shall also derive an explicit form of the conformal factor $\Phi(x,\theta)$ of (2.37)–(2.38).

2.3 $AdS_3\times S^3$ superspaces

The $N=2$, $D=6$ super Poincare group with **16** supercharges is a subsupergroup of the superconformal group in six dimensions $U_\alpha(4|2;H) = \mathrm{OSp}(8^*|4)$ which has **32** supercharges and $O^*(8)\times\mathrm{USp}(4)$ as the bosonic subgroup (see, e.g. [18, 19, 20] for a list of corresponding Lie superalgebras, superconformal algebras and their subalgebras). A superspace whose bosonic body is $AdS_3\times S^3$ is the supercoset $(\mathrm{SU}(1,1|2)\times\mathrm{SU}(1,1|2))/(\mathrm{SO}(1,2)\times\mathrm{SO}(3))$, its isometry supergroup being the direct product of two supergroups $\mathrm{SU}(1,1|2)$. The bosonic subgroup of this isometry supergroup is $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$, and the fermionic sector consists of **16** generators (supercharges). This supercoset is a solution of $N=(2,0)$, $D=6$ supergravity. But, as in the case of $AdS_2\times S^2$ considered above, the isometry supergroup $\mathrm{SU}(1,1|2)\times\mathrm{SU}(1,1|2)$ cannot be embedded as the appropriate subgroup into the $D=6$ superconformal group $\mathrm{OSp}(8^*|4)$. Instead, the latter contains as such a subgroup the supergroup $\mathrm{OSp}(4^*|2)\times\mathrm{OSp}(4^*|2)$. Thus, it is the coset superspace $(\mathrm{OSp}(4^*|2)\times\mathrm{OSp}(4^*|2))/(\mathrm{SO}(1,2)\times\mathrm{SO}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2))$, also having $AdS_3\times S^3$ as the bosonic subspace and **16** fermionic directions, which is superconformal. We have presented here only group-theoretical arguments, but by analogy with the $AdS_2\times S^2$ case one can also demonstrate this by analyzing the relevant Maurer-Cartan equations or applying the "bottom-up" approach of sections 3,4.

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2.4 $AdS_5 \times S^5$ superspace

The super $AdS_5 \times S^5$ background is a maximally supersymmetric solution of type-IIB $D = 10$ supergravity. If it were superconformally flat, then the superconformally flat ansatz for $AdS_5 \times S^5$ would solve the type-IIB $D = 10$ supergravity constraints (which are equivalent to the superfield supergravity equations of motion).

We shall again see that the constraints which describe $AdS_5 \times S^5$ superspace do not have a superconformally flat solution.[7] The reason is that the isometry supergroup SU(2, 2|4) of super $AdS_5 \times S_5$ is *not a sub-supergroup* of the superconformal group OSp(2|32; R) or OSp(1|64; R) in ten dimensions.

The Maurer-Cartan structure equations for the relevant supercoset SU(2, 2|4)/(SO(1, 4) ×SO(5)) coincide with the type-IIB supergravity constraints [21] restricted to the $AdS_5 \times S^5$ superbackground [22] ($\underline{a} = 0, 1, \ldots, 9,\ \underline{\alpha} = 1, \ldots 16,\ I = 1, 2$)

$$\begin{aligned} T^{\underline{a}} &\equiv dE^{\underline{a}} - E^{\underline{b}} \wedge w_{\underline{b}}{}^{\underline{a}} = -i\, E^{\underline{\alpha}I} \wedge E^{\underline{\delta}I}\, \sigma^{\underline{a}}_{\underline{\alpha\delta}} \\ &\equiv -i\, \left(E^{\underline{\alpha}1} \wedge E^{\underline{\delta}1} + E^{\underline{\alpha}2} \wedge E^{\underline{\delta}2}\right)\, \sigma^{\underline{a}}_{\underline{\alpha\delta}}\,, \end{aligned} \tag{2.42}$$

$$T^{\underline{\alpha}I} \equiv dE^{\underline{\alpha}I} - E^{\underline{\beta}I} \wedge w_{\underline{\beta}}{}^{\underline{\alpha}} = \frac{1}{R} E^{\underline{a}} \wedge E^{\underline{\beta}J}\, \epsilon_{IJ}\, f_{\underline{a}\underline{b}_1 \cdots \underline{b}_4} (\sigma^{\underline{b}_1 \cdots \underline{b}_4})_{\underline{\beta}}{}^{\underline{\alpha}}\,, \tag{2.43}$$

$$R^{\underline{ab}} \equiv dw^{\underline{ab}} - w^{\underline{ac}} \wedge w_{\underline{c}}{}^{\underline{b}} = -\frac{1}{R^2} E^{\underline{a}} \wedge E^{\underline{b}} - \frac{4i}{R^2} E^{\underline{\alpha}I} \wedge E^{\underline{\delta}J} \epsilon_{IJ} f^{\underline{abc}_1\underline{c}_2\underline{c}_3} \sigma_{\underline{c}_1\underline{c}_2\underline{c}_3\underline{\alpha\delta}}\,, \tag{2.44}$$

where

$$w_{\underline{\beta}}{}^{\underline{\alpha}} \equiv \frac{1}{4} w^{\underline{ab}} \sigma_{\underline{ab}\underline{\beta}}{}^{\underline{\alpha}}\,, \tag{2.45}$$

and $\sigma^{c_1 \cdots c_p}$ are antisymmetrized products of p ten-dimensional “Pauli” matrices.

The relative coefficients in (2.42)–(2.44) are dictated by the Bianchi identities

$$\mathcal{D}\bar{T}^{\underline{a}} = -\bar{E}^{\underline{b}} \wedge R_{\underline{b}}{}^{\underline{a}}\,, \tag{2.46}$$

$$\mathcal{D}T^{\underline{\alpha}} = -E^{\underline{\beta}} \wedge R_{\underline{\beta}}{}^{\underline{\alpha}}\,, \tag{2.47}$$

$$\mathcal{D}R_{\underline{b}}{}^{\underline{a}} = 0\,. \tag{2.48}$$

Here $E^{\underline{a}} = (E^{\hat{a}}, E^i)$, where $\hat{a} = 0, 1, \ldots, 4$ is the vector index of the tangent space of AdS_5, and $i = 1, \ldots, 5$ is the vector index of the tangent space of S^5, $f_{\underline{a}_1 \cdots \underline{a}_5}$ is a constant self-dual tensor of the following form

$$f_{\hat{a}_1 \ldots \hat{a}_5} = c\varepsilon_{\hat{a}_1 \ldots \hat{a}_5}\,, \qquad f_{i_1 \ldots i_5} = c\varepsilon_{i_1 \ldots i_5}\,, \tag{2.49}$$

with all other components vanishing. The constant c is proportional to the radius of AdS_5 or S^5 (these radii are equal).

We now check whether the superconformally flat ansatz for the $AdS_5 \times S^5$ superbackground

$$E^{\underline{a}} = e^{2W} \Pi^{\underline{a}}\,, \qquad \Pi^{\underline{a}} \equiv dx^{\underline{a}} - i d\theta^{\underline{\alpha}I}\, \sigma^{\underline{a}}_{\underline{\alpha\beta}} \theta^{\underline{\beta}I}\,, \tag{2.50}$$

is compatible with eqs. (2.42)–(2.44).

[7] Superconformally flat solutions of type-IIB supergravity equations requires a non-trivial axion-dilaton background which is zero in the $AdS_5 \times S^5$ superspace.

Substituting (2.50) into (2.42) and analyzing the lower dimensional components of the latter

$$T_{\underline{\alpha} I\, \underline{\beta} J}{}^{a} = -2i\delta_{IJ}\sigma^{a}_{\underline{\alpha\beta}}\,, \qquad T_{\underline{\alpha} I\, b}{}^{a} = 0\,,$$

one finds that the most general form of the fermionic supervielbein consistent with both (2.50) and (2.42) is

$$E^{\underline{\alpha} I} = e^{W}\left(d\Theta^{\underline{\alpha} J} - i\,\Pi^{\underline{a}}\tilde{\sigma}_{\underline{a}}^{\underline{\alpha\beta}} D_{\underline{\alpha} J} W\right)\, h_J^{\;I}\,, \tag{2.51}$$

where

$$h_J^{\;I} = \frac{1}{\sqrt{1+a^2}}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \equiv \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{2.52}$$

is an SO(2) matrix constructed from a real superfield $a(x,\theta)$, with $\alpha(x,\theta) = \arccos(1/\sqrt{1+a^2})$.

However, as can be shown by a straightforward though tedious calculation, the lowest dimensional component of eq. (2.43), $T_{\underline{\gamma} K\, \underline{\beta} J}^{\;\underline{\alpha} I} = 0$, implies

$$D_{\underline{\alpha} I} W = 0\,, \qquad D_{\underline{\alpha} I}\; \alpha = 0\,, \tag{2.53}$$

which requires the background superspace to be flat.

We have already mentioned that the negative answer for the case of $AdS_5 \times S^5$ (as well as for the conventional $AdS_2 \times S^2$ and $AdS_3 \times S^3$ superspaces) is explained by the following fact. Though the type-IIB $D = 10$ super Poincaré group (with tensorial charges) is a subgroup of generalized simple superconformal groups $\mathrm{OSp}(2|32;R)$ and $\mathrm{OSp}(1|64;R)$ in ten dimensions, the isometry supergroup $\mathrm{SU}(2,2|4)$ of super $AdS_5 \times S^5$ is *not a sub-supergroup* of the superconformal group. At the same time, for the background to possess a (super)conformal structure it is crucial that both its (super)isometry group and (super)isometry of the flat (super)space are subgroups of some encompassing (super)conformal group.

A simple and transparent way to see that $\mathrm{SU}(2,2|4)$ cannot be a subgroup of $\mathrm{OSp}(2|32;R)$ and/or of $\mathrm{OSp}(1|64;R)$ is as follows. A fundamental representation of $\mathrm{SU}(2,2|4)$ is a complex supertwistor [23]

$$Z^{A} = (\lambda^{\alpha},\, \bar{\mu}_{\dot\alpha},\, \psi^{i})\,, \tag{2.54}$$

where two Grassmann-even Weyl spinors λ^{α} and $\bar{\mu}_{\dot\alpha}$ form a Dirac spinor with respect to the $D = 4$ Lorentz transformations, and the complex Grassmann-odd components ψ^{i} ($i = 1,2,3,4$) are in the fundamental representation of the R-symmetry subgroup SU(4). Therefore the Grassmann-odd part of the supertwistor has eight real components. (There are no representation with a less number of Grassmann-odd components).

If the supertwistor obeys the following commutation relations

$$\left[Z^{A},\, \bar{Z}_{B}\right]_{\mp} = \delta^{A}_{B}\,, \quad \left[Z^{A},\, Z^{B}\right]_{\mp} = 0\,, \quad \bar{Z}_{B} = \left(\mu^{\beta},\, \bar{\lambda}_{\dot\beta},\, \bar{\psi}^{i}\right) \tag{2.55}$$

([, $]_+$ stands for the anticommutator of the Grassmann-odd components), the generators of SU$(2,2|4)$ can be realized as the hermitean bilinear combination

$$M^A_B = Z^A \bar{Z}_B \,. \tag{2.56}$$

Fundamental representations of OSp$(2|32;R)$ and OSp$(1|64;R)$ are real supertwistor (or supersingleton) representations

$$\Lambda_{\hat{A}} = \left(\lambda_{\hat{\alpha}}, \chi^I\right), \tag{2.57}$$

where now $\lambda_{\hat{\alpha}}$ is a Grassmann-even 32-component real spinor or a 64-component real spinor and χ^I denotes the Grassmann-odd real components of the supertwistor. The OSp$(2|32;R)$ supertwistor contains two real Grassmann-odd components, which form a vector of an SO(2) subgroup of OSp$(2|32;R)$, and the OSp$(1|64;R)$ supertwistor contains only one real Grassmann-odd component.

If the components of (2.57) satisfy the (anti)commutation relations

$$\left[\Lambda_{\hat{A}}, \Lambda_{\hat{B}}\right]_{\mp} = C_{\hat{A}\hat{B}}\,, \tag{2.58}$$

where $C_{\hat{A}\hat{B}} = (C_{\hat{\alpha}\hat{\beta}}, \delta_{IJ})$ is an OSp-invariant constant matrix, the generators of OSp$(2|32;R)$ and OSp$(1|64;R)$ can be realized as the bilinear combination of the supertwistor (2.57) components

$$M_{\hat{A}\hat{B}} = \Lambda_{(\hat{A}}\Lambda_{\hat{B}]}\,, \tag{2.59}$$

where the matrix $M_{\hat{A}\hat{B}}$ is symmetric in the indices $\hat{\alpha}$, $\hat{\beta}$ and antisymmetric in the indices I, J.

Now, if SU$(2,2|4)$ were a subgroup of OSp$(2|32;R)$ or OSp$(1|64;R)$, the supersingleton representation (2.57), having been decomposed into irreps of this subgroup, would contain the supertwistor representation (2.54). But this is obviously not the case since (2.54) has eight real Grassmann-odd components and (2.57) has only one or two. The above reasoning suggests that the minimal simple supergroups which can simultaneously contain both OSp$(2|32;R)$ and SU$(2,2|4)$ as sub-supergroups are OSp$(8|32;R)$ and SU$(16,16|4)$. The type-IIB $D = 10$ super Poincaré group with a "central" extension, as well as special superconformal transformations are contained in OSp$(2|32;R)$, but the fermionic sector of SU$(2,2|4)$ is certainly not. Thus even within these much larger supergroups[8] we are not able to relate (in the superconformal sense discussed in sections 3 and 4) the flat D=10 superspace and the $AdS_5 \times S^5$ superspace. This explains why the superconformally flat ansatz is inconsistent with the $AdS_5 \times S^5$ solution of the type-IIB supergravity constraints. One may ask whether there exists a different $AdS_5 \times S^5$ supercoset, analogous to the ones of $AdS_2 \times S^2$ and $AdS_3 \times S^3$ of subsections 2.2 and 2.3, which is superconformal. We have looked through the list of the real forms of the classical Lie superalgebras ([19]) and have not found another simple superalgebra, different from su$(2,2|4)$, which could be the isometry of such a supercoset. Thus it does not exist.[9]

[8] Note that these supergroups have 256 Grassmann-odd generators, which is four times as much as the number of spinor charges one assumes to be present in superconformal groups associated with M-theory (i.e. 64 [13, 15, 16]).

[9] We would like to thank Paul Sorba for the discussion of this point.

3. Superconformal flatness of AdS superspaces. "Bottom-up" approach

3.1 The idea of the method

The general strategy of proving superconformal flatness for AdS superspaces and finding the relevant superconformal factors which we shall follow in this section applies to the cases when both the Poincaré supersymmetry and AdS supersymmetry, with the equal number of supercharges and the equal number D of translation generators in flat and AdS spaces, form two subgroups of the superconformal group acting in a Minkowski superspace of the bosonic dimension D. One starts from a coordinate realization of this superconformal group in the Minkowski superspace [23] and studies the transformation properties of the relevant flat covariant differential of x under this realization. Generically, this differential is multiplied by a scalar weight factor and undergoes some induced (super coordinate dependent) Lorentz transformation. Then one singles out the AdS supersymmetry transformations from the superconformal transformations as a linear combination of those of Poincaré supersymmetry and those of the special conformal supersymmetry. As the next step, one constructs, out of the original flat superspace coordinates, the appropriate scalar density compensating the weight part of the transformation of the flat covariant differential under the AdS supersymmetry. The flat covariant differential of x multiplied by this factor undergoes only induced Lorentz transformations under the AdS supersymmetry transformations and so it is the sought covariant differential of the AdS supersymmetry (the vector Cartan form E^a of the previous consideration). By construction, the corresponding interval is conformal to the super Poincaré group covariant interval, which proves the superconformal flatness of the AdS superspace. In other words, one starts from the parametrization of the AdS superspace by the coordinates in which the super Poincaré subgroup of the underlying superconformal group has the "canonical" manifest realization. The precise relation of such a parametrization to the parametrization where the AdS subgroup has the "canonical" realization (corresponding, e.g. to the exponential parametrization of the AdS super coset element) can be fairly complicated and we are not going to discuss this point here. Also, we shall not give the precise form of theAdS-covariant differentials of Grassmann coordinates (spinorial Cartan forms) in this approach. Once we are aware of the superconformal factor, they can be restored, up to a Lorentz rotation, by general formulas (1.2), (2.6), (2.7), (2.38) and (2.38).

We shall start from the purely bosonic $D = 4$ case and then consider the cases of $N = 1$ and $N = 2$ AdS supersymmetries in $D = 4$. The case of general N is considered in subsection 3.5 and the $AdS_2 \times S^2$ case in section 4.

3.2 Toy example: bosonic AdS_4

The special conformal transformations of the $D = 4$ Minkowskian coordinates x^m ($m = 0, \ldots, 3$) are as follows:

$$\delta_K x^m = 2(b \cdot x)x^m - b^m x^2 \,, \tag{3.1}$$

where b^m is a constant parameter. The covariant differential dx^m is transformed as follows:

$$\delta_K dx^m = l^{[mn]} dx_n + 2(b \cdot x) dx^m \,, \qquad l^{[mn]} = 2(x^m b^n - x^n b^m) \,. \tag{3.2}$$

The AdS subgroup of $D = 4$ conformal group is singled out via the identification

$$\delta_{AdS} = \delta_P + \delta_K \,,$$

with

$$b^m = m^2 a^m$$

and

$$\delta_{AdS} x^m = a^m + m^2 \left[2(a \cdot x) x^m - a^m x^2 \right], \tag{3.3}$$

$$\delta_{AdS} dx^m = \tilde{l}^{[mn]} dx_n + 2m^2 (a \cdot x) dx^m \,, \qquad \tilde{l}^{[mn]} = 2m^2 (a^n x^m - a^m x^n) \,, \tag{3.4}$$

where m is proportional to the inverse AdS radius (the case of $-m^2$ would correspond to dS subgroup). Now we wish to construct the scalar factor $f(x)$ which compensates for the weight factor $2m^2(a \cdot x)$ in (3.4). By Lorentz covariance, it can depend only on $x^2 \equiv X$, $f(x) = f(X)$, and should have the following transformation properties under (3.3):

$$\delta_{AdS} f(X) = -2m^2 (a \cdot x) f(X) \,. \tag{3.5}$$

From the last relation one obtains the simple equation for $f(X)$:

$$f' = -\frac{m^2}{1 + m^2 x^2} \, f \,, \tag{3.6}$$

which, up to a constant normalization factor, is solved by

$$f(x^2) = \frac{1}{1 + m^2 x^2} \,. \tag{3.7}$$

Hence, the object

$$\mathcal{D} x^m = \frac{1}{1 + m^2 x^2} \, dx^m \tag{3.8}$$

undergoes only induced Lorentz rotation under the AdS translations and so it is the AdS-covariant differential. Its square

$$ds^2 = \mathcal{D} x^m \mathcal{D} x_m = \frac{1}{(1 + m^2 x^2)^2} \, dx^m dx_m$$

is the AdS interval which in this parametrization is explicitly conformal to the Minkowski interval $dx^m dx_m$. Just this parametrization of AdS_4 was used e.g. in [37].

3.3 $N = 1$, $D = 4$ AdS superspace

The $N = 1$, $D = 4$ superspace coordinates $(x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\beta}})$ are transformed in the following way under the special supersymmetry of $N = 1, D = 4$ superconformal group (see e.g. [23]):[10]

$$\begin{aligned}
\delta_\eta x^{\alpha\dot{\alpha}} &= \eta^\lambda \theta^\alpha x_\lambda{}^{\dot{\alpha}} + \frac{i}{2} \eta^\alpha \bar{\theta}^{\dot{\alpha}} \theta^2 - \bar{\eta}^{\dot{\rho}} \bar{\theta}^{\dot{\alpha}} x^\alpha{}_{\dot{\rho}} + \frac{i}{2} \bar{\eta}^{\dot{\alpha}} \theta^\alpha \bar{\theta}^2 \,, \\
\delta_\eta \theta^\rho &= \eta^\rho \theta^2 - i \bar{\eta}^{\dot{\rho}} x^\rho{}_{\dot{\rho}} - \bar{\eta}^{\dot{\rho}} \bar{\theta}_{\dot{\rho}} \theta^\rho \,, \\
\delta_\eta \bar{\theta}^{\dot{\rho}} &= \bar{\eta}^{\dot{\rho}} \bar{\theta}^2 + i \eta^\rho x_\rho{}^{\dot{\rho}} + \eta^\rho \theta_\rho \bar{\theta}^{\dot{\rho}} \,.
\end{aligned} \tag{3.9}$$

[10] $\theta^\alpha = \epsilon^{\alpha\beta} \theta_\beta$, $\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma$, $\theta^2 = \theta^\alpha \theta_\alpha$, $\bar{\theta}^2 = \overline{\theta^2} = \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$.

It is enough to consider transformations with odd parameters since all the remaining superconformal transformations are contained in the closure of this conformal supersymmetry and the standard Poincaré supersymmetry:

$$\delta_\epsilon \theta^\alpha = \epsilon^\alpha\,, \qquad \delta_\epsilon \bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha}\,, \qquad \delta_\epsilon x^{\rho\dot\rho} = -i(\epsilon^\rho \bar\theta^{\dot\rho} + \bar\epsilon^{\dot\rho}\theta^\rho)\,. \tag{3.10}$$

The flat covariant differential of x,

$$\Pi^{\alpha\dot\alpha} = dx^{\alpha\dot\alpha} - i(d\theta^\alpha \bar\theta^{\dot\alpha} + d\bar\theta^{\dot\alpha}\theta^\alpha)\,, \tag{3.11}$$

is evidently invariant under (3.10) and is transformed in the following way under (3.9)

$$\delta\Pi^{\alpha\dot\alpha} = 2\eta^{(\alpha}\theta^{\lambda)}\Pi_\lambda{}^{\dot\alpha} - 2\bar\eta^{(\dot\alpha}\bar\theta^{\dot\lambda)}\Pi^\alpha{}_{\dot\lambda} + \left(\eta\theta + \bar\eta\bar\theta\right)\Pi^{\alpha\dot\alpha}\,. \tag{3.12}$$

The first two terms are the induced Lorentz rotation, while the last term is the weight transformation. Now we single out the AdS_4 supertranslations as the linear combination of (3.10) and (3.9) with

$$\eta^\alpha \longrightarrow m\epsilon^\alpha\,, \qquad \bar\eta^{\dot\alpha} \longrightarrow m\bar\epsilon^{\dot\alpha}\,,$$

where m is an arbitrary real parameter of the contraction to the Poincaré supersymmetry. Like in the $N = 0$ case we need to find the real scalar factor

$$A(x,\theta,\bar\theta) = \bar A\,,$$

such that under the AdS supersymmetry it has the following transformation property

$$\delta_{AdS} A = -m\left(\epsilon\theta + \bar\epsilon\bar\theta\right)A\,. \tag{3.13}$$

To preserve Lorentz covariance, it can depend only on the following invariants

$$x^2 \equiv x^{\alpha\dot\alpha}x_{\alpha\dot\alpha}\,, \quad \theta^2\,,\ \bar\theta^2\,,\ x^{\alpha\dot\alpha}\theta_\alpha\bar\theta_{\dot\alpha}\,.$$

Fortunately, the terms proportional to the last nilpotent invariant vanish (it can be directly checked) and A has the following θ expansion

$$A = a(x^2) + c(x^2)\theta^2 + \bar c(x^2)\bar\theta^2 + d(x^2)\theta^2\bar\theta^2\,. \tag{3.14}$$

From the condition (3.13) we get the set of equations for the coefficients in (3.14)

$$\begin{aligned} a' - mc = a' - m\bar c &= 0 \longrightarrow c = \bar c\,,\\ mx^2 a' + ma + 2c &= 0\,,\\ ma' - 2c' + 2md &= 0\,,\\ mx^2 c' + 3mc + 2d &= 0\,. \end{aligned} \tag{3.15}$$

The first two equations yield

$$a'\left(1 + \frac{m^2}{2}x^2\right) = -\frac{m^2}{2}a\,, \tag{3.16}$$

i.e., the same equation as (3.6) (up to rescaling of m). Thus,

$$a=\frac{\beta}{1+(m^2/2)x^2}\,, \tag{3.17}$$

where β is an arbitrary integration constant. The rest of equations allows one to uniquely restore other functions, and the resulting expression for A proves to be as follows:

$$A=\frac{\beta}{(1+(m^2/2)x^2)}\left[1-\frac{m}{2}\frac{1}{(1+(m^2/2)x^2)}(\,\theta^2+\bar\theta^2\,)+\frac{m^2}{4}\frac{3+(m^2/2)x^2}{(1+(m^2/2)x^2)^2}\,\theta^2\bar\theta^2\right]. \tag{3.18}$$

Hereafter we choose $\beta=1$.

One can obtain more elegant expression for A in terms of

$$x_L^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}+i\theta^\alpha\bar\theta^{\dot\alpha}\,,\qquad x_R^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}-i\theta^\alpha\bar\theta^{\dot\alpha}\,.$$

The factor A has the following product structure

$$A\,=\,f(x_L,\theta)\bar f(x_R,\bar\theta)\;,\quad \bar f=\overline{(f)}\,, \tag{3.19}$$

$$f(x_L,\theta)\,=\,\frac{1}{\sqrt{1+(m^2/2)x_L^2}}\left[1-\frac{m}{2}\,\frac{1}{\left(1+(m^2/2)x_L^2\right)}\,\theta^2\right]. \tag{3.20}$$

The $N=1$ covariant AdS x-differential is now defined as

$$\mathcal{D}x^{\alpha\dot\alpha}=f(\zeta_L)\bar f(\zeta_R)\Pi^{\alpha\dot\alpha}\quad\left(\zeta_L=(x_L,\theta),\;\zeta_R=\overline{(\zeta_L)}\right). \tag{3.21}$$

It undergoes only an induced Lorentz transformation under the AdS_4 supertranslations (and the appropriate one under the AdS_4 translations). Thus the $N=1$ super AdS_4 invariant interval is given by

$$ds^2=\mathcal{D}x^{\alpha\dot\alpha}\mathcal{D}x_{\alpha\dot\alpha}=f^2\bar f^2\Pi^{\alpha\dot\alpha}\Pi_{\alpha\dot\alpha}\,, \tag{3.22}$$

and it is manifestly conformal to the Poincaré SUSY invariant interval, in full correspondence with the derivation based on the Maurer-Cartan equations. The above chiral factors coincide with the factors entering the superfield Weyl transformations defined in [24, 37].

To establish a contact with the consideration in section 2 , one can check that the chiral multipliers $f(x_L,\theta),\bar f(x_R,\bar\theta)$ of the superconformal factor obey the equation

$$\bar D_{\dot\alpha}\bar D^{\dot\alpha}\bar f^2=4mf^4$$

and its conjugate (in the left-chiral parametrization $D_\alpha=(\partial/\partial\theta^\alpha)+2i\bar\theta^{\dot\beta}\partial^L_{\alpha\dot\beta}$, $\bar D_{\dot\alpha}=-\partial/\partial\bar\theta^{\dot\alpha}$). One can reduce this equation just to (2.17) by making a proper complex rescaling of $f,\bar f$. Also, it is straightforward to check the relation

$$f\,\partial^L_{\alpha\dot\beta}\bar f-\bar f\,\partial^L_{\alpha\dot\beta}f=iD_\alpha f\,\bar D_{\dot\beta}\bar f\,,$$

which amounts to eq. (2.19). Thus the "bottom-up" method directly yields the precise form of the required particular Lorentz invariant solution of (2.17), (2.19). Note that the linear eqs.(3.15) (or their analogs for f or $\bar f$) are obviously easier to SOLVE than the non-linear eqs. (2.17) and (2.19).

3.4 $N = 2,\ D = 4\ AdS$ superspace

In this case it is more convenient to work directly in the chiral basis

$$x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + 2i\theta_i^\alpha\bar\theta^{\dot\alpha i}, \qquad \theta_i^\alpha,\ \bar\theta^{\dot\alpha i}\,.$$

Under the special conformal and Poincaré $N = 2$ supersymmetries these coordinates transform as follows:

$$\begin{aligned}
\delta_\eta x_L^{\alpha\dot\alpha} &= -4ix_L^{\beta\dot\alpha}\eta^i_\beta\theta_i^\alpha\,,\\
\delta_\eta\theta_i^\alpha &= -2i\eta_{\beta i}\theta_k^\beta\theta^{\alpha k} - 2i\eta_k^\alpha\theta^\beta_{(i}\theta^{k)}_\beta + \bar\eta_{\dot\beta i}x_L^{\dot\beta\alpha}\,, \quad \delta_\eta\bar\theta^{\dot\alpha i} = \overline{(\delta_\eta\theta_i^\alpha)}\,, \qquad (3.23)\\
\delta_\epsilon x_L^{\alpha\dot\alpha} &= -4i\bar\epsilon^{\dot\alpha i}\theta_i^\alpha\,, \quad \delta_\epsilon\theta_i^\alpha = \epsilon_i^\alpha\,, \quad \delta_\epsilon\bar\theta^{\dot\alpha i} = \bar\epsilon^{\dot\alpha i}\,, \qquad (3.24)\\
\bar\epsilon^{\dot\alpha i} &= \overline{(\epsilon_i^\alpha)}\,, \quad \bar\eta_i^{\dot\alpha} = \overline{(\eta^{\alpha i})}\,.
\end{aligned}$$

Up to induced Lorentz rotations, the flat covariant differential

$$\Pi^{\alpha\dot\alpha} = dx^{\alpha\dot\alpha} - 2i\left(d\theta_i^\alpha\bar\theta^{\dot\alpha i} + d\bar\theta^{\dot\alpha i}\theta_i^\alpha\right) \qquad (3.25)$$

is transformed as

$$\delta_\eta\Pi^{\alpha\dot\alpha} = 2i(\eta^{\rho i}\theta_{\rho i})\,\Pi^{\alpha\dot\alpha} + \text{c.c.}\,. \qquad (3.26)$$

A new feature compared to the previous case is that the $N = 2\ AdS_4$ supersymmetry is extracted via the following identification

$$\delta_{AdS} = \delta_\epsilon + \delta_\eta\,, \qquad \eta^i_\rho = c^{(ik)}\epsilon_{\rho k}\,, \quad \bar\eta^i_{\dot\rho} = -c^{(ik)}\bar\epsilon_{\dot\rho k}\,, \qquad \overline{(c^{(ik)})} = \epsilon_{il}\epsilon_{ks}c^{(ls)}\,, \quad (3.27)$$

where a constant real vector $c^{(ik)}$ breaks the internal symmetry subgroup SU(2) of the $N = 2$ superconformal group SU(2,2|2) down to SO(2) which is the internal symmetry subgroup of the $N = 2\ AdS_4$ supergroup OSp(2|4; R).

Thus, under the $N = 2\ AdS$ supersymmetry the flat supercovariant differential $\Pi^{\alpha\dot\alpha}$ is transformed (modulo induced Lorentz rotations) as

$$\delta_{AdS}\Pi^{\alpha\dot\alpha} = -2ic^{(ik)}\epsilon_{\rho k}\theta_i^\rho\,\Pi^{\alpha\dot\alpha} + \text{c.c.}\,. \qquad (3.28)$$

In order to find the compensating scalar factor, we assume that, similar to the previous case, it is factorized into the product of chiral and anti-chiral (conjugate) factors. So we need to find a complex factor $B(x_L, \theta)$, such that it transforms under the $N = 2\ AdS$ supertranslations as follows:

$$\delta_{AdS}B = 2ic^{(ik)}\epsilon_{\rho k}\theta_i^\rho\,B\,. \qquad (3.29)$$

The general θ expansion of B consistent with the Lorentz and SO(2) invariances reads

$$B = b_0(y) + \theta^{\alpha(i}\theta_\alpha^{k)}c_{ik}b_1(y) + \theta^{\alpha(i}\theta_\alpha^{k)}\theta^\beta_{(i}\theta_{\beta k)}b_2(y)\,, \quad y \equiv x_L^2\,. \qquad (3.30)$$

As in the previous example, the transformation rule (3.29) amounts to a set of the first-order linear differential equations for the coefficients in (3.30)

$$\begin{aligned}
&(\text{a})\ 2i\,yb_0' + ib_0 + b_1 = 0\,,\\
&(\text{b})\ 8ib_0' - c^2b_1 = 0\,,\\
&(\text{c})\ 4ib_1' - 3b_2 = 0\,,\\
&(\text{d})\ ic^2b_1 + \frac{2i}{3}c^2yb_1' + 2b_2 = 0\,, \qquad (3.31)
\end{aligned}$$

where $c^2 = c^{ik}c_{ik}$. The following identities are useful while extracting the independent structures in (3.29) in the course of deriving (3.31):

$$c^{mn}\epsilon^\alpha_n\theta_{\alpha m}\phi = \frac{1}{3}c^2\epsilon^\alpha_n\theta_{\alpha m}\phi^{(mn)}\,, \quad \phi \equiv c^{ik}\theta^\beta_i\theta_{\beta k}\,,\ \phi^{(mn)} \equiv \theta^{\beta m}\theta^n_\beta\,,$$
$$\phi\phi = \frac{1}{3}c^2\phi^{(ik)}\phi_{(ik)}\,.$$

Equations (3.31a), (3.31b), (3.31c) allow one to find b_0, b_1 and b_2:

$$b_0 = \frac{1}{\sqrt{1+(1/4)c^2y}}\,, \quad b_1 = -i\frac{1}{(1+(1/4)c^2y)^{3/2}}\,, \quad b_2 = -\frac{1}{2}\frac{c^2}{(1+(1/4)c^2y)^{5/2}}\,. \tag{3.32}$$

Equation (3.31d) is then satisfied identically.

The final answer for B is

$$\begin{aligned} B &= \frac{1}{\sqrt{1+(1/4)c^2x_L^2}}\left[1 - i\phi\,\frac{1}{(1+(1/4)c^2x_L^2)} - \frac{1}{2}\phi^{ik}\phi_{ik}\frac{c^2}{(1+(1/4)c^2x_L^2)^2}\right] \\ &= \frac{1}{\sqrt{1+(1/4)c^2x_L^2}}\left[1 - i\phi\,\frac{1}{(1+(1/4)c^2x_L^2)} - \frac{3}{2}\phi^2\frac{1}{(1+(1/4)c^2x_L^2)^2}\right]. \end{aligned} \tag{3.33}$$

The $N=2$ AdS-covariant differential reads

$$\mathcal{D}x^{\alpha\dot\alpha} = B\bar{B}\,\Pi^{\alpha\dot\alpha} \tag{3.34}$$

and the invariant interval is

$$ds^2 = B^2\bar{B}^2\,\Pi^{\alpha\dot\alpha}\Pi_{\alpha\dot\alpha}\,. \tag{3.35}$$

3.5 The case of arbitrary N

The above construction works for any N in $D=4$. In the generic case of arbitrary N the special conformal and Poincaré supersymmetry transformations (both embedded into $\mathrm{SU}(2,2|N)$) of the coordinates of the N-extended Poincaré superspace in the left-chiral parametrization are given by [23]

$$\begin{aligned} \delta_\eta x^{\alpha\dot\alpha}_L &= -4ix^{\beta\dot\alpha}_L\eta^i_\beta\theta^\alpha_i\,, \\ \delta_\eta\theta^\alpha_i &= 4i\eta^k_\beta\theta^\beta_i\theta^\alpha_k + \bar\eta_{\dot\beta i}x^{\dot\beta\alpha}_L\,, \quad \delta_\eta\bar\theta^{\dot\alpha i} = \overline{(\delta_\eta\theta^\alpha_i)}\,, \qquad (3.36) \\ \delta_\epsilon x^{\alpha\dot\alpha}_L &= -4i\bar\epsilon^{\dot\alpha i}\theta^\alpha_i\,, \quad \delta_\epsilon\theta^\alpha_i = \epsilon^\alpha_i\,, \quad \delta_\epsilon\bar\theta^{\dot\alpha i} = \bar\epsilon^{\dot\alpha i}\,, \qquad (3.37) \\ \bar\epsilon^{\dot\alpha i} &= \overline{(\epsilon^\alpha_i)}\,, \quad \bar\eta^{\dot\alpha}_i = \overline{(\eta^{\alpha i})}\,. \end{aligned}$$

Now the indices $i,k,\ldots$ run from 1 to N (they correspond to the fundamental representation of $\mathrm{SU}(N)$) and the objects with the upper-case and lower-case indices are no longer equivalent to each other (as distinct from the special $N=2$ case, no analog of ϵ_{ik} exists). The super Poincaré invariant differential is still given by eq. (3.25).

The super AdS_4 subgroup $\mathrm{OSp}(N|4;R)$ is singled out from $\mathrm{SU}(2,2|N)$ via the following identification:

$$\delta_{AdS} = \delta_\epsilon + \delta_\eta\,, \qquad \eta^i_\alpha = C^{(ik)}\epsilon_{\alpha k}\,,$$

where $C^{(ik)}$ is a constant symmetric tensor which breaks SU(N) down to SO(N). Using the SU(N)/SO(N) freedom, one can always bring $C^{(ik)}$ to the diagonal form, $C^{(ik)} = const_i \times \delta^{ik}$. We shall use the normalization

$$C^{(ik)}\bar{C}_{(ij)} = \frac{1}{N}\delta^k_j c^2 \,. \tag{3.38}$$

It is easy to check that in the generic case the differential $\Pi^{\alpha\dot{\alpha}}$ is transformed under the η transformations, modulo induced Lorentz rotations, by the same law (3.28), now with $i = 1, \ldots, N$. Thus we should now try to construct general left-chiral superfunction $B(x_L, \theta)$ with the transformation law

$$\delta_{AdS}B = 2iC^{(ik)}\epsilon_{\rho k}\theta^\rho_i \, B \,. \tag{3.39}$$

Let us take the following general ansatz for B:

$$B = b_0(y) + \phi b_1(y) + \phi^2 b_2(y) + \cdots + \phi^n b_n(y) + \cdots + \phi^N b_N(y) \,, \tag{3.40}$$

where

$$\phi = C^{(ik)}\theta^\alpha_i\theta_{\alpha k} \,, \quad y = x^2_L \,.$$

The latter invariants have the following transformation properties under the AdS supertranslations:

$$\begin{aligned}
\delta_{AdS}\phi &= 4iC^{(ik)}\epsilon^\alpha_i\theta_{\alpha k}\phi + \frac{2}{N}c^2\bar{\epsilon}^i_{\dot{\alpha}}\theta_{\alpha i}x^{\dot{\alpha}\alpha}_L + 2C^{(ik)}\epsilon^\alpha_i\theta_{\alpha k} \,, \\
\delta_{AdS}x^2_L &= -8i\bar{\epsilon}^{\dot{\alpha}i}\theta^\alpha_i x_{L\alpha\dot{\alpha}} + 4ix^2_L C^{(ik)}\epsilon^\alpha_i\theta_{\alpha k} \,.
\end{aligned} \tag{3.41}$$

A thorough inspection of the conditions imposed on the coefficients in the expansion (3.40) by requiring B to transform according to (3.39) shows that one always gets two independent equations for each two consecutive coefficients, e.g. for b_0 and b_1, b_1 and b_2, b_2 and b_3, etc. For each pair these equations (obtained by putting to zero the coefficients of ϵ and $\bar{\epsilon}$ in the appropriate variations) form a closed set and determine the relevant coefficients up to integration constants, being different for different pairs. Since each coefficient (except for b_0) appears within two adjacent pairs, there arise relations between these integration constants which finally fix B up to an overall constant which we choose, as in the previous particular cases, equal to 1.

Leaving the detailed calculations for an inquisitive reader (they are much like to those in the previous subsections), let us give the final surprisingly simple answer for $B(x_L, \theta)$:

$$B = \frac{1}{\sqrt{1 + \frac{c^2}{2N}x^2_L + 2i\phi}} \,. \tag{3.42}$$

Its expansion in powers of $\phi = C^{(ik)}\theta^\alpha_i\theta_{\alpha k}$ automatically terminates at ϕ^N due to the evident Grassmann property $\phi^{N+1} = 0$. It can be checked that at $N = 1, 2$ eq. (3.42) reproduces the chiral superconformal factors obtained in previous subsections.

The AdS-covariant differential $\mathcal{D}x^{\alpha\dot{\alpha}}$ and the invariant interval are defined in the same way as above, i.e. by eqs. (3.34) and (3.35).

4. $AdS_2 \times S^2$ superspace

4.1 Bosonic case

The manifold $AdS_2 \times S^2$ is a coset $(\mathrm{SO}(2,1)/\,\mathrm{SO}(1,1)) \times (\mathrm{SO}(3)/\,\mathrm{SO}(2))$. The mutually commuting sets of the $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3)$ generators are singled out in the set of generators of the M_4 conformal group $\mathrm{SO}(2,4)$ in the following way (in the notation of (subsection 3.2)

$$\begin{aligned} \mathrm{SO}(2,1):& \quad \left(P_{a'} + m^2 K_{a'} \ , \ M_{a'b'}\right), \\ \mathrm{SO}(3):& \quad \left(P_{\hat a} - m^2 K_{\hat a} \ , \ M_{\hat a \hat b}\right), \quad a' = 0,3\,, \eta^{a'b'} = \mathrm{diag}(1,-1)\,, \quad \hat a = 1,2\,. \end{aligned} \tag{4.1}$$

This choice is unique up to an $\mathrm{SO}(3)$ rotation of the space-like indices $1,2,3$ and it is most convenient for supersymmetrizing. It is easy to check that these two sets of the $\mathrm{SO}(2,4)$ generators indeed commute with each other; the first one comes from the AdS_4 subgroup $\mathrm{SO}(2,3)$ while the second one from the dS_4 subgroup $\mathrm{SO}(1,4)$. It is important to note that the commutativity is possible only with the same contraction parameter m^2 in both the AdS_2 and S^2 translation generators, which amounts to the equal radii of the AdS_2 and S^2 from the geometric point of view.

On the original Minkowskian set of coordinates $x^m \equiv (x^{a'}\ , y^{\hat a})$ the translation generators $P_{a'} + m^2 K_{a'}$ and $P_{\hat a} - m^2 K_{\hat a}$ of these two subgroups act in the following way (we denote the relevant variations by the indices 1 and 2, respectively, and the parameters by $a^{a'}$ and $c^{\hat a}$)

$$\begin{aligned} \delta_1 x^{a'} &= a^{a'}\left[1 - m^2\left(x^2 - y^2\right)\right] + 2m^2 (a\cdot x) x^{a'}\,, \quad \delta_1 y^{\hat a} = 2m^2 (a\cdot x) y^{\hat a}\,, \\ \delta_2 y^{\hat a} &= c^{\hat a}\left[1 + m^2\left(x^2 - y^2\right)\right] + 2m^2 (c\cdot y) y^{\hat a}\,, \quad \delta_2 x^{a'} = 2m^2 (c\cdot y) x^{a'}\,. \end{aligned} \tag{4.2}$$

Hereafter, $x^2 \equiv (x^0)^2 - (x^3)^2$, $y^2 \equiv (y^1)^2 + (y^2)^2$ and analogously for $(a\cdot x)$ and $(c\cdot y)$.

We see that $x^{a'}$ and $y^{\hat a}$ do not form closed sets under these two commuting groups. The irreducible sets $z^{a'}$ and $t^{\hat a}$ can be defined by the following invertible relations

$$\begin{aligned} z^{a'} &= \frac{2}{A + \sqrt{A^2 + 4m^2 x^2}}\, x^{a'}\,, \quad A = 1 - m^2\left(x^2 - y^2\right), \\ t^{\hat a} &= \frac{2}{B + \sqrt{B^2 + 4m^2 y^2}}\, y^{\hat a}\,, \quad B = 1 + m^2\left(x^2 - y^2\right), \end{aligned} \tag{4.3}$$

$$x^{a'} = \frac{1 + m^2 t^2}{1 - m^2 z^2 t^2} z^{a'}\,, \quad y^{\hat a} = \frac{1 + m^2 z^2}{1 - m^2 z^2 t^2} t^{\hat a}\,, \tag{4.4}$$

$$\begin{aligned} \delta_1 z^{a'} &= a^{a'}\left(1 - m^2 z^2\right) + 2m^2 (a\cdot z) z^{a'}\,, \quad \delta_1 t^{\hat a} = 0\,, \\ \delta_2 t^{\hat a} &= c^{\hat a}\left(1 - m^2 t^2\right) + 2m^2 (c\cdot t) t^{\hat a}\,, \quad \delta_2 z^{a'} = 0\,. \end{aligned} \tag{4.5}$$

The covariant differentials of $z^{a'}$ and $t^{\hat a}$ can now be found by the method of subsection 3.2:

$$\mathcal{D} z^{a'} = \frac{1}{1 + m^2 z^2}\, dz^{a'}\,, \qquad \mathcal{D} t^{\hat a} = \frac{1}{1 + m^2 t^2}\, dt^{\hat a}\,. \tag{4.6}$$

They merely undergo induced $\mathrm{SO}(1,1)$ and $\mathrm{SO}(2)$ rotations in their indices, so

$$\mathcal{D} z^{a'} \mathcal{D} z_{a'} = \frac{1}{(1 + m^2 z^2)^2} dz^{a'} dz_{a'}\,,$$

and

$$\mathcal{D}t^{\hat{a}}\mathcal{D}t^{\hat{a}} = \frac{1}{(1+m^2t^2)^2}dt^{\hat{a}}dt^{\hat{a}}$$

are the corresponding invariant intervals.

Now, using the relations

$$1+m^2z^2 = \frac{2\sqrt{A^2+4m^2x^2}}{A+\sqrt{A^2+4m^2x^2}}\,, \qquad 1+m^2t^2 = \frac{2\sqrt{B^2+4m^2y^2}}{B+\sqrt{B^2+4m^2y^2}}\,,$$

and

$$A^2+4m^2x^2 = B^2+4m^2y^2 \equiv F^2(x^2,y^2) = 1+2m^2(x^2+y^2)+m^4(x^4+y^4-2x^2y^2)\,, \quad (4.7)$$

it is straightforward to show that

$$\mathcal{D}z^{a'}\mathcal{D}z_{a'} - \mathcal{D}t^{\hat{a}}\mathcal{D}t^{\hat{a}} = F^{-2}(x^2,y^2)\left(dx^2-dy^2\right) = F^{-2}\,dx^m dx_m\,. \quad (4.8)$$

This relation demonstrates in which sense $AdS_2 \times S^2$ is conformal to the Minkowski space M_4.

Another way to derive (4.8) (which directly applies to the supersymmetry case) does not require passing to the coordinates $\tilde{x}^m = (z^{a'}, t^{\hat{a}})$. It is as follows. One first examines how the differentials dx^m are transformed under (4.2)

$$\begin{aligned}
\delta_1 dx^{a'} &= 2m^2(a\cdot x)dx^{a'} - 2m^2(a^{a'}x^{b'} - x^{a'}a^{b'})dx_{b'} + 2m^2a^{a'}y^{\hat{a}}dy^{\hat{a}}\,, \\
\delta_1 dy^{\hat{a}} &= 2m^2(a\cdot x)dy^{\hat{a}} + 2m^2y^{\hat{a}}x^{a'}dx_{a'}\,, \\
\delta_2 dy^{\hat{a}} &= 2m^2(c\cdot y)dy^{\hat{a}} - 2m^2(c^{\hat{a}}y^{\hat{b}} - y^{\hat{a}}c^{\hat{b}})dy^{\hat{b}} + 2m^2c^{\hat{a}}x^{a'}dx_{a'}\,, \\
\delta_2 dx^{a'} &= 2m^2(c\cdot y)dx^{a'} + 2m^2x^{a'}c^{\hat{a}}dy^{\hat{a}}\,.
\end{aligned} \quad (4.9)$$

One observes that, similar to other examples, dx^m undergoes an x^m-dependent SO$(1,3)$ transformation and an x^m-dependent rescaling

$$\delta dx^m = 2m^2\left[(a\cdot x)+(c\cdot y)\right]dx^m + L^{[mn]}_{(ind)}dx_n\,. \quad (4.10)$$

Then one constructs a “semi-covariant” differential $\mathcal{D}x^m$ which undergoes only the induced Lorentz rotation

$$\mathcal{D}x^m = f(x^2,y^2)dx^m\,, \quad \delta f = -2m^2[(a\cdot x)+(c\cdot y)]f\,. \quad (4.11)$$

One gets a simple differential equation for f, which is solved, up to an overall integration constant, by

$$f = F^{-1} \equiv \left[1+2m^2(x^2+y^2)+m^4(x^4+y^4-2x^2y^2)\right]^{-\frac{1}{2}}\,, \quad (4.12)$$

where F is the quantity defined in (4.7). Thus we once again come to the conformal flatness relation (4.8).

In conclusion of this subsection, let us notice that all above formulas are equally valid for an arbitrary D-dimensional Minkowski space M_D. They establish the conformal flatness, in the above sense, of the product spaces $AdS_m \times S^n = (z^{a'}, t^{\hat{a}})$, $m+n=D$, with $a' = 0, 1, \ldots m-1$, $\hat{a} = m, \ldots D-1$. The property that AdS_D is conformal to M_D is a corollary of this general statement. The necessary condition of the conformal flatness is the equality of the AdS_m and S^n radii. It is automatically satisfied when the $AdS_m \times S^n$ isometry group is embedded into the conformal group of M_D.

4.2 Supersymmetrization

Let us now pass to the supersymmetric case. We shall consider only the four-dimensional case. First of all, we should identify the $D=4$ superconformal group, which might contain as a subgroup the isometry supergroup $\mathrm{SU}(1,1|2)$ or $\mathrm{OSp}(4^*|2)$ of the $AdS_2 \times S^2$ superspaces considered in subsection 2.2. It cannot be the $N=1$, $D=4$ superconformal group $\mathrm{SU}(2,2|1)$, since $\mathrm{SU}(1,1|2)$, $\mathrm{OSp}(4^*|2)$ and $\mathrm{SU}(2,2|1)$ have the same number (eight) of spinor generators, and they are obviously different.

The simplest $D=4$ superconformal algebra which may contain $\mathrm{su}(1,1|2)$ or $\mathrm{osp}(4^*|2)$ is the $N=2$ superconformal algebra $\mathrm{su}(2,2|2)$, its non-vanishing (anti)commutation relations relevant to our study being

$$
\begin{aligned}
\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}k}\} &= 2\delta^i_k \sigma^m_{\alpha\dot{\alpha}} P_m\,, \quad \{S_{\alpha k}, \bar{S}^i_{\dot{\alpha}}\} = 2\delta^i_k \sigma^m_{\alpha\dot{\alpha}} K_m\,, \\
\{Q^i_\alpha, S^{\beta k}\} &= \epsilon^{ik}(\sigma^{mn})^\beta_\alpha M_{mn} + 2i\epsilon^{ik}\delta^\beta_\alpha (D+iR) - 4i\delta^\beta_\alpha T^{(ik)}\,, \quad \text{and c.c.}\,, \qquad (4.13) \\
\left[K_m, Q^i_\alpha\right] &= (\sigma_m)_{\alpha\dot{\alpha}} \bar{S}^{\dot{\alpha}i}\,, \quad [P_m, S_{\alpha i}] = (\sigma_m)_{\alpha\dot{\alpha}} \bar{Q}^{\dot{\alpha}}_i\,, \qquad (4.14) \\
[M_{mn}, P_s] &= i(\eta_{ns}P_m - \eta_{ms}P_n)\,, \quad \left[M_{mn}, Q^i_\alpha\right] = -\frac{1}{2}(\sigma_{mn})^\beta_\alpha Q^i_\beta\,, \qquad (4.15) \\
\left[T^{ij}, T^{lk}\right] &= -i(\epsilon^{il}T^{jk} + \epsilon^{jk}T^{il})\,. \qquad (4.16)
\end{aligned}
$$

The conjugation rules are as follows: $\overline{Q^i_\alpha} = \bar{Q}_{\dot{\alpha}i}$, $\overline{S_{\alpha i}} = \bar{S}^i_{\dot{\alpha}}$.

Now we shall show that $\mathrm{osp}(4^*|2)$ is a subalgebra of $\mathrm{su}(2,2|2)$, while no an appropriate subalgebra $\mathrm{su}(1,1|2)$ can be found. It is straightforward to check that the anticommutators of the following generators

$$
\begin{aligned}
\hat{Q}^i_1 &= Q^i_1 + m\,S^i_1\,, \quad \bar{\hat{Q}}_{\dot{1}i} = \bar{Q}_{\dot{1}i} - m\,\bar{S}_{\dot{1}i}\,, \\
\hat{Q}^i_2 &= Q^i_2 - m\,S^i_2\,, \quad \bar{\hat{Q}}_{\dot{2}i} = \bar{Q}_{\dot{2}i} + m\,\bar{S}_{\dot{2}i} \qquad (4.17)
\end{aligned}
$$

produce just the generators (4.1)

$$
\begin{aligned}
\{\hat{Q}^i_1, \bar{\hat{Q}}_{\dot{1}k}\} &= 2\delta^i_k \left[(P_0+P_3) + m^2(K_0+K_3)\right], \\
\{\hat{Q}^i_2, \bar{\hat{Q}}_{\dot{2}k}\} &= 2\delta^i_k \left[(P_0-P_3) + m^2(K_0-K_3)\right], \\
\{\hat{Q}^i_1, \bar{\hat{Q}}_{\dot{2}k}\} &= 2\delta^i_k \left[(P_1-iP_2) - m^2(K_1-iK_2)\right], \text{ and c.c.}\,, \\
\{\hat{Q}^i_1, \hat{Q}^k_1\} &= \{\hat{Q}^i_2, \hat{Q}^k_2\} = 0, \ \{\hat{Q}^i_1, \hat{Q}^k_2\} \\
&= -4im\left[\epsilon^{ik}(M_{03}+iM_{12}) + 2T^{(ik)}\right], \text{ and c.c.}\,. \qquad (4.18)
\end{aligned}
$$

For establishing relation with the description of the $AdS_2 \times S^2$ cosets in subsection 2.2, it is instructive to rewrite (4.18) in a $D = 4$ Lorentz "covariant" fashion

$$\begin{aligned}
\{\hat{Q}^i_\alpha, \bar{\hat{Q}}_{\dot{\alpha}k}\} &= 2\delta^i_k \left[\sigma^{a'}_{\alpha\dot{\alpha}}(P_{a'} + m^2 K_{a'}) + \sigma^{\hat{a}}_{\alpha\dot{\alpha}}(P_{\hat{a}} - m^2 K_{\hat{a}})\right], \\
\{\hat{Q}^i_\alpha, \hat{Q}^k_\beta\} &= -4im\left[\epsilon_{\alpha\beta}\epsilon^{ik}\,(M_{03} + iM_{12}) + 2(1-i)f_{\alpha\beta}T^{(ik)}\right], \text{ and c.c.}, \qquad (4.19)
\end{aligned}$$

where $a' = 0, 3$ and $\hat{a} = 1, 2$ are the AdS_2 and S^2 vector indices, respectively.

The commutators of the spinor charges with the generators of SO$(2,1)$ and SO(3) are

$$\begin{aligned}
\left[P_{a'} + m^2\, K_{a'}, \hat{Q}^i_\alpha\right] &= m\,\epsilon_{a'b'}\sigma^{b'}_{\alpha\dot{\alpha}}\bar{\hat{Q}}^{i\dot{\alpha}}\,, \qquad \left[P_{\hat{a}} - m^2\, K_{\hat{a}}, \hat{Q}^i_\alpha\right] = mi\,\epsilon_{\hat{a}\hat{b}}\sigma^{\hat{b}}_{\alpha\dot{\alpha}}\bar{\hat{Q}}^{i\dot{\alpha}}\,, \\
\left[M_{03}, \hat{Q}^i_\alpha\right] &= \sigma^{3\ \beta}_{\alpha}\hat{Q}^i_\beta, \qquad \left[M_{12}, \hat{Q}^i_\alpha\right] = -i\sigma^{3\ \beta}_{\alpha}\hat{Q}^i_\beta, \quad \text{and c.c.}. \qquad (4.20)
\end{aligned}$$

Note that in addition to the generators of so$(2,1)$ and so(3) the right hand side of (4.19) also contains su(2) generators $T^{(ik)}$. The subalgebra of su$(2,2|2)$ defined by (4.17) and (4.19) is a superalgebra osp$(4^*|2)$. The bosonic sector of this subalgebra is so$^*(4) \oplus$ $usp(2) \sim$ so$(1,2) \oplus$ so$(3) \oplus$ su(2). The group corresponding to the latter su(2) acts only on the fermionic coordinates of the relevant superspace (as in the $N = 2$, $D = 4$ Minkowski superspace), it commutes with special conformal transformations and, hence, does not affect the geometry of the bosonic manifold which is always $AdS_2 \times S^2$. We thus conclude that the $AdS_2 \times S^2$ superspace can be realized as a supercoset OSp$(4^*|2)/($SO$(1,1) \times$SO$(2) \times$SU$(2))$. The su$(1,1|2)$ superalgebra is not a subalgebra of osp$(4^*|2)$ and it cannot be obtained from the latter by any contraction. We conclude that no su$(1,1|2)$ subalgebra exists in su$(2,2|2)$, such that its bosonic subalgebra so$(2,1) \oplus$ so(3) lies in the bosonic conformal subalgebra so$(2,4)$ of su$(2,2|2)$.[11]

Let us now present the basic anticommutation relations (4.18) in a form where the so$(2,1) \oplus$ so$(3) \oplus$ su(2) structure is manifest. We introduce

$$Q^{Aii'}\,, \quad \overline{(Q^{Aii'})} = \epsilon_{ik}\epsilon_{i'k'}Q^{Akk'}\,,$$

such that

$$Q^{A=1,i,i'=1} \equiv -i\hat{Q}^i_2\,, \qquad Q^{A=2,i,i'=2} = -\hat{Q}^i_1\,, \qquad Q^{1i2} = i\bar{\hat{Q}}^i_{\dot{2}}, \qquad Q^{2i1} = \bar{\hat{Q}}^i_{\dot{1}}\,, \qquad (4.21)$$

where A, i and i' are spinor indices of SL$(2,R) \sim$ SO$(2,1)$ and of two SU(2) groups, respectively. Then (4.18) can be rewritten in the following concise form

$$\{Q^{Aii'}, Q^{Bkk'}\} = -4m(\,\epsilon^{AB}\epsilon^{ik}T_1^{i'k'} + \epsilon^{ik}\epsilon^{i'k'}T_2^{AB} - 2\epsilon^{AB}\epsilon^{i'k'}T^{ik}\,)\,, \qquad (4.22)$$

where all bosonic generators satisfy the same commutation relations (4.16), T_2^{AB} and $T_1^{i'k'}$

[11]Of course there is an obvious embedding of su$(1,1|2)$ in su$(2,2|2)$ such that their internal su(2) sectors coincide. However, this embedding does not suit our purposes.

being generators of SO(2,1) and SO(3), respectively. They are related to the original ones as

$$
\begin{aligned}
T_1^{11} &= \frac{i}{2m}[(P_1 + iP_2) - m^2(K_1 + iK_2)]\,, \\
T_1^{22} &= (T_1^{11})^\dagger\,,\; T_1^{12} = -iM_{12}\,, \\
T_2^{11} &= -\frac{1}{2m}[(P_0 - P_3) + m^2(K_0 - K_3)]\,, \\
T_2^{22} &= -\frac{1}{2m}[(P_0 + P_3) + m^2(K_0 + K_3)]\,, \\
T_2^{12} &= M_{03}\,.
\end{aligned}
\tag{4.23}
$$

In this notation it is easy to see that osp$(4^*|2)$ is a particular case of a real form of the exceptional Lie superalgebra $D(2,1;\alpha)$. The basic anticommutation relation of the latter [19] can be obtained by replacing the coefficients before the generators T_1 and T in the r.h.s. of (4.22) by numerical parameters α and $-(1+\alpha)$, respectively,

$$
\{Q^{Aii'}, Q^{Bkk'}\} = -4m(\,\alpha\,\epsilon^{AB}\epsilon^{ik}T_1^{i'k'} + \epsilon^{ik}\epsilon^{i'k'}T_2^{AB} - (1+\alpha)\,\epsilon^{AB}\epsilon^{i'k'}T^{ik}\,)\,. \tag{4.24}
$$

The superalgebra osp$(4^*|2)$ is recovered from (4.24) with the choice $\alpha = 1$, while the choices $\alpha = 0$ or $\alpha = -1$ lead to two isomorphic superlagebras, each being a semidirect sum of a superalgebra su$(1,1|2)$ and an external automorphism algebra su(2). The su$(1,1|2)$ superalgebra corresponding to $\alpha = -1$ is obtained from (4.19) by skipping the SU(2) generators T^{ij} and changing the sign in front of the SO(3) generators $P_{\hat a} - m^2 K_{\hat a}$ and M_{12}, the commutation relations (4.20) being unchanged. This form of the su$(1,1|2)$ superalgebra corresponds to the form of the Maurer-Cartan equations analyzed in subsection 2.2.

To learn how the conformal flatness relation (4.8) generalizes to the supersymmetry case, we need, before all, to have the realization of the subalgebra (4.18) on the coordinates of the $N = 2$, $D = 4$ superspace. We shall proceed from the $N = 2$, $D = 4$ superconformal transformations (3.23), (3.24), (3.26) in the left-chiral parametrization. It will be convenient to relabel the coordinates and $N = 2$ supersymmetry parameters as follows:

$$
\begin{aligned}
&x_L^{1\dot 1} \equiv x^{++}\,, \quad x_L^{2\dot 2} \equiv x^{--}\,, \quad x_L^{2\dot 1} \equiv z\,, \quad x_L^{1\dot 2} \equiv \bar z\,, \\
&\theta_i^1 \equiv \theta_i^+\,, \quad \theta_i^2 \equiv \theta_i^-\,, \quad \bar\theta^{\dot 1 i} = \bar\theta^{+i}\,, \quad \bar\theta^{\dot 2 i} \equiv \bar\theta^{-i}\,, \\
&\epsilon_i^1 \equiv \epsilon_i^+\,, \quad \epsilon_i^2 \equiv \epsilon_i^-\,, \quad \bar\epsilon^{\dot 1 i} = \bar\epsilon^{+i}\,, \quad \bar\epsilon^{\dot 2 i} \equiv \bar\epsilon^{-i}\,.
\end{aligned}
\tag{4.25}
$$

The subalgebra (4.17) is singled out by the following identification of the parameters of the special superconformal and Poincaré supersymmetry transformations (this choice properly breaks $D = 4$ Lorentz invariance):

$$
\eta_i^1 = -m\epsilon_i^+\,, \quad \eta_i^2 = m\epsilon_i^-\,, \quad \bar\eta^{\dot 1 i} = m\bar\epsilon^{+i}\,, \quad \bar\eta^{\dot 2 i} = -m\bar\epsilon^{-i}\,. \tag{4.26}
$$

The corresponding transformations are given by

$$
\begin{aligned}
\delta x^{++} &= -4i\bar\epsilon^+\theta^+ - 4im\left[x^{++}(\epsilon^-\theta^+) + z\,(\epsilon^+\theta^+)\right], \\
\delta x^{--} &= -4i\bar\epsilon^-\theta^- - 4im\left[x^{--}(\epsilon^+\theta^-) + \bar z\,(\epsilon^-\theta^-)\right],
\end{aligned}
$$

$$
\begin{aligned}
\delta z &= -4i\bar{\epsilon}^{+}\theta^{-} - 4im\left[x^{++}\left(\epsilon^{-}\theta^{-}\right) + z\left(\epsilon^{+}\theta^{-}\right)\right], \\
\delta \bar{z} &= -4i\bar{\epsilon}^{-}\theta^{+} - 4im\left[x^{--}\left(\epsilon^{+}\theta^{+}\right) + \bar{z}\left(\epsilon^{-}\theta^{+}\right)\right], \\
\delta\theta_i^{+} &= \epsilon_i^{+} + 2im\left[\epsilon_i^{-}\left(\theta^{+}\right)^2 - 2(\epsilon^{+}\theta^{+})\theta_i^{-}\right] - m\left(\bar{\epsilon}_i^{-}x^{++} + \bar{\epsilon}_i^{+}\bar{z}\right), \\
\delta\theta_i^{-} &= \epsilon_i^{-} + 2im\left[\epsilon_i^{+}\left(\theta^{-}\right)^2 - 2(\epsilon^{-}\theta^{-})\theta_i^{+}\right] - m\left(\bar{\epsilon}_i^{+}x^{--} + \bar{\epsilon}_i^{-}z\right)
\end{aligned} \tag{4.27}
$$

(for brevity, we have omitted the chiral index “L” on the coordinates and denoted $a^i b_i \equiv (ab)$). Note an asymmetry in the transformation laws of coordinates are not mutually conjugate in the complex chiral z and $\bar{z}$ which is of course related to the fact that these basis. The transformation of the super Poincaré covariant differential $\Pi^{\alpha\dot{\alpha}}$ under this subgroup, up to an induced Lorentz rotation, is as follows:

$$
\delta\Pi^{\alpha\dot{\alpha}} = -\left[2im\left(\epsilon^{-}\theta^{+} + \epsilon^{+}\theta^{-}\right) + \text{c.c.}\right]\Pi^{\alpha\dot{\alpha}}. \tag{4.28}
$$

As in the bosonic case, in order to construct the “semi-covariant” differential which would undergo only coordinate dependent Lorentz rotations under the action of the $AdS_2 \times S^2$ supergroup, one should construct a density $B(x, \theta, \bar{\theta})$ which compensates the weight factor in (4.28),

$$
\delta B = \left[2im\left(\epsilon^{-}\theta^{+} + \epsilon^{+}\theta^{-}\right) + \text{c.c.}\right] B. \tag{4.29}
$$

In analogy to the previous supersymmetric examples, we assume B to have the product structure

$$
B(x, \theta, \bar{\theta}) = B_L(x_L, \theta) B_R(x_R, \bar{\theta}), \quad \delta B_L = 2im\left(\epsilon^{-}\theta^{+} + \epsilon^{+}\theta^{-}\right) B_L, \; \delta B_R = \overline{\delta B_L}. \tag{4.30}
$$

The most general θ expansion of B_L compatible with the $AdS_2 \times S^2$ isotropy group $\mathrm{SO}(1,1) \times \mathrm{SO}(2)$ is as follows:

$$
\begin{aligned}
B_L &= a(X,Y) + b(X,Y)\theta^{+}\theta^{-} + d(X,Y)x^{--}z(\theta^{+})^2 + c(X,Y)x^{++}\bar{z}(\theta^{-})^2 + \\
&\quad + f(X,Y)(\theta^{+})^2(\theta^{-})^2,
\end{aligned} \tag{4.31}
$$

$$
X \equiv x^{++}x^{--}, \qquad Y = z\bar{z}. \tag{4.32}
$$

Requiring B_L to have the transformation rule (4.30) under (4.27), we find quite a lot of equations for the coefficient functions in (4.31). But only few of them are essential

$$
\begin{aligned}
4i\frac{\partial a}{\partial X} + mb + 2mYd &= 0, \\
4i\frac{\partial a}{\partial Y} + mb + 2mXd &= 0, \\
4im\left(X\frac{\partial a}{\partial X} + Y\frac{\partial a}{\partial Y}\right) - b + 2ima &= 0, \\
4im\left(\frac{\partial a}{\partial X} + \frac{\partial a}{\partial Y}\right) - 2d &= 0, \\
i\frac{\partial b}{\partial X} - 2iY\frac{\partial d}{\partial Y} - 2id - mf &= 0, \\
c &= d,
\end{aligned} \tag{4.33}
$$

while the rest of equations become identities on the solutions of these basic ones (and so serve as self-consistency conditions). It is straight forward to find that (up to an arbitrary overall constant) the general solution of (4.33) is given by

$$
\begin{aligned}
a(X,Y) &= \frac{1}{[1+2m^2(X+Y)+m^4(X-Y)^2]^{1/4}}\,,\\
b(X,Y) &= 2im\frac{1+m^2(X+Y)}{[1+2m^2(X+Y)+m^4(X-Y)^2]^{5/4}}\,,\\
c(X,Y) &= d(X,Y) = -2im^3\,\frac{1}{[1+2m^2(X+Y)+m^4(X-Y)^2]^{5/4}}\,,\\
f(X,Y) &= -\frac{m^2}{[1+2m^2(X+Y)+m^4(X-Y)^2]^{5/4}}\,.
\end{aligned}
\tag{4.34}
$$

Thus

$$\mathcal{D}x^{\alpha\dot\alpha} = B_L B_R \Pi^{\alpha\dot\alpha} \tag{4.35}$$

and the supersymmetric analog of the bosonic conformal flatness relation (4.8) is

$$\mathcal{D}s^2 = \mathcal{D}x^{\alpha\dot\alpha}\mathcal{D}x_{\alpha\dot\alpha} = (B_L)^2(B_R)^2(\Pi^{\alpha\dot\alpha}\Pi_{\alpha\dot\alpha})\,. \tag{4.36}$$

We have thus found the explicit superconformal factors in the expressions for the supervielbeins of the $AdS_2\times S^2$ conformally flat superspace OSp($4^*|2$)/(SO(1,1) $\times$ SO(2) $\times$ SU(2)) discussed in section 2. These factors are defined by eqs. (4.31), (4.32), (4.34) and their conjugate.

5. Applications—an outlook

We shall now demonstrate, using the results obtained, that in the superconformally flat superbackgrounds the classical dynamics of a massless particle is (conformally) equivalent to the classical dynamics of a superparticle in flat superspace.[12]

The classical action of a massless superparticle in a D-dimensional supergravity background parametrized by supercoordinates $z^M = (x^m, \theta^\mu, \bar\theta^{\dot\mu})$ and described by supervielbeins $dz^M\,E_M^A(x,\theta,\bar\theta)$, $A=(a,\alpha,\dot\alpha)$, has the following form

$$S = \frac{1}{2}\int d\tau\,\frac{1}{e(\tau)}\dot z^M\,\dot z^N\,E_M^a\,E_N^b\,\eta_{ab}\,, \tag{5.1}$$

where $\dot z^M = dz^M/d\tau$ and $e(\tau)$ is a lagrange multiplier which insures the mass shell condition $\dot z^M\,E_M^a\,\dot z^N\,E_N^b\eta_{ab}=0$.

The action (5.1) is invariant under worldline reparametrizations and under a local κ-symmetry, provided the supervielbeins satisfy appropriate supergravity constraints (for a review of superparticle models see [25]). In particular, the superbackground can be of the

[12] In [38] the equivalence of the dynamics of a massless bosonic particle propagating in conformally flat backgrounds has been demonstrated in a twistor-like formulation. Using the results of this section one can show that the equivalence established in [38] can be extended to massless AdS $D=4$ superparticles formulated in the supertwistor framework.

$AdS \times S$ types discussed in sections 2, 3 and 4. Then the supervielbeins $dz^M E_M^a$ are as in equations (1.2), (2.4), (2.5) and (2.37), and the action (5.1) takes the form

$$S = \frac{1}{2} \int d\tau \, \frac{e^{4\Phi(z)}}{e(\tau)} \Pi^a \Pi^b \, \eta_{ab} \,. \tag{5.2}$$

Note that the action (5.2) is invariant with respect to the superconformal transformations of the target superspace coordinates, provided the lagrange multiplier $e(\tau)$ (einbein) gets rescaled in an appropriate way. The property of the actions of massless bosonic particles, spinning particles and superparticles to be target-space (super)conformal invariant is actually well known and has been extensively discussed in the literature (see e.g. [26]–[32]). Our observation is that if the superbackground is superconformally flat, the conformal factor can be absorbed into the redefined einbein

$$e'(\tau) = e^{-4\Phi(z)} e(\tau) \,, \tag{5.3}$$

and the action reduces to that of a massless particle in a flat superbackground. We thus conclude that, for instance, the dynamics of massless superparticles propagating on the AdS_4 coset superspace $\mathrm{OSp}(2|4;R)/(\mathrm{SO}(1,3) \times \mathrm{SO}(2))$, on the $AdS_2 \times S^2$ coset superspace $\mathrm{OSp}(4^*|2)/(\mathrm{SO}(1,1) \times \mathrm{SO}(2) \times \mathrm{SU}(2))$ and in flat $N = 2$, $D = 4$ superspace are classically equivalent (at least locally), the superconformal group being $\mathrm{SU}(2,2|2)$ for all three cases. This essentially simplifies the analysis of the superparticle mechanics.

At the quantum level, because of the operator ordering ambiguity problem and of the non-trivial topological structure of the $AdS \times S$ manifolds, these cases should show up differences from the flat one. For instance, the operator ordering for each of these cases should be fixed (at least partially) by the requirement that the quantum constraints ($\sim$ superfield equations) are covariant with respect to the superconformal transformations and, moreover, possess a symmetry associated with the isometry of the specific AdS superbackground.[13] Such a requirement will obviously result in different equations of motion for superfields which describe the first quantized state vectors. In addition, the definition of energy and of the mass of states on AdS-manifolds is subtle (see Duff et al. [2] for a review).

As an example let us consider in more details the operator ordering procedure for the canonical quantization of a bosonic massless particle in flat and AdS spaces.

From the bosonic counterpart of the action (5.2) it follows that, classically, in both cases we have a single first-class constraint $p_m p^m = 0$, p_m being the canonical momentum of the particle. The classical equations of motion of the massless particle in flat space and in AdS can be made equivalent by rescaling the einbein $e(\tau)$. And in this sense the dynamics in flat space and in AdS space are equivalent. What is different is the symmetry of the dynamical systems in flat and AdS spaces, because of different geometrical properties of these backgrounds.

[13]This requirement is well known both in the conventional quantum field theory (see e.g. [33]) and quantum string theory [34]. It is usually applied to justify the consistency of the quantum theory by constructing the generators of all the symmetries in terms of quantum variables (fields) and verifying that they still satisfy the commutation relations characteristic of the symmetries of the classical theory.

When we quantize these systems we must respect these symmetries, and this is a criterion for the choice of operator ordering. Let us consider how it works. We start with the constraint $p_m p^m = 0$, which upon quantization we would like to apply on the wave function and to obtain the field equation of motion.

If we are in flat space, we directly convert $p_m p^m$ into the Klein-Gordon operator by replacing p_m by $-i\partial_m$. In this way we get the correct Klein-Gordon equation in flat space which respects the Poincaré symmetry of the initial classical system.

However, if we are in an AdS background, we cannot proceed in the same way, since $\partial_m \partial^m$ is not invariant under isometries of the AdS space. To obtain the correct AdS-invariant Klein-Gordon equation we should use a different operator ordering procedure based on the insertion of the conformal factor $\exp\,(2(D-2)\phi(x))$. To understand how it works let us start with the result. Upon quantization we should get the Klein-Gordon equation in the form

$$D_m g^{mn} \partial_n V(x) = 0\,,$$

where D_m is the AdS covariant derivative and g^{mn} is the inverse AdS metric. The above equation can be rewritten in the equivalent form as

$$g^{-\frac{1}{2}} \partial_m \left(g^{\frac{1}{2}} g^{mn} \partial_n V(x) \right) = 0\,, \quad g \equiv \det g_{mn}\,. \tag{5.4}$$

Now let $g_{mn} = e^{4\phi(x)} \eta_{mn}$. Then the eq. (5.4) reduces to

$$e^{-2D\phi} \partial_m \left(e^{2(D-2)\phi} \partial^m V(x) \right) = 0\,, \tag{5.5}$$

where D is the dimension of the AdS background.

The form of these equations suggests which kind of the operator ordering procedure we should follow when quantizing the AdS particle. We should rewrite $p_m p^m$ in the following classically equivalent form:

$$\eta^{mn} p_m p_n \equiv p_m p^m = e^{-2(D-2)\phi} p_m e^{2(D-2)\phi} p^m\,. \tag{5.6}$$

Upon quantization we shall impose the first class constraint (5.6) on the state vector $V(x)$ in the coordinate representation ($x^m \to x^m$, $p_m \to -i\partial_m$). Taking the operator ordering of the constraint as in the right hand side of (5.6), we get the AdS-covariant field equation, which is obviously different from the flat space Klein-Gordon equation. This equation differs from (5.5) by the factor $e^{-4\phi}$ which can be restored already at the classical level from the requirement of the invariance (versus covariance) of the constraints with respect to the isometries of the AdS space, $p_m p^m = 0 \to e^{-4\phi} p_m p^m = 0$.

By making a different momentum ordering in (5.6) such that for $D > 2$

$$\begin{aligned} p_m p^m &= e^{-2(D-2)\phi} p_m e^{2(D-2)\phi} p^m - \\ &\quad - 4c\,\frac{D-1}{D-2} \left[e^{-2(D-2)\phi} p_m e^{2(D-2)\phi} p^m - e^{-(D-2)\phi} p_m p^m e^{(D-2)\phi} \right] \end{aligned} \tag{5.7}$$

and for $D = 2$

$$p_m p^m = p_m p^m - c\,\left[e^{-\phi} p_m e^{\phi} p^m - p_m e^{-\phi} p^m e^{\phi} \right], \tag{5.8}$$

one gets a contribution to the Klein-Gordon equation (5.4) proportional to the scalar curvature $\mathcal{R}$ of the conformally flat manifold

$$(D_m g^{mn} \partial_n - c\mathcal{R}) V(x) = 0 \, . \tag{5.9}$$

The term in the square brackets of eqs. (5.7) and (5.8), which classically equals to zero, produces the scalar curvature term $\mathcal{R}$ when acting on $V(x)$ upon quantization. In the case of the AdS spaces with constant $\mathcal{R}$ such a term modifies the mass operator of the fields. The arbitrary constant c can be fixed by requiring eq. (5.9) to be conformally invariant [39], for instance $c = 1/6$ in $D = 4$.

Hence the quantum dynamics of the massless particle in flat space and in AdS space are not equivalent.

The same reasoning applies to the supersymmetric case. Note that at the classical level not only the mass shell condition $p_m p^m = 0$, but also the fermionic constraints $\pi_\alpha - ip_m \gamma^m_{\alpha\beta}\theta^\beta = 0$ (where π_α is the momentum conjugate to θ^α) are equivalent for the massless superparticles (5.2) moving in different superconformally flat backgrounds.

Quantum equivalence of the dynamics of massless (super)particles in different conformally flat (super)backgrounds can be also analyzed in the framework of path integral quantization. In a simpler model of non-relativistic bosonic particles moving in a curved Riemann space with a metric g_{mn} one performs quantization by taking the path integral over the trajectories $x_m(t)$ with a functional measure $(\det g_{mn})^{1/2}$ invariant under target space diffeomorphisms (see e.g. [40, 41]). Such a covariant measure in the coordinate path integral can be derived from the path integral over phase space trajectories by integrating over the momentum paths. For the relativistic superparticle (5.1) moving in a curved superbackground described by the supervierbein E_M^A, because of the presence of constraints, the quantum theory will be defined by a generalized path integral over a Batalin-Fradkin-Vilkovisky-extended phase space. The functional integral over superspace trajectories $z_M(\tau)$ invariant under the target space superdiffeomorphisms should emerge upon doing an integral over the BFV-extended momentum trajectories, which should provide the measure with a covariant factor $\mathrm{Ber}(E_M^A)$. The derivation of this measure is an interesting and still unsolved problem. The above reasoning suggests that the functional integration measure of the quantum supersymmetric theory based on the classical action (5.1) includes $\mathrm{Ber}(E_M^A)$. At the next step, respecting the symmetries of the classical model in each of the superconformally flat backgrounds, one should choose a suitable regularization prescription, e.g. a suitable discretization of superpaths, and operator ordering in the definition of this formal functional integral (for non-supersymmetric case see [42, 43]).

In the case of massive superparticles, superstrings and superbranes the action does not have invariances corresponding to *superconformal* symmetries of target superspace [29]. It depends not only on the vector components of the target space supervielbeins but also on the superform gauge fields (C_1 for massive superparticles, B_2 for superstrings, B_2, C_0, C_2, C_4 for super-Dp-branes with $p = 1, 3$, etc.), whose pull-back enters the Wess-Zumino term. Their field strengths are expressed through bosonic and fermionic supervielbeins (1.2) as a consequence of relevant target space supergravity constraints. It would be of interest to examine whether the superconformally flat structure of the target space supervielbeins may result in a simplification of actions for such objects propagating in AdS supermanifolds.

Superconformal quantum mechanics (of multi-black holes) with a $D(2,1;\alpha)$ (and, in particular, $\mathrm{OSp}(4^*|2)$) superconformal symmetry has been considered in [44] as a generalization of the $\mathrm{SU}(1,1|2)$ superconformal mechanics in the background of a single Reissner-Nordström black hole. We therefore see that the supergroups $D(2,1;\alpha)$ have appeared in physical applications. Hence, it would be of interest to study whether they are relevant to supergravity theories and, in particular, whether coset superspaces $D(2,1;\alpha)/(\mathrm{SO}(1,1)\times \mathrm{SO}(2)\times \mathrm{SU}(2))$ and $(D(2,1;\alpha)\times D(2,1;\alpha))/(\mathrm{SO}(1,2)\times \mathrm{SO}(3)\times \mathrm{SU}(2)\times \mathrm{SU}(2))$ can be recovered as solutions of some N-extended (conformal) supergravities with local $\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times \mathrm{SU}(2)$ R-symmetries.

6. Conclusion

We have analyzed the superconformal structure of a class of supermanifolds with the $AdS\times S$ bosonic body and proposed a recipe for deriving the exact form of the conformal factors of the supervielbeins. In particular, we have demonstrated that the AdS_4 and $AdS_2\times S^2$ superspaces whose superconformal group is $\mathrm{SU}(2,2|2)$ are superconformally flat and, hence, conformally-equivalent to $N=2$, $D=4$ flat superspace. This is also the case for the $AdS_3\times S^3$ superspace associated with the supercoset $(\mathrm{OSp}(4^*|2)\times \mathrm{OSp}(4^*|2))/(\mathrm{SO}(1,2)\times \mathrm{SO}(3)\times \mathrm{SU}(2)\times \mathrm{SU}(2))$ which is conformally equivalent to flat $N=(2,0)$, $D=6$ superspace. However, the conventional $AdS_{D/2}\times S^{D/2}$ superspaces which are maximally supersymmetric solutions of classical $N=2$, $D=4,6,10$ supergravity constraints, are not conformal supermanifolds, since for the reasons explained in sections 2 and 4, the isometry supergroups of these supermanifolds are not subgroups of the superconformal groups in $D=4,6$ and 10 dimensions, respectively. Therefore, these $AdS\times S$ vacuum configurations are not superconformal and the issue of their stability under higher order corrections to the quantum effective action of the complete supersymmetric theory should be revised (as e.g. in [12]).

Let us note that the "pure" N-extended AdS_2 coset superspaces, i.e. the ones without "S-factors" are always superconformal. This is because the isometry supergroups of such supercosets can always be embedded into an appropriate N-extended $D=2$ superconformal group, which is in agreement with the conclusion of [17] about the superconformally flat structure of $D=2$ supergravities.

"Pure" N-extended AdS_3 coset superspaces are superconformal when their isometries can be embedded into the N-extended $D=3$ superconformal group $\mathrm{OSp}(N|4;R)$. Examples are the $N=1$ supercoset $(\mathrm{OSp}(1|2;R)\times \mathrm{SO}(1,2))/\mathrm{SO}(1,2)$, $N=2$ supercosets $(\mathrm{OSp}(1|2;R)\times \mathrm{OSp}(1|2;R))/\mathrm{SO}(1,2)$ and $(\mathrm{OSp}(2|2;R)\times \mathrm{SO}(1,2))/(\mathrm{SO}(1,2)\times \mathrm{SO}(2))\sim (\mathrm{SU}(1,1|1)\times \mathrm{SL}(2;R))/(\mathrm{SL}(2;R)\times \mathrm{U}(1))$, and, in general, the supercosets $(\mathrm{OSp}(N-n|2;R)\times \mathrm{OSp}(n|2;R))/(\mathrm{SO}(1,2)\times \mathrm{SO}(N-n)\times \mathrm{SO}(n))$.

The AdS_5 coset superspace $\mathrm{SU}(2,2|N)/(\mathrm{SO}(1,4)\times \mathrm{U}(N))$ is superconformal only for $N=1$, since only $\mathrm{SU}(2,2|1)$ can be embedded as a subgroup into the unique $D=5$ superconformal group $F(4;2)$, with R-symmetry being $\mathrm{USp}(2)\sim \mathrm{SU}(2)$ [14]. For $D\geq 5$ we have not found AdS_D superspaces, whose isometries could be embedded into the corresponding superconformal group. For instance, the AdS_6 supercoset $F(4;2)/(\mathrm{SO}(1,5)\times\mathrm{USp}(2))$ is not superconformal, since $F(4;2)$ is not a subgroup of the $D=6$ superconformal group $\mathrm{OSp}(8^*|4)$ [14].

One may conjecture that in higher dimensions, and in particular in $D=10$, there exist generalized $AdS_m\times S^n$ superspaces enlarged with tensorial charge coordinates and, possibly, with additional Grassmann-odd coordinates, which can presumably be superconformally flat with respect to generalized superconformal groups. Thus the study of superconformal structure of higher dimensional superspaces brings us to supersymmetric models with tensorial central charge coordinates (see e.g. [15, 35, 45, 46]) which previously have already found various other motivations for their consideration, including exotic BPS configurations which preserve more than one half supersymmetry [36, 38, 47, 48] and the theory of higher spin fields [49]. This point still requires a detailed analysis.

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