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Abstract: The worldvolume field equations of M-branes and D-branes are known to admit p-brane soliton solutions. These solitons are shown to saturate a BPS-type bound on their p-volume tensions, which are expressed in terms of central charges that are expected to appear in the worldvolume supertranslation algebra. The cases we consider include vortices, 'BIons', instantons and dyons (both abelian and non-abelian), and the string boundaries of M-2-branes in the M-5-brane.

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Contents

1 Introduction

In the past few years many supersymmetric field theories have been reformulated as world-volume field theories on branes or on their intersections. One aspect of this reformulation of a supersymmetric field theory is that its 'BPS-saturated' states, some of which correspond to 1/2 supersymmetric classical solutions, acquire a spacetime interpretation as intersections with other branes. The prototype is an electric charge on a Type II D-brane, which acquires an interpretation as the endpoint of a 'fundamental' Type II string [1]. This has its M-theory analogue in the interpretation of a self-dual string in the M-5-brane as the boundary of an M-2-brane [2, 3]. Other examples are vortices on M-2-branes or M-5-branes, which acquire an interpretation as 0-brane or 3-brane intersections with a second M-2-brane or M-5-brane [4]. In all these cases, the 1/2 supersymmetric solutions of the worldvolume field theory of a single D-brane or M-brane have now been found [5, 6, 7, 8].

A remarkable feature of these 'worldvolume solitons' is that, while ostensibly just solutions of some (p+1)-dimensional field theory, they in fact suggest their own 10 or 11 dimensional *spacetime* interpretation. This arises from the fact that the world-volume scalars determine the spacetime embedding. In a recent paper it was pointed out that the spacetime interpretation is already implicit in the central charge structure of the world-volume supersymmetry algebra [9]. An example is the D=6 (2,0) worldvolume supersymmetry algebra of the M-5-brane. Allowing for all possible p-form charges we have [8]

$$\{Q^I_\alpha, Q^J_\beta\} = \Omega^{IJ} P_{[\alpha\beta]} + Y^{[IJ]}_{[\alpha\beta]} + Z^{(IJ)}_{(\alpha\beta)}, \tag{1.1}$$

where $\alpha, \beta = 1, \ldots, 4$ is an index of $SU^*(4) \cong Spin(5,1)$ and $I = 1, \ldots, 4$ is an index of $Sp(2)$, with Ω^{IJ} being its invariant antisymmetric tensor. The Y-charge, satisfying $\Omega_{IJ} Y^{[IJ]} = 0$, is a worldvolume 1-form carried by worldvolume strings and the Z-charge is a worldvolume self-dual 3-form carried by worldvolume 3-branes. The representations of the R-symmetry group $Sp(2)$ encode the possible interpretations of the solitons carrying these charges as intersections of other objects with the M-5-brane [9]. For example, the string charge, being a 5-vector of $Spin(5)$, can be viewed as a 1-form in the 5-space transverse to the M-5-brane worldvolume in spacetime. It therefore defines a direction in this space which may be identified as the direction in which an M-2-brane 'leaves' the M-5-brane, consistent with the interpretation of the worldvolume string as an M-2-brane boundary. Similarly, the 3-brane charge can be viewed as a transverse 2-form, consistent with its spacetime interpretation as the intersection with another M-5-brane.

The results we report on here arose from a consideration of the way in which the magnitudes of the p-form charges carried by worldvolume solitons are expressed as integrals of charge densities constructed from worldvolume fields. As we shall argue, the correct expressions for the magnitudes of the charges carried by the string and 3-brane in the M-5-brane are

$$Y = \int dX \wedge H \qquad Z = \frac{i}{2} \int dU \wedge d\bar{U}\,, \tag{1.2}$$

where $H = dA$ is the worldvolume three form field strength and X and U are, respectively, real and complex worldvolume scalar fields describing fluctuations transverse to the M-5-brane worldvolume in spacetime. The integrals are over the subspaces of the 5-dimensional worldspace transverse to the p-brane solitons. Note that Y is not given simply by an integral of H over a 3-sphere surrounding the string in the M-5-brane, as one might naively have expected. It also includes a dependence on one scalar field, as required by its identification with the magnitude of a 1-form in the space transverse to the M-5-brane.[1] Similarly, the dependence of the 3-brane charge Z on a complex scalar is required by its interpretation as a transverse 2-form.

To establish the correctness of the above expressions, and similar expressions for the magnitudes of charges carried by p-branes in other M-theory and Type II branes, one could explicitly construct the supersymmetry generators as Noether charges and determine their algebra directly.[2] Here we shall take an alternative path by showing that the

[1]The string charge Y also appears in [10], but in the context of the linearised five-brane theory in the light-cone gauge. Note also that the strings considered there are non-self-dual dipole strings with vanishing Y charge.

[2]For M-branes, the result of such a calculation is in principle implied by the results of [11, 12]. One should consider a combination of p-form charges associated to a $\frac{1}{4}$ supersymmetric configuration of intersecting M-branes and then project onto the subspace spanned by the supercharges linearly realized on one brane.

p-volume tensions of worldvolume p-brane solitons are bounded from below by expressions that are precisely of the form (1.2). The method is similar to that employed in [13] but differs in essential respects owing to the fact that brane actions are generally non-quadratic in derivatives. However, the configurations saturating the bound are precisely those satisfying first-order BPS-type equations, the solutions of which are (in this context) the worldvolume solitons found in [5, 6, 7, 8] which are known to preserve 1/2 of the supersymmetry.[3]

In the case of a single D-brane the worldvolume hamiltonian involves the Born-Infeld (BI) $U(1)$ 2-form field strength F. The particles on the D-brane that carry the electric $U(1)$ charges were called 'BIons' in [7], and we shall adopt this terminology here for the supersymmetric solutions. BIons satisfy a BPS-type equation involving F, which we shall call the 'abelian BIon' equation, but there is a natural non-abelian extension as appropriate to multiple coincident D-branes. There is also a natural non-abelian extension of the D-brane hamiltonian, natural in the sense that the energy bound of the abelian case continues to hold but is saturated by solutions of the non-abelian BIon equations. One can simply take this as the definition of the non-abelian BI theory in each case of interest. One then finds, for example, that the monopole and dyon solutions of (3+1)-dimensional supersymmetric non-abelian gauge theories are also solutions of the non-abelian D-3-brane worldvolume equations. This approach is physically compelling and is simpler than previous investigations of the effects of BI 'corrections' on BPS monopoles [15], but it remains to be seen whether our definition of the non-abelian BI theory accords with other definitions e.g. [16].

The worldvolume field equations of M-theory or Type II branes depend on the M-theory or Type II background. The worldvolume solitons found in [5, 6, 7, 8] are 1/2 supersymmetric solutions of the worldvolume field equations in an M-theory or Type II spacetime vacuum, by which we mean D=11 or D=10 Minkowski space with all other space-time fields vanishing. Since an M-theory or Type II brane preserves half the supersymmetry of the spacetime vacuum, its worldvolume supersymmetry algebra has 16 supercharges.[4] Each 1/2 supersymmetric worldvolume soliton must therefore correspond to some p-form charge in one of the supersymmetry algebras with 16 supersymmetries considered in [9]. The converse is not true, however, since some charges correspond to brane boundaries, which are normally determined by imposing boundary conditions rather than by solving field equations. An interesting exception to this is provided by the endpoints of multiple coincident D-strings on D-3-branes, which are determined by a solution of Nahm's equations [17].

Each BPS-type equation can occur as the condition for the saturation of a bound on the tension of more than one worldvolume brane, since many of the latter are related by duality. We shall therefore order the presentation of our results in terms of the type of BPS equation, choosing the simplest case to derive the bound that its solutions saturate.

[3]The solution in [5, 7] of relevance here was shown to be supersymmetric in [6, 14].

[4]The 16 non-linearly realised supersymmetries will not play a direct role in this paper.

We shall also consider them roughly in order of increasing complexity. We conclude with a brief discussion of some unresolved puzzles.

2 Abelian vortices

We shall consider first a 0-brane soliton in the M-2-brane, arising from intersections with other M-2-branes. This is associated with a scalar central charge Z in the M-2-brane's worldvolume supersymmetry algebra. It should be possible to express Z as an integral over the 2-dimensional 'worldspace' of a two-form constructed from the two worldvolume scalars defining the 2-plane of the second M-2-brane. Let U be a complex coordinate for this 2-plane. The (real) charge Z must then take the form

$$Z = \frac{i}{2}\int_{M2} dU \wedge d\bar{U}\,. \tag{2.1}$$

We shall now confirm the relevance of this charge by deriving a bound on the energy of 0-branes within the M-2-brane.

The phase space Lagrangian density for the M-2-brane, in the M-theory vacuum and omitting fermions, is [18]

$$\mathcal{L} = P\cdot\dot{X} - s^a P\cdot\partial_a X - \frac{1}{2}v(P^2 + \det g)\,, \tag{2.2}$$

where all fields depend on the worldvolume coordinates (t,σ^a) $(a=1,2)$. The Lagrange multiplier fields s^a and v impose the 'worldspace' diffeomorphism and hamiltonian constraints, respectively. In the 'physical' or 'static' gauge $(X^0=t, X^a=\sigma^a)$ the diffeomorphism constraint reduces to

$$P_a = \mathbf{P}\cdot\partial_a\mathbf{X}\,, \tag{2.3}$$

where $\mathbf{X}$ are the eight worldvolume scalars describing transverse fluctuations and $\mathbf{P}$ are their conjugate momenta. In this gauge the induced worldspace metric is

$$g_{ab} = \delta_{ab} + \partial_a\mathbf{X}\cdot\partial_b\mathbf{X}\,. \tag{2.4}$$

If we now restrict ourselves to static configurations, for which $\mathbf{P}=\mathbf{0}$, then $P_a=0$ too and so all components of the 11-momentum density P vanish except $P^0=\mathcal{E}$, the energy density. The hamiltonian constraint now reduces to

$$\begin{aligned}\mathcal{E}^2 &= \det g\\ &= 1+|\partial_1\mathbf{X}|^2+|\partial_2\mathbf{X}|^2+|\partial_1\mathbf{X}|^2|\partial_2\mathbf{X}|^2-(\partial_1\mathbf{X}\cdot\partial_2\mathbf{X})^2\,.\end{aligned} \tag{2.5}$$

Since we expect only two scalar fields to be of relevance we shall simplify our task by setting to zero all but two of the eight scalars $\mathbf{X}$, in which case[5]

$$\begin{aligned}\mathcal{E}^2 &= 1+|\vec{\nabla}X|^2+|\vec{\nabla}Y|^2+(\vec{\nabla}X\times\vec{\nabla}Y)^2\\ &= (1\pm\vec{\nabla}X\times\vec{\nabla}Y)^2+|\vec{\nabla}X\mp\star\vec{\nabla}Y|^2\,,\end{aligned} \tag{2.6}$$

[5] Retention of the others six scalars leads to additional positive semidefinite terms which vanish when the six scalars are constants.

where X and Y are the two scalars, and we use standard vector calculus for E^2, with

$$\vec{\nabla} = (\partial_1, \partial_2) \qquad \star\vec{\nabla} = (\partial_2, -\partial_1)\,. \tag{2.7}$$

We deduce that

$$\mathcal{E} - 1 \geq |\vec{\nabla}X \times \vec{\nabla}Y|\,, \tag{2.8}$$

with equality when

$$\vec{\nabla}X = \pm\star\vec{\nabla}Y\,. \tag{2.9}$$

We may define the total energy relative to the worldvolume vacuum as

$$E = \int_{M2} (\mathcal{E} - 1)\,. \tag{2.10}$$

The bound on $\mathcal{E}$ implies the bound[6]

$$E \geq |Z| \tag{2.11}$$

where Z is the topological charge

$$Z = \int_{M2} dX \wedge dY\,. \tag{2.12}$$

This bound will be saturated by solutions of (2.9) if, for these solutions, the charge density is (positive or negative) semi-definite. This condition is satisfied because for solutions of (2.9) the charge density equals $\pm|\vec{\nabla}X|^2$. Thus, the total energy is bounded by the magnitude of the charge Z and is equal to $|Z|$ for solutions of (2.9).

The equations (2.9) are equivalent to the Cauchy-Riemann equations for the complex function $U = X + iY$ of the complex variable $\sigma^1 + i\sigma^2$. In other words, the energy bound is saturated by holomorphic functions U, and the energy is then the magnitude of a charge of the form (2.1), as claimed. Singularities of the holomorphic function U represent 'vortices' on the M-2-brane. These have been discussed in detail in in [5, 7]; we shall call them 'abelian vortices' for reasons that will become apparent later. Essentially the same solution was used in [8] as the M-5-brane worldvolume field configuration representing a 3-brane, which therefore accounts for the form of the 3-brane charge given in (1.2). The fact that the same solution serves in both contexts is to be expected from the equivalence under spacetime duality of the intersecting brane configurations associated with the M-2-brane vortex and the M-5-brane 3-brane. There is, however, an additional feature of the energy bound in the M-5-brane context. One must introduce constant worldvolume vector fields associated with the 3-brane worldvolume directions; one then finds that the saturation of the bound requires the vanishing of the derivatives of all worldvolume fields in these 3-brane directions. This point will be illustrated later with the M-5-brane string soliton so we pass over the details here.

[6] A related bound was discussed in [7].

3 D-brane solitons

The phase-space form of the DBI worldvolume Lagrangian density, in the D=10 spacetime vacuum and omitting fermions, is [19]

$$\begin{aligned}\mathcal{L} = & \; P \cdot \dot{X} + E^a \dot{V}_a + V_t \partial_a E^a - s^a (P \cdot \partial_a X + E^b F_{ab}) \\ & - \frac{1}{2} v [P^2 + E^a E^b g_{ab} + \det(g + F)] \, ,\end{aligned} \tag{3.1}$$

where E^a is the worldspace electric field and F_{ab} the worldspace magnetic 2-form for the BI gauge potential $V = dtV_t + d\sigma^a V_a$. The component V_t imposes the Gauss law constraint on the electric field. The Lagrange multipliers s^a and v impose the worldspace diffeomorphism and hamiltonian constraints, respectively.

In static gauge the s-constraint becomes

$$P_a = -\mathbf{P} \cdot \partial_a \mathbf{X} - E^b F_{ab} \, , \tag{3.2}$$

where $\mathbf{X}$ are the worldvolume scalars transverse to the D-brane and $\mathbf{P}$ their conjugate momenta. In addition, the worldspace metric g_{ab} again reduces to the form given in (2.4). We shall now restrict to static configurations for which $\dot{\mathbf{X}} = 0$ and $\mathbf{P} = 0$. In this case the 10-vector P is

$$P = (\mathcal{E}, -E^b F_{ab}, \mathbf{0}) \, , \tag{3.3}$$

where $\mathcal{E}$ is the energy density, and the hamiltonian constraint yields

$$\mathcal{E}^2 = E^c E^d F_{ac} F_{bd} \delta^{ab} + E^a E^b g_{ab} + \det(g + F) \, . \tag{3.4}$$

We now consider how this formula may be rewritten as a sum of squares, in various special cases.

3.1 BIons

For purely electric configurations on the D-brane worldvolume we set $F_{ab} = 0$ to get

$$\mathcal{E}^2 = E^a E^b g_{ab} + \det g \, . \tag{3.5}$$

We expect purely electric solutions to arise as endpoints of strings, the string specifying a direction in the transverse space with coordinates $\mathbf{X}$. We therefore set all but one of these transverse fluctuations to zero, in which case

$$g_{ab} = \delta_{ab} + \partial_a X \partial_b X \, , \tag{3.6}$$

where X is the one non-zero scalar. Then $\det g = 1 + (\partial X)^2$ and we can rewrite (3.5) as

$$\mathcal{E}^2 = (1 \pm E^a \partial_a X)^2 + (E \mp \partial X)^2 \, . \tag{3.7}$$

We deduce that

$$\mathcal{E} - 1 \geq |E^a \partial_a X| \, , \tag{3.8}$$

with equality when

$$E_a = \pm \partial_a X\,. \tag{3.9}$$

The total energy E relative to the worldvolume vacuum is therefore subject to the bound

$$E \geq |Z_{el}| \tag{3.10}$$

where Z_{el} is the charge

$$Z_{el} = \int_W E^a \partial_a X\,, \tag{3.11}$$

where W is the D-brane worldspace. The form of this charge is expected from the fact that electric scalar charges in the worldvolume supersymmetry algebra are transverse 1-forms (arising from the reduction of the 10-momentum of the N=1 D=10 supersymmetry algebra). The bound (3.10) on the energy is saturated by solutions of (3.9) since the charge density is clearly semi-definite for such solutions.

Because of the Gauss law constraint on electric field, solutions of (3.9) correspond to solutions of $\nabla^2 X = 0$, i.e. to harmonic functions on worldspace. Isolated singularities of X are the charged particle solutions found in [5, 7]. For D-P-branes with $P \geq 3$ the simplest solution is

$$X = \frac{q}{\Omega_{P-1} r^{P-2}}\,, \tag{3.12}$$

corresponding to a charge q at the origin, where Ω_P is the volume of the unit P-sphere. Gauss's law allows us to write the energy as an integral over a (hyper)sphere of radius ϵ surrounding the charge. Since $X = X(\epsilon)$ is then constant over the integration region we have

$$\begin{aligned} E &= \lim_{\epsilon \to 0} |X(\epsilon) \int_{r=\epsilon} d\vec{S} \cdot \vec{E}| \\ &= q \lim_{\epsilon \to 0} X(\epsilon)\,. \end{aligned} \tag{3.13}$$

As pointed out in [5, 7], the energy is infinite since $X \to \infty$ as $\epsilon \to 0$, but the infinity has a physical explanation as the energy of an infinite string of finite, and constant, tension q.

The D-string is a special case of particular interest. In this case[7]

$$Z = \int_{D1} EX' \tag{3.14}$$

Gauss's law implies that the electic field E is locally constant but it must have a discontinuity at the endpoint of the F-string (F for 'Fundamental') because this endpoint carries electric charge. We may suppose that this charge is at the origin $\sigma = 0$ on the D-string and that $E = 0$ when $\sigma < 0$. If we further suppose that $X(0) = 0$ then

$$Z = E \lim_{L \to \infty} X(L)\,, \tag{3.15}$$

[7] To avoid possible confusion with the total energy, we remark that the letter E is reserved exclusively for the electric field in the passage to follow.

where E is the constant value of the electric field for $\sigma > 0$. This is formally infinite because $X(L)$ grows linearly with L, but the infinity again has a physical interpretation as the energy in an infinite string of tension E. Since $E = dX$ we see that the D-string configuration with energy $|Z|$ is

$$X(\sigma) = \begin{cases} 0 & \sigma < 0 \\ E\sigma & \sigma > 0 \end{cases} , \tag{3.16}$$

so that a IIB F-string ending on a D-string produces, literally, a kink in the latter. This point (in the context of the leading terms in the expansion of the BI action) has recently been made independently in an interesting paper [20] which we became aware of while writing up this article.

3.2 BI Instanton

Let us consider the case of the D-4-brane. The worldvolume superalgebra allows scalar central charges in the $\mathbf{1} \oplus \mathbf{5}$ representations of the $Spin(5)$ R-symmetry group, which we interpret as the transverse rotation group. The scalars in the $\mathbf{5}$ representation can be interpreted as the endpoints of fundamental strings on the D-4-brane, which is the D-4-brane subcase of the purely electric case considered above. The $Spin(5)$ singlet scalar is a magnetic charge, so we set the electric field to zero. The magnetic charge on the D-4-brane corresponds to a spacetime configuration in which a D-0-brane 'intersects' a D-4-brane. Because it is a singlet its charge cannot depend on any of the transverse scalars $\mathbf{X}$. We therefore set these to zero. We then have

$$\begin{aligned} \mathcal{E}^2 &= \det(\delta_{ab} + F_{ab}) \\ &= (1 \mp \frac{1}{4}\mathrm{tr}F\tilde{F})^2 - \frac{1}{4}\mathrm{tr}(F \mp \tilde{F})^2 \,, \end{aligned} \tag{3.17}$$

where $\tilde{F}$ is the worldspace Hodge dual of F. The trace is over the 'worldspace' indices, i.e. $\mathrm{tr}F^2 = F_{ab}F^{ba}$, but we can suppose it to include a trace over $u(n)$ indices in the case of n coincident D-4-branes. In either case we deduce that

$$\mathcal{E} \geq 1 \mp \frac{1}{4}\mathrm{tr}F\tilde{F} \,, \tag{3.18}$$

with equality when $F = \pm\tilde{F}$. The total energy E, relative to the worldvolume vacuum is therefore subject to the bound

$$E \geq |Z| \tag{3.19}$$

where Z is the topological charge

$$Z = \frac{1}{4}\int_{D4} \mathrm{tr}F\tilde{F} \,, \tag{3.20}$$

with equality when F satisfies

$$F = \pm\tilde{F} \,. \tag{3.21}$$

In the non-abelian case this is solved by (multi) instanton configurations. In the abelian case, any solution must involve a singular BI gauge potential, but the energy will remain finite as long as the charge is finite.

3.3 BI dyons

Consider now the D-3-brane. In this case, the endpoint of a (p,q) string will appear in the worldvolume as a dyon. We should therefore allow for both magnetic and electric fields, and one non-zero scalar. Thus, we have

$$\mathcal{E}^2 = E^a E^b(\delta_{ab} + \partial_a X \partial_b X) + \det(\delta_{ab} + \partial_a X \partial_b X + F_{ab}) + E^a E^b F_{ac} F_{bd}\, \delta^{cd} . \tag{3.22}$$

Defining

$$B^a = \frac{1}{2}\varepsilon^{abc} F_{bc} \tag{3.23}$$

and expanding the 3×3 determinant, we can rewrite this in standard vector calculus notation as

$$\begin{aligned} \mathcal{E}^2 &= 1 + |\vec{\nabla} X|^2 + |\vec{E}|^2 + |\vec{B}|^2 + (\vec{E}\cdot\vec{\nabla} X)^2 + (\vec{B}\cdot\vec{\nabla} X)^2 + |\vec{E}\times\vec{B}|^2 \\ &= (1 + \sin\vartheta\ \vec{E}\cdot\vec{\nabla} X + \cos\vartheta\ \vec{B}\cdot\vec{\nabla} X)^2 + |\vec{E} - \sin\vartheta\ \vec{\nabla} X|^2 + |\vec{B} - \cos\vartheta\ \vec{\nabla} X|^2 \\ &\quad + |\cos\vartheta\ \vec{E}\cdot\vec{\nabla} X - \sin\vartheta\ \vec{B}\cdot\vec{\nabla} X|^2 + |\vec{E}\times\vec{B}|^2 , \end{aligned} \tag{3.24}$$

where the second equality is valid for arbitrary angle ϑ. We therefore deduce that

$$\mathcal{E}^2 \geq (1 + \sin\vartheta\ \vec{E}\cdot\vec{\nabla} X + \cos\vartheta\ \vec{B}\cdot\vec{\nabla} X)^2 \tag{3.25}$$

for arbitrary ϑ. Taking the square root and then integrating over the worldvolume of the D-3-brane we deduce that the total energy, relative to the D-3-brane vacuum, satisfies the bound

$$E \geq \sqrt{Z_{el}^2 + Z_{mag}^2} , \tag{3.26}$$

where

$$Z_{el} = \int_{D3} \vec{E}\cdot\vec{\nabla} X \qquad Z_{mag} = \int_{D3} \vec{B}\cdot\vec{\nabla} X . \tag{3.27}$$

The bound is saturated when

$$\vec{E} = \sin\vartheta\ \vec{\nabla} X , \qquad \vec{B} = \cos\vartheta \vec{\nabla} X , \tag{3.28}$$

with

$$\tan\vartheta = Z_{el}/Z_{mag} . \tag{3.29}$$

Since both $\vec{E}$ and $\vec{B}$ are divergence free (the latter as a consequence of the Bianchi identity), X must be harmonic, i.e.

$$\nabla^2 X = 0 . \tag{3.30}$$

Given a harmonic function X, the electric and magnetic fields are then determined by (3.28). A BI dyon is then an isolated singularity of X. Again, all these formula have a natural non-abelian extension, but we shall continue to assume a $U(1)$ group for ease of presentation.

For simplicity, let us choose the solution for which

$$\cos\vartheta X = \frac{g}{4\pi r}\,, \tag{3.31}$$

where r is the radial coordinate for E^3. Then the fact that $\vec{E}$ and $\vec{B}$ are divergence free allows us to rewrite each charge as the $\epsilon \to 0$ limit of an integral over a sphere of radius ϵ centred on the origin. Since X is constant over this sphere we have

$$Z_{el} = e \lim_{\epsilon\to 0} X(\epsilon) \qquad Z_{mag} = g \lim_{\epsilon\to 0} X(\epsilon)\,, \tag{3.32}$$

where we have set $e = g\tan\vartheta$. Thus

$$E = \sqrt{e^2+g^2}\,\lim_{\epsilon\to 0} X(\epsilon)\,. \tag{3.33}$$

This is infinite since X increases without bound as $\epsilon \to \infty$, but the infinity has a physical interpretation as the total energy of an infinite string of finite tension

$$T = \sqrt{e^2+g^2}\,, \tag{3.34}$$

as one expects for a (p,q) string emanating from the D-3-brane.

3.4 Non-abelian vortex

The BPS equations for the BI magnetic monopole in the D-3-brane are just the dimensional reduction of the self-duality equations for the BI instanton in the D-4-brane. This follows from the fact that a D-0-brane in a D-4-brane is T-dual to a D-string ending on a D-3-brane. If the starting point is a single D-4-brane, with $U(1)$ gauge potential, then a further dimensional reduction yields the abelian vortex equations discussed previously, corresponding to the fact that the D-string ending on a D-3-brane is T-dual to two D-2-branes intersecting at a point. This configuration is the obvious reduction of the similar one involving two M-2-branes. However, we could also start from a multi D-4-brane worldvolume with non-abelian gauge potential. In this case T-duality takes the non-abelian instanton into a non-abelian monopole. Further T-duality leads to what we may call a non-abelian vortex. We may obtain the expression for the energy density by dimensional reduction of (3.17). To this end, we define

$$V_3 = X \qquad V_4 = Y \tag{3.35}$$

both in the adjoint representation of some non-abelian Lie algebra. Then

$$\mathcal{E}^2 = \Big[1 \pm \mathrm{tr}(\,\star F[X,Y] - DX\times DY)\Big]^2 + \mathrm{tr}|DX \pm \star DY|^2 + \mathrm{tr}(\star F \mp [X,Y])^2\,, \tag{3.36}$$

so that we deduce the bound

$$\mathcal{E} - 1 \geq \pm(\,\star F[X,Y] - DX\times DY)\,, \tag{3.37}$$

which is saturated when

$$DX = \mp \star DY \qquad \star F = \pm [X, Y] \,, \tag{3.38}$$

which are the equations studied by Hitchin [21]. The total energy is therefore subject to the bound

$$E \geq |Z| \,, \tag{3.39}$$

where Z is the topological charge

$$Z = \int_{D2} \mathrm{tr}(\frac{1}{2}DX^2 + \frac{1}{2}DY^2 + [X, Y]^2) \,. \tag{3.40}$$

3.5 Nahm's equations

A further T-duality of the two intersecting D-2-branes leads us back to a D-string ending on a D-3-brane, but we now find BPS-type equations for the *string's* worldvolume field. These are just the dimensionally reduced version of Hitchin equations (3.38). Equivalently, they are the self-duality equations for non-abelian F on the D-4-brane, dimensionally reduced in three orthogonal directions. Let $X_i = V_i$, $i = 1, 2, 3$ (the components of the Lie-algebra valued gauge potential in the three directions) and define D to be the covariant derivative in the 4-direction. Then we arrive at

$$DX_i = \pm \frac{1}{2} \varepsilon_{ijk} [X_j, X_k], \tag{3.41}$$

which are Nahm's equations. Solutions of these equations are associated with an energy given by

$$E = \frac{1}{2} | \int_{D1} \varepsilon_{ijk} tr(DX_i [X_j, X_k]) | \,. \tag{3.42}$$

and represent the intersection of multiple D-strings with D-3-branes. Since the D-strings end on the D-3-branes we must impose suitable boundary conditions on the worldline fields. In [22] it was argued in the context of the leading order terms in the BI action that the appropriate supersymmetric boundary conditions are precisely those that lead to the Nahm description of the moduli spaces of $SU(k)$ monopoles, where k is the number of D-3-branes. That the arguments in [22] are valid for the full BI gauge theory is supported by our results.

A further dimensional reduction leads to BI quantum mechanics for $U(n)$ matrices describing the dynamics of n D-0-branes. The energy bound becomes

$$E \geq \frac{1}{2} | \varepsilon_{ijkl} tr(X_i X_j X_k X_l) | \,, \tag{3.43}$$

where here $X_i = V_i$, $i = 1, \ldots, 4$, and the bound is saturated for matrices satisfying

$$[X_i, X_j] = \frac{1}{2} \varepsilon_{ijkl} [X_k, X_l] \,. \tag{3.44}$$

Using the cyclic property of the trace we see that the right hand side of (3.43) vanishes for finite dimensional matrices. Note that for infinite dimensional matrices such charges correspond to longitudinal fivebranes in the matrix theory approach to M-theory [23, 24].

4 String-in-fivebrane

The phase space Lagrangian density for the M-5-brane, in the M-theory vacuum and omitting fermions, is [25]

$$\begin{aligned}\mathcal{L} = & P \cdot \dot{X} + \Pi^{ab} \dot{A}_{ab} + \lambda_a \partial_b \Pi^{ab} - s^a (P \cdot \partial_a X - V_a) \\ & + \sigma_{ab} (\Pi^{ab} + \frac{1}{4} \tilde{\mathcal{H}}^{ab}) - \frac{1}{2} v [(P - g^{ab} V_a \partial_b X)^2 + \det(g + \tilde{H})] ,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}\tilde{\mathcal{H}}^{ab} &= \frac{1}{6} \varepsilon^{abcde} H_{cde} \\ \tilde{H}_{ab} &= \frac{1}{\sqrt{\det g}} g_{ac} g_{bd} \tilde{\mathcal{H}}^{cd} \\ V_f &= \frac{1}{24} \varepsilon^{abcde} H_{abc} H_{def} ,\end{aligned} \tag{4.2}$$

with ε the invariant worldspace tensor density (such that $\varepsilon^{12345} = 1$). Note that λ_a imposes the Gauss law constraint on the electric 2-form Π. This becomes equivalent to the Bianchi identity $dH = 0$ upon using the constraint imposed by the Lagrange multiplier σ_{ab}.

In static gauge, and restricting to static configurations, the s-equation implies $P_a = V_a$, so the hamiltonian constraint becomes

$$\mathcal{E}^2 = V_a V_b m^{ab} + \det(g_{ab} + \tilde{H}_{ab}) , \tag{4.3}$$

where

$$m^{ab} = g^{aa'} g^{bb'} [\partial_{a'} \mathbf{X} \cdot \partial_{b'} \mathbf{X} + (\partial_{a'} \mathbf{X} \cdot \partial_c \mathbf{X}) \delta^{cd} (\partial_{b'} \mathbf{X} \cdot \partial_d \mathbf{X})] . \tag{4.4}$$

The expansion of the determinant leads to terms quartic in $\tilde{H}$, but the identity

$$\det(\tilde{H}) \equiv V_a V_b g^{ab} \tag{4.5}$$

allows these terms to be combined with the other terms quadratic in V, leading to the result:

$$\mathcal{E}^2 = \det g + \frac{1}{2} \tilde{\mathcal{H}}^{ac} \tilde{\mathcal{H}}^{bd} g_{ab} g_{cd} + V_a V_b \delta^{ab} . \tag{4.6}$$

We shall now set all but one of the transverse scalars to zero, in which case

$$\mathcal{E}^2 = 1 + (\partial X)^2 + \frac{1}{2} |\tilde{\mathcal{H}}|^2 + |\tilde{\mathcal{H}} \cdot \partial X|^2 + |V|^2 . \tag{4.7}$$

where X is the one non-zero scalar field and

$$\begin{aligned}|\tilde{\mathcal{H}}|^2 &= \mathcal{H}^{ab} \mathcal{H}^{cd} \delta_{ac} \delta_{bd} \\ |\tilde{\mathcal{H}} \cdot \partial X|^2 &= \tilde{\mathcal{H}}^{ab} \tilde{\mathcal{H}}^{cd} \partial_b X \partial_d X \delta_{ac} \\ |V|^2 &= V_a V_b \delta^{ab} .\end{aligned} \tag{4.8}$$

We can rewrite this as

$$\begin{aligned}\mathcal{E}^2 &= |\zeta^a \pm \tilde{\mathcal{H}}^{ab}\partial_b X|^2 + 2\left|\partial_{[a}X\zeta_{b]} \pm \frac{1}{2}\delta_{ac}\delta_{bd}\tilde{\mathcal{H}}^{cd}\right|^2 \\ &\quad + (\zeta^a\partial_a X)^2 + |V|^2 \,, \end{aligned} \tag{4.9}$$

where ζ is a unit length worldspace 5-vector, i.e.

$$\zeta^a\zeta^b\delta_{ab} = 1\,. \tag{4.10}$$

We may choose $\zeta^5 = 1$ and $\zeta^{\hat{a}} = 0$, $(\hat{a} = 1,2,3,4)$, in which case we deduce the inequality

$$\mathcal{E} - 1 \geq \pm\frac{1}{6}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} H_{\hat{a}\hat{b}\hat{c}}\partial_{\hat{d}} X\,, \tag{4.11}$$

with equality when

$$\partial_5 X = 0 \qquad H_{5\hat{a}\hat{b}} = 0 \tag{4.12}$$

and

$$H = \pm\star dX\,, \tag{4.13}$$

where, in the last equation, H is restricted to the 4-dimensional subspace of worldspace orthogonal to ζ, which we shall call w_4, and $\star$ is the Hodge dual of w_4.

Imposing periodic boundary conditions in the 5-direction so that the vector field ζ has orbits of length L, we see that the total energy satisfies the bound

$$E \geq L \times |Z|\,, \tag{4.14}$$

where Z is the topological charge

$$Z = \int_{w_4} H \wedge dX\,. \tag{4.15}$$

The tension $T = E/L$ is therefore bounded by Z, as claimed in the introduction, with equality for configurations satisfying (4.13). Because H is closed, this equation implies that X is harmonic. Singularities of X are the strings found in [6]. The simplest solution is obtained by choosing a single isolated point singularity at the origin. In this case the energy integral can be rewritten as the small radius limit of a surface integral over a 3-sphere surrounding the origin. Since X is constant on this surface we deduce that the string tension is given by

$$T = \mu \lim_{\epsilon\to 0} X(\epsilon)\,, \tag{4.16}$$

where

$$\mu = \int_{S^3} H \tag{4.17}$$

is the 'naive' string charge defined by the 3-form flux through a 3-sphere surronding the string in the 5-brane. The tension of this string is formally infinite since $X(\epsilon)$ increases without bound as $\epsilon \to 0$, but this is precisely what is expected if the string is the boundary of a semi-infinite membrane.

5 Discussion

There is a class of brane intersections for which no worldvolume soliton is known. This class includes the case of two M-5-branes intersecting on a string, two D-4-branes intersecting on a point, and a D-0-brane in a D-8-brane. In the latter case, the 0-brane charge is expected to be

$$Z = \int_{D8} F \wedge F \wedge F \wedge F \,. \tag{5.1}$$

The charge in the two M5-brane case should be similar, and the other cases, which are related by duality, are presumably obtained by dimensional reduction. By analogy with the instanton case, we would expect a purely magnetic solution with all scalars constant. For such configurations the energy density is given by $\mathcal{E}^2 = \det(\delta + F)$ which can be rewritten as

$$\mathcal{E}^2 = (1 \pm \star F^4)^2 + \frac{1}{2}(F \mp \star F^3)^2 + \frac{1}{2.4!}(F^2 \pm \star F^2)^2 \,, \tag{5.2}$$

where

$$\begin{aligned}
\star F^4 &= \frac{1}{2^4 4!}\varepsilon^{ijklmnpq} F_{ij}F_{kl}F_{mn}F_{pq} \\
(\star F^3)^{ij} &= \frac{1}{2^3 3!}\varepsilon^{ijklmnpq} F_{kl}F_{mn}F_{pq} \\
(\star F^2)^{ijkl} &= \frac{1}{8}\varepsilon^{ijklmnpq} F_{mn}F_{pq} \\
(F^2)_{ijkl} &= 3F_{[ij}F_{kl]} \,.
\end{aligned} \tag{5.3}$$

We conclude that

$$E \geq |Z| \,, \tag{5.4}$$

but the bound cannot be saturated because the equations

$$F = \pm \star F^3 \qquad F^2 = \mp \star F^2 \tag{5.5}$$

have no simultaneous solutions.[8] We suspect that a resolution of this problem will involve the VF^4 Chern-Simons term for the BI field that is required in a massive IIA background [27], but we leave this to future investigations.

We should not conclude without a comment on the remarkable fact that the BPS-type equations for (supersymmetric) worldvolume solitons are *linear*, despite the non-linearity of the action. The seems to be a special feature of M-brane and D-brane worldvolume actions.

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[8] If the signs were different a solution of the form considered in [26] would be possible.

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