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Piezoelectric control of the static behaviour of flextensional actuators with constricted hinges

Jacek Przybylski

Czestochowa University of Technology, Dabrowskiego 69, 42-201 Czestochowa, Poland

E-mail: jacek.pr@imipkm.pcz.pl

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Abstract

The objective of this paper is to present the mathematical modelling and computational testing of the static operational performance and effectiveness of flextensional actuators comprised of two rectilinear or initially deflected beams placed equidistantly from a centrally located piezoceramic stack in the form of a rod. The beams are mounted by stiff links with an offset to a piezoelectric transformer. A monolithic hinge lever mechanism is applied by cutting constricted hinges at the links to generate and magnify the in-plane displacement created by the application of a voltage to the piezorod. Structures of such a type have been commonly used as passive or active actuators since the manufacturing of the mechanism’s prototypes in the form of Moonie or cymbal actuators. An analytical model of the actuator is developed on the basis of stationary values of the total potential energy principle with the use of the von Kármán non-linear strains theory. During the numerical computations, the deflection and internal axial force generated by both the externally distributed load and the application of an electric field are determined by changing the actuator properties such as the distance between the beams and the rod, the amplitude of the beam’s initial displacement as well as the stiffness of the constricted hinges. Additionally, the application of structure prestressing is considered to avoid an undesired stretching of the piezo stack. It has been shown that for the flextensional actuator with a very high flexibility of constricted hinges, the generated transverse displacement is limited by the maximum electric field as the characteristic property for each piezoceramic material. A vast number of numerical results exhibit the mechanical responses of the transducer of different geometrical and physical properties to piezoelectric stimulation; this has potential applications in the design process of such actuators.

Keywords: piezoelectric actuator, geometrical nonlinearity, static behaviour, initial deflection

1. Introduction

Flextentional actuators and sensors originate from acoustic foghorns, which have been used on ships for navigation purposes since the 1920s. The operating principle of such transducers consists in the amplification of the longitudinal in-plane strains in a piezoceramic disc or bar into the out-of-plane bending of a metal shell or beam. That concept was extended by Newnham et al., who developed the Moonie actuator [1] and who with Dogan developed its improved version called the cymbal actuator [2]. Contemporary producers offer a large number of mechanically amplified piezoelectric actuators of different forms, dimensions and geometries related to industrial applications, which are characterized by large deformation and large strokes both in static
and dynamic conditions, including resonance. The attributes of these actuators, such as compact structure, very precise movement, high force generation and quick response time with low energy consumption, make them appropriate for micro positioning, structure shaping, structural health monitoring, vibration control, cancellation and generation, fluid control functions in valves, dispensers and micropumps, and for use in various domains such as automotive, life and medical sciences, aviation, aerospace, optics and electronics applications.

During the construction process of flexure transducers, a monolithic hinge lever mechanism was developed by cutting constricted hinges in the body of the device [3]. The idea behind that solution was to reduce the bending stiffness at the hinges, while maintaining enough axial (extensional) rigidity of the active parts (beams) of the actuator and, as a result, to obtain greater magnification of the out-of-plane displacement. Flexure hinges have been applied in other types of actuators e.g. for nanoscale positioning [4] and in the parallel kinematic manipulator [5]. According to Xu and King [6], who investigated the performance of different flexure hinge profiles in terms of flexibility and accuracy, the flexure hinge displacement amplifier has numerous advantages, such as smoothness of movement, no need for lubrication and zero backlash. It should be noted here that the transfer of energy from the actuator to the mechanical system is maximized when the stiffness of the actuator and the mechanical system are matched [7]. Horizontal beams with hinges and a piezoelectric bar were parts of the actuator-compliant mechanism amplifier studied theoretically and experimentally by Bharti and Frecker [8]. The authors used the topology optimization method to obtain the initial topology of the compliant mechanism, followed by detailed finite element analysis. The effect of the geometry parameters, material selection, and epoxy bonding layers in the piezoelectric actuator was revealed in detail. A thorough study on actuators destined for the active control of a helicopter rotor through the use of trailing edge flaps was conducted by Mangeot et al [9]. Among the number of presented concepts of mechanical amplification was one where the strain is concentrated at specific hinge points through the use of a deformable frame. Zhou and Henson [10] first reviewed different mechanical amplifier mechanisms that increase the displacement of the piezo actuator, called diamond-shaped mechanical amplifiers. In the theoretical analysis, the authors derived expressions for the forces, displacements and stresses produced by the amplifier. They were compared with the experimental results obtained during the testing of a prototype to find that the displacement of the prototype amplifier does not deviate more than 7% from the theoretical value when close to maximum displacement. Another design of the displacement amplifier based on the honeycomb link mechanism was described by Muroaka and Sanada [11]. The amplifier emulates the in-plane deformation of a specific honeycomb with bow-tie-type reentrant cells. When a multilayer piezoelectric actuator causes the elongation of a cell of the amplifier in the longitudinal direction, the cell causes expansion in the transverse direction with amplification depending on its geometry.

The authors discussed theoretical formulas which are used in the amplification of actuators, and demonstrated the compactness of an amplified piezoelectric actuator based on the proposed mechanism using prototypes of uniaxial actuators and XY stages.

Accepting the crucial role of flextensional piezoceramic–metal composite actuators in various applications, tailoring their electro-mechanical performance has received the attention of several authors for, first of all, Moonie and cymbal type transducers. Erhart and Panos [12], after modifying a Moonie transducer within the finite element method model in order to use shear stress in PZT ceramics, stated that due to the complicated transfer of the hydrostatic pressure to the shear stress, the shear-Moonie design had much lower performance than that of a conventional Moonie. After their investigation, the authors declared that the Moonie is one of the most successful piezoelectric hydrophone transducers. A simplified model for the analysis of Moonie actuator functioning was developed by Lalande et al [13]. The model consisted of two beams symmetrically arranged around the actuator neutral axis and attached to the actuator at the ends with an offset. The authors found that the offset distance of the beams had a large impact on the behaviour of the system. A finite element analysis was employed to study the properties of cymbal-type flextensional transducers [14, 15]. Tressler et al [14] investigated the effect of material properties and dimensional changes on the dynamic properties of the cymbal actuator to calculate its vibration mode shapes, resonance frequencies and admittance spectra. The computations showed that the fundamental resonance frequency can be easily manipulated by changing the cymbal cap material or dimensions. Dogan et al [15] stated that the underwater performance of piezocomposite cymbal transducers can be tailored by changing the ceramic driving element and the end cap material because that material has a strong effect on resonance frequency. Moreover, changing the PZT leads to changes in the sensitivity and transmitting voltage of underwater transducers. By combining the geometric parameter studies with those results, the cymbal transducer requirements for specific applications can also be optimized. Ochoa et al [16–18] performed a thorough study on the modelling of the coupled electrical–mechanical characteristics of cymbal ceramic–metal composites. The first problem concerned the electric potential and mechanical stress distributions of the cymbal actuator for both the resonant and off-resonant modes. In order to reduce the stress concentration, a new strategy was investigated by finite element analysis leading to the removal of a portion of the ceramic just below the point where the maximum stress concentration was observed, or by differentiating the thickness and distribution of the bonding layer [18]. The influence of the stiffness of the metallic cap, the piezoelectric coefficients of the ceramics and the characteristics of the epoxy bond on cymbal actuator performance has been evaluated by Fernandez et al [19]. It was found that the higher the transverse piezoelectric coefficient, the higher the displacement of the actuator. The stiffness of the metal reduces the displacement but allows the composite to support higher loads. Taking into account the thermal factor, the
authors discovered that by selecting appropriate materials, it is possible to avoid this thermally induced displacement.

Modifications and development in the cymbal transducer design have also been reported in [20–22]. Lam et al [20] proposed a lead-free piezoelectric ceramic as an active element material instead of the standard lead zirconate titanate (PZT), the most commonly used material in cymbals for actuation purposes. Both the electrical and mechanical properties of the lead-free piezoceramic cymbal were compared with that of the lead zirconate titanate device to discover whether the performance of both actuators is similar, as the lead-free material has reasonable piezoelectric coefficients and low density. Using FEA analysis, Lam et al [21] investigated a piezoelectric–metal–cavity cymbal type actuator (PMC), which was designed especially to exhibit a large flexural displacement. This new actuator consisted of a metal ring sandwiched between two identical piezoelectric unimorphs. The metal cavity played the role of the circumferential coupling of the piezoelectric unimorphs. It was found that the effective longitudinal piezoelectric coefficient of the PMC actuator could be greatly enhanced, compared with that of the cymbal actuator with comparative dimensions. A very interesting approach to the displacement amplification in flextensional actuators was devised by Gilbert and Austin [22], who included the PZT polarization vector as a design variable in the design of a transducer. They presented two examples based on the cymbal actuator: one using a simplified model to justify off-angle polarization and another one using the polarization vector as a design variable to optimize the topology of a compliant mechanism. The exemplary results showed that both the free displacement and blocked force varied widely with the polarization angle of the PZT and the geometry of the structure. Moreover, the authors stated that despite not having the assurance of obtaining a truly optimal structure, increases in performance could be obtained by including the polarization vector in the design domain and experimenting with various initial guesses.

In order to derive the governing equations, one can treat the stack as a monolithic bar, which results from its manufacturing process, as co-firing is a practical way to assemble and wire a large number of piezo layers into one monolithic structure. Determination of the effective elastic, dielectric, piezoelectric and coupling constants from the complex impedance spectra of piezo stacks was described by Sherrit et al [23, 24] who performed impedance-based measurements. This identification technique lies in its simplicity and the determined constants were in good agreement with the data obtained by the manufacturers [25, 26] who, presenting the specifications of piezo stacks, also treat those products as monoliths rendering their fundamental physical parameters (Young’s modulus, density, Poisson’s ratio, compressive and tensile stress) as is usually done for solid homogeneous actuators. The idea of implementing piezoelectric transducers (in different shapes and dimensions) discretely mounted to the host structure to enhance its performance has strong motivation from studies presented, e.g. in [27, 28]. Such fitting of the piezoceramic actuator was postulated by Chaudhry and Rogers [27] who, when performing theoretical and experimental studies on the pre-buckling of a simply supported beam with a piezoelectric rod attached at two points, pointed out that actuators like ceramic piezoelectric PZT and electrostrictive devices possess flexural stiffness that could be comparable to that of the substructure. Przybylski and Sokół in [28] investigated the usefulness of an actuator in the form of a piezoceramic rod discretely attached to a host column for the purpose of suppressing the lateral deflection of the eccentrically loaded structure. The eccentricity was treated as an undesired effect of imprecise manufacturing or an improper assembling process of supporting and loading heads. A thorough discussion of the results led to the conclusion that the piezo actuation achieved by means of a discretely attached piezoceramic rod seems to be a very effective way of controlling the shape of the considered type of structures. On the contrary, in this study the piezo stack has a reverse role to play, i.e. it is a tool for enforcing controlled deflection of the beams.

It should be noted here that piezoelectric materials, despite their advantages, exhibit inherent nonlinearities such as creep and hysteresis, which, when incorporated into actuators, reduce their positioning performance. To cope with such problems, compensation methods are proposed in the literature. Gu et al employed an ellipse-based mathematical model [29] or an effective informed adaptive particle swarm optimization algorithm [30] to identify the model parameters. Then, the identified models were implemented into a real-time feedforward controller for fast inverse hysteresis compensation. As a result of the performed experiments the authors reported up to an 88% reduction in the hysteresis error [29].

Having the knowledge of realizable control of the static behaviour of slender systems via piezoactuation, the main purpose of this work is to estimate the influence of the piezoelectric force on the performance of flextensional actuators of different geometric shapes and mechanical parameters. In the proposed design, the hinges are placed in links which can be made of a fatigue resistant material and are not prone to high stress concentration. The flexural stiffness of the hinges and the bending stiffness of the beams may be mutually tuned to enhance the actuator amplification efficiency. In this paper, the static behaviour of two types of flextensional actuators distinguished by the geometrical shape of two metal beams, mounted by means of two hinged links with an offset to a centrally located piezoceramic stack in the form of a rod, has been analysed to estimate the effectiveness of the design parameters on the electro-mechanical performance of such a device. In the first model of the transformer, the beams are rectilinear; in the second model, more advanced one, the beams are initially deflected. Generally, the actuator under consideration can be treated as a metal frame wrapped around the PZT stack and is constructionally similar to those that were experimentally investigated by Cross et al [31]. The models of transducers presented in this literature survey concern other types of flextensional actuators, the behaviour of which was primarily examined on the basis of finite element modeling [17, 20, 21, 32]. The most comparable to the studied model are those prepared by Fernadez et al [19] and Gilbert and Austin [22] who introduced two-dimensional
analytical models of a cymbal transducer, in which the end-caps and piezoelement were represented by compliant segments or members connected by pins.

2. Problem formulation

The simplified model of the transducer, whose upper symmetric half with regard to two constructional variants is shown in figure 1, has been developed to evaluate the characteristics of the device as a function of the three main parameters: the offset distance ($e^*$) of the metal beams with regard to the piezoceramic stack, the initial deflection of a beam and the flexural stiffness of the hinges. In the model, the hinges are represented by pins strengthened by linear rotational springs of stiffness ($C$). A uniform external transverse load $q^*$ can be applied towards or away from the beams. In the real structures, the constricted hinges are usually machined to obtain the required shape and dimensions and, as a result, the expected flexural stiffness. The upper part of the right end section of the actuator with one hinged link is presented in figure 2.

In a flextensional actuator, depending on the direction of the electric field applied to the piezo stack, the beams can deflect inwards or outwards. As the main static features of the actuator are both its flexural displacement and the internal axial force that directly induces that displacement, those two quantities being determined and analysed in this work, are taken as the fundamental measures for application purposes. The structural design makes the application of the electric voltage to the piezo stack, due to the longitudinal $d_{33}$-effect, lead to its contraction or extension depending on the direction of the electric field vector, and finally to the flexural displacement of both beams. During the deformation of the transducer, the rod strains axially, whereas the beams deform transversally with respect to their rectilinear or bent axes. To describe the static behaviour of the system, the von Kármán theory, with a nonlinear strain-displacements relation, is proposed. This effect appears when a beam has its ends restrained to remain a fixed distance apart. In the considered case, the beam ends cannot move freely because they are clamped by the links to the central rod.

Here, the assumptions for the mathematical model are as follows:
- both elements of the actuator have rectangular cross sections and are prismatic; the beam is initially straight or has a deflected shape;
- the material of the beam is homogeneous and isotropic and that of the multilayer stack transformer is homogeneous;
- the slenderness of the beam qualifies it for application of the Bernoulli–Euler theory;
- due to the geometry of the piezoelectric stack, the longitudinal effect $d_{33}$ is taken as dominant, whereas the transverse effect $d_{31}$ is disregarded;
- the non-linear kinematic relations correspond to small strains but moderate rotations.

Due to the large deflection of the beam, description of the problem is made on the basis of the von Kármán theory, according to which the total strain of the beam $\epsilon(x)$ can be presented as the sum of the mid-plane strain $\epsilon_1(x)$ and the strain of the beam layer $\epsilon_2(x)$ located at a distance $z$ from the symmetry axis:

$$\epsilon(x) = \epsilon_1(x) + \epsilon_2(x)$$  \hspace{1cm} (1)

where

$$\epsilon_1(x) = \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2$$

$$+ \frac{dW_0(x)}{dx} \frac{dW(x)}{dx},$$  \hspace{1cm} (2a)

$$\epsilon_2(x) = -z \frac{d^2W(x)}{dx^2},$$  \hspace{1cm} (2b)

$U(x)$ and $W(x)$ are the axial and transverse displacements of the beam, respectively, and $W_0(x)$ is a function describing the initial imperfections of the beam; for the actuator with rectilinear beams $W_0(x) = 0$.

The constitutive equation for the piezoelectric material of the rod in the one-dimensional problem may be presented as:

$$\sigma_p = E_p \varepsilon_p(x) - \frac{\varepsilon_{33}}{t_p} V$$  \hspace{1cm} (3)

where $E_p$ denotes the Young’s modulus of the piezo-material along the strain axis, $\varepsilon_{33}$ is the piezoelectric constant ($\varepsilon_{33} = d_{33} E_p$), $V$ is the operating voltage applied to the piezoelectric stack, $t_p$ is the thickness of the piezoelectric layer and $\varepsilon_p(x)$ is the strain resulting from the axial displacement of the rod:

$$\varepsilon_p(x) = \frac{dU_p(x)}{dx}$$  \hspace{1cm} (4)

Because a stack transformer has $n$ layers of piezoelectric material, the length of the actuator may be expressed using the relationship $L = nt_p$.

When the piezo stack is supplied with an equal homogenous electric field $E_3 = V/t_p$ of the vector parallel to the longest axis of the stack (figure 3) and the beam is under a distributed load $q^*(x)$, the total potential energy of the actuator, with a term concerning the electro-mechanical coupling, is as follows:

$$V_{poe} = \frac{1}{2} A_p \left[ \int_{\alpha_{p1}}^{\alpha_{p2}} \sigma_p E_p d\alpha_{p1} + \int_{\alpha_{p2}}^{\alpha_{p1}} \sigma E d\alpha + \int \sigma d\Omega + C \left( \frac{dW(x)}{dx} \right)^2 \right] - \int_{0}^{L} q^*(x) W(x) dx$$  \hspace{1cm} (5)

The electric displacement $D_3$ caused by the electric field applied to the piezo stack in the direction of its polarization (longitudinal mode) is expressed as

$$D_3 = \varepsilon_{33} E_p + \xi_{33} E_3,$$  \hspace{1cm} (6)

where $\xi_{33}$ stands for the permittivity coefficient of the piezoelectric material.

After necessary elaboration, the potential energy of the stack and the beam takes the form:

$$V_{poe} = \frac{1}{2} A_p \left[ E_p \int_{0}^{L} \varepsilon^2_p dx - 2 \varepsilon_{33} E_3 \int_{0}^{L} \varepsilon_1 dx - \xi_{33} E_3^2 \right]$$

$$+ \int_{0}^{L} \left[ A_1 \int_{0}^{L} \varepsilon_1^2 dx - 2 B_1 \int_{0}^{L} \varepsilon_1 \frac{dW(x)}{dx} dx \right]$$

$$+ D_1 \int_{0}^{L} \left( \frac{d^2W(x)}{dx^2} \right)^2 dx$$

$$= \frac{1}{2} A_p \left[ E_p \int_{0}^{L} \left( \frac{dU_p(x)}{dx} \right)^2 dx - 2 \varepsilon_{33} V/t_p \int_{0}^{L} \frac{dU_p(x)}{dx} dx \right]$$

$$- \xi_{33} E_3^2 \right] + \frac{1}{2} b \left[ A_1 \int_{0}^{L} \left( \frac{dU(x)}{dx} \right) dx \right.$$

$$+ D_1 \int_{0}^{L} \left( \frac{d^2W(x)}{dx^2} \right)^2 dx$$

$$+ D_1 \int_{0}^{L} \left( \frac{d^2W(x)}{dx^2} \right)^2 dx$$

$$+ \frac{1}{2} b \left[ A_1 \int_{0}^{L} \left( \frac{dU(x)}{dx} \right) dx \right.$$

$$+ D_1 \int_{0}^{L} \left( \frac{d^2W(x)}{dx^2} \right)^2 dx$$

$$+ D_1 \int_{0}^{L} \left( \frac{d^2W(x)}{dx^2} \right)^2 dx$$

where $b$ is the common width of the beam and the piezo rod,
is the Young’s modulus of the beam material and

$$ A_{11}, B_{11}, D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2) Edz. \quad (8) $$

In order to apply the principle of stationary value of total potential energy, after performing integration by parts in equation (5) with the assumption that the applied voltage \( V \) has a constant value, one can write

$$ A_p \left\{ \left[ E_p \frac{dU_p(x)}{dx} - e_{33} I_p \right] \delta U_p(x) \frac{L}{b} \right. $$

$$ - E_p \int_{0}^{\frac{L}{2}} dU_p(x) \frac{dU_p(x)}{dx} \delta U_p(x) dx \right\} $$

$$ + EA \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ \times \left[ \delta U(x) \frac{L}{b} + \left( \frac{dW(x)}{dx} + \frac{dW_0(x) dW(x)}{dx} \right) \delta W(x) \frac{L}{b} \right] $$

$$ - EA \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ \left\{ \left( \frac{dW_0(x) dW(x)}{dx} \right) \delta W(x) \frac{L}{b} \right\} $$

$$ + \frac{dW_0(x) dW(x)}{dx} \delta W(x) \frac{L}{b} $$

$$ + \frac{dW_0(x) dW(x)}{dx} \delta W(x) \frac{L}{b} $$

$$ \right\} \delta W(x) dx $$

$$ - EA \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ \left\{ \frac{dW_0(x) dW(x)}{dx} \right\} \delta W(x) dx $$

$$ + EI \left\{ \frac{d^3W(x)}{dx^3} \frac{L}{b} - \frac{d^3W(x)}{dx^3} \delta W(x) \frac{L}{b} \right\} $$

$$ \left( \frac{dW_0(x) dW(x)}{dx} \right) \delta W(x) dx $$

$$ \frac{dW(x)}{dx} \left|_{x=\frac{L}{2}} \right. $$

$$ \frac{d^2W(x)}{dx^2} \left|_{x=0} \right. $$

$$ + C \left( \frac{dW(x)}{dx} \left|_{x=\frac{L}{2}} \right. - \frac{dW(x)}{dx} \left|_{x=\frac{L}{2}} \right. \right) $$

$$ - \frac{\int_{0}^{\frac{L}{2}} q^*(x) \delta W(x) dx = 0}{9} $$

In equation (9) \( A_p \) denotes the cross-section area of the piezo stack, and \( A \) and \( I \) are the cross-section area and the area moment of inertia of the beam cross section, respectively. Collecting the various coefficients of \( \delta U_p(x,t), \delta U(x,t) \) and \( \delta W(x,t) \) leads to the following equations

$$ \frac{d^2U_p(x)}{dx^2} = 0 \quad (10) $$

$$ EA \frac{d}{dx} \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ + \frac{dW_0(x) dW(x)}{dx} \right\} = 0 $$

$$ (11) $$

$$ EI \frac{d^4W(x)}{dx^4} - EA \frac{d}{dx} \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ \times \left( \frac{dW_0(x) dW(x)}{dx} \right) = q^*(x) $$

$$ (12) $$

Under the assumption that the piezoelectric force

$$ F = -e_{33} A_p \frac{V}{I_p} \quad (13) $$

creates a compressing axial force along the beam \( S \), which after mathematical elaboration of equation (11) can be defined as

$$ S = -EA \left[ \frac{dU(x)}{dx} + \frac{1}{2} \left( \frac{dW(x)}{dx} \right)^2 + \frac{dW_0(x) dW(x)}{dx} \right] $$

$$ + \frac{dW_0(x) dW(x)}{dx} \right\} $$

$$ (14) $$

and a tensile force along the rod \( \left( S_p \right) \)

$$ S_p = E_p A_p \frac{dU_p(x)}{dx} $$

$$ (15) $$

the natural and geometrical boundary conditions are as follows:

$$ W(x) \left|_{x=\frac{L}{2}} = 0, \right. $$

$$ \frac{dW(x)}{dx} \left|_{x=0} = \frac{d^2W(x)}{dx^2} \left|_{x=0} \right. = 0, \right. $$

$$ \left( \frac{dW_0(x) dW(x)}{dx} \right) \delta W(x) \frac{L}{b} - Se^* = 0 $$

$$ \left. U_p(x) \right|_{x=\frac{L}{2}} = U(x) \left|_{x=\frac{L}{2}} = e^* \frac{dW(x)}{dx} \left|_{x=\frac{L}{2}} \right. \right. $$

$$ S_p + F - S = 0 $$

$$ (16 (a-f)) $$
The equations are made dimensionless through the substitutions
\[ \xi = \frac{x}{L}, \quad w(\xi) = \frac{W(x)}{L}, \quad w_p(\xi) = \frac{W_p(x)}{L}, \quad u(\xi) = \frac{U(x)}{L}, \]
\[ u_p(\xi) = \frac{U_p(x)}{L}, \quad s^2 = \frac{SL^2}{EI}, \quad f^2 = \frac{FL^2}{EI}, \quad q(\xi) = \frac{q^e(x)L^2}{EI}, \]
\[ c = \frac{CL}{EI}, \quad e = \frac{e^s}{L}, \quad \lambda = \frac{AL^2}{I}, \quad \alpha = \frac{E_s}{E}, \quad \beta = \frac{A_p}{A}, \quad (17(a-m)) \]

which, after insertion into the original equations and then performing basic symbolic transformation, yields
\[ \frac{w''(\xi)}{c} + s^2 \left[ w''(\xi) + w_p''(\xi) \right] = q, \quad (18) \]
\[ u_p''(\xi) = 0, \quad (19) \]

where:
\[ s^2 = -\lambda \left\{ u'(\xi) + \frac{1}{2} \left[ w'(\xi) \right]^2 + w_p'(\xi) w'(\xi) \right\}, \quad (20) \]

and the Roman numerals denote the order of derivation with respect to the space variable.

3. Initial curvature of a beam and actuator prestressing

After applying the shape of beam initial curvature within the range \( \xi \in \left\{ 0, \frac{1}{\xi} \right\} \) as described by the cosine function
\[ w_0(\xi) = -\mathcal{A} \cos \pi \xi, \quad (21) \]

where \( \mathcal{A} \) is the beam midspan initial rise, the solution of equation (18) in the form of
\[ w(\xi) = C_1 \sin(s \xi) + C_2 \cos(s \xi) + C_3 + \frac{q^e}{2s^2} + \mathcal{A} - \frac{2}{s^2 - \pi^2} \cos \pi \xi, \quad (22) \]

when introduced into four boundary conditions (16(a)–(d)), written in the non-dimensional version, leads to a system of four inhomogeneous linear algebraic equations with respect to the integration constants \( (C_1, C_2, C_3, C_4) \). That system cannot be solved due to the unknown axial force parameter \( s \). The fifth supplementary equation is derived by using boundary condition (16(e)), to which the solutions of equations (18) and (19) are inserted. That procedure leads to the following dimensionless equation
\[ \frac{s^2}{\lambda} \left[ 1 + \frac{1}{a\beta} \right] - \frac{f^2}{2a\beta} + \int_0^{a_0} \left\{ \frac{1}{2} \left[ w'(\xi) \right]^2 + w_p'(\xi) w'(\xi) \right\} d\xi + 2e \left[ w'(\xi) \right]_{\xi=a_0} = 0, \quad (23) \]

which determines the axial force coefficient \( s \) in the beam as a function of the piezoelectric force coefficient \( f \) under an imposed geometric condition establishing the relation between the axial displacements of the actuator components. Numerical analysis of the static problem makes it possible to investigate the electro-mechanical performance of the actuator for its different geometric and physical parameters.

Some manufacturers of flextensional actuators recommend prestressing to avoid undesirable stretching of the piezoelectric stack. Prestress of the item may be introduced when the length of the stack is intentionally oversized when compared with the inner dimension of the frame. In the considered model, the prestress is represented by the non-dimensional force
\[ p^2 = \frac{PL^2}{EI}, \quad (24) \]

which after introduction into equation (23) gives
\[ \frac{s^2}{\lambda} \left[ 1 + \frac{1}{a\beta} \right] + \frac{f^2}{2a\beta} + \int_0^{a_0} \left\{ \frac{1}{2} \left[ w'(\xi) \right]^2 + w_p'(\xi) w'(\xi) \right\} d\xi + 2e \left[ w'(\xi) \right]_{\xi=a_0} = 0, \quad (25) \]

As a result of such a routine, the beam is preloaded with the tensile force and the stack is under an initial compressing force of the same amplitude. To complete the role of prestressing, the opposite preloading creating the stack stretching and beam compressing needs to be also considered. Additionally, to discuss the problem for the second configuration of the actuator relating to the opposite sense of the electric field vector providing stretching of the beam, an analogous derivation has to be performed. In such a case, the signs in equations (14) and (15) defining the axial forces must be taken as opposite, which changes only the equations of the beam and rod equilibrium as well as the postulated forms of their solutions, leaving the whole mathematical elaborations unchanged. In consequence, during the numerical analysis four sets of equations are taken for computation corresponding to possible ways of prestressing and piezoelectric force actions.

4. Results of numerical calculations and discussion

The elastic brass beam of Young’s modulus \( E = 103 \) GPa and piezoelectric material P61 for the stack of Young’s modulus \( E_p = 87.4 \) GPa have been taken for numerical analysis, hence
\[ \alpha = \frac{E_p}{E} = 0.8484. \]

Assuming that the width of both elements of the actuator is identical, the ratio of the rectangular cross section of the rod to the beam equals the ratio of particular depths of those elements \( \beta = A_p/A = h_p/h = 2 \). For the same reason, the slenderness beam parameter is independent of the width of both components \( (\lambda = AL^2/I = 12(L/h)^2) \).

Assuming that the value of beam depth \( h = \frac{1}{2}L \), the slenderness parameter \( \lambda = 6912 \). In the calculations, a change in cross section area ratio \( \beta \) implies a change in the depth of
piezorod $h_p$, while the depth of beam $h$ is kept constant, which together with a constant value of length $L$ makes it possible to operate with a constant value of $\lambda$.

The applied mathematical model makes it possible to operate on both types of actuators, i.e. actuators with rectilinear and curvilinear beams. The adaptation of formulas derived in this work for the latter case consists of eliminating function $w_0(x)$ expressing the initial deflection of the beams. Due to the intention of studying the particular effects on the static behaviour of the transducer, its model is developed gradually. First, the externally unloaded actuator with rectilinear beams is considered to show the influence of piezoelectric actuation on the generated axial force and the beam’s transversal displacement (figure 1(b)). Next, the effect of external constant load on the stated objectives is taken into account for the same device. Finally, the complete model of the initially prestressed transducer with curvilinear beams under uniform transverse load is analyzed (figure 1(a)). Respective analysis is provided in the forthcoming subsections, the headings of which correspond to the initial shape of the actuator. The numerical analysis is divided into two parts —the first one concerns the influence of the piezoelectric force on the behaviour of the actuator with rectilinear beams. The two main and the most characteristic parameters for this type of actuator are taken into consideration as design variables, i.e. the dimensionless offset $e$ representing the distance between the axis of the rod and the beam, and the non-dimensional stiffness of the rotational spring $c$ strengthening the fulcrum of the beam. In the numerical analysis, those two parameters have been given the following values: $e=0.05, 0.1, 0.2, 0.5$ and $c=0.1, 10, 50, 1000$. After presentation of the results concerning the influence of the chosen design parameters on the stated objectives, the effect of the distributed load in deviating piezoelectrically generated forces and displacements is taken into account.

The second part of the analysis is aimed at the static behavior of the prestressed transducer with initially curved beams loaded by a constant distributed load. The initial deflection amplitude can be treated as an additional design parameter, being an object of numerical analysis together with the level of external load and internal prestressing.

### 4.1. Analysis of actuator with rectilinear beams

To illustrate the static behaviour of the device, introductory research concerning the externally unloaded actuator comprised of rectilinear beams has been conducted (figure 1(b)). Figures 4–6 illustrate the relationship between the piezoelectric force $f$, generated by the electric field applied in the direction parallel to the piezo stack long axis, and the axial force $s$ along the beam for $c=0.1, 10$ and 1000, respectively. The change in the direction of the electric vector field is reflected by the change in the sign of the force $f$. Negative values of $f$ create such a stress state in the structure, as a result of which, the beam is eccentrically compressed ($s>0$) and the stack transformer is stretched, whereas a positive $f$ results in an opposite way of loading the transducer components. In some works, instead of the nondimensional piezoelectric force $f$, the nondimensional voltage is described as a stimulating factor. Although those two quantities can be treated alternatively on the basis of equation (13), the piezoforce has a more general sense, because it covers the participation not only of the voltage, but also the piezoelectric parameter $e_{33}$ and the thickness of a piezo layer, all of them being the influential components of similar importance. Four curves have been plotted in each of figures 4–6 exemplifying four different values of the distance parameter ($e=0.05, 0.1, 0.2, 0.5$).

Applying the same value of $f$ from the considered range (e.g. $f \in \{-10, 10\}$) for each value of $c$, one can notice that the greater the spring stiffness, the greater the absolute value of the axial force $s$. Due to an eccentricity $e$ of force $s$ generated by the piezostack, the beam is compressed or stretched by this force and simultaneously bent by a couple of a
magnitudes of the product of $s$ and $e$. As the axial force or eccentricity increases, the beam bends further. For each value of spring stiffness $c$, when the electric field vector causes the beam to compress, the beam may be compressed to such a level ($s_c$) that its deflection goes to infinity. That phenomenon in regard to the considered spring stiffnesses is depicted in detail in figures 7–9. The value of $s_c$ is identical to the critical load of a column with both ends supported by pin joints strengthened by rotational springs of stiffness $c$, e.g., for $c = 1000$, the critical force $s_c$ necessary to destabilize the column equals 6.2706. In the considered transformer, when the axial force $s$ asymptotically reaches such a value as a result of piezo actuation (figure 6), the beam midpoint transversal displacement tends to infinity, as shown in figure 9. Practically, such a great axial force, undesirable in this type of device, can be attained only in actuators with very slender beams and a relatively thick piezo stack due to limitations determined by the maximal electric field as the property characterized by each piezoelectric material. For each value of spring stiffness $c$, the increasing distance $e$, between the axis of the transformer and the beam, leads to a decrease in the absolute value of the axial force $s$ generated by the same value of the piezoelectric force $f$.

Due to the beam eccentricity, the axial force $s$ can deflect the rectilinear beam away from the actuator when it compresses the beam ($s > 0$), or towards the actuator when it stretches the beam ($s < 0$). According to the coordinate system shown in figure 1(b), the outward deflection is recognized by a negative value of transverse displacement and the inward one by a positive value of that displacement. The transverse displacement of the beam midpoint $w(0) = w_{0,5}$ is shown plotted against the $s$-force in figures 7–9 for $c = 0.1, 10$ and $1000$, respectively. All the plots have been made for the four

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**Figure 6.** Axial internal beam force versus piezoelectric force for different eccentricities $e$ and rotational spring stiffness $c = 1000$. The remaining system parameters are $\alpha = 0.848$, $\beta = 2$.

**Figure 7.** Beam midpoint transversal displacement as function of axial force for different eccentricities $e$ and rotational spring stiffness $c = 0.1$. The remaining system parameters are $\alpha = 0.848$, $\beta = 2$.

**Figure 8.** Beam midpoint transversal displacement as function of the axial force for different eccentricities $e$ and rotational spring stiffness $c = 10$. The remaining system parameters are $\alpha = 0.848$, $\beta = 2$.

**Figure 9.** Beam midpoint transversal displacement as function of the axial force for different eccentricities $e$ and rotational spring stiffness $c = 1000$. The remaining system parameters are $\alpha = 0.848$, $\beta = 2$. 
considered values of the beam offset $e$ to demonstrate that independent of the spring stiffness $c$, the greater the eccentricity $e$, the greater the displacement $w_{0.5}$ for the same level of $s$-force. Spring stiffness $c$ has a great influence on the efficiency of the electromechanical transduction—for each value of eccentricity $e$, the more rigid the support $c$, the less deflection is produced by the same $s$-force. It is shown, however, in figures 7–9, that for any $c$, when the beam is compressed by the $s$-force reaching its critical $s_c$, the displacement of the midpoint goes to infinity.

Two effects of increasing the offset distance $e$ illustrated in figures 4–6 and 7–9, related reciprocally for any value of $c$, need to be carefully considered during the design process of actuators. An increase in eccentricity $e$ produces both a suitable augmentation of the midspan displacement of the beam and an undesirable increase in the piezoelectric force, which needs to be generated. This is especially noticed for a smaller stiffness of the beam fulcrum. Therefore, the optimization procedure should be included in the design process to reconcile the need for the demanded deflection and the potentiality of the piezoelectric force induction. Considering the results presented and taking into account the realistic geometry of the actuator, one can notice that the transducer with offset distance $e = 0.1$ and a beam fulcrum stiffness relating to $c \geq 10$ fulfils the design expectations.

4.2. Actuator under uniform transverse load

The value of the uniform transverse load $q$ taken for computation has been applied as that which creates a midspan deflection not greater than 0.025 of the thickness $h$ of a rectilinear beam with both ends clamped. Hence, the maximal dimensional deflection is

$$w_{\text{midspan}} = \frac{1}{384} \frac{q^* L^4}{EI} = 0.025h \quad (26)$$

Assuming that beam thickness $h = \frac{1}{24}L$, (e.g. $h = 5$ mm and the beam length is $L = 120$ mm), the maximal non-dimensional uniform load on the basis of (26) and (17h) is $q = 0.4$.

When a beam of the device is under a distributed load of constant intensity $q$ applied over its entire length, the beam is elongated with a force $s$, the value of which rises with growing values of intensity $q$. That is depicted in figure 10 in the form of points $A_i$ ($i = 1, 2, 3$) along ordinate $s$. In that and in the forthcoming figures, the range of change of force $f$ has been limited to a level adequate for the electromechanical features of the most common piezoelectric materials in regard to the considered stacks.

Application of the electric field to the stack modifies force $s$ depending on the direction of the field vector. The introduction of beam compression by piezoelectric force $f (f < 0)$ can reduce the internal force in the system to the value of zero, which is reflected by points $B_i$ ($i = 1, 2, 3$). A further increase in the absolute value of the $f$-force brings the beam to behave as described for an unloaded actuator (figures 4–6). The courses of the dotted lines in figure 10 (quadrant IV) indicate that the beam stretching force can be enlarged when the electric field vector changes its sense, which is demonstrated by positive values of $f (f > 0)$. The graphical distinction of the presented curves with respect to particular quadrants has been provided to show the nature of actuator internal forces when one takes into account a simultaneous deflection adjustability, because the control regime of the device by

Figure 10. Influence of the piezoelectric force on the internal axial force in the actuator under a distributed load $q$. The remaining system parameters are $c = 10$, $e = 0.1$, $a = 0.848$, $\beta = 2$.

Figure 11. Influence of piezoelectric force on the beam midpoint transversal displacement for different values of the distributed load $q$. The remaining system parameters are $c = 10$, $e = 0.1$, $a = 0.848$, $\beta = 2$.
using the piezoelectric effect also covers the influence of the piezoelectric force \( f \) on the lateral displacement of the beam midpoint \( w_{0.5} \), which is illustrated in figure 11.

The pictured curves for different values of load \( q \) manifest the applicability of piezo actuation for creating the demanded beam deflection in both directions: towards or away from the device. In this figure, points \( A_i \) (\( i=0,1,2,3 \)) exhibit a level of lateral displacement originating from the external distributed load. Points \( C_i \) (\( i=1,2 \)) placed on the abscissa demonstrate the necessary magnitudes of force \( f \) for the complete reduction of the beam deflection created by load \( q \). At those points, the beam changes its inward deflection to an outward deflection or vice versa depending on the direction of the piezoelectric force.

A characteristic feature of the actuator can be observed in figures 10 and 11. The reduction of the internal force \( s \) and the lateral beam deflection \( w_{0.5} \) needs the reciprocal piezo actuations at certain intervals of \( f \) force. Piezo tension of the beam diminishes its flexural displacement from points \( A_i \) to the value of zero at points \( C_i \) (along the dotted lines in figure 11), but simultaneously enlarges the absolute value of internal force \( s \) (dotted lines in figure 10). Compression of the beam by the force \( f \) reduces the absolute value of internal force, as can be observed along segments \( A_i-B_i \) of particular curves (drawn with solid lines) in figure 10, increasing the coincidently lateral displacement illustrated in figure 11.

**4.3. Actuator with curvilinear beams under uniform transverse load**

To study the actuator with curved sinusoidal beams of beam midspan initial rise \( \mathcal{A} \), which represents an imperfection amplitude (equation (21)), the following load and physical parameters of the device shown in figure 1(a) have been taken for computation: \( q=0.4, \ c=10, \ e=0.1, \ \alpha=0.848, \ \beta=2 \). The effect of the piezoelectric force \( f \) on the axial force \( s \) and the midspan displacement \( w_{0.5} \) for four values of amplitude \( \mathcal{A} \) \( (\mathcal{A}=0,0.1,0.3,0.5) \) is shown in figures 12 and 13, respectively. In both figures, the graphical distinction of the curves in regard to particular quadrants has remained. The scale of the co-ordinate systems used in figures 10 and 12 is the same and the curve concerning zero initial amplitude \( (\mathcal{A}=0) \) is repeated in figure 12 for comparison purposes. The greater the

![Figure 12. Influence of the piezoelectric force on the internal axial force in the actuator with different amplitudes of beam initial imperfections \( \mathcal{A} \). The remaining system parameters are \( q=0.4, c=10, e=0.1, \alpha=0.848, \beta=2 \).](image1)

![Figure 13. Influence of the piezoelectric force on the beam midpoint transversal displacement for different amplitudes of beam initial imperfections \( \mathcal{A} \). The remaining system parameters are \( q=0.4, c=10, e=0.1, \alpha=0.848, \beta=2 \).](image2)

![Figure 14. Influence of the piezoelectric force on the internal axial force in the actuator with different internal prestressing \( p \). The remaining system parameters are \( \mathcal{A}=0.05, q=0.4, c=10, e=0.1, \alpha=0.848, \beta=2 \).](image3)
initial curvature of the beam, the lesser internal force is created by the same external load $q$, which is exhibited by the location of points $A_i$ ($i=0, 1, 2, 3$) on the $s$-axis. For that reason, a smaller compensation via piezo actuation is needed to cancel force $s$ (points $B_i$ on the $f$-axis). On the other hand, when there is no axial force in the beam, its lateral displacement caused by the same load $q$ is identical independently of the level of imperfection amplitude $\mathcal{A}$, which is depicted in figure 13 (points $B_i$ ($i=0, 1, 2, 3$) have the same ordinate).

Without piezo actuation, the greater the imperfection amplitude, the greater the flexural displacement created by $q=0.4$, illustrated by points $A_i$ on the $w_0.5$-axis of figure 13. Elimination of this displacement by means of the piezoelectric force needs a greater value of the force, the greater initial curvature the beam has (points $C_i$ ($i=0, 1, 2$)).

4.4. Actuator with internal prestressing

The role of prestressing in regard to the actuator with beams of initial curvature described by amplitude ($\mathcal{A}=0.05$) and loaded by $q=0.4$ is demonstrated in figures 14 and 15, to show its effect on the piezo force–axial force and the piezo force–lateral displacement relations, respectively. The applied convention in graphical presentation indicates the way of prestressing: a positive $p$-force or to keep it negative. This condition secures the working regime of the actuator with a compressed piezo stack. One can notice, observing the curves in figure 15, the working regime of the actuator with a compressed piezo stack. One can observe the curves in figure 15, that the prestress only slightly moderates the effect of the piezo force on the lateral displacement, keeping almost constant the gradient of the drawn curves.

5. Summary and conclusions

This work is devoted to the development of a mathematical model, theory and numerical analysis of the static behaviour of a flextensional actuator composed of two rectilinear or curvilinear metal beams connected by means of rigid links to a centrally placed piezoceramic stack in the form of a rod. In the proposed design solution, depending on the direction of the electric field, both inwards and outwards deflection of the beams can be attained.

For the considered structure, three parameters have been taken as design variables, i.e. the offset distance between the beam and the rod, the non-dimensional stiffness of the rotational spring strengthening the fulcrum of the beam and the initial deflection of the beams. A change in the design parameters has a great influence on the internal axial forces in the rod and the beam, which have the same absolute value and opposite directions, as well as on the transverse displacement of the beam generated by the electric field applied to the piezoelectric stack.
The greater the offset distance between the axis of the transformer and the beam, the more the axial force bends the beam, but an increase in that distance leads to a decrease in the absolute value of the axial force generated by the same value of the piezoelectric force. Generation of the axial force is also strictly dependent on the rigidity of the constricted beam, but an increase in that distance leads to a decrease in the transformer and the beam, the more the axial force bends the beam. Taking into account such a duality, the scale of the actuation must be controlled due to the maximum electric field as the quantity that is characterized by each piezoelectric material. Hence, optimization needs to be performed during the design process of such actuators to balance both the stiffness of the beam supports and the offset distance.

It has been proved that the greater the external load, the greater the piezoelectric force that needs to be induced to eliminate the internal axial force or the deflection caused by the mechanical loading. It should be added that an opposite electric field is required for elimination of the axial load or the lateral displacement.

Transversal displacement control, with a simultaneous requirement of keeping the piezo rod compressed, is possible for any value of prestressing and can be realised by applying an adequate piezoelectric force. Changes in the piezo force can cause a change in beam shapes, which, being concave due to their external load, bulge gradually to become convex, or vice versa, ensuring in this way the demanded stroke.

Future works may focus on other designs of flextensional actuators with a greater number of hinges and different types of external loads to optimise the device for particular application purposes. Taking into account the composite structure of the transducer, its static behaviour can be enhanced through the selection of materials. The next extension of the present work is to study the dynamic performance of the structure with respect to its design in the sense of geometry as well as electromechanical parameters of the components.

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