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# Nucleation of the phase of a finite Josephson junction

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**Abstract.** We present the results of a study of thermal and quantum nucleation in an overlap Josephson junction with a finite length. There is a critical length $L_c$ that marks the boundary between nucleation with a uniform phase across the junction and nucleation that is concentrated at one of the ends of the junction. In the thermal activation regime, it is shown to be $L_c = \pi\Lambda_J$, while at absolute zero, in the quantum tunnelling regime, it is given by $L_c = 2\pi\Lambda_J/\sqrt{5}$. Here, $\Lambda_J$ is the effective Josephson penetration depth. The rates for nucleation at the ends of the junction are given for junctions in the thermal activation regime for all lengths, and for junctions undergoing quantum nucleation at absolute zero temperature for lengths less than or equal to the critical length.

## 1. Introduction

Several years ago, we considered the quantum and thermal nucleation of the phase of an overlap Josephson junction (JJ) when the size of the junction is much larger than the effective Josephson penetration depth $\Lambda_J$ [1]. (See the next section.) The nucleation rates were given for zero and high temperatures in terms of experimentally determined parameters. The very large size of the junction that was considered was, strictly speaking, infinite, and nucleation takes place at any point along the homogeneous junction. The results, therefore, gave us the upper limit of the nucleation rate. On the other hand, we realize that, experimentally, the larger the junction the more difficult it is to make the junction homogeneous. Therefore, it would be interesting to study nucleation of the phase of a junction of finite size, especially a junction of size about equal to the Josephson penetration depth.

For a finite junction, as was stated in reference [1], nucleation takes place predominantly at the boundaries of the junction. The purpose of the present paper is to give results for thermal and quantum nucleation at various lengths. An interesting feature is the existence of a critical length at which there is singular behaviour: in particular, below $L_c$ the phase at nucleation is constant over the length of the JJ and the energy barrier for thermal nucleation is strictly proportional to the length $L$ of the junction. Just above $L_c$, the phase at nucleation is not uniform but is concentrated over a distance $\Lambda_J$ at one of the boundaries. In the thermal activation regime, the critical length $L_c$ is given by $\pi\Lambda_J$ [2]. In the quantum nucleation regime at absolute zero, the critical length is given by $L_c = 2\pi\Lambda_J/\sqrt{5}$. For lengths very much larger than $L_c$ (describable by nucleation within an infinite JJ [1]), nucleation at various positions along the interior of the JJ ('interior nucleation') can compete with nucleation at the boundaries ('boundary nucleation'). The length $L_0$ above which interior nucleation is more probable than boundary nucleation can be estimated as follows. In the Arrhenius factor, the energy barrier for interior nucleation $E_{IB}$ is exactly twice as large as the energy barrier for boundary nucleation $E_{BB}$. Hence, the Arrhenius factor for interior nucleation is exponentially smaller than that for boundary nucleation. However, the phase space is much larger for interior nucleation than for boundary nucleation by a factor of about $L/\Lambda_J$. Hence the ratio of nucleation rates is

$$\frac{\Gamma_I}{\Gamma_B} \sim \frac{L}{\Lambda_J} \exp(-E_{BB}/kT). \tag{1}$$

Setting $\Gamma_I$ equal to $\Gamma_B$ gives us $L_0$ equal to $\Lambda_J \exp(E_{BB}/kT)$. If $E_{BB}/kT = 23$ and $\Lambda_J = 10^{-3}$ cm, we obtain $L_0$ equal to 100 km! Therefore, in a typical perfect sample, boundary nucleation will predominate. However, in real, imperfect JJs, nucleation will usually take place around a sample inhomogeneity within the interior.

We will present the results for low temperatures as well as high temperatures above the crossover temperature $T_0$ between quantum and thermal nucleation. All of the basic parameters are experimentally determinable in the well understood classical thermal activation regime, so the predictions in the quantum tunnelling regime can be checked. Our work extends the macroscopic quantum phenomena [3, 4] in the Josephson junction systems.

## 2. Basic theory

We consider an *overlap* Josephson junction with a finite length $L$, but small width $W \ll \lambda_J$, where $\lambda_J$ is the Josephson penetration depth. The junction is current biased with a uniform current density $J$ and has a critical current density $J_c$ [5]. The self-field effect causes the phase of the junction to be space dependent [1, 5]. The imaginary-time action for the phase $\varphi$ of the junction at a finite temperature $T$ is given by [1]

$$\begin{aligned} S[\varphi(x,\tau)] = W \int_{-L/2}^{L/2} \mathrm{d}x \int_{-\beta\hbar/2}^{\beta\hbar/2} \mathrm{d}\tau \left[ \frac{\mathcal{C}}{2}\left(\frac{\hbar}{2e}\right)^2 \left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{{\lambda_J}^2}{2}\left(\frac{\hbar J_c}{2e}\right)\left(\frac{\partial\varphi}{\partial x}\right)^2 + U(\varphi)\right] \\ + \frac{2\hbar}{L} \int_{-L/2}^{L/2} \mathrm{d}x \int_{-\beta\hbar/2}^{\beta\hbar/2} \mathrm{d}\tau \int_{-\beta\hbar/2}^{\beta\hbar/2} \mathrm{d}\tau' \, \alpha(\tau-\tau') \\ \times \sin^2\left[\{\varphi(x,\tau) - \varphi(x,\tau')\}/4\right] \end{aligned} \tag{2}$$

where the potential energy is given as

$$U(\varphi) = -\frac{\hbar J_c}{2e}(\cos\varphi + \mu\varphi) \tag{3}$$

and $\mu \equiv J/J_c$. $\beta = 1/k_B T$ and $k_B$ is the Boltzmann's constant. $e$ is the absolute value of the charge of an electron and $\hbar$ is Planck's constant divided by $2\pi$. $\mathcal{C}$ is the capacitance of the junction *per unit area*. The kernel $\alpha(\tau)$ represents dissipation and is related to the $I$–$V$ characteristic of the junction, so it can be written in terms of experimentally determined parameters [1, 6].

The function $\alpha(\tau)$ can be written as a Fourier series [1, 6]:

$$\alpha(\tau) = \frac{1}{\beta\hbar} \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(\mathrm{i}\frac{2\pi n\tau}{\beta\hbar}\right). \tag{4}$$

In the following, we will be interested in the case of an unshunted junction, for which the $\alpha_n$ are effectively given in terms of experimentally determined parameters as [1, 6]

$$\alpha_n = -\frac{e}{\hbar^2}\int_0^\infty \frac{\mathrm{d}V}{\pi}\ \frac{V}{(eV/\hbar)^2 + (2\pi n/\beta\hbar)^2}\ I_{dc}(V) \tag{5}$$

where $I_{dc}(V)$ is the dc $I$–$V$ curve of the junction. For low $T$, we will use the approximate form [6]

$$\begin{aligned} I &= \frac{V}{R_q} \qquad \text{for } 0 < V < \frac{2\Delta}{e} \\ I &= \frac{V}{R_N} \qquad \text{for } V > \frac{2\Delta}{e} \end{aligned} \tag{6}$$

where $R_q \equiv R_q(T)$ is the temperature-dependent quasi-particle resistance and $R_N$ is the normal resistance of the junction. $2\Delta$ is the energy gap of the superconductors of the junction, which are considered identical, for simplicity. With this approximation, we have [6]

$$\alpha_n = -\frac{2nk_BT}{e^2}\left[\frac{1}{R_q}\cot^{-1}\left(\frac{\pi nk_BT}{\Delta}\right) + \frac{1}{R_N}\tan^{-1}\left(\frac{\pi nk_BT}{\Delta}\right)\right]. \tag{7}$$

For convenience, we define the dimensionless variables $t \equiv \Omega\tau$, $y \equiv x/\Lambda_J$, and $\ell \equiv L/\Lambda_J$. $\Omega$ is the frequency of small oscillations about a minimum in the potential energy and is given by

$$\Omega = \left(\frac{2eJ_c}{\hbar\mathcal{C}}\right)^{1/2}(1-\mu^2)^{1/4}. \tag{8}$$

The effective Josephson penetration depth $\Lambda_J$ is given by [1]

$$\Lambda_J = (1-\mu^2)^{-1/4}\lambda_J. \tag{9}$$

We define the reduced phase $\phi$ as

$$\phi \equiv \frac{\mu}{3(1-\mu^2)^{1/2}}(\varphi - \sin^{-1}\mu). \tag{10}$$

We will be interested in the case of a current density $J$ close to the critical current density $J_c$, in which case in equation (2) we approximate $\sin x$ by $x$ and the potential term by its quadratic plus cubic terms (after removing a constant). With this approximation throughout the paper, we replace equation (2) by

$$\begin{aligned} S[\phi(y,t)] = S_0\int_{-\ell/2}^{\ell/2}\mathrm{d}y\int_{-\beta\hbar\Omega/2}^{\beta\hbar\Omega/2}\mathrm{d}t\ \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 + u(\phi)\right] \\ + \frac{1}{2}S_0\int_{-\ell/2}^{\ell/2}\mathrm{d}y\int_{-\beta\hbar\Omega/2}^{\beta\hbar\Omega/2}\mathrm{d}t\int_{-\beta\hbar\Omega/2}^{\beta\hbar\Omega/2}\mathrm{d}t'\ \eta(t-t')\left[\phi(y,t)-\phi(y,t')\right]^2 \end{aligned} \tag{11}$$

where

$$S_0 = \frac{9(1-\mu^2)W\lambda_J\hbar}{4\mu^2}\left(\frac{2\hbar J_c\mathcal{C}}{e^3}\right)^{1/2} \tag{12}$$

and

$$u(\phi) = \frac{1}{2}\phi^2(1-\phi). \tag{13}$$

$\eta(t)$ is given by

$$\eta(t) = \frac{1}{\beta\hbar\Omega} \sum_{n=-\infty}^{\infty} \eta_n \exp(\mathrm{i}\nu_n t) \tag{14}$$

where $\nu_n \equiv 2\pi n/\beta\hbar\Omega$ and [1]

$$\eta_n = -\frac{2nk_B T}{\hbar W L \mathcal{C} \Omega^2} \left[ \frac{1}{R_q} \cot^{-1}\left( \frac{\pi n k_B T}{\Delta} \right) + \frac{1}{R_N} \tan^{-1}\left( \frac{\pi n k_B T}{\Delta} \right) \right]. \tag{15}$$

## 3. Thermal nucleation

We start by considering thermal nucleation in a junction with finite length at temperatures very much above the crossover temperature $T_0$ (see below) between quantum and thermal nucleation. In this case, the phase $\phi(y, t)$ is $t$-independent and the usual method of path integration with the action of equation (11) gives the thermal nucleation rate at $T \gg T_0$ as [1]

$$\Gamma = \frac{\ell k_B T_0}{\hbar} \left[ \prod_{n=1;\alpha=0}^{\infty} \frac{\lambda^0_{n\alpha}}{\lambda_{n\alpha}} \right] \left[ \prod_{\alpha=1} \frac{\lambda^0_{0\alpha}}{\lambda_{0\alpha}} \right]^{1/2} \times \left[ \frac{\beta\Omega S_0}{2\pi} \int_{-\ell/2}^{\ell/2} \mathrm{d}y \left( \frac{\mathrm{d}\phi_c(y)}{\mathrm{d}y} \right)^2 \right]^{1/2} \exp\left[ -(S[\phi_c(y)])/\hbar \right] \tag{16}$$

where

$$S[\phi_c(y)] = \beta\hbar\Omega S_0 \int_{-\ell/2}^{\ell/2} \mathrm{d}y \left[ \frac{1}{2} \left( \frac{\mathrm{d}}{\mathrm{d}y} \phi_c(y) \right)^2 + u(\phi_c) \right]. \tag{17}$$

Note that $S[\phi_c(y)]/\hbar$ is simply $E_n/kT$, where $E_n$ is the energy of a nucleation taking place at the top of the energy barrier in the function space of the phase $\phi(y)$.

The equation of motion for $\phi_c(y)$ is given by $\delta S[\phi_c(y)] = 0$, where it follows that

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2} \phi_c(y) + \phi_c(y) \left( 1 - \frac{3}{2} \phi_c(y) \right) = 0 \tag{18}$$

with the boundary conditions

$$\left. \frac{\mathrm{d}\phi_c(y)}{\mathrm{d}y} \right|_{-\ell/2} = \left. \frac{\mathrm{d}\phi_c(y)}{\mathrm{d}y} \right|_{\ell/2} = 0. \tag{19}$$

The eigenvalues $\lambda^0_{n,\alpha}$ and $\lambda_{n,\alpha}$ in equation (16) are respectively

$$\lambda^0_{n,\alpha} = {\nu_n}^2 - 2\eta_n + k^0_\alpha \qquad \lambda_{n,\alpha} = {\nu_n}^2 - 2\eta_n + k_\alpha \qquad n = 0, \pm 1, \ldots \tag{20}$$

where $k^0_\alpha$ and $k_\alpha$ satisfy

$$\left[ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 1 \right] Q^0_\alpha(y) = k^0_\alpha Q^0_\alpha(y) \tag{21}$$

$$\left[ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 1 - 3\phi_c(y) \right] Q_\alpha(y) = k_\alpha Q_\alpha(y). \tag{22}$$

The crossover temperature $T_0$ is given by the equation ${\nu_1}^2 - 2\eta_1 - 1 = 0$ (see reference [6] and also below), so we obtain

$$1 = \left( \frac{2\pi k_B T_0}{\hbar\Omega} \right)^2 + \frac{4k_B T_0}{\hbar W L \mathcal{C} \Omega^2} \left[ \frac{1}{R_q} \cot^{-1}\left( \frac{\pi k_B T_0}{\Delta} \right) + \frac{1}{R_N} \tan^{-1}\left( \frac{\pi k_B T_0}{\Delta} \right) \right]. \tag{23}$$

For $k_B T_0 \ll 2\Delta$ (i.e., $T_0 \ll T_c$, where $T_c$ is the critical temperature of the superconductors), we can neglect the $R_q$-term and also approximate $\tan^{-1} x$ by $x$. In this approximation, we have

$$k_B T_0 \approx \frac{\hbar\Omega}{2\pi}\left[1+\frac{\hbar}{\pi W L \mathcal{C} R_N \Delta}\right]^{-1/2}. \tag{24}$$

We now need to find the solution $\phi_c(y)$ for the exponent of the rate. There are two solutions of $\phi_c(y)$ in equation (18) and they represent the two possible nucleations at the two respective boundaries of the junction located at $y = \pm\ell/2$. These solutions are given respectively by

$$\phi_c(y) = \phi_1 - (\phi_1 - \phi_2)\mathrm{sn}^2\left(\frac{1}{2}\left(y-\frac{\ell}{2}\right)(\phi_1-\phi_3)^{1/2}\right) \tag{25}$$

and

$$\phi_c(y) = \phi_1 - (\phi_1 - \phi_2)\mathrm{sn}^2\left(\frac{1}{2}\left(y+\frac{\ell}{2}\right)(\phi_1-\phi_3)^{1/2}\right) \tag{26}$$

where $\mathrm{sn}(x)$ is the Jacobi sine-amplitude function [7] and the constants $\phi_i$ with $\phi_1 > \phi_2 > \phi_3$ are the roots of $\phi^2 - \phi^3 = E(\ell)$ with $E(\ell)$ the constant of motion of equation (18). The roots also satisfy the condition

$$\frac{2}{\sqrt{\phi_1-\phi_3}}K(k) = \ell \tag{27}$$

where $K(k)$ is the complete elliptic integral of the first kind [7], and $k$ is given by

$$k = \left(\frac{\phi_1-\phi_2}{\phi_1-\phi_3}\right)^{1/2}. \tag{28}$$

We have found the solution $\phi_c(y)$ for a junction with finite length. On the other hand, when the junction is small, $\phi_c(y)$ is $y$-independent. Therefore, there exists a critical length that separates a small and a large junction. This critical length is given when the solution $\phi_c(y)$ is uniform, in which case $\phi_c(y) = \phi_1 = \phi_2 = 2/3$. The reduced critical length $\ell_c$ is determined from equation (27) as

$$\ell_c = \pi. \tag{29}$$

The actual critical length of the junction is given by $L_c = \pi\Lambda_J$. Our goal now is to find the rates of nucleation for various finite junction lengths.

Let us now consider the exponent $S[\phi_c(y)]$. First, for a small junction with $0 \leqslant \ell \leqslant \ell_c$ the solution $\phi_c(y) = \phi_1 = \phi_2 = 2/3$ gives the exponent $S\left[\phi_c(y)\right]$ in equation (17) as

$$S\left[\phi_c(y)\right] = \frac{2}{27}\beta\hbar\Omega S_0\ell = \frac{(1-\mu^2)^{5/4}}{6\mu^2}\left(\frac{2\hbar J_c\mathcal{C}}{e^3}\right)^{1/2} WL\frac{\hbar^2\Omega}{k_B T} = \frac{(2\epsilon)^{3/2}}{3}\frac{\hbar^2 J_c}{ek_B T}WL \tag{30}$$

where $\epsilon \equiv 1-\mu \ll 1$. Thus, we have reproduced the result of reference [6].

Now, with the solution in equation (25) (or, equation (26)), we find the exponent $S[\phi_c(y)]$ in equation (17) for a junction with length $L \geqslant L_c$. We obtain

$$S\left[\phi_c(y)\right] = \beta\hbar\Omega S_0\left[\frac{E(\ell)}{2}\ell + \frac{\pi}{8}(\phi_1-\phi_2)^2(\phi_1-\phi_3)^{1/2}\ F\left(-\frac{1}{2},\frac{3}{2},3,k^2\right)\right] \tag{31}$$

where $F(a,b,c,z)$ is the hypergeometric function [7]. This result reduces to equation (30) when $L = L_c$. With the results in equations (30) and (31) we have plotted the reduced exponent $S[\phi_c(y)]/\beta\hbar\Omega S_0$ as a function of $\ell$ in figure 1. As $\ell \to \infty$, $S[\phi_c(y)]/\beta\hbar\Omega S_0 \to 4/15$.

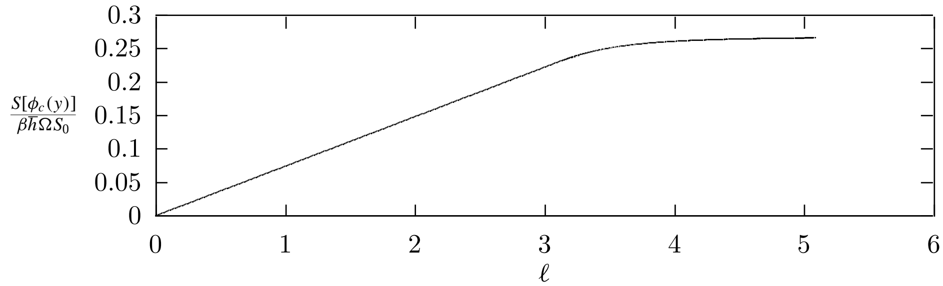


**Figure 1.** The reduced exponent $S[\phi_c(y)]/\beta\hbar\Omega S_0$ as a function of the reduced length $\ell$.

For the integral in the prefactor in equation (16), we obtain

$$\int_{-\ell/2}^{\ell/2} \mathrm{d}y \left(\frac{\mathrm{d}\phi_c(y)}{\mathrm{d}y}\right)^2 = \frac{\pi}{8}(\phi_1-\phi_2)^2(\phi_1-\phi_3)^{1/2} F\left(-\frac{1}{2}, \frac{3}{2}, 3, k^2\right). \tag{32}$$

We have not been able to solve for the product of eigenvalues in equation (16) for general $L$. However, we can deal with the case of thermal nucleation in a junction with length about equal to the critical length (i.e., $L \sim L_c$) at high temperatures. In this case $\phi_c(y) \sim \phi_{\max} = 2/3$ where $\phi_{\max}$ is the position of the maximum of the potential $u(\phi)$.

For the action in equation (11), the method of path integration will give us the thermal nucleation rate for $T \gg T_0$ and $L \sim L_c$ as

$$\Gamma = \frac{k_B T_0}{\hbar} \exp[-2S_0\Omega\ell/27k_B T] \left[\prod_{n=1;\alpha=0}^{\infty} \frac{\lambda_{n\alpha}^0}{\lambda_{n\alpha}}\right] \left[\prod_{\alpha=2} \frac{\lambda_{0\alpha}^0}{\lambda_{0\alpha}}\right]^{1/2}$$
$$\times \left(\frac{\lambda_{00}^0}{|\lambda_{00}|}\right)^{1/2} \left(\frac{S_0\lambda_{01}^0}{2\pi\hbar}\right)^{1/2} \frac{g}{\sqrt{2}} \exp(z) K_{1/4}(z) \tag{33}$$

where $K_\nu(z)$ is the modified Bessel function [7]. Here, we also have $\lambda_{n\alpha}^0$ and $\lambda_{n\alpha}$ as in equation (20) but now with $\phi_c(y) = \phi_{\max} = 2/3$, i.e.,

$$\lambda_{n\alpha} = {\nu_n}^2 - 2\eta_n + \left(\frac{\pi\alpha}{\ell}\right)^2 - 1 \qquad \begin{cases} n = 0, \pm 1, \ldots \\ \alpha = 0, 1, \ldots \end{cases} \tag{34}$$

$$\lambda_{n\alpha}^0 = {\nu_n}^2 - 2\eta_n + \left(\frac{\pi\alpha}{\ell}\right)^2 + 1 \qquad \begin{cases} n = 0, \pm 1, \ldots \\ \alpha = 0, 1, \ldots . \end{cases} \tag{35}$$

In equation (33), we have $z$ and $g$ respectively as

$$z = \frac{{\lambda_{01}}^2 S_0 \beta\Omega\ell}{36(1/|\lambda_{00}| - 1/2\lambda_{02})} = \frac{S_0\Omega\ell}{18k_B T}\left(\frac{4\pi^2-\ell^2}{8\pi^2-3\ell^2}\right)\left(\frac{\pi^2}{\ell^2}-1\right)^2 \tag{36}$$

and

$$g = \frac{(\lambda_{01}\beta\hbar\Omega\ell)^{1/2}}{3(1/2|\lambda_{00}| - 1/4\lambda_{02})^{1/2}} = \frac{2}{3}\left(\frac{\hbar\Omega\ell}{k_B T}\right)^{1/2}\left(\frac{\pi^2}{\ell^2}-1\right)^{1/2}\left(\frac{4\pi^2-\ell^2}{8\pi^2-3\ell^2}\right)^{1/2}. \tag{37}$$

Thus, equation (33) becomes

$$\Gamma = \frac{k_B T_0}{\hbar} \exp[-2S_0\Omega\ell/27k_B T] \left[\prod_{n=1;\alpha=0}^{\infty} \frac{\lambda_{n\alpha}^0}{\lambda_{n\alpha}}\right] \left[\prod_{\alpha=2} \frac{\lambda_{0\alpha}^0}{\lambda_{0\alpha}}\right]^{1/2}$$
$$\times \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \left[\left(\frac{\pi}{\ell}\right)^2 + 1\right]^{1/2} \frac{g}{\sqrt{2}} \exp(z) K_{1/4}(z). \tag{38}$$

Since $\lambda_{01} \to 0$ when $\ell \to \ell_c$, we can simplify equation (38) so that the thermal nucleation rate for $T \gg T_0$ and $L \sim L_c$ becomes

$$\Gamma \approx \frac{k_B T_0}{\hbar} \exp[-2S_0\Omega\ell/27k_B T] \left[\prod_{n=1;\alpha=0}^{\infty} \frac{\lambda_{n\alpha}^0}{\lambda_{n\alpha}}\right] \left[\prod_{\alpha=2} \frac{\lambda_{0\alpha}^0}{\lambda_{0\alpha}}\right]^{1/2} \times \left[\left(\frac{\pi}{\ell}\right)^2 + 1\right]^{1/2} \left[\frac{16S_0\Omega\ell}{9k_B T}\left(\frac{4\pi^2 - \ell^2}{8\pi^2 - 3\ell^2}\right)\right]^{1/4} \frac{\Gamma(1/4)}{2\sqrt{2\pi}} \times \exp\left[\frac{S_0\Omega\ell}{18k_B T}\left(\frac{4\pi^2 - \ell^2}{8\pi^2 - 3\ell^2}\right)\left(\left(\frac{\pi}{\ell}\right)^2 - 1\right)^2\right] \tag{39}$$

where $\Gamma(x)$ is the Gamma function [7].

It remains for us to find the product of eigenvalues in equation (39). With the eigenvalues in equations (34) and (35) it is difficult to calculate the product for general dissipation. However, for low dissipation, which is usually the case, we again neglect the $R_q$-term in $\eta_n$ and approximate $\tan^{-1} x$ with $x$ so that the product over $n$ gives us

$$\left[\prod_{n=1;\alpha=0}^{\infty} \frac{\lambda_{n\alpha}^0}{\lambda_{n\alpha}}\right] \left[\prod_{\alpha=2} \frac{\lambda_{0\alpha}^0}{\lambda_{0\alpha}}\right]^{1/2} \approx \frac{D}{\pi} \; \frac{\sinh(\pi/D)}{\sin(\pi/D)} \; \frac{\sinh((\pi/D)\sqrt{(\pi/\ell)^2+1})}{\sqrt{(\pi/\ell)^2+1}} \times \prod_{\alpha=2}^{\alpha_c} \left[\frac{\sinh((\pi/D)\sqrt{(\pi\alpha/\ell)^2+1})}{\sinh((\pi/D)\sqrt{(\pi\alpha/\ell)^2-1})}\right]. \tag{40}$$

Here we have

$$D \equiv \left(1 + \frac{\hbar}{\pi W L \mathcal{C} R_N \Delta}\right)^{1/2} \left(\frac{2\pi k_B T}{\hbar\Omega}\right) \tag{41}$$

and we have introduced a cut-off $\alpha_c$ on the product. The need for this cut-off is to avoid the infrared divergence as a result of the breakdown of the model Lagrangian at small wavelength [1]. This short wavelength is given by the coherence length $\xi$ of the superconductors of the junction, so the cut-off is given by

$$\alpha_c = \frac{\ell}{\pi} \frac{1}{(2\epsilon)^{1/4}} \frac{\lambda_J}{\xi} \;\sim\; \frac{1}{(2\epsilon)^{1/4}} \frac{\lambda_J}{\xi}. \tag{42}$$

The remaining product in equation (40) can now be computed numerically for a given set of junction parameters.

## 4. Quantum nucleation

We now move on to the case of quantum nucleation in a junction with length $L \sim L_c$ (see below) at low temperatures. It is difficult to solve the nucleation rate for arbitrary dissipation and temperature. However, for low dissipation, which is usually the case, the effect of dissipation is to renormalize the capacitance [1] to the undamped case, so we can set $\eta(t) \equiv 0$ but replace $\mathcal{C}$ by $\mathcal{C}^*$ as [1]

$$\mathcal{C}^* = \mathcal{C} + \frac{\hbar}{\pi R_N \Delta W L}. \tag{43}$$

The method of path integration [8] will lead us to the quantum nucleation rate as

$$\Gamma = \frac{\Omega^* S_0}{2\pi\hbar} \exp(-S[\phi_c(t)]/\hbar) \left[\ell \int_{-\beta'/2}^{\beta'/2} \mathrm{d}t \; \left[\frac{\mathrm{d}\phi_c(t)}{\mathrm{d}t}\right]^2\right]^{1/2}$$

$$\times \frac{g}{\sqrt{2}} \mathrm{e}^{z} K_{1/4}(z) \Bigg[ \prod_{n=0;\alpha=0} \lambda^{0}_{n\alpha} \Bigg/ \underbrace{\prod_{n=0;\alpha=0} |\lambda_{n\alpha}|}_{(n,\alpha)\neq(1,0),(0,1)} \Bigg]^{1/2} \tag{44}$$

where $\beta' \equiv \beta\hbar\Omega^*$ with $\Omega^*$ given by equation (8) but with $\mathcal{C}$ replaced by $\mathcal{C}^*$, and the parameters $g$ and $z$ are given below. The action $S[\phi_c(t)]$ now becomes

$$S[\phi_c(t)] = S_0\ell \int_{-\beta'/2}^{\beta'/2} \mathrm{d}t \, \left[ \frac{1}{2}\left(\frac{\mathrm{d}\phi_c}{\mathrm{d}t}\right)^2 + u(\phi_c) \right] \tag{45}$$

and $\phi_c(t)$ is periodic (with period $\beta'$) and satisfies the equation of motion $\delta S[\phi_c(t)] = 0$, which is like equation (18) but with the variable $y$ replaced by $t$.

The eigenvalues $\lambda^0_{n\alpha}$ and $\lambda_{n\alpha}$ are given by [8]

$$\lambda^0_{n\alpha} \equiv \gamma^0_n + \left(\frac{\pi\alpha}{\ell}\right)^2 \qquad \lambda_{n\alpha} \equiv \gamma_n + \left(\frac{\pi\alpha}{\ell}\right)^2 \qquad \begin{cases} n = 0, 1, \ldots \\ \alpha = 0, 1, \ldots \end{cases} \tag{46}$$

where $\gamma^0_n$ and $\gamma_n$ and their corresponding periodic eigenfunctions $Q^0_n(t)$ and $Q_n(t)$ satisfy the eigenvalue equations like those in equations (21) and (22) but with the variable $y$ replaced by $t$. In equation (44), we have used the notation $\lambda_{00}$ and $\lambda_{10}$ for the negative and zero eigenvalues, respectively. We now have

$$z = \frac{{\lambda_{01}}^2 S_0\ell}{4\hbar[P(\beta')]^2(1/|\lambda_{00}| - 1/2\lambda_{02})} \tag{47}$$

and

$$g = \frac{(\lambda_{01}\ell)^{1/2}}{|P(\beta')|(1/2|\lambda_{00}| - 1/4\lambda_{02})^{1/2}} \tag{48}$$

where

$$P(\beta') \equiv \int_{-\beta'/2}^{\beta'/2} \mathrm{d}t \; u'''(\phi_c(t))[Q_0(t)]^3. \tag{49}$$

The solution of $\phi_c(t)$ for finite $T$ is known [9]. At $T = 0$, $\phi_c(t) = \mathrm{sech}^2(t/2)$. For $T \ll \hbar\Omega^*/k \ln 64 \sim T_0$, the correction to the solution at $T = 0$ is exponentially small. We will therefore use the $T = 0$ solution for such small $T$, obtaining for the exponent at low temperatures

$$S[\phi_c(t)] = \frac{8}{15} S_0\ell[1 - 60\mathrm{e}^{-\hbar\Omega^*/kT}]. \tag{50}$$

It is difficult to find the eigenvalues for arbitrary temperatures. However, with the exponentially small correction to the exponent, we may now approximate the prefactor of the rate with its value at zero temperature. In this case, the use of $\phi_c(t) = \mathrm{sech}^2(t/2)$ gives $\lambda_{00} = -5/4$ and $Q_0(t) = (15/32)^{1/2}\mathrm{sech}^3(t/2)$, so we obtain $P(\infty) = -(105\pi/64)(15/32)^{3/2}$. We also have

$$z = \frac{4\pi^2}{21}\left(\frac{32}{15}\right)^3 \left[\left(\frac{\pi}{\ell}\right)^2 - \frac{5}{4}\right]^2 \left[1 - \frac{1}{2}\left(\frac{4}{5}\left(\frac{2\pi}{\ell}\right)^2 - 1\right)^{-1}\right]^{-1} \frac{S_0\ell}{\hbar} \tag{51}$$

and

$$g = \frac{1}{\pi}\left(\frac{32}{15}\right)^2 \left(\frac{5\ell}{7}\right)^{1/2} \left[\left(\frac{\pi}{\ell}\right)^2 - \frac{5}{4}\right]^{1/2} \left[1 - \frac{1}{2}\left[\frac{4}{5}\left(\frac{2\pi}{\ell}\right)^2 - 1\right]^{-1}\right]^{-1/2}. \tag{52}$$

The critical length is obtained by setting $\lambda_{01} = 0$, which leads to $\ell_c = \pi/\sqrt{|\lambda_{00}|}$. Thus the critical length at $T = 0$ becomes

$$L_c = \frac{2\pi}{\sqrt{5}}\Lambda_J. \tag{53}$$

The quantum nucleation rate at low temperatures becomes approximately

$$\begin{aligned}\Gamma \approx 4\Omega^*\left(\frac{S_0\ell}{2\pi\hbar}\right)^{1/2} g K_{1/4}(z)\exp(z)\exp(-S[\phi_c(t)]/\hbar)\\ \times\left[\frac{(\sqrt{(\pi/\ell)^2+1}\ +\frac{1}{2})(\sqrt{(\pi/\ell)^2+1}\ +1)}{(\sqrt{(\pi/\ell)^2+1}\ -\frac{1}{2})(\sqrt{(\pi/\ell)^2+1}\ -1)}\right]^{1/2}\\ \times\left(\sqrt{(\pi/\ell)^2+1}\ +\frac{3}{2}\right)Q^{1/2}\end{aligned} \tag{54}$$

$$\begin{aligned}\Gamma \approx \frac{2}{\sqrt{\pi}}\Gamma(1/4)\Omega^*\left(\frac{S_0\ell}{2\pi\hbar}\right)^{1/2}\left(\frac{S_0\ell}{\hbar\ a}\right)^{1/4}\exp(z)\exp(-S[\phi_c(t)]/\hbar)\\ \times\left[\frac{(\sqrt{(\pi/\ell)^2+1}\ +\frac{1}{2})(\sqrt{(\pi/\ell)^2+1}\ +1)}{(\sqrt{(\pi/\ell)^2+1}\ -\frac{1}{2})(\sqrt{(\pi/\ell)^2+1}\ -1)}\right]^{1/2}\\ \times\left(\sqrt{(\pi/\ell)^2+1}+\frac{3}{2}\right)Q^{1/2}.\end{aligned} \tag{55}$$

$S[\phi_c(t)]$ is given by equation (50) and the factor $a$ is

$$a = \frac{1}{10}\left(\frac{105\pi}{64}\right)^2\left(\frac{15}{32}\right)^3\left(1-\frac{5\ell^2}{32\pi^2-10\ell^2}\right) \tag{56}$$

and the factor $Q$ is

$$Q = \prod_{\alpha=2}^{\alpha_c}\left[\frac{(2\sqrt{(\pi\alpha/\ell)^2+1}+1)(2\sqrt{(\pi\alpha/\ell)^2+1}+2)(2\sqrt{(\pi\alpha/\ell)^2+1}+3)}{(2\sqrt{(\pi\alpha/\ell)^2+1}-1)(2\sqrt{(\pi\alpha/\ell)^2+1}-2)(2\sqrt{(\pi\alpha/\ell)^2+1}-3)}\right] \tag{57}$$

where $\alpha_c$ is again the cut-off in equation (42). Keeping only the highest contribution from the cut-off, we may approximate $Q$ with

$$\ln Q \approx \frac{6\ell}{\pi}\ln\alpha_c. \tag{58}$$

A further temperature correction to the quantum nucleation rate at low temperatures would be to multiply equation (55) (or equation (54)) by

$$1 + 30\beta\hbar\Omega^*\exp(-\beta\hbar\Omega^*).$$

## 5. Summary

We have calculated the thermal and quantum nucleation rates in a finite Josephson junction for various lengths relative to the Josephson penetration depth. The results are given for low and high temperatures relative to the crossover temperature $T_0$ between quantum and thermal nucleation. The critical length for thermal nucleation is $L_c = \pi\Lambda_J$, and $L_c = 2\pi\Lambda_J/\sqrt{5}$ for quantum nucleation at $T = 0$. It would be most interesting to study the quantum tunnelling regime below $T_0$ for $\ell > \ell_c$. This problem is made difficult by the need to obtain the bounce trajectory in both space and time. Finally, it should be recognized that while our results have been discussed within the framework of nucleation along a Josephson junction, our results can be applied to other nucleation processes, such as nucleation of a magnetic domain wall along a wire, by replacing some of the parameters and the potential.

## References

[1] Simanjuntak H and Gunther L 1990 *Phys Rev.* B **42** 930
[2] Simanjuntak H and Gunther L 1990 *Bull. Am. Phys. Soc.* **35** 628
The critical length $\pi\Lambda_J$ in the thermal activation regime was discussed at the talk delivered by LG at the APS meeting in Anaheim, CA, in March, 1990.
[3] Leggett A J 1992 *Quantum Tunnelling in Condensed Media* ed Yu Kagan and A J Leggett (Amsterdam: North-Holland)
[4] Leggett A J, Chakravarty S, Dorsey A, Fisher M P A, Garg A and Zwerger W 1987 *Rev. Mod. Phys.* **59** 1
[5] Barone A and Paterno G 1982 *Physics and Applications of the Josephson Effect* (New York: Wiley) ch 5
[6] Simanjuntak H and Gunther L 1990 *J. Low. Temp. Phys.* **78** 131
[7] *Table of Integrals, Series and Products* 1980 ed I S Gradshteyn and I M Ryzhik (New York: Academic)
[8] With $\phi(y,t) = \phi_c(t) + \sum_{n,\alpha} Q_n(t)\cos(\alpha\pi(y+\ell/2)/\ell)$; this work will be published elsewhere (HPS).
[9] Larkin A I and Ovchinnikov Yu N 1983 *Sov. Phys.–JETP* **58** 867