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Tunable ground states in helical $p$-wave Josephson junctions

Qiang Cheng$^{1,6}$, Kunhua Zhang$^2$, Dongyang Yu$^3$, Chongju Chen$^4$, Yinhan Zhang$^{5,6}$ and Biao Jin$^{4,6}$

$^1$ School of Science, Qingdao Technological University, Qingdao, Shandong 266520, People’s Republic of China
$^2$ ICQD, Hefei National Laboratory for Physical Sciences at Microscale, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China
$^3$ Department of Physics, Renmin University of China, Beijing 100872, People’s Republic of China
$^4$ School of Physics, University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
$^5$ International Center for Quantum Materials, Peking University, Beijing 100871, People’s Republic of China

E-mail: chengqiang07@mails.ucas.ac.cn, zhangyinhan2008@gmail.com and biaojin@ucas.ac.cn

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Abstract

We study new types of Josephson junctions composed of helical $p$-wave superconductors with $k_x \hat{x} \pm k_y \hat{y}$ and $k_x \hat{x} \pm k_y \hat{y}$-pairing symmetries using quasi-classical Green’s functions with generalized Riccati parametrization. The junctions can host rich ground states: $\pi$ phase, $0 + \pi$ phase, $j_0$ phase and $j_1$ phase. The phase transition can be tuned by rotating the magnetization in the ferromagnetic interface. We present the phase diagrams in the parameter space formed by the orientation of the magnetization or by the magnitude of the interfacial potentials. The selection rules for the lowest order current which are responsible for the formation of the rich phases are summarized from the current-phase relations based on the numerical calculation. We construct a Ginzburg–Landau type of free energy for the junctions with $d$-vectors and the magnetization, which not only reveals the interaction forms of spin-triplet superconductivity and ferromagnetism, but can also directly lead to the selection rules. In addition, the energies of the Andreev bound states and the novel symmetries in the current-phase relations are also investigated. Our results are helpful both in the prediction of novel Josephson phases and in the design of quantum circuits.

Keywords: Josephson junctions, helical superconductivity, ferromagnetism

(Some figures may appear in colour only in the online journal)

1. Introduction

Josephson junctions have been the subject of continuously growing interest because of rich ground states in these systems and their potential applications in superconducting electronics [1–5]. The ground states can be classified into 0 phase, $\pi$ phase, $\varphi_0$ phase and $\varphi$ phase according to the number and the position of the energy minimum within $2\pi$ intervals of the superconducting phase $\varphi$ across the junctions. The junctions in the 0 phase and $\pi$ phase, which have been realized experimentally [6–9], have an energy minimum at $\varphi = 0$ and $\varphi = \pi$ [10, 11], respectively, while the $\varphi_0$ junctions have a single energy minimum at $\varphi = \varphi_0 \neq 0, \pi$ as predicted in [12]. The $\varphi$-phase, unlike the other phases, is a doubly degenerate state which possesses two energy minima at $\varphi = \pm \varphi$ [13–19]. Several schemes to realize the $\varphi_0$ phase and the $\varphi$ phase have been proposed [20–22]. For example, it is expected that the latter
phase can be realized in periodic alternating 0 and \( \pi \) junction structures [23]; recently the evidence of this phase has been found experimentally [24]. In fact, Josephson junctions can also host a mixture of the states, such as \( \varphi_0 \pm \varphi \) phases proposed by Goldobin et al [21] more recently.

The formation of rich phases in Josephson junctions is based on the current-phase relations (CPRs). Generally, the Josephson current can be expressed as the composition of the harmonics \( \sin n\phi \) and \( \cos n\phi \), in which the integer number \( n \) denotes the \( n \)-th order contribution. It is demonstrated that the lowest order current (LOC) with \( n = 1 \), \( \sin \phi \) or \( \cos \phi \), is absent in spin-singlet superconductor|spin-triplet superconductor junctions due to the orthogonality of the Cooper-pair wave functions [25]. However, the situation will change, as predicted in [25], when the interface is magnetically active. For example, the interface which is a ferromagnetic barrier or of spin–orbit coupling can lead to a Josephson current proportional to \( \cos \phi \) when the triplet superconductor is in the chiral \( p \)-wave state [26, 27]. Furthermore, the dependence of CPRs on the magnetization in the barrier can bring different phases in spin-triplet Josephson junctions with \( p \)-wave pairing. The \( 0-\pi \) transition has been found when the superconductor is characterized by the \( d \)-vector with a uniform direction [28, 29]. Nevertheless, since the direction of the \( d \)-vector is independent of wavevectors, more phases cannot be expected in the junctions, although the interplay between ferromagnetism and triplet superconductivity can give many interesting and important physical results [30–32].

In this paper, we propose a concise scheme to realize rich ground states in Josephson junctions consisting of helical \( p \)-wave superconductors (HPSs) with pairing symmetries \( k_x \hat{x} \pm k_y \hat{y} \) and \( k_x \hat{x} \pm k_y \hat{y} \) and a ferromagnet (F). We are interested in these helical superconducting states for many reasons. The states, with \( d \)-vectors pinned in the crystallographic \( ab \)-plane, are candidates for pairing in Sr$_2$RuO$_4$ [33–35] and the triplet part of the order parameter in the non-centrosymmetric superconductor CePt$_3$Si [34, 36]. Further, \( k_x \hat{x} + k_y \hat{y} \) is the two-dimensional analog of the Balian–Werthamer state (B phase) in $^3$He [33, 34]; \( k_x \hat{x} - k_y \hat{y} \) is analogous to the quantum spin Hall system [37]. Recently, new symmetries of charge conductance in F|HPS junctions [38] and peculiar features of spin accumulation in spin-singlet superconductor|HPS junctions [39] were found. The selection rules for LOC in spin-singlet superconductor|HPS junctions are also summarized [40] and are distinct from those in the junctions involving a triplet superconductor described by a uniform \( d \)-vector [41]. As a result, it is reasonable to expect anomalous Josephson effects in the helical \( p \)-wave Josephson junctions. How CPRs depend on the orientation of the magnetization and which phases the junctions can host are questions which remain to be answered.

In the present work, we systematically study CPRs and ground states of HPS|F/HPS junctions using the method of quasi-classical Green’s functions with generalized Riccati parametrization [42, 43]. In order to conveniently describe the anisotropic superconductor in the junctions, we show explicitly the diagrammatic representation of the boundary conditions for this method. Through numerical calculations, we find the junctions can host the 0 phase, 0 + \( \pi \) phase, \( \pi \) phase, \( \varphi_0 \) phase and \( \varphi \) phase, where the 0 + \( \pi \) phase is a new ground state in which the free energy has two minima at \( \phi = 0 \) and \( \phi = \pi \). The transition from one phase to another can be realized through controlling the direction of the magnetization with a weak external field. The phase diagrams are presented in the orientation space of the magnetization or in the space spanned by the magnitude of the magnetization and the non-magnetic potential. The selection rules for LOC are derived from CPRs which are responsible for the formation of rich phases. In order to explain the rules, we construct a Ginzburg–Landau type of free energy of the junctions with \( d \)-vectors in HPSs and the magnetization in F, which reveals the interaction mechanism between the helical \( p \)-wave superconductivity and ferromagnetism. We also clarify the Andreev bound states (ABS) formed at the interface and the novel symmetries in CPRs.

The paper is organized as follows. In section 2, we establish the theoretical framework which will be used to obtain the results. In section 3, we present the detailed numerical results for the junctions. The features of CPRs and phase diagrams are also covered. In section 4, we further discuss the selection rules for LOC from the viewpoint of free energy. Section 5 concludes the work.

2. Quasi-classical Green’s function formalism

We consider the Josephson junctions in the clean limit as shown in figure 1. The barrier, located at \( x = 0 \), with its interface along the \( y \)-axis, is modeled by a delta function 
U(x) = (U_0 + M \cdot \hat{\sigma})\delta(x)

in which \( U_0 \) and \( M \cdot \hat{\sigma} \) denote the non-magnetic potential and the ferromagnetic term, respectively. The magnetization \( M = M_0 (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m) \) where \( \theta_m \) is the polar angle and \( \phi_m \) is the azimuthal angle.
which span the orientation space of the magnetization. For the superconductors, we consider the following helical states,

\[
\begin{align*}
\mathbf{d}_1 &= \Delta_0(k_x + k_y), \\
\mathbf{d}_2 &= \Delta_0(k_x - k_y), \\
\mathbf{d}_3 &= \Delta_0(k_x + k_y), \\
\mathbf{d}_4 &= \Delta_0(k_x - k_y),
\end{align*}
\]

with \(\Delta_0\) the temperature-dependent gap magnitude which is determined by the Bardeen–Cooper–Schrieffer-type equation. For simplicity, we use \(\mathbf{HPS}\) to denote the helical \(p\)-wave superconductor with the \(\mathbf{d}_i\)-vector.

The HPS can be described by the quasi-classical Green’s function \(g\), a \(2 \times 2\) matrix in Keldysh space, which is solution of the Eilenberger equation with the normalization condition \(g \otimes g = -\pi^2 i\). For the physical quantities involved in this paper, it is sufficient to obtain the retarded Green’s function \(g^R\), which is the upper-left element of \(g\). The retarded Green’s function \(g^R\), a \(4 \times 4\) matrix in spin\(\otimes\)particle-hole space, can be written as \[42\]

\[
g^R = -2\pi i \begin{pmatrix} g & f \\ -f & -g \end{pmatrix} + i\pi \tilde{n},
\]

with the parametrization

\[
g = (1 - \gamma \tilde{\gamma})^{-1}, \quad f = (1 - \gamma \tilde{\gamma})^{-1}\gamma, \quad \tilde{g} = (1 - \tilde{\gamma} \gamma)^{-1}, \quad \tilde{f} = (1 - \tilde{\gamma} \gamma)^{-1}\tilde{\gamma},
\]

in which \(\gamma\) and \(\tilde{\gamma}\) are the retarded coherence functions. Physically, \(\gamma\) (\(\tilde{\gamma}\)) describes the probability amplitude for conversion of a hole (particle) to a particle (hole). The coherence functions, \(2 \times 2\) matrices in spin space, are a generalization of the so-called Riccati amplitudes. For simplicity, we have omitted the superscript \(R\) for the retarded functions \(g, f, \tilde{g}, \tilde{f}, \gamma\) and \(\tilde{\gamma}\).

The coherence functions obey the Riccati-type transport equations

\[
\begin{align*}
(i\hbar \mathbf{v}_F \cdot \nabla + 2\xi\varepsilon) \gamma &= \gamma \Delta \gamma - \Delta, \\
(i\hbar \mathbf{v}_F \cdot \nabla - 2\xi\varepsilon) \tilde{\gamma} &= \tilde{\gamma} \Delta \tilde{\gamma} - \Delta,
\end{align*}
\]

with boundary (initial) conditions, which are numerically stable. Here, \(\mathbf{v}_F\) is the Fermi velocity, \(\varepsilon\) the quasi-particle energy measured from the Fermi energy, and \(\Delta\) the energy-gap matrix with the relation \(\Delta(k) = [\Delta(-k)]^T\). As in [42], we use \(\gamma, \tilde{\gamma}\) and \(\Gamma, \tilde{\Gamma}\) in the following to denote the incoming and outgoing quantities, respectively. The quasi-classical Green’s function characterized by the Fermi momentum \(p_F\) is composed of both incoming and outgoing quantities. The solutions for \(\gamma_1, \tilde{\gamma}_1\) and \(\gamma_2, \tilde{\gamma}_2\) in the left (subscript 1) and the right (subscript 2) superconductor are stable when integrating the equations from the bulk to the interface; the initial conditions are their bulk values in the superconductor (see appendix A). The solutions for \(\tilde{\Gamma}_1, \tilde{\Gamma}_1^*\) and \(\tilde{\Gamma}_2, \tilde{\Gamma}_2^*\) are stable when integrating the equations from the interface to the bulk; the initial conditions are their values at the interface which can be expressed by the incoming quantities and the scattering matrix \(\tilde{S}\) in the normal state. For example, \(\tilde{\Gamma}_1\) can be written as

\[
\tilde{\Gamma}_1 = \gamma_{11} + \gamma_{12}(1 - \gamma_{22})^{-1}\gamma_{21},
\]

where the scattering processes are contained in \(\gamma_{\alpha\beta}\) with \(\alpha, \beta = 1, 2\).

The scattering matrix \(\tilde{S}\) is diagonal in the particle-hole space, i.e., \(\tilde{S} = \text{diag}(\tilde{S},\tilde{S})\) with

\[
\tilde{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix}
\]

The \(2 \times 2\) matrices \(S_{11}(S_{22})\) and \(S_{21}(S_{12})\) in the spin space represent the electron reflection in the left-hand (right-hand) metal and the electron transition from the left-hand (right-hand) metal to the right-hand (left-hand) one, respectively. The hole reflection and transition are represented by the matrices \(\tilde{S}_{11}(\tilde{S}_{22})\) and \(\tilde{S}_{21}(\tilde{S}_{12})\). Generally, \(\tilde{S}\) depends on the direction of the incident particles, such as the scattering at the spin–orbit coupling interface. The explicit expression of the matrix \(\tilde{S}\) for the ferromagnetic interface considered in this paper is given in appendix A. In the expression, we have defined the effective magnetization magnitude \(X = \frac{M_a}{\hbar^2 v_F}\), the effective non-magnetic potential \(Z = \frac{U_m}{\hbar^2 v_F}\) and \(k_F' = \frac{k_F}{k_F}\) where \(k_F\) is the Fermi wavevector.

For anisotropic superconductors, the pair potential and hence the bulk solutions of \(\gamma_{12}\) and \(\gamma_{21}\) are also dependent on the direction of the momentum of the quasi-particles. In order to show the scattering processes at the interface clearly and to write the momentum-dependent quantities conveniently and correctly, it is necessary to explicitly give the diagrammatic representation of \(\gamma_{\alpha\beta}\) and \(\tilde{\gamma}_{\alpha\beta}\), in which the directions of the momenta contained in the coherence functions and the scattering matrices are specific. We adopt the diagrammatic symbols for \(\gamma, \tilde{\gamma}, \gamma\) and \(\tilde{\gamma}\) as shown in figure 2. The diagrams for \(\tilde{\gamma}_{\alpha\beta}\) are given in figure 3. For simplicity, we do not show the diagrams for \(\gamma_{\alpha\beta}\) which can be given in a similar way. Along the reverse direction of the arrow, we can write the expressions of \(\tilde{\gamma}_{\alpha\beta}\) as

\[
\begin{align*}
\tilde{\gamma}_{11} &= \tilde{S}_{11} \gamma_{11} S_{11} + \tilde{S}_{12} \gamma_{22} S_{21}, \\
\tilde{\gamma}_{12} &= \tilde{S}_{11} \gamma_{12} S_{12} + \tilde{S}_{12} \gamma_{21} S_{22}, \\
\tilde{\gamma}_{21} &= \tilde{S}_{21} \gamma_{11} S_{11} + \tilde{S}_{22} \gamma_{22} S_{21}, \\
\tilde{\gamma}_{22} &= \tilde{S}_{21} \gamma_{12} S_{12} + \tilde{S}_{22} \gamma_{22} S_{22}.
\end{align*}
\]

Ignoring proximity effect, the retarded Green’s function \(g^R\) in the left-hand superconductor can be obtained by

Figure 2. Diagrammatic symbols of \(\gamma, \tilde{\gamma}, S\) and \(\tilde{S}\). \(\gamma\) describes the conversion of a hole (blue dashed line) to a particle (orange solid line); \(\tilde{\gamma}\) describes the conversion of a particle to a hole. \(\tilde{S}(\tilde{S})\) denotes the scattering of a particle (hole). Note, the arrow represents the momentum direction of a particle and the opposite direction of the momentum of a hole.

\[\text{Figure 2. Diagrammatic symbols of } \gamma, \tilde{\gamma}, S \text{ and } \tilde{S}. \gamma \text{ describes the conversion of a hole (blue dashed line) to a particle (orange solid line); } \tilde{\gamma} \text{ describes the conversion of a particle to a hole. } \tilde{S}(\tilde{S}) \text{ denotes the scattering of a particle (hole). Note, the arrow represents the momentum direction of a particle and the opposite direction of the momentum of a hole.}\]
substituting $\tilde{g}$ into equation (3). The Josephson current density can be found from

$\mathbf{3. Results and discussion}$

$\mathbf{3.1. HP_{1S}|HP_{2S} junction}$

In our calculations, the temperature is taken as $T = 0.3 T_C$. Firstly, we consider the CPRs for $X = 0$. There is no magnetic potential in the interfacial barrier. The $HP_{1S}|HP_{2S}$ junction

$$j(w_n, \theta) = \text{Tr}[\tilde{g} \tilde{g}^\dagger],$$

with $j(w_n, \theta) = \text{Tr}[\tilde{g} \tilde{g}^\dagger]$. $N(0)$ is the density of states at the Fermi level in the normal state; the Fermi surface average is only over positive directions. The Matsubara frequency $w_n = 2\pi T (n + \frac{1}{2})$ with $n$ an integer number and $\theta$ the angle between the normal to the interface and the momentum of the incident particle. The dimensionless Josephson current denoted by $I_J$ can be expressed as $I_J = \frac{e d n}{k_B T}$, where $d = I_A$ is the current for junctions with interface area $A$, $R_N$ the resistance for junctions in the normal state and $T_C$ the critical temperature of the superconductor.

$$j(w_n, \theta) = \frac{8\pi k^2 \gamma^2 \sin \phi}{Z^2 (1 + \gamma^2)^2 + k^2 \gamma^2 (1 + \gamma^4) - 2k^2 \gamma^2 \cos \phi},$$

with $\gamma$ defined in appendix A, which gives the sinusoidal form of the CPRs as shown in figure 4(a) with $Z = 0$, 1 and 5. When writing the effective expression of $j(w_n, \theta)$, we used the relation $\gamma^*(w_n) = 1/\gamma(-w_n)$, with $\gamma^*$ the complex conjugate of $\gamma$, and canceled the terms which have no contribution to the current density $J$. The critical current for the tunneling limit with $Z = 5$ is larger than that of the transparent limit where $Z = 0$. The dependence of the critical current on the barrier height is different from that of the $s$-wave Josephson junction which possesses the sinusoidal CPR, and the suppressed critical current with increasing $Z$ [10]. For the $s$-wave situation, the energies of the ABS are $E = \pm D \sqrt{1 - D \sin^2 \phi} / 2$ with $D$ the transmission coefficient, which applies to the point contact or short junction [44]. The zero-energy level appears when $D = 1$ for the transparent limit and will disappear where $D < 1$. However, this is not the case for the $HP_{1S}|HP_{2S}$ junction, as shown in figure 4(b). When $\theta = 0$, the zero-energy level always exists irrespective of the barrier height. The energies of ABS can be expressed as

$\text{Figure 3.}$ The scattering processes involved in $\tilde{g}_{1,2}$, which conserve the momentum component parallel to the interface. $\tilde{g}_{1,2}$ gives two processes where an incident particle from the left-hand (right-hand) superconductor is converted into a hole moving into the same superconductor. $\tilde{g}_{1,2}$ gives two processes where an incident particle from the right-hand (left-hand) superconductor is converted into a hole moving into the superconductor on the opposite side.

$\text{Figure 4.}$ (a) The CPRs of the $HP_{1S}|HP_{2S}$ junction for $X = 0$ with $Z = 0$, 1 and 5. (b) The corresponding energies of the ABS for $\theta = 0$.
\[ E = \pm \Delta_0 \sqrt{D} \cos \phi / 2 \] with \( D = \frac{1}{4 + 2^2} \), the transition coefficient for the normal incidence of the quasi-particles, which is simply the square of the modulus of the diagonal element of \( S_{12} \) or \( S_{21} \).

When \( X = 0 \), the CPR strongly depends on the orientation of the magnetization. Figure 5(a) gives the CPRs for \( X = 1 \) and \( Z = 0 \). We take the azimuthal angle \( \phi_m = 0 \). For \( \theta_m = 0 \) (\( \mathbf{M} || \hat{z} \)), we have the sin \( \phi \)-dominated CPR. The free energy of the junction, given by \( \frac{\hbar}{2e} \int_0^\phi J_I(\psi) d\psi \), has a minimum at \( \phi = 0 \) with no current across the junction. When the relative angle between the magnetization and the \( z \)-axis is increased, such as \( \theta_m = 0.3\pi \) or \( 0.5\pi \) (\( \mathbf{M} \perp \hat{z} \)), the current curve crosses the horizontal line with \( I_J = 0 \) at a position in between \( \phi = 0 \) and \( \phi = \pi \). The free-energy-phase relation has two minima at \( \phi = 0 \) and \( \phi = \pi \); the junction is in the \( 0 + \pi \) phase. The energies of ABS for \( \theta_m = 0 \) and \( \theta_m = 0.5\pi \) are presented in figure 5(b). The presence of the \( x \)-component of the magnetization leads to the splitting of the energies.

From figure 5(a), we can find that the rotation of the magnetization can tune the HP[Si]HP junction between two states: the \( 0 \) phase in which the free-energy minimum is obtained at \( \phi = 0 \) and the \( 0 + \pi \) phase in which the free-energy minima are obtained at \( \phi = 0 \) and \( \pi \). For clarity, we show in figure 5(c) the phase diagram for the states in the orientation space of the magnetization. There are two characteristics: (a) The \( 0 + \pi \) phase can be realized in two circle-like zones with centers located at the points \( (\theta_m, \phi_m) = (0.5\pi, 0) \) and \( (0.5\pi, \pi) \), respectively. The ‘diameter’ of the zones is about 0.36\( \pi \) long. (b) The phase diagram is symmetric about the axes \( \theta_m = 0.5\pi \), \( \phi_m = 0.5\pi \) and \( \phi_m = \pi \), which is a reflection of the symmetries of the CPRs about the direction of the magnetization. They are \( I_J(\theta_m, \phi_m) = I_J(\pi - \theta_m, \phi_m) = I_J(\theta_m, \pi - \phi_m) = I_J(\theta_m, \pi + \phi_m) \). It is interesting to compare the CPRs with those of spin-triplet Josephson junctions characterized by \( \mathbf{d} \)-vectors with uniform directions [45].

There, when the \( \mathbf{d} \)-vectors are both along the \( z \)-axis, \( I_J \) is independent of the azimuthal angle of the magnetization. As a result, the orientation space will be divided into rectangular zones by different phases.

Now, we turn to the CPRs for \( X = 0 \) and \( Z = 0 \). Figure 6(a) shows the currents with \( \phi_m = 0 \) at \( X = 1 \) and \( Z = 1 \). The CPR for \( \theta_m = 0.5\pi \), see figure 5(a), evolves into the sin \( \phi \)-dominated line shape with a negative critical current, see figure 6(a), as the non-magnetic potential \( Z \) increases from 0 to 1. The free energy of the junction in this case has a minimum at \( \phi = \pi \); the junction is in the \( \pi \) phase. The energy of the ABS for \( \theta_m = 0.5\pi \) is given in figure 6(b). From the phase diagram in figure 6(c), we can find the zones for the \( \pi \) state are located in the ellipse-like zones for the \( 0 + \pi \) state. They possess the same centers: the ‘diameter’ of the \( \pi \) zones is about 0.36\( \pi \) long; the major (minor) axis of the \( 0 + \pi \) zones is about 0.54\( \pi \) (0.4\( \pi \)) long. If one continues to increase the values of \( X \) and \( Z \) and simultaneously keeps \( X = Z \), another new state will emerge at the upper and the lower edges of the \( 0 + \pi \) zones. Figure 7(a) and (b) plot the CPRs and the free energies for the edge point \( (\theta_m, \phi_m) = (0.5\pi, 0.25\pi) \) at various values of \( X \) and \( Z \). As shown in the figures, the energy minima of the new state are realized at the location in between \( \phi = 0 \) and \( \phi = \pi \) and its symmetric location in between \( \phi = \pi \) and \( \phi = 2\pi \). This new state is the so-called \( \varphi \) phase. From
Figure 6. (a) The CPRs of the HP1S$|\Phi|$HP2S junction for $Z=1$, $X=1$ and $\phi_m=0$. (b) The corresponding energies of the ABS. (c) The phase diagram for 0 phase, $0+\pi$ phase and $\pi$ phase in the orientation space at $Z=1$ and $X=1$.

Figure 7. (a) The CPRs of the HP1S$|\Phi|$HP2S junction for $Z=X$ when $\phi_m=0.25\pi$ and $\theta_m=0.5\pi$. (b) The corresponding free-energy-phase relations. (c) The phase diagram for 0 phase, $0+\pi$ phase, $\pi$ phase and $\varphi$ phase in the orientation space at $Z=X=3$. 
In figure 7(b), we find that the locations will tend to $\phi = \pi$ when $X$ and $Z$ are increased. Figure 7(c) shows the phase diagram for the $0 + \pi$, $\pi$ and $\varphi$ phases in the orientation space when $X = 3$

The magnitude of the magnetization and the non-magnetic potential are important parameters in the realization of different phases. In figure 8, we give the phase diagrams in the $X$-$Z$ plane for three representative points in the orientation space which are denoted by the coordinate $(\phi_m, \theta_m)$. Figure 8(a) is the diagram for the point $(0, 0.5\pi)$ which is the center of the zones for the $0 + \pi$ and $\pi$ states. When $X < 0.4$, one can only obtain the $0$ phase; when $X > 3$, one can only obtain the $\pi$ phase. For the moderate values of $X$ with $1 < X < 3$, the $0 + \pi$ phase exists as nearly a boundary line between the $0$ phase and the $\pi$ phase. The parameters $Z$ and $X$ play roughly opposite roles in the formation of the $0$ phase and the $\pi$ phase. This is qualitatively similar to the spin-triplet Josephson junctions with unitary equal-spin-pairing states considered in [28]. Figure 8(b) is the diagram for the point $(0, 0.5\pi)$. It is found that when the point deviates from the center $(0, 0.5\pi)$, the domain of the $\pi$ phase is decreased compared to that in figure 8(a). Figure 8(c) shows the diagram for $(0.25\pi, 0.5\pi)$ which is an edge point of the $0$ phase and $\pi$ phase in figure 7(c). From the diagram, we can find the condition for the formation of the $\varphi$ phase which is that $Z$ and $X$ must have large enough values (larger than about 2) and satisfy $Z \approx X$.

Finally, we briefly discuss the presence of the lowest order current, the harmonics $\sin \phi$ and $\cos \phi$, in the CPRs. There are two main features: (a) the $\sin \phi$-type current always exists, both for the non-magnetic interface and the magnetic case; (b) no matter how one changes the magnitude of the potentials and the direction of the magnetization, the $\cos \phi$-type current will not be obtained. The two features will be further analyzed in section 4.

3.2. HP$_1$S|HP$_2$S junction

For $X = 0$, the effective expression of $j(w_n, \theta)$ can be written as:

$$j(w_n, \theta) = -4\pi k_x^2 \cos \phi \times \left[ \frac{\gamma^2}{Z^2(1 + \gamma^2)^2 + k_x^2(1 + \gamma^4)} - 2k_x^2 \gamma^2 \sin \phi \right]$$

(10)

The CPRs are shown in figure 9(a) with $Z = 0, 1$ and 5. In contrast to the HP$_1$S|HP$_2$S junction, there is no LOC in junction HP$_1$S|HP$_2$S. The current with the $\sin 2\phi$ form dominates the CPRs. The energies of ABS with $\theta = 0$ are given by $E = \pm \Delta_0 \sqrt{1 + \sin \phi}/2(1 + Z^2)$ as shown in figure 9(b). One must remember that $I_f \propto \sin 2\phi$ is the typical CPR for spin-singlet/spin-triplet superconductor junctions. The absence of LOC in these junctions originates from the orthogonality of the order parameters. For the junctions with the chiral $p$-wave state in triplet superconductors [26], the energies of the ABS are given by $E = \pm \Delta_0 \sqrt{(1 + Z^2) \pm \sqrt{(1 + Z^2)^2 - \sin^2 \phi}^2}/2(1 + Z^2)$.

Figure 10(a) plots the CPRs for $X = 1$ and $Z = 0$. For $\phi_m = 0$, the variation of the polar angle $\theta_m$ only changes the value of the critical current; the CPRs keep the $\sin 2\phi$ form. That is to say, when $\textbf{M}$ is in the $xz$-plane, one cannot expect the presence of LOC. The situation will be changed when $\phi_m$ deviates from 0 as given in figure 10(b) with $\phi_m = 0.25\pi$. As $\theta_m$ is increased from zero, the harmonic $\sin \phi$ emerges and soon dominates the CPR. The junction changes its state from the $0 + \pi$ phase to the $\pi$ phase accordingly. The phase diagram in the orientation space is presented in figure 10(c) which is invariant under a reflection about $\theta_m = 0.5\pi$ or under a $\pi$ translation of $\phi_m$. The invariances of the diagram are the results of symmetries of the current, i.e. $I_f(\theta_m, \phi_m, \phi)$.
For the HP1S junction, the phase cannot be achieved in the current and black lines with $\theta_m = n\pi$ for $n$ is an integer number in the diagram. For these values, the term $\sin\phi$ is absent in the current and $I_j \propto \sin 2\phi$ as shown in figure 10(a).

The CPRs for $X = 0$ and $Z = 0$ are presented in figure 11 with $X = 1$ and $Z = 1$. It is found from figure 11(a) that for $\theta_m = 0$, LOC with the harmonic $\cos\phi$ dominates the CPR. The corresponding free energy has a single minimum at $\phi \approx 1.5\pi$, as given in figure 11(d), which indicates the junction is in the so-called $\phi_0$ phase. As $\theta_m$ is increased, LOC is weakened and will disappear when $\theta_m = 0.5\pi$. The junction changes its state from the $\phi_0$ phase to the $0 + \pi$ phase accordingly. For $\phi_m = 0.25\pi$ in figure 11(b), as $\theta_m$ is increased to $0.5\pi$, $I_j \propto \cos\phi$ with negative critical current will dominate the CPR. The junction changes its state from the $\phi_0$ phase to the $\pi$ phase accordingly as shown in figure 11(e). In contrast, for $\phi_m = 0.75\pi$, $I_j \propto \sin\phi$ with positive critical current will dominate the CPR when $\theta_m$ is increased to $0.5\pi$. The junction changes its state from the $\phi_0$ phase to the $0$ phase accordingly, as shown in figure 11(f). For $Z = 1$ and $X = 1$, we also have symmetries of $I_j$ such as $I_j (\theta_m, \phi_m, \phi) = -I_j (\pi - \theta_m, \phi_m, 2\pi - \phi)$, $I_j (\theta_m, \phi_m, 0) = I_j (\theta_m, \pi + \phi_m, \phi)$ and $I_j (\theta_m, n\pi/2, \phi) = I_j (\theta_m, (n + 1)\pi/2, \phi)$. From the numerical results, we find the $\phi_0$ phase can exist in the HP$_3$S4HP$_3$S junction except for $\theta_m = 0.5\pi$. It is worth noting that the phase cannot be achieved in the HP$_3$S4HP$_3$S junction due to the absence of the $\cos\phi$-type current in their CPRs.

From the above results, we can summarize the features of CPRs in the HP$_3$S4HP$_3$S junction which are as follows: (a) when $X = 0$, LOC is absent. (b) for $X = \pi$, one can obtain the $\sin\phi$-type current as long as $\theta_m = n\pi/2$ and $\theta_m = n\pi$. (c) for $X = 0$ and $Z = 0$, one can obtain the $\cos\phi$-type current as long as $\theta_m = 0.5\pi$. We will give the physical explanations of the features in section 4.

3.3. HP$_3$S4HP$_3$S junction

The CPRs for $X = 0$ are presented in figure 12; they also satisfy $I_j \propto \sin 2\phi$ as those in the HP$_3$S4HP$_3$ junction do. One cannot obtain LOC when the magnetic potential is absent in the interface. The effective expression of $j(w_m, \theta)$ is given by

$$j(w_m, \theta) = \frac{-8\pi K_0^2 |\gamma|^3 \sin 2\phi}{[K_0^2(1 + |\gamma|^4) + Z^2(1 + 2|\gamma|^2\mu^2) - 4K_0^2|\gamma|^4 \sin^2\phi]} \quad (11)$$

We do not show the energies of the ABS because they are the same as those for the HP$_3$S4HP$_3$ junction. Figure 13 plots the CPRs for $X = 2$ and $Z = 0$. For $\phi_m = 0$, as shown in figure 13(a), the increment in the value of $\theta_m$ only suppresses the critical current. The magnetization in the $xz$-plane will not bring about the LOC. For $\phi_m = 0.25\pi$, as shown in figure 13(b), as $\theta_m$ is increased from 0, the $\sin\phi$-type current soon begins to dominate the CPRs. The junction changes its state from the $0 + \pi$ phase to the phase. Since we have $I_j (\theta_m, \phi_m, \phi) = -I_j (\theta_m, \pi - \phi_m, \pi - \phi)$, the harmonic $\sin\phi$ with negative critical current will dominate the CPRs for $\phi_m = 0.75\pi$ when $\theta_m$ is increased from 0. In this case, the junction changes its state from the $0 + \pi$ phase to the $\pi$ phase. The phase diagram for 0 phase, 0 + $\pi$ phase and $\pi$ phase are presented in figure 13(c). The symmetries of the diagram are the results of the relations $I_j (\theta_m, \phi_m, \phi) = -I_j (\theta_m, \pi - \phi_m, \pi - \phi)$ and $I_j (\theta_m, \phi_m, \phi) = I_j (\theta_m, \phi_m, \pi + \phi_m)$. There are also some black lines with $\theta_m = n\pi/2$ or $\phi_m = n\pi$ in the diagram. For these values, we have $I_j \propto \sin 2\phi$ with no LOC. Figures 14(a)–(c) show the CPRs for $X = 1$ and $Z = 2$. For $\phi_m = 0$ in figure 14(a), the $\cos\phi$-type CPR evolves into the $\sin\phi$ form as $\theta_m$ is increased. The junction changes its state from the $\phi_0$ phase with $\phi_0 \approx 0.5\pi$ to the $0 + \pi$ phase accordingly as shown in figure 14(d). However, for $\phi_m = 0.25\pi$ in figure 14(b), the $\cos\phi$-type CPR will evolve into the $\sin\phi$ form as $\theta_m$ is increased. The junction changes its state from the $\phi_0$ phase to the $0$ phase accordingly, as shown in figure 14(e). For $\phi_m = 0.75\pi$ in figure 14(c), the CPR will evolve into the $\sin\phi$ form with a negative critical current. The junction changes its state from the $\phi_0$ phase to the $\pi$ phase accordingly, as given in figure 14(f). For $\theta_m = 0.5\pi$, the phase will evolve into the $\sin\phi$ form with a negative critical current. The features of CPRs in the HP$_3$S4HP$_3$S junction are the same as those in the HP$_3$S4HP$_3$S junction which have been summarized in section 3.2. Finally, we briefly discuss
the CPRs in other types of helical junctions. For the junctions with the symmetric geometry such as the HP₆SHP₆S junction, we have trivial CPRs which are dominated by the harmonic sin φ. For other asymmetric junctions, their CPRs can be derived from the junctions we have considered. For example, I J (φ) in junction HP₆SHP₆S is identical to I J (π − φ) in the HP₆SHP₆S junction.

4. Free energy and selection rules

Now, we explain the features of the CPRs of the helical Josephson junctions through constructing the free energy of junctions. The selection rules for LOC will be obtained. Firstly, we consider the non-magnetic junctions with X = 0. In this case, there are two relevant vectors in each junction, i.e. d₁ and d₂, with α = 2, 3 or 4. We calculate the scalar product of the vectors:

\[
\begin{align*}
\langle d_1 \cdot d_2 \rangle_{k_i} & = \frac{1}{3} \Delta_0^2, \\
\langle d_1 \cdot d_3 \rangle_{k_i} & = 0, \\
\langle d_4 \cdot d_4 \rangle_{k_i} & = 0,
\end{align*}
\]

in which \(\langle \cdots \rangle_{k_i}\) denotes the average over the momentum parallel to the interface. The vanishing of the average value implies the ‘orthogonality’ of the superconducting states. As a result, LOC will be absent in the non-magnetic junctions HP₆SHP₆S and HP₆SHP₆S. In contrast, the harmonic sin φ dominates the CPR in the HP₆SHP₆S junction due to the finite average value. This indicates a contribution to the free energy, \(\langle d_3 \cdot d_3 \rangle_{k_i} \cos \phi\), for the non-magnetic junctions. The Josephson current, as the derivative of the free energy with respect to φ, is proportional to \(\langle d_3 \cdot d_3 \rangle_{k_i} \sin \phi\). Hence, the selection rule is just the non-zero condition for the current, i.e. \(\langle d_3 \cdot d_3 \rangle_{k_i} \neq 0\).

Secondly, we consider the magnetic case. There are three relevant vectors, i.e. M, d₁, and d₄ with α = 2, 3 or 4, in each junction. In order to include the interaction between the magnetization and the helical superconductivity, we calculate the following scalar product of the vectors,

\[
\begin{align*}
\langle d_{1M} \cdot d_{2M} \rangle_{k_i} & = \frac{1}{4} \Delta_0^2 \left[ \frac{1}{3} (7 + \cos \theta_m) - 2 \sin^2 \theta_m \cos 2 \phi_m \right], \\
\langle d_{1M} \cdot d_{3M} \rangle_{k_i} & = -\frac{1}{2} \Delta_0^2 \sin^2 \theta_m \sin 2 \phi_m, \\
\langle d_{4M} \cdot d_{4M} \rangle_{k_i} & = \frac{1}{6} \Delta_0^2 \sin^2 \theta_m \sin 2 \phi_m,
\end{align*}
\]

in which \(d_{\alpha M}\) with α = 1, 2, 3, 4 denotes the d-vectors written in the spin space of the magnetization which can be obtained by performing unitary transformations (see appendix B). The averages are determined by the orientation of the magnetization. Since \(\langle d_{1M} \cdot d_{2M} \rangle_{k_i} > 0\) holds in the whole space of the orientation, the sin φ-type current always exists in the junction HP₆SHP₆S irrespective of \(\theta_m\) and \(\phi_m\). The condition for \(\langle d_{1M} \cdot d_{3M} \rangle_{k_i} = 0\) \(\sin \theta_m = 0\) and \(\sin 2 \phi_m = 0\), therefore one can expect the sin φ-type current when \(\theta_m = n\pi\) and \(\phi_m = n\pi/2\) in the junction HP₆SHP₆S. This implies a contribution to the free energy, \(\langle d_{\alpha M} \cdot d_{4M} \rangle_{k_i} \cos \phi\), for the magnetic junctions. Accordingly, the Josephson current is proportional to \(\langle d_{\alpha M} \cdot d_{4M} \rangle_{k_i} \sin \phi\).

Figure 10. (a) The CPRs of the HP₆SHP₆S junction for Z = 0, X = 1 and φₘ = 0. (b) The CPRs for Z = 0, X = 1 and φₘ = 0.25π. (c) The phase diagram for 0 phase, 0 + π phase and π phase in the orientation space at Z = 0 and X = 1.
Thirdly, for the magnetic case, we can also construct another scalar quantity involving both the magnetization \( \mathbf{M} \) and two \( \mathbf{d} \)-vectors. The averages of the quantity for different junctions are given by

\[
\langle \mathbf{M} \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \rangle_{k_x} = 0,
\]

\[
\langle \mathbf{M} \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \rangle_{k_x} = \frac{1}{3} \Delta_3^2 \cos \theta_m,
\]

\[
\langle \mathbf{M} \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \rangle_{k_x} = -\Delta_0^2 \cos \theta_m.
\]

For the junction \( \text{HP}_3 \text{SHP}_1 \text{S} \), the value of the average is zero for all \( \theta_m \) and \( \phi_m \); one cannot find the cos \( \phi \)-type current in the junction. For the junction \( \text{HP}_3 \text{SHP}_1 \text{S} \), the vanishing of the average happens only at \( \theta_m = \pi/2 \); one can obtain the cos \( \phi \)-type current so long as \( \theta_m = \pi/2 \) when \( Z = 0 \) and \( X \neq 0 \). This implies another contribution to the free energy, \( \langle \mathbf{M} \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \rangle_{k_x} \) \( \cos \phi \), for the magnetic junctions. The term \( \langle \mathbf{M} \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \rangle_{k_x} \) \( \cos \phi \) contributes to the Josephson current accordingly. The selection rules for the magnetic case are also the non-zero conditions for the current.

The complete expressions of the free energy and the Josephson current are very complicated: they are functions of temperature, the non-magnetic potential, the magnitude and the direction of magnetization and the superconducting phase \( \phi \).

Here, we try to give qualitative explanations of the formation of various phases in helical junctions on the basis of the constructed free energy and the corresponding current. For the \( \text{HP}_3 \text{SHP}_1 \text{S} \) junction, there is no cos \( \phi \)-type LOC. The current is the composition of

\[
\langle \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3) \rangle_{k_x} \sin \phi \sin 2 \phi.
\]

The second order harmonic \( \sin 2 \phi \) originates from the coherent tunneling of an even number of Cooper pairs. In this case, the smaller the value of \( \langle \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3) \rangle_{k_x} \sin \phi \sin 2 \phi \), the more easily the \( 0 + \pi \) phase comes into being. As shown in figure 15(a), \( \langle \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3) \rangle_{k_x} \sin \phi \) obtains its minimum value at two points in the orientation space of magnetization. The orientation specified by \( (\theta_m, \phi_m) \) in the zones around the points will lead to the formation of the \( 0 + \pi \) phase which corresponds to the phase diagram given in figure 5. The \( \pi \) phase and the \( \varphi \) phase are results of the sign reversal of the current when \( X \) or \( Z \) is changed. Note, the \( \varphi_0 \) phase does not exist in the junction due to the absence of the cos \( \phi \)-type LOC.

For the junction \( \text{HP}_3 \text{SHP}_1 \text{S} \) with \( Z = 0 \), the current is the composition of \( \langle \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3) \rangle_{k_x} \) \( \sin \phi \sin 2 \phi \). The positive (negative) value of \( \langle \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3) \rangle_{k_x} \) is favorable to the
formation of the 0 (π) phase. \((\mathbf{d}_m \cdot \mathbf{d}_{3M})_k\) possesses two peaks with the positive maximum value and two valleys with the negative minimum value, as shown in figure 15(b). The values of \((\mathbf{d}_m \cdot \mathbf{d}_{3M})_k\) around the peaks and the valleys help to form the π phase and the 0 phase, respectively, which leads to the phase diagram in figure 10. The black lines in the diagram are the results of the absence of \(\sin \phi\) when \((\mathbf{d}_m \cdot \mathbf{d}_{3M})_k = 0\). For the HP\(_1\)SHP\(_3\)S junction with \(Z = 0\), the presence of the \(\cos \phi\)-type LOC for \(\theta_m = \pi/2\) is helpful in the formation of the \(\varphi_0\) phase. In this situation, the current is

Figure 13. The CPRs of the HP\(_1\)SHP\(_3\)S junction for \(Z = 0, X = 2\) and (a) \(\phi_m = 0\); (b) \(\phi_m = 0.25\pi\). The corresponding phase diagram for 0 phase, 0 + π phase and π phase in the orientation space at \(Z = 0\) and \(X = 2\).

Figure 14. The CPRs of the HP\(_1\)SHP\(_3\)S junction with \(Z = 2, X = 1\) for (a) \(\phi_m = 0\), (b) \(\phi_m = 0.25\pi\) and (c) \(\phi_m = 0.75\pi\). The corresponding free energy is presented in (d)–(f), respectively.
the composition of the helical p-wave Josephson junctions using the quasi-classical Green’s function technology with diagrammatic representation of the boundary conditions. Various CPRs are found in the junctions due to the interfacial potential-dependent current, which lead to rich phase diagrams. The presence of LOC plays an important role in the formation of different phases. In order to reveal the laws for the occurrence of LOC, we construct two kinds of scalar quantities with magnetization and \( \mathbf{d} \)-vectors which reflect the interplay of ferromagnetism and helical superconductivity. The non-zero condition for the averages of the quantities will directly lead to the selection rules for LOC. In fact, from our analysis, we can also infer some results for the CPRs in the junctions described by \( \mathbf{d} \)-vectors with uniform directions. For example, one will not find LOC in the non-magnetic junctions when two \( \mathbf{d} \)-vectors are perpendicular to each other; LOC will not be found in the junctions in which one vector is proportional to \( \mathbf{k}_l \) and the other is proportional to \( \mathbf{k}_r \).

5. Conclusions

In this paper, we calculate the current in the helical p-wave Josephson junctions using the quasi-classical Green’s function technology with diagrammatic representation of the boundary conditions. Various CPRs are found in the junctions due to the interfacial potential-dependent current, which lead to rich phase occurrence of LOC, we construct two kinds of scalar quantities with magnetization and \( \mathbf{d} \)-vectors which reflect the interplay of ferromagnetism and helical superconductivity. The non-zero condition for the averages of the quantities will directly lead to the selection rules for LOC. In fact, from our analysis, we can also infer some results for the CPRs in the junctions described by \( \mathbf{d} \)-vectors with uniform directions. For example, one will not find LOC in the non-magnetic junctions when two \( \mathbf{d} \)-vectors are perpendicular to each other; LOC will not be found in the junctions in which one vector is proportional to \( \mathbf{k}_l \) and the other is proportional to \( \mathbf{k}_r \).

Figure 15. (a) The normalized \( \langle \mathbf{d}_{1M} \cdot \mathbf{d}_{3M} \rangle_\kappa_s \) as a function of \( \theta_m \) and \( \phi_m \). (b) The normalized \( \langle \mathbf{d}_{1M} \cdot \mathbf{d}_{3M} \rangle_\kappa_s \) as a function of \( \theta_m \) and \( \phi_m \).

\[
\gamma_1 = \begin{pmatrix} \gamma^* & 0 \\ 0 & \gamma \end{pmatrix} \mathrm{e}^{i\phi}, \quad \gamma_1 = -\begin{pmatrix} \gamma^* & 0 \\ 0 & \gamma \end{pmatrix} \mathrm{e}^{-i\phi}, \quad (A.1)
\]

with \( \gamma = \frac{i\Delta_{0}\mathrm{e}^{-i\phi}}{v_0 + \sqrt{\Delta_0^2 + \Delta_{0}^2}} \).

The bulk values of \( \gamma_2 \) and \( \bar{\gamma}_2 \) in the right-hand superconductor are given by

\[
\gamma_2 = \begin{pmatrix} \gamma^* & 0 \\ 0 & \gamma \end{pmatrix}, \quad \bar{\gamma}_2 = \begin{pmatrix} \gamma^* & 0 \\ 0 & \gamma \end{pmatrix} \quad \text{for HP}_2\text{S},
\]

\[
\gamma_2 = i\begin{pmatrix} \gamma^* & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \bar{\gamma}_2 = i\begin{pmatrix} \gamma^* & 0 \\ 0 & -\gamma \end{pmatrix} \quad \text{for HP}_3\text{S},
\]

\[
\gamma_2 = i\begin{pmatrix} \gamma^* & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \bar{\gamma}_2 = i\begin{pmatrix} 0 & -\gamma^* \\ \gamma^* & 0 \end{pmatrix} \quad \text{for HP}_2\text{S} \quad (A.2)
\]

When we write the expressions, we have taken the directions of wavevectors (as shown in figure 3) into account.

For the interface with the ferromagnetic potential, the explicit expressions of the scattering matrices can be given by

\[
S_{11} = \begin{pmatrix}
\frac{Z^2 - X^2 - ik'_z(Z - X\cos\theta_m)}{X^2 + (k'_z + iZ)^2} & \frac{ik'_zX\sin\theta_m e^{-i\phi_m}}{X^2 + (k'_z + iZ)^2} \\
\frac{ik'_zX\sin\theta_m e^{i\phi_m}}{X^2 + (k'_z + iZ)^2} & \frac{Z^2 - X^2 - ik'_z(Z + X\cos\theta_m)}{X^2 + (k'_z + iZ)^2}
\end{pmatrix}, \quad (A.3)
\]

\[
S_{22} = S_{11}, \quad S_{12} = S_{21} = \bar{1} + S_{11}, \quad \bar{S}_{11} = \bar{S}_{22} = S_{11}^\dagger \quad \text{and} \quad \bar{S}_{12} = \bar{S}_{21} = S_{12}.
\]
Appendix B. The transformation of d-vectors

The energy-gap matrix in the coordinate of spin space in F can be obtained by performing unitary transformation:

$$\Delta_M = U^\dagger \Delta U^*$$  \hspace{1cm} (B.1)

with

$$U = \begin{pmatrix}
\frac{\theta_m}{2} e^{-i\phi_m/2} & -\frac{\theta_m}{2} e^{i\phi_m/2} \\
\frac{\theta_m}{2} e^{i\phi_m/2} & \frac{\theta_m}{2} e^{-i\phi_m/2}
\end{pmatrix}.$$  \hspace{1cm} (B.2)

Using the relation between the d-vector and the energy-gap matrix given by

$$\Delta = \begin{pmatrix}
-d_x + i d_y & d_z \\
d_z & d_x + i d_y
\end{pmatrix},$$  \hspace{1cm} (B.3)

we obtain the vectors $\mathbf{d}_{\alpha, m}$ which can be written as

$$\mathbf{d}_{1M} = \Delta_0 [\cos \theta_m \cos (\theta - \phi_m) \hat{x} + \sin (\theta - \phi_m) \hat{y} + \sin \theta_m \cos (\theta - \phi_m) \hat{z}],$$

$$\mathbf{d}_{2M} = \Delta_0 [\cos \theta_m \cos (\theta + \phi_m) \hat{x} - \sin (\theta + \phi_m) \hat{y} + \sin \theta_m \cos (\theta + \phi_m) \hat{z}],$$

$$\mathbf{d}_{3M} = \Delta_0 [\cos \theta_m \sin (\theta + \phi_m) \hat{x} + \cos (\theta + \phi_m) \hat{y} + \sin \theta_m \sin (\theta + \phi_m) \hat{z}],$$

$$\mathbf{d}_{4M} = \Delta_0 [\cos \theta_m \sin (\theta - \phi_m) \hat{x} - \cos (\theta - \phi_m) \hat{y} + \sin \theta_m \sin (\theta - \phi_m) \hat{z}].$$  \hspace{1cm} (B.4)

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