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Polynomial integrals for a Hamiltonian system and breakdown of smooth solutions for quasi-linear equations

Misha Bialy†

Raymond and Beverly Sackler Faculty of Exact Sciences, School of Mathematical Sciences, Tel Aviv University, Ramat-Aviv, 69978 Israel

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Abstract. The purpose of this paper is to relate the non-existence of polynomial integrals for a Hamiltonian system to the breakdown phenomenon of smooth solutions in quasi-linear equations. Using this relation it is shown that for the classical Hamiltonian system with 1.5 degrees of freedom there are no non-trivial third power integrals of motion. The main tool used in the proof is the Lax analysis on formation of singularities in quasi-linear equations. Some results and perspectives for the case of higher degrees are discussed.

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1. Introduction and main result

A Hamiltonian system generally has no integrals of motion additional to the energy, i.e. functions on the phase space having constant values on the phase trajectories. It is an extremely interesting problem to find new integrable Hamiltonians or to understand obstructions to integrability. The reader can consult the survey papers [AKN, DKN, K] for the details of different aspects of the problem.

Surprisingly, for all known integrable Hamiltonians of classical mechanics, the additional integrals are polynomial in the momentum variables. For example, an integral which is linear in momentum appears if the system admits a one-parametric group of symmetries (Noether's Theorem). An integral which is quadratic in momentum exists if there is a separation of variables in the Hamilton–Jacobi equation. However there are integrable systems whose integrability is not related to any obvious symmetry. A remarkable example provides the Hamiltonian of the three-particle Toda lattice, where the additional integral is of third degree in momentum (see [LL] for an explicit expression).

In the present paper we study the existence of polynomial integrals for a system with $1\frac{1}{2}$ degrees of freedom with the Hamiltonian $H = \frac{1}{2}p^2 + u(q,t)$, where u is a smooth potential 1-periodic in both variables. The phase space of the system is $T^*S^1 \times S^1 \simeq \mathbb{T}^2 \times \mathbb{R}$ and the dynamics is governed by the Hamiltonian equations

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -u_q(q,t) \ . \end{aligned} \tag{1}$$

† e-mail: bialy@math.tau.ac.il

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For such a system a function $F : T^*S^1 \times S^1 \to \mathbb{R}$ is an integral if it satisfies the following equation:

$$F_t + pF_q - u_q F_p = 0 \,. \tag{2}$$

If such an integral exists then each non-singular level of it is a 2-torus invariant under the Hamiltonian flow and carrying quasi-periodic motion. However, it can be shown using variational analysis that for some potentials u there are no invariant tori in a certain region of the phase space [M].

In the following two cases (corresponding to those mentioned above) there exist solutions of (2) that are linear and quadratic in p. In the first case u is a constant and in the second u has a form $f(kq + \ell t)$, where f is a 1-periodic function of one variable and k, ℓ are integers. Let me remark that equations (1) and (2) are not changed if one adds to u an arbitrary function of t.

Question. Do there exist other cases when (1) has a polynomial integral of degree n?

It turns out that this question is closely related to the problem of existence of smooth solutions for a quasi-linear system of equations of the form

$$U_t + A(U)U_q = 0 \tag{3}$$

where $A(U)$ is $(n-1) \times (n-1)$ matrix. With this language the two cases of integrability mentioned above correspond to the simple wave solutions of (3). If $n = 3$ the question can be completely answered.

Theorem 1. *Let u be a smooth (of class C^2) 1-periodic potential. System (1) has an integral F polynomial of degree 3 with 1-periodic coefficients if and only if u= constant.*

Remark. Here and below the equality u = constant is understood modulo an arbitrary function of t.

The main tools in the proof of theorem 1 are Hopf's strong maximum principle and Lax's method of formation of singularities in genuinely nonlinear hyperbolic systems. The proof is given in section 2.

In the case $n > 3$ the problem becomes much more complicated. Using a completely different method by Hopf, which was used in his paper on Riemannian metrics without conjugate points, we show that there are no non-constant solutions of (3) for which the matrix $A(U)$ is elliptic. This is done in section 3. In the case $n > 3$, even if one assumes the strict hyperbiolicity of the matric $A(U)$, this does not yet imply genuine nonlinearity as was the case for $n = 3$. Therefore, the method by John [J] is not applicable at least formally in this case. Nevertheless the system (3) has remarkable properties. It turns out that it can be written as a system of conservation laws. Moreover it can be shown that in this form it is a Hamiltonian system of hydrodynamic type (see [DKN]) and has infinitely many additional conservation laws. So far it is unclear as to how it can be used to give a complete answer to the question.

2. Cubic integrals

Let us first derive the quasi-linear system (3). Let $F = \sum_{i=0}^{n} a_i(q,t)p^i$ be a polynomial integral for (1). Recall that u, a_i are assumed to be 1-periodic in q, t. Since F satisfies (2), we immediately obtain the following equations

$$(a_n)_q = 0 \qquad (a_n)_t + (a_{n-1})_q = 0$$
$$(a_i)_t + (a_{i-1})_q - (i+1)u_q a_{i+1} = 0 \qquad i = 0, \ldots, n-1$$

with the convention $a_{-1} \equiv 0$.

The first three equations immediately give the following. The function a_n is a constant and we will assume $a_n = \frac{1}{n}$ without loss of generality. In addition, $a_{n-1} = \mu = \text{constant}$, $a_{n-2} = u + f(t)$. We can assume without loss of generality that $f(t) \equiv 0$, since $f(t)$ does not change equations (1) and (2). Using this information, we are left with the following system of quasi-linear equations

$$(a_i)_t + (a_{i-1})_q - (i+1)u_q a_{i+1} = 0 \qquad i = 0, \ldots, n-2$$

where $a_{-1} \equiv 0$, $a_{n-2} = u$, $a_{n-1} = \mu = \text{constant}$. This is exactly the form (3), where

$$U = (u, a_{n-3}, \ldots a_0) .$$

For $n = 3$ this system takes a very simple form

$$\begin{cases} u_t + v_q - 2\mu u_q = 0 \\ v_t - u u_q = 0 \end{cases} \tag{4}$$

where $v = a_0$. The proof of theorem 1 follows immediately from the following

Theorem 2. *There are no smooth solutions of (4) periodic in q and t except constants.*

Proof. Note first, that u has to satisfy the following equation

$$u_{tt} - 2\mu u_{qt} + u u_{qq} + u_q^2 = 0 \tag{5}$$

We claim that if a non-constant u satisfies (5) then

$$\max u \leqslant \mu^2 .$$

Indeed, if $\max u > \mu^2$ then in the domain $\{u > \mu^2\}$ the equation (5) is elliptic and hence Hopf's strong maximum principle implies $u = \text{constant}$.

Now suppose that u is non-constant and $u \leqslant \mu^2$ everywhere. We now apply Lax's analysis (see [L]) of formation of singularities in a quasi-linear 2×2 system taking special care on the points of degeneracy where $u = \mu^2$. The eigenvalues of the matrix $A(U) = \begin{pmatrix} -2\mu & 1 \\ -u & 0 \end{pmatrix}$ are

$$\lambda = -\mu - \sqrt{\mu^2 - u} \qquad \rho = -\mu + \sqrt{\mu^2 - u} \qquad \lambda \leqslant \rho .$$

The right and left eigenvectors of A are given by the formulae

$$\begin{aligned} \ell_\rho &= (\rho, 1) \qquad \ell_\lambda = (\lambda, 1) \\ r_\rho &= (1, -\lambda)^T \qquad r_\lambda = (1, -\rho)^T . \end{aligned}$$

Let us note that for $u < \mu^2$ the system is genuinely nonlinear

$$d\lambda(r_\lambda) = \frac{1}{2\sqrt{\mu^2 - u}} > 0 .$$

Moreover we see that the nonlinearity increases if u approaches μ^2. This enables one to carry out the method of Lax. In what follows, we apply Lax's analysis [L] adapted for our case.

Let us now compute Riemann invariants that are the functions w, z on u, v satisfying the equations

$$dw(r_\lambda) = 0 \qquad dz(r_\rho) = 0 .$$

The functions w, z can be easily computed

$$\begin{aligned} w &= -v - \mu u - \tfrac{2}{3}(\mu^2 - u)^{3/2} \\ z &= -v - \mu u + \tfrac{2}{3}(\mu^2 - u)^{3/2} . \end{aligned}$$

Note that w, z are C^1-functions on the whole q, t space and the ρ, λ are Lipshitz and bounded. The functions w, z are constants along ρ and λ-characteristics respectively

$$\begin{aligned} w' &= w_t + \rho w_q = 0 \\ \dot{z} &= z_t + \lambda z_q = 0 \end{aligned} \tag{6}$$

where $'$ and $\dot{}$ denote differentiation along the ρ and λ characteristics respectively. Note that λ, ρ are Lipshitz and hence the equations of characteristics $\frac{dq}{dt} = \lambda$, $\frac{dq}{dt} = \rho$ have no pathologies.

Now we obtain the equation of evolution of $\alpha = w_q$ along the ρ-characteristic. This equation is valid whenever $u < \mu^2$

$$w_{tq} + \rho w_{qq} + \rho_w w_q^2 + \rho_z w_q z_q = 0 .$$

Now $z - w = \frac{4}{3}(\mu^2 - u)^{3/2}$ and hence

$$\rho = -\mu + \left[\frac{3}{4}(z - w)\right]^{1/3}$$

$$\rho_w = -\frac{1}{4}\left[\frac{3}{4}(z - w)\right]^{-2/3} \qquad \rho_z = -\rho_w .$$

By the definition of z we have $z_q = \frac{z'}{\rho - \lambda}$. So we obtain

$$\alpha' + \rho_w \alpha^2 + \frac{\rho_z}{\rho - \lambda} z' \alpha = 0 .$$

Denote by $h(w,z)$ a function which satisfies $h_z = \frac{\rho_z}{\rho-\lambda}$, i.e. $h_z = \frac{1}{6(z-w)}$. Then $h = \frac{1}{6}\ln(z-w)$. This choice gives $h' = h_z z' = \frac{\rho_z}{\rho-\lambda}z'$. So we obtain $\alpha' + \rho_w\alpha^2 + h'\alpha = 0$. Denote by $\beta = \alpha e^h$, $k = e^{-h}\rho_w$. We obtain the following equation on β

$$\beta' + k\beta^2 = 0 \tag{7}$$

where

$$\beta = (z-w)^{1/6} w_q$$

$$k = -\frac{1}{4}\left(\frac{3}{4}\right)^{-\frac{2}{3}}(z-w)^{-5/6} = -k_0(\mu^2 - u)^{-5/4} .$$

Note that β is a continuous bounded function on the q,t space. On the other hand, the solutions of (7) are given by the formula

$$\beta(t) = \frac{\beta(t_0)}{1 + \beta(t_0)\cdot K(t)} \qquad K(t) = \int_{t_0}^{t} k(t)\mathrm{d}t .$$

We now claim that β remains bounded only if $\beta(t_0) = 0$. Suppose for example that $\beta(t_0) > 0$. If the ρ-characteristic passing through the point (q_0, t_0) remains the whole time in region $\{u < \mu^2\}$, then for some positive $\bar{t}$ the quantity $1 + \beta(t_0)K(\bar{t})$ vanishes and this gives the blowup of β. In the case when the ρ-characteristic passing through (q_0, t_0) reaches the region $\{u = \mu^2\}$ at time t^* for the first time, then we will use the following.

Lemma. *Integral $\int_{t_0}^{t^*} k(t)\mathrm{d}t$ diverges to $-\infty$.*

The lemma implies that also in this case there exists $\bar{t}$ such that $1 + \beta(t_0)K(\bar{t}) = 0$. This proves the claim. This claim implies that $w_q \equiv 0$ for all q,t where $u < \mu^2$. However, equation (6) then implies that $w_t \equiv 0$ and thus w are constant. Analogously z is constant and hence u is constant in the domain $\{u < \mu^2\}$. But this means that $u =$ constant everywhere. This completes the proof of theorem 2. □

Proof of the lemma. This proceeds as follows. Let $q = f(t)$ be the ρ-characteristic passing through (q_0, t_0), $t_0 \leqslant t \leqslant t^*$. We assume that $u(f(t), t) < \mu^2$ for $t_0 \leqslant t < t^*$ and $u(f(t^*), t^*) = \mu^2$. We have to show that

$$\int_{t_0}^{t^*} k(t)\mathrm{d}t = -k_0 \int_{t_0}^{t^*} \left[\mu^2 - u(f(t), t)\right]^{-5/4} \mathrm{d}t = -\infty.$$

Indeed $\mu^2 - u(q,t)$ can be estimated in a neighbourhood of the point $(f(t^*), t^*)$

$$\mu^2 - u(q,t) \leqslant m_1\left((q - f(t^*))^2 + (t - t^*)^2\right)$$

for some positive constant m_1. On the other hand $\dot{f}(t) = -\mu + \sqrt{\mu^2 - u(f(t), t)} = \rho$ and hence

$$|f(t) - f(t^*)| \leqslant m_2|t - t^*| .$$

These two inequalities imply

$$\mu^2 - u(f(t), t) \leqslant m_3(t - t^*)^2 .$$

The integral

$$\int_{t_0}^{t^*} \left[\mu^2 - u(f(t), t)\right]^{-5/4} \geqslant m_4 \int_{t_0}^{t^*} (t - t^*)^{-10/4}\mathrm{d}t = +\infty .$$

This proves the lemma. □

3. Elliptic case for $n > 3$

In this section we explain the geometric meaning of the eigenvalues of $A(U)$ and, using an idea by Hopf from Riemannian geometry (see [H]), we prove the following statement.

Theorem 3. *If all eigenvalues of $A(U)$ are non-real for all q, t then $U = constant$.*

Note that for $n = 3$ we used Hopf's strong maximum principle.

Proof. We begin with the following lemma which is easily achieved by a straightforward computation.

Lemma. *The characteristic polynomial of the matrix $A(U)$ coincides with the p-derivative of F.*

This lemma implies that with the assumptions of the theorem $F_p(p, q, t)$ does not vanish. This means that all level sets of F are invariant 2-tori which are *graphs* of functions on q, t. Introduce a function

$$\omega(p, q, t) = -\frac{F_q}{F_p}$$

which is well-defined on the phase space and speaking geometrically, measures the 'slope' of the invariant tori. It is easily checked that ω satisfies the following equation

$$\dot{\omega} + \omega^2 + u_{qq} = 0 . \tag{8}$$

We claim that $\omega \equiv 0$. This implies that $u = f(t)$ and finishes the proof.

To prove the claim, integrate the equation (8) over the region of the phase space $\Omega_M = \{|F| \leqslant M\}$ with respect to the measure $\mathrm{d}\mu = \mathrm{d}p\mathrm{d}q\mathrm{d}t$ which is invariant under the flow of (1). We obtain

$$\int_{\Omega_M} \omega^2 \mathrm{d}\mu + \int_{\Omega_M} u_{qq} \mathrm{d}\mu = 0 .$$

On the other hand note that $\lim_{M\to+\infty} \int_{\Omega_M} u_{qq} \mathrm{d}\mu \longrightarrow 0$. Indeed this follows from the fact that for large M the tori $\{F = \pm M\}$ differ very little from the tori $p =$ constant, because the slope ω decays as p^{-1} uniformly in q, t. This implies the result. □

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