Nonlinearity

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To cite this article: Alberto Della Vedova *et al* 2016 *Nonlinearity* **29** 2715

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IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **29** (2016) 2715–2754 doi:10.1088/0951-7715/29/9/2715

Deformations of non-semisimple Poisson pencils of hydrodynamic type

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Received 24 June 2015, revised 16 May 2016
Accepted for publication 27 June 2016
Published 1 August 2016

Recommended by Professor Tamara Grava

Abstract

We study the deformations of two-component non-semisimple Poisson pencils of hydrodynamic type associated with Balinskiĭ–Novikov algebras. We show that in most cases the second order deformations are parametrized by two functions of a single variable. We find that one function is invariant with respect to the subgroup of Miura transformations, preserving the dispersionless limit, and another function is related to a one-parameter family of truncated structures. In two exceptional cases the second order deformations are parametrized by four functions. Among these two are invariants and two are related to a two-parameter family of truncated structures. We also study the lift of the deformations of n-component semisimple structures. This example suggests that deformations of non-semisimple pencils corresponding to the lifted invariant parameters are unobstructed.

Keywords: bi-Hamiltonian structures, Balinskĭ–Novikov algebras, complete lift
Mathematics Subject Classification numbers: 37K05, 37K10, 37K25, 53D45

1. Introduction

Systems of hydrodynamic type are systems of quasilinear evolutionary partial differential equations (PDEs) of the form

$$u^i_t = V^i_j(\mathbf{u})\, u^j_x, \qquad i = 1, .., n, \tag{1.1}$$

where $\mathbf{u} = (u^1, ..., u^n)$ and the summation convention is used to sum over the repeated index j. Equations of this form have many applications in mathematics and physics, such as in gas dynamics, fluid mechanics, plasma physics, magnetohydrodynamics, the Whitham averaging method, two-dimensional topological field theory, etc.

The theory of *integrable Hamiltonian* quasilinear systems of PDEs of the form (1.1) started in the 1980s and was originally motivated by the study of Whitham modulation equations [15]. The Hamiltonian formalism for systems (1.1) was studied by Dubrovin and Novikov who introduced the notion of Hamiltonian operators of hydrodynamic type. They are first order differential operators of the form

$$P^{ij} = g^{ij}(u)\partial_x - g^{il}\Gamma^j_{lk}(u)u^k_x = g^{ij}(u)\partial_x + b^{ij}_k(u)u^k_x, \tag{1.2}$$

where g is a flat pseudo-metric and Γ^j_{lk} are the Christoffel symbols corresponding to g. Hamiltonian operators of the form (1.2) are sometimes called *Dubrovin–Novikov Hamiltonian operators* and sometimes called *Hamiltonian operators of differential geometric type* due to the geometric meaning of their coefficients. The system (1.1) is said to be Hamiltonian with respect to a Dubrovin–Novikov Hamiltonian operator if there exists a local functional $H = \int h(u)\,\mathrm{d}x$ such that the right-hand side of (1.1) can be written as

$$V^i_j u^j_x = P^{ij}\frac{\delta H}{\delta u^j} = (g^{ij}(u)\partial_x + b^{ij}_k(u)u^k_x)\frac{\partial h}{\partial u^j}. \tag{1.3}$$

Recall that the system (1.1) is called *hyperbolic* if all the eigenvalues of the affinor V^i_j are real and the eigenvectors are linearly independent, and *strictly hyperbolic* if the eigenvalues are real and pair-wise distinct. Under some additional assumptions (in the strictly hyperbolic case, the vanishing of the Haantjes tensor) a hyperbolic system of hydrodynamic type can be diagonalized. This means that there exist special coordinates $\{r^1, ..., r^n\}$ called *Riemann invariants* where the system takes diagonal form

$$r^i_t = v^i(r)\, r^i_x, \qquad i = 1, .., n. \tag{1.4}$$

In the case of complex eigenvalues, allowing complex change of coordinates, one can introduce complex Riemann invariants. Novikov conjectured that *diagonalizable* Hamiltonian systems of hydrodynamic type are always integrable. The general theory of diagonalizable integrable systems of hydrodynamic type was developed by Tsarev [39] who proved that the integrability conditions are given by a set of PDEs for the *characteristic velocities* v^i of the system (1.4)

$$\partial_k\left(\frac{\partial_j v^i}{v^j - v^i}\right) = \partial_j\left(\frac{\partial_k v^i}{v^k - v^i}\right), \qquad i \neq j \neq k \neq i. \tag{1.5}$$

The above set of integrability conditions was called the *semi-Hamiltonian condition* since they are automatically satisfied by diagonalizable *Hamiltonian systems* as conjectured by Novikov. Tsarev's integrability conditions imply the existence of a family of symmetries and a family of conservation laws depending on functional parameters. The characteristic velocities of these symmetries and the densities of these conservation laws are obtained as solutions of systems of linear PDEs. Tsarev also discovered that knowledge of the symmetries allows one to write the general solution of the system (1.4) in implicit form. In other words, for systems of hydrodynamic type integrability means linearizability. Later it was proved by Sevennec [37] that semi-Hamiltonian systems coincide with diagonalizable systems of conservation laws and it was conjectured by Ferapontov that all semi-Hamiltonian systems are Hamiltonian with respect to a suitable (in general non-local) Hamiltonian operator [18].

In general, it might be very difficult to solve linear systems providing symmetries and conservation laws and, in practice, it is very useful to have some additional structure that provides a recursive procedure to obtain a countable subset of solutions. For instance, this can be done in the case of bi-Hamiltonian systems by means of classical Lenard–Magri recursive relations [31].

Bi-Hamiltonian systems are systems of differential equations which are Hamiltonian with respect to a pair (P_1, P_2) of compatible Hamiltonian operators. The compatibility of P_1 and P_2 means that the pencil $P_\lambda = P_2 - \lambda P_1$ is a Poisson pencil, i.e. it defines a Hamiltonian operator for any value of the parameter.

In the case of systems of hydrodynamic type (1.1) the bi-Hamiltonian structure is defined by a pair of Hamiltonian operators of differential geometric type (1.2) and the compatibility means that the associated pencil of the contravariant metric $g_\lambda = g_2 - \lambda g_1$ is flat for any λ and that the pencils of the contravariant Christoffel symbols $b^{ij}_{2;k} - \lambda b^{ij}_{1;k}$ define the controvariant Christoffel symbols of the pencil g_λ [10].

The *semisimple case* has been studied in detail. In this case there exists a special set of coordinates, the roots $(r^1, ..., r^n)$ of the equation $\det g_\lambda = 0$, such that both metrics of the pencil g_λ take a diagonal form [10, 19]. The coordinates $(r^1, ..., r^n)$ are called *canonical coordinates* and provide a set of Riemann invariants for the associated bi-Hamiltonian systems. They are usually assumed to be real. However this assumption can be relaxed [11].

Systems of hydrodynamic type are often obtained as the dispersionless limit of systems of evolutionary equations of the form

$$u^i_t = V^i_j(u)u^j_x + \epsilon(A^i_j u^j_{xx} + B^i_{jk}u^j_x u^k_x) + \epsilon^2(C^i_j u^j_{xxx} + D^i_{jk}u^j_x u^k_{xx} + E^i_{jkl}u^j_x u^k_x u^l_x) + \mathcal{O}(\epsilon^3), \tag{1.6}$$

where $A^i_j, B^i_{jk}, ...$ are functions of $(u^1, ..., u^n)$. Starting from this, Dubrovin and Zhang proposed a perturbative approach to the study of integrable bi-Hamiltonian systems of this form [16]. In their approach the full system (1.6) is obtained via a bi-Hamiltonian deformation procedure from the dispersionless limit $\epsilon \to 0$. Instead of deforming the system Dubrovin and Zhang deform the bi-Hamiltonian structure. Since the leading term of (1.6) is a system of hydrodynamic type, the leading term of the full Poisson pencil is a Poisson pencil of hydrodynamic type:

$$\omega^{ij}_2 - \lambda\omega^{ij}_1 = g^{ij}_2\partial_x + b^{ij}_{2;k}u^k_x - \lambda(g^{ij}_1\partial_x + b^{ij}_{1;k}u^k_x). \tag{1.7}$$

Moreover, since the leading term of the system (1.6) is assumed to be diagonalizable the Poisson pencil (1.7) is assumed to be semisimple. Under this additional assumption the classification of the deformed bi-Hamiltonian structures was completely solved. It turned out that bi-Hamiltonian deformations are parametrized by a finite number of functions of a single variable, called *central invariants* (see the next section for a short review of the semisimple case). For the non-semisimple case no results are available. The aim of this paper is to start the study of the classification problem in the non-semisimple case. As the dispersionless limit we consider two examples of non-semisimple bi-Hamiltonian structures: a class of two-component structures related to the so-called Balinskii–Novikov algebras and the lift on the tangent bundle of a semisimple bi-Hamiltonian structure. The first example is important as it shows that in addition to the non-semisimple analogue of the central invariants there is a new set of functional parameters related to a family of truncated structures. The second example is important as it suggests that, as in the semisimple case, the deformations related to the non-semisimple analogue of the central invariants might be unobstructed.

2. The semisimple case

Let

$$\Pi_\lambda^{ij} = \omega_2^{ij} + \sum_{k\geqslant 1}\epsilon^k \sum_{l=0}^{k+1} A_{2;k,l}^{ij}(u, u_x, \dots, u_{(l)})\partial_x^{(k-l+1)}$$
$$-\lambda\left(\omega_1^{ij} + \sum_{k\geqslant 1}\epsilon^k \sum_{l=0}^{k+1} A_{1;k,l}^{ij}(u, u_x, \dots, u_{(l)})\partial_x^{(k-l+1)}\right), \tag{2.1}$$

($A_{1;k,l}^{ij}$ and $A_{2;k,l}^{ij}$ are homogeneous differential polynomials of degree l) be a deformation of a semisimple Poisson pencil of hydrodynamic type. Two deformations Π_λ and $\tilde{\Pi}_\lambda$ of the same pencil are considered equivalent if they are related by a Miura transformation of the form

$$\tilde{u}^i = u^i + \sum_{k\geqslant 1}\epsilon^k F_k^i(u, u_x, \dots, u_{(k)}), \tag{2.2}$$

where $F_k^i(u, u_x, \dots, u_{(k)})$ are differential polynomials of degree k. This means that two pencils belonging to the same class are related by

$$\tilde{\Pi}_\lambda^{ij} = L_k^{*i}\Pi_\lambda^{kl}L_l^j,$$

where

$$L_k^i = \sum_s(-\partial_x)^s\frac{\partial\tilde{u}^i}{\partial u^{(k,s)}}, \qquad L_k^{*i} = \sum_s\frac{\partial\tilde{u}^i}{\partial u^{(k,s)}}\partial_x^s.$$

Dubrovin, Liu and Zhang proved that the equivalence classes are labelled by n functions $c^i(r^i)$ called *central invariants* [11, 27]. These functions are obtained by expanding the roots λ^i of the equation

$$\det\left(g_2^{ij} - \lambda g_1^{ij} + \sum_{k\geqslant 1}(A_{2;k,0}^{ij}(u) - \lambda A_{1;k,0}^{ij}(u))p^k\right) = 0,$$

near $\lambda^i = r^i$:

$$\lambda^i = r^i + \sum_{k=1}^{\infty}\lambda_{2k}^i p^{2k}, \tag{2.3}$$

and selecting the coefficient of p^2. The central invariants are then defined as [12, 27]:

$$c^i = \frac{1}{3}\frac{\lambda_2^i}{g_1^{ii}} = \frac{1}{(f^i)^2}\left(Q_2^{ii} - r^iQ_1^{ii} + \sum_{k\neq i}\frac{(P_2^{ki} - r^iP_1^{ki})^2}{f^k(r^k - r^i)}\right), \quad i = 1, \dots, n,$$

where f^i are the diagonal components of the contravariant metric g_1 in canonical coordinates and

$$P_\theta^{ij}(u) = A_{\theta;1,2}^{ij}(u), \quad Q_\theta^{ij}(u) = A_{\theta;2,3}^{ij}(u), \quad i,j = 1, \dots, n, \quad \theta = 1, 2.$$

They can also be defined by (see [17])

$$c^i = -\frac{1}{3f^i}\mathrm{Res}_{\lambda=r^i}\mathrm{Tr}\,[g_\lambda^{-1}(Q_\lambda^{ij} + (g_\lambda^{-1})_{lk}P_\lambda^{li}P_\lambda^{kj})]\,,$$

where $Q^{ij}_\lambda = Q^{ij}_2 - \lambda Q^{ij}_1$ and $P^{ij}_\lambda = P^{ij}_2 - \lambda P^{ij}_1$.

In this framework the following facts should be mentioned:

- Each function c^i depends only on the corresponding canonical coordinate r^i and is invariant with respect to Miura transformations (2.2) [27].
- Two deformations (of the same pencil) belong to the same class of equivalence if and only if they have the same central invariants [11].
- For any choice of the dispersionless limit and of the central invariants the equivalence classes are not empty. This fact, suggested by some computations (for the scalar case see [2, 30]), has been proved only recently, by Liu and Zhang in the scalar case [29] and by Carlet, Posthuma and Shadrin in the general semisimple case [8]. The proof is based on the vanishing of certain cohomology groups introduced in [27].
- Fixing the dispersionless limit as ω_λ and the central invariants as $c^i(r^i)$ there exists a Miura transformation (2.2) reducing the pencil to the standard form [27]

$$\begin{aligned}\Pi_\lambda &= \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_{X_{(c_1,\dots,c_n)}}\omega_1 + \epsilon^4\Pi_4 + \epsilon^6\Pi_6 + \dots\\ &= \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_{Y_{(c_1,\dots,c_n)}}\omega_2 + \epsilon^4\Pi_4 + \epsilon^6\Pi_6 + \dots\end{aligned}$$

  where the *polynomial* vector fields $X_{(c_1,\dots,c_n)}$ and $Y_{(c_1,\dots,c_n)}$ can be written as the difference of two Hamiltonian vector fields

$$X_{(c_1,\dots,c_n)} = \omega_2\,\delta H - \omega_1\,\delta K, \qquad Y_{(c_1,\dots,c_n)} = \omega_2\,\delta H' - \omega_1\,\delta K'$$

  with *non-polynomial* Hamiltonian densities:

$$\begin{aligned}H[r] &= \sum_{i=1}^n \int c^i(r^i) r^i_x \log r^i_x \,\mathrm{d}x, \quad K[r] = \sum_{i=1}^n \int r^i c^i(r^i) r^i_x \log r^i_x \,\mathrm{d}x.\\ H'[r] &= \sum_{i=1}^n \int \frac{c^i(r^i)}{r^i} r^i_x \log r^i_x \,\mathrm{d}x, \quad K'[r] = \sum_{i=1}^n \int c^i(r^i) r^i_x \log r^i_x \,\mathrm{d}x.\end{aligned} \tag{2.4}$$

- The coefficients $F_k(u, u_x, \dots, u_{(k)})$ of the Miura transformation (2.2) are assumed to depend *polynomially* on the derivatives of u^i. Removing this assumption the classification problem becomes 'trivial': all deformations turn out to be equivalent to their dispersionless limit. This remarkable property of the deformations was discovered in [11] and is called *quasitriviality*. For instance, it is easy to check that the *canonical quasi-Miura transformation* generated by the Hamiltonian H defined in the formula (2.4) reduces the pencil Π^{ij}_λ to the form $\omega^{ij}_2 - \lambda\omega^{ij}_1 + \mathcal{O}(\epsilon^4)$.

3. Outline of the results of the paper

In the present paper we start the study of the non-semisimple case. Whereas the semisimple case is fairly well understood, the non-semisimple case is wide open. In addition to computational difficulties, the lack of canonical coordinates, or at least of a normal form theorem for non-semisimple pencils, makes a unified approach to the problem very difficult to obtain. For this reason in this paper we try to obtain some information on the general case focusing on two special subcases where computations are feasible:

The deformations of Poisson pencils related to two-dimensional Balinskiĭ–Novikov algebras [6] *and the associated invariant bilinear forms*. These are linear Poisson pencils of the form

Table 1. Pair of metrics of bi-Hamiltonian structures.

Type	Linear metric g	Constant metric η	Affinors $L = g\eta^{-1}$
(T3)	$\begin{pmatrix} 0 & -u^1 \\ -u^1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$	$\begin{pmatrix} -\frac{u^1}{\eta^{12}} & 0 \\ \frac{\eta^{22}u^1}{(\eta^{12})^2} & -\frac{u^1}{\eta^{12}} \end{pmatrix}$
(N5)	$\begin{pmatrix} 0 & u^1 \\ u^1 & 2(u^1+u^2) \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$	$\begin{pmatrix} \frac{u^1}{\eta^{12}} & 0 \\ \frac{2(u^1+u^2)}{\eta^{12}} - \frac{\eta^{22}u^1}{(\eta^{12})^2} & \frac{u^1}{\eta^{12}} \end{pmatrix}$
(N3, N4, N6)	$\begin{pmatrix} 0 & (1+\kappa)u^1 \\ (1+\kappa)u^1 & 2u^2 \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$	$\begin{pmatrix} \frac{(1+\kappa)u^1}{\eta^{12}} & 0 \\ \frac{2u^2}{\eta^{12}} - \frac{(1+\kappa)\eta^{22}u^1}{(\eta^{12})^2} & \frac{(1+\kappa)u^1}{\eta^{12}} \end{pmatrix}$

$$\omega_2^{ij} - \lambda\omega_1^{ij} = g^{ij}\partial_x + b^{ij}_k u^k_x - \lambda\eta^{ij}\partial_x, \quad i,j,k = 1,\ldots,n, \tag{3.1}$$

where g is a linear metric, that is $g^{ij} = (b^{ij}_k + b^{ji}_k)u^k$, and η is a constant metric. As proved by Balinskiĭ and Novikov in [6] the numbers b^{ij}_k are the structure constants of an algebra B satisfying the following identities

$$a \cdot (b \cdot c) = b \cdot (a \cdot c),$$
$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = (a \cdot c) \cdot b - a \cdot (c \cdot b).$$

We refer to them as *Balinskiĭ–Novikov algebras*, even if in the literature they are often called Novikov algebras (following [36]). A first approach to the study of such algebras was made by Zelmanov [43]. In low dimensions the problem of classification was addressed by Bai and Meng [3, 5] and recently by Burde and de Graaf [7], resulting in a complete description of one-, two- and three-dimensional Balinskiĭ–Novikov algebras. Unfortunately, the full classification of these structures of dimension four and higher is far from complete.

It has been recently proved by Strachan and Szablikowski that special deformations of such structures, associated with second and third order co-cycles of B, naturally arise in the study of multi-component generalisations of the Camassa–Holm equation [38].

In this paper we focus on two-component ($n = 2$) non-degenerate ($\det g^{ij} \neq 0$, $\det \eta^{ij} \neq 0$) structures of the form (3.1), which are non-semisimple. These structures have already been classified by Bai and Meng [3], and are summarized in table 1 (the full list obtained in [3] is recalled later in section 4, table 2).

We prove that in the cases T3, N3 (corresponding to $\kappa = 1$), N5 and N6 with $\kappa \neq 0, -1, -2$ the deformations are quasi-trivial and can be reduced to the form

$$\Pi_\lambda = \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_{X_{(F_1,F_2)}}\omega_2 + \mathcal{O}(\epsilon^3)$$

with $X_{(F_1,F_2)} = \omega_1\,\delta H - \omega_2\,\delta K$ where

$$H[u] = \int \sum_{i,j} (h_{ij} u^i_x \log u^j_x)\,\mathrm{d}x, \qquad K[u] = \int \sum_{i,j} (f_{ij} u^i_x \log u^j_x)\,\mathrm{d}x,$$

and the functions h_{ij} and f_{ij} are uniquely determined by two arbitrary functions F_1, F_2. Moreover both functions F_1 and F_2 depend only on the eigenvalue of the affinor L.

The cases N4 (corresponding to $\kappa = 0$) and N6 with $\kappa = -2$ are more involved and there are four functions labelling non-Miura equivalent deformations (still depending on the eigenvalue of the affinor L).

In all cases one half of the arbitrary functions parametrizing the deformations (one in the two-parameter case, two in the four-parameter case) is related to a family of truncated structures and one half is invariant with respect to the Miura transformations that preserve the dispersionless limit. The invariant functions are related to the first coefficients of the expansion (2.3) (in the second case also the odd powers of p appear in this expansion): the coefficients of p^2 in the case of the algebras T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$ and the coefficients of p and p^2 in the case of the algebras N4 and N6 with $\kappa = -2$. Moreover our computations suggest that in exceptional cases the generic deformations are not quasi-trivial. This fact is rather unexpected and deserves deeper investigation.

The lift of deformations of semisimple structures. These are obtained using an infinite-dimensional version of the complete lift introduced by Yano and Kobayashi [40]. The main point is to view the tangent bundle TM of a given manifold M as a manifold itself and to lift tensor fields and affine connections from M to TM via the projection map of the bundle. Such a construction can be extended to the case where the base manifold is a loop space $\mathcal{L}(M)$ and the lifted structures then are defined on $\mathcal{L}(TM)$. In particular one can lift Poisson structures of hydrodynamic type as well as Frobenius structures. The relevance of this construction for our purposes is due to the fact that a lifted Poisson structure turns out to be non-semisimple despite the semisimplicity of the starting one.

Although elementary, this case is important as it provides examples of full deformations of non-semisimple structures depending on functional parameters. By construction, all deformations of an n-component semisimple structure can be lifted to deformations of a $2n$-component non-semisimple structure. This means that the deformations of the lifted Poisson pencils contain n functional parameters *at least*.

4. Linear Poisson bi-vectors of hydrodynamic type

Let us introduce a Poisson bi-vector of hydrodynamic type on the loop space $\mathcal{L}(M)$. The tangent space to $\mathcal{L}(M)$ at a loop $\gamma : S^1 \to M$ is naturally identified with the space $\Gamma(S^1, \gamma^*TM)$ of vector fields along γ. On the other hand, (a subspace of) the cotangent space to $\mathcal{L}(M)$ at γ is identified with the space $\Gamma(S^1, \gamma^*T^*M)$ of co-vector fields along γ, and the pairing between a tangent vector X and a co-vector ξ is just $\int_{S^1} \xi(X)\,\mathrm{d}x$.

Let g be a pseudo-metric on M with Levi-Civita connection ∇. For any co-vector $\xi \in \Gamma(S^1, \gamma^*T^*M)$, let $X_\xi \in \Gamma(S^1, \gamma^*TM)$ be the point-wise metric dual of ξ. Given two co-vectors $\xi, \eta \in \Gamma(S^1, \gamma^*T^*M)$, letting

$$P(\xi, \eta) = \int_{S^1} \xi(\nabla_{\dot\gamma} X_\eta)\,\mathrm{d}x$$

defines a bi-vector on $\mathcal{L}(M)$. As shown by Dubrovin and Novikov, P is a Poisson structure on $\mathcal{L}(M)$ if and only if ∇ is flat [13, 14]. In local coordinates u^i on M and x on S^1 the Poisson tensor P is represented by a differential operator of the form (1.2).

In this paper we will study linear Hamiltonian operators

$$P^{ij} = (b^{ij}_k + b^{ji}_k)u^k\partial_x + b^{ij}_k u^k_x,$$

related to *Balinskiĭ–Novikov algebras* of structure constants b^{ij}_k.

Table 2. Two-dimensional Balinskiĭ–Novikov algebras and invariant bilinear forms.

Type	Characteristic matrix $e^i \cdot e^j$	Linear Poisson structure	Invariant bilinear forms
T1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & \eta^{12} \\ \eta^{21} & \eta^{22} \end{pmatrix}$
T2	$\begin{pmatrix} e^2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2u^2\partial_x+u^2_x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & \eta^{12} \\ \eta^{12} & 0 \end{pmatrix}$
T3	$\begin{pmatrix} 0 & 0 \\ -e^1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -u^1\partial_x \\ -u^1\partial_x-u^1_x & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$
N1	$\begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix}$	$\begin{pmatrix} 2u^1\partial_x+u^1_x & 0 \\ 0 & 2u^2\partial_x+u^2_x \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & 0 \\ 0 & \eta^{22} \end{pmatrix}$
N2	$\begin{pmatrix} e^1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2u^1\partial_x+u^1_x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & 0 \\ 0 & \eta^{22} \end{pmatrix}$
N3	$\begin{pmatrix} e^1 & e^2 \\ e^2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2u^1\partial_x+u^1_x & 2u^2\partial_x+u^2_x \\ 2u^2\partial_x+u^2_x & 0 \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & \eta^{12} \\ \eta^{12} & 0 \end{pmatrix}$
N4	$\begin{pmatrix} 0 & e^1 \\ 0 & e^2 \end{pmatrix}$	$\begin{pmatrix} 0 & u^1\partial_x+u^1_x \\ u^1\partial_x & 2u^2\partial_x+u^2_x \end{pmatrix}$	$\begin{pmatrix} \eta^{11} & \eta^{12} \\ \eta^{21} & \eta^{22} \end{pmatrix}$
N5	$\begin{pmatrix} 0 & e^1 \\ 0 & e^1+e^2 \end{pmatrix}$	$\begin{pmatrix} 0 & u^1\partial_x+u^1_x \\ u^1\partial_x & 2(u^1+u^2)\partial_x+u^2_x+u^1_x \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$
N6	$\begin{pmatrix} 0 & e^1 \\ \kappa e^1 & e^2 \end{pmatrix}$ $\kappa \neq 0, 1$	$\begin{pmatrix} 0 & (1+\kappa)u^1\partial_x+u^1_x \\ (1+\kappa)u^1\partial_x+\kappa u^1_x & 2u^2\partial_x+u^2_x \end{pmatrix}$	$\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}$

4.1. Invariant bilinear forms and bi-Hamiltonian structures

Given a Balinskiĭ–Novikov algebra B, as observed in [38], any invariant bilinear symmetric form on it gives rise to a bi-Hamiltonian structure in a canonical way. For convenience of the reader let us briefly recall how they are defined. Let $e^1, \ldots, e^n$ be a basis of B, and let b^{ij}_k be the corresponding structure constants. A bilinear form $\eta : B \times B \to F$ is called *invariant* if and only if

$$\eta(e^i \cdot e^j, e^k) = \eta(e^i, e^k \cdot e^j).$$

Bai and Meng classified these invariant bilinear forms on two- and three-dimensional Balinskiĭ–Novikov algebras in [3, 4]. For future reference we recall the two-dimensional classification in table 2.

Remark. Note that the cases N1 and N4 with $\eta^{11} \neq 0$ are semisimple, and therefore they are covered by the Dubrovin–Liu–Zhang theory. Thus, for this reason in the case N4 we will always assume $\eta^{11} = 0$. Moreover, the cases N3 and N4 can be considered a subclasses of N6, if we remove the constraints $\kappa \neq 0, 1$. Indeed, for $\kappa = 0$ we easily obtain N4 (with $\eta^{11} = 0$), while N3 is equivalent to the case $\kappa = 1$ after the change of coordinates $\tilde{u}^1 = u^2, \tilde{u}^2 = u^1$. According to [3], this distinction is due to different algebraic properties: the cases N3 and N4 are characterized by the associativity of the algebra, while this is not the case for N6 with $\kappa \neq 0, 1$. However, for our purposes, we do not need to distinguish these cases.

Let us point out that adding the constraint $\eta^{21} = \eta^{12}$ in T1 and N4, the bilinear invariant forms associated with two-dimensional Balinskiĭ–Novikov algebra become symmetric. As observed by Strachan and Szablikowski in [38] the associated Hamiltonian operator $\eta^{ij}\partial_x$ is compatible with the linear Hamiltonian operator defining the Balinskiĭ–Novikov algebra.

Remark. Pairs of compatible flat metrics play a key role in the theory of multi-dimensional Poisson structures of hydrodynamic type [34, 35]. In two dimensions, such structures are given by

$$P^{ij} = g_1^{ij}\partial_x + b_{1;k}^{ij}u_x^k + g_2^{ij}\partial_y + b_{2;k}^{ij}u_y^k,$$

where g_1 and g_2 are compatible flat metrics satisfying some additional constraints, coming from the skew-symmetry and the Jacobi condition for P^{ij}: in flat coordinates of g_1, the metric g_2 must be a linear (symmetric) Killing tensor of g_1 with vanishing Nijenuis torsion [20]. Since the vanishing of Nijenuis torsion is a necessary condition for the compatibility, the structures coming from Balinskiĭ–Novikov algebras and their associated invariant bilinear forms can be interpreted as two-dimensional Poisson structures of hydrodynamic type if the Killing conditions hold. It is remarkable that in two-component structures only the case N6 with $\kappa = -2$ satisfies this additional constraint.

4.2. Classification results

In this section we provide a classification of second order deformations of Poisson pencils coming from Balinskiĭ–Novikov algebras.

By definition, a kth deformation of a Poisson pencil of hydrodynamic type (1.7) is a deformation (2.1) such that $[\tilde{\Pi}_\lambda, \tilde{\Pi}_\lambda] = \mathcal{O}(\epsilon^{k+1})$. Here $\tilde{\Pi}_\lambda^{ij}$ denotes the distribution

$$\begin{aligned}\tilde{\Pi}^{ij} = \omega_2^{ij} + \sum_{k\geqslant 1}\epsilon^k\sum_{l=0}^{k+1}A_{2;k,l}^{ij}(u, u_x, \dots, u_{(l)})\delta^{(k-l+1)}(x-y)\\ -\lambda\left(\omega_1^{ij} + \sum_{k\geqslant 1}\epsilon^k\sum_{l=0}^{k+1}A_{1;k,l}^{ij}(u, u_x, \dots, u_{(l)})\delta^{(k-l+1)}(x-y)\right),\end{aligned}$$

and the Schouten bracket is defined as follows [16]:

$$\begin{aligned}&[\tilde{\Pi}_\lambda, \tilde{\Pi}_\lambda]^{ijk}(x, y, z)\\ &= 2\frac{\partial\tilde{\Pi}_\lambda^{ij}(x, y)}{\partial u_{(s)}^l(x)}\partial_x^s\tilde{\Pi}_\lambda^{lk}(x, z) + 2\frac{\partial\tilde{\Pi}_\lambda^{ki}(z, x)}{\partial u_{(s)}^l(z)}\partial_z^s\tilde{\Pi}_\lambda^{lj}(z, y) + 2\frac{\partial\tilde{\Pi}_\lambda^{jk}(y, z)}{\partial u_{(s)}^l(y)}\partial_y^s\tilde{\Pi}_\lambda^{li}(y, x).\end{aligned}$$

We have to distinguish two cases:

1. The cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$ where second order deformed structures depend on two functions.
2. The remaining cases N4 (which corresponds to $\kappa = 0$) and N6 with $\kappa = -2$, namely

$$g_1 = \begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & \pm u^1 \\ \pm u^1 & 2u^2 \end{pmatrix},$$

where second order deformed structures depend on four functions.

Theorem 1. *In the cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$, second order deformations can be reduced by a Miura transformation to the form*

$$\Pi_\lambda = \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_{X_{(F_1,F_2)}}\omega_2 + \mathcal{O}(\epsilon^3)$$

with $X_{(F_1,F_2)} = \omega_1\,\delta H - \omega_2 \delta K$ where

$$H[u] = \int \sum_{i,j} (h_{ij} u_x^i \log u_x^j)\,\mathrm{d}x, \qquad K[u] = \int \sum_{i,j} (k_{ij} u_x^i \log u_x^j)\,\mathrm{d}x,$$

and the functions h_{ij} and k_{ij} are uniquely determined in terms of two arbitrary functions F_1, F_2 depending only on the eigenvalue of the affinor $L = g_2 g_1^{-1}$. Calling $\mathcal{K} = (k_{ij})$ and $\mathcal{H} = (h_{ij})$, we have $\mathcal{K} = L^T\mathcal{H}$, where L^T means the transpose of L and $\mathcal{H}$ is given, respectively, for each case by

- *T3: $h_{12} = h_{22} = 0$ and*

$$h_{11} = \frac{\mathrm{e}^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}}}{3\eta^{12}}\left(\eta^{22}u^1 F_2' + \frac{\eta^{12}u^2 + \eta^{22}u^1}{u^1} F_2\right) - F_1, \quad h_{21} = -\frac{\mathrm{e}^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}}}{3} F_2.$$

- *N5: $h_{12} = h_{22} = 0$ and*

$$h_{11} = \frac{\sqrt{2\eta^{12}(u^1+u^2) - \eta^{22}u^1}\,F_2'}{3\eta^{12}} + \frac{(2\eta^{12} - \eta^{22})F_2}{6\eta^{12}\sqrt{2\eta^{12}(u^1+u^2) - \eta^{22}u^1}} + \frac{F_1}{2\eta^{12}},$$

$$h_{21} = \frac{1}{3\sqrt{2\eta^{12}(u^1+u^2) - \eta^{22}u^1}} F_2.$$

- *N3, N6 ($\kappa \neq 0, -1, -2$): $h_{12} = h_{22} = 0$ and*

$$h_{11} = \frac{(2\eta^{12}u^2 - (\kappa+1)\eta^{22}u^1)^{\frac{\kappa+1}{2}} F_2'}{3(\kappa+1)^2\eta^{12}} - \frac{\eta^{22}(2\eta^{12}u^2 - (\kappa+1)\eta^{22}u^1)^{\frac{\kappa-1}{2}} F_2}{6\eta^{12}} + \frac{F_1}{\eta^{12}\kappa(\kappa+2)},$$

$$h_{21} = \frac{(2\eta^{12}u^2 - (\kappa+1)\eta^{22}u^1)^{\frac{k-1}{2}}}{3(\kappa+1)} F_2.$$

Here $F_i = F_i(u^1)$, $i = 1, 2$.

In the case N4, namely

$$g_2 = \begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix},$$

the second order deformations can be reduced by a Miura transformation to the form

$$\Pi_\lambda = \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_X \omega_2 + \mathcal{O}(\epsilon^3)$$

where

$$X^i = X^i_1 u^1_{xx} + X^i_2 (u^1_x)^2 + X^i_3 u^1_x u^2_x + X^i_4 (u^2_x)^2 + X^i_5 u^2_{xx},$$

with

$$\begin{aligned} X^1_1 &= 0, \\ X^1_2 &= \theta F_1, \\ X^1_3 &= \partial_1(\theta F_2) \\ X^1_4 &= \partial_2(\theta F_2), \\ X^1_5 &= \theta F_2, \\ X^2_1 &= 0, \\ X^2_2 &= \theta F_3, \\ X^2_3 &= \partial_1\left(\theta \tfrac{1}{2} F_4 - \frac{\partial_1 F_2}{\eta^{12}}\right), \\ X^2_4 &= \partial_2\left(\theta \tfrac{1}{2} F_4 - \frac{\partial_1 F_2}{\eta^{12}}\right), \\ X^2_5 &= \theta \tfrac{1}{2} F_4 - \frac{\partial_1 F_2}{\eta^{12}}. \end{aligned}$$

In the above formulas F_i *are* 4 *arbitrary functions of* u^1 *and* $\theta = (\eta^{22}u^1 - 2\eta^{12}u^2)^{-1}$.
*In the case N*6 *with* $\kappa = -2$, *namely*

$$g_1 = \begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & -u^1 \\ -u^1 & 2u^2 \end{pmatrix},$$

the second order deformations can be reduced by a Miura transformation to the form

$$\Pi_\lambda = \omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_X \omega_2 + \mathcal{O}(\epsilon^3) \tag{4.1}$$

where

$$X^i = X^i_1 u^1_{xx} + X^i_2 (u^1_x)^2 + X^i_3 u^1_x u^2_x + X^i_4 (u^2_x)^2 + X^i_5 u^2_{xx},$$

with

$$
\begin{aligned}
X_1^1 &= 0,\\
X_2^1 &= 2\eta^{22}\theta\left(\theta^{\frac{3}{2}}F_4 - \frac{\partial_1(\theta^2 F_2)}{\eta^{12}}\right) + \theta F_1,\\
X_3^1 &= 2\eta^{12}\theta^{\frac{5}{2}}F_4 - \partial_1(\theta^3 F_2),\\
X_4^1 &= -4\eta^{12}\theta^4 F_2,\\
X_5^1 &= \theta^3 F_2,\\
X_1^2 &= 0,\\
X_2^2 &= F_3,\\
X_3^2 &= \partial_1(\theta^{\frac{3}{2}}F_4) - \frac{\partial_1^2(\theta^2 F_2)}{\eta^{12}},\\
X_4^2 &= 4\partial_1(\theta^3 F_2) + \partial_2(\theta^{\frac{3}{2}}F_4),\\
X_5^2 &= \theta^{\frac{3}{2}}F_4 - \frac{\partial_1(\theta^2 F_2)}{\eta^{12}}.
\end{aligned}
$$

In the above formulas F_i are four arbitrary functions of u^1 and $\theta = (2\eta^{12}u^2 + \eta^{22}u^1)^{-1}$. Moreover all parameters F_i are essential in the sense that they cannot be removed by means of a Miura transformation.

Due to its technical nature, we postpone the proof to appendix A.

Corollary 2. *In the cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$, all second order deformations are quasi-trivial.*

Proof. By construction, the canonical quasi-Miura transformation generated by $H[u]$ reduces the pencil to its dispersionless limit up to terms of order $\mathcal{O}(\epsilon^4)$. ∎

Remark. General Miura transformations have the form

$$u^i \to \tilde{u}^i = f^i(u) + \sum_{k\geqslant 1} \epsilon^k F_k^i(u, u_x, \ldots, u_{(k)}).$$

where $\det\frac{\partial f^i}{\partial u^j} \neq 0$. In this paper we are interested in Miura transformations preserving the disperionless limit and for this reason we consider the subgroup

$$u^i \to \tilde{u}^i = u^i + \sum_{k\geqslant 1} \epsilon^k F_k^i(u, u_x, \ldots, u_{(k)}).$$

Indeed, the only diffeomorphism preserving both metrics of the pencil is the identity.

4.3. Invariants of bi-Hamiltonian structures

As already mentioned in the introduction, the central invariants for deformations of semisimple Poisson pencils of hydrodynamic type (2.1) are related to the roots of the equation

$$\det\left(g_2^{ij} - \lambda g_1^{ij} + \sum_{k\geqslant 1}(A_{2;k,0}^{ij}(u) - \lambda A_{1;k,0}^{ij}(u))p^k\right) = 0.$$

Expanding these roots near $\lambda^i = r^i$ one obtains a series:

$$\lambda^i = r^i + \sum_{k=1}^{\infty} \lambda^i_k p^k, \tag{4.2}$$

whose coefficients are invariants (up to permutations) with respect to Miura transformations as shown by Dubrovin, Liu and Zhang in [12].

Due to the skew-symmetry of the pencil, the sum and product of the roots contain only even powers of p. In the semisimple case the expansions (4.2) of the roots also contain only even powers of p, while in the non-semisimple case in general odd powers are also allowed. For instance, in the case of deformations of non-semisimple pencils associated with Balinskiĭ–Novikov algebras one obtains the expansions

$$\lambda^1 = u^1 + \sum_{k=1}^{\infty} \lambda^1_k p^k, \qquad \lambda^2 = u^1 + \sum_{k=1}^{\infty} \lambda^2_k p^k. \tag{4.3}$$

where, due to skew-symmetry:

$$\lambda^1_{2k+1} + \lambda^2_{2k+1} = 0, \qquad \lambda^1_{2k} - \lambda^2_{2k} = 0. \tag{4.4}$$

Thus it is natural to divide Poisson pencils associated with Balinskiĭ–Novikov algebras in two classes: those admitting as invariants $\lambda^1_1 = -\lambda^2_1$ and $\lambda^1_2 = \lambda^2_2$ (and eventually higher order coefficients of the expansions (4.3)) and those admitting as invariants only $\lambda^1_2 - \lambda^2_2$ (and eventually higher order coefficients of the expansions (4.3)).

4.3.1. The cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$. In the cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$, the expansions of λ^i do not contain the linear term in p and the coefficients of the quadratic terms $\lambda^1_2 = \lambda^2_2$ are related to the functional parameter F_2.

Theorem 3. *Let $\omega_\lambda = \omega_2 - \lambda\omega_1$ be a bi-Hamiltonian structure corresponding to one of the Balinskiĭ–Novikov algebras T3, N3, N5 or N6 with $\kappa \neq 0, -1, -2$ and the associated symmetric bilinear invariant form η. Let us consider bi-Hamiltonian structures Π_λ of the form (2.1) with leading term ω^{ij}_λ. Then the coefficients λ^1_2 and λ^2_2 of the expansion (4.2) coincide and they are related to the functional parameter F_2 by the formulas:*

- *T3:* $\lambda^i_2 = \frac{u^1}{\eta^{12}} e^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}} F_2(u^1)$.
- *N5:* $\lambda^i_2 = -\frac{u^1 F_2(u^1)}{\eta^{12}\sqrt{2\eta^{12}(u^1+u^2) - \eta^{22}u^1}}$.
- *N3, N6 with* $\kappa \neq 0, -1, -2$: $\lambda^i_2 = -\frac{(\kappa+1)u^1(2\eta^{12}u^2 - (\kappa+1)\eta^{22}u^1)^{\frac{\kappa-1}{2}}}{\eta^{12}} F_2(u^1)$.

Proof. We are going to prove this statement in the case T3 with $\eta^{22} \neq 0$. In this case the dispersionless limit is given by

$$\omega^{ij}_1 = \begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix} \partial_x, \qquad \omega^{ij}_2 = \begin{pmatrix} 0 & -u^1 \\ -u^1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 0 \\ -u^1_x & 0 \end{pmatrix}.$$

If we write the pencil in the standard form

$$\Pi^{ij}_\lambda = \omega^{ij}_\lambda + \sum_{k=1}^{2} \epsilon^k \sum_{l=0}^{k+1} (A^{ij}_{2;k,l}(u,\dots,u_{(l)}) - \lambda A^{ij}_{1;k,l}(u,\dots,u_{(l)}))\partial_x^{(k-l+1)} + \mathcal{O}(\epsilon^3),$$

the first two terms of the expansion (4.2) are

$$\lambda_1^i = 0, \tag{4.5}$$

$$\lambda_2^i = \frac{1}{\eta^{12}}\left(Q_2^{12} + \frac{(P_2^{12})^2}{u^1} + \frac{\eta^{22}Q_2^{11}}{2\eta^{12}} + \frac{u^1Q_1^{12} + P_1^{12}P_2^{12}}{\eta^{12}}\right), \tag{4.6}$$

where

$$P_\theta^{ij}(u) = A_{\theta;1,2}^{ij}(u), \quad Q_\theta^{ij}(u) = A_{\theta;2,3}^{ij}(u), \quad i,j = 1,\dots,n, \quad \theta = 1,2.$$

We know from general theory that these coefficients are invariant up to permutations. The condition $\lambda_{2n}^1 = \lambda_{2n}^2$ implies that they are genuine invariants.

Using this proof is a straightforward computation: substituting the relations

$$P_1 = P_2 = Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & u^1 e^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}} F_2(u^1) \\ u^1 e^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}} F_2(u^1) & * \end{pmatrix},$$

in the formula (4.6) we obtain the result. The remaining cases can be proved following the same procedure. ■

Remark. The invariant λ_2^i can be also written as

$$\lambda_2^i = -\frac{1}{2}\mathrm{Res}_{\lambda=\hat{\lambda}}\mathrm{Tr}(g_\lambda^{-1}\Lambda_\lambda)$$

where $\hat{\lambda}$ is the eigenvalue of the affinor $L = g_2g_1^{-1}$ and $\Lambda_\lambda^{ij} = Q_\lambda^{ij} + \frac{1}{2}(g_\lambda^{-1})_{lk}P_\lambda^{li}P_\lambda^{kj}$.

4.3.2. The cases N4 and N6 with $\kappa = -2$. In the remaining cases the expansion of λ^i also contains the linear term in p and the invariants $\lambda_1^1 = -\lambda_1^2$ and $\lambda_2^1 = \lambda_2^2$ are related to the functional parameters F_2 and F_4, respectively.

Theorem 4. *Let $\omega_\lambda = \omega_2 - \lambda\omega_1$ be a bi-Hamiltonian structure corresponding to one of the Balinskiĭ–Novikov algebras N4 or N6 with $\kappa = -2$ and the associated symmetric bilinear invariant form η. Let us consider a bi-Hamiltonian structures Π_λ of the form (2.1) with leading term ω_λ^{ij}. Then, the invariants $(\lambda_1^i)^2$ and λ_2^i are related to the functional parameters F_2 and F_4 through the formulas:*

- *N4:*

$$(\lambda_1^i)^2 = \frac{2u^1F_2}{(\eta^{12})^3},$$

$$\lambda_2^i = \frac{\partial_1(u^1F_2)}{(\eta^{12})^2} - \frac{u^1F_4}{\eta^{12}\sqrt{-2\eta^{12}u^2 + \eta^{22}u^1}}.$$

- *N6, $\kappa = -2$:*

$$(\lambda_1^i)^2 = \frac{2u^1 F_2}{(\eta^{12})^3(2\eta^{12}u^2 + \eta^{22}u^1)^2},$$
$$\lambda_2^i = \frac{u^1 F_4}{\eta^{12}(2\eta^{12}u^2 + \eta^{22}u^1)^{3/2}} - \frac{(2\eta^{12}u^2 - \eta^{22}u^1)F_2 + u^1 F_2'}{(\eta^{12})^2(2\eta^{12}u^2 + \eta^{22}u^1)^3}.$$

Proof. We outline the proof in the case N4 (corresponding to $\kappa = 0$). In this case, the standard form of the pencil is

$$\tilde{\Pi}_\lambda^{ij} = \omega_\lambda^{ij} + \epsilon^2\Theta^{ij} + \mathcal{O}(\epsilon^3) = \omega_\lambda^{ij} + \epsilon^2(\Theta_{(3)}^{ij}\partial_x^3 + \Theta_{(2)}^{ij}\partial_x^2 + \Theta_{(1)}^{ij}\partial_x + \Theta_{(0)}^{ij}) + \mathcal{O}(\epsilon^3),$$

where

$$\omega_\lambda^{ij} = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix}\partial_x + \begin{pmatrix} 0 & u_x^1 \\ 0 & u_x^2 \end{pmatrix} - \lambda\begin{pmatrix} 0 & \eta^{12} \\ \eta^{12} & \eta^{22} \end{pmatrix}\partial_x.$$

and

$$\Theta_{(3)} = \begin{pmatrix} \dfrac{2u^1F_2}{2\eta^{12}u^2 - \eta^{22}u^1} & \dfrac{u^1F_2'}{\eta^{12}} - \dfrac{u^1F_4}{\sqrt{-2\eta^{12}u^2 + \eta^{22}u^1}} + \dfrac{2u^2F_2}{2\eta^{12}u^2 - \eta^{22}u^1} \\ \dfrac{u^1F_2'}{\eta^{12}} - \dfrac{u^1F_4}{\sqrt{-2\eta^{12}u^2 + \eta^{22}u^1}} + \dfrac{2u^2F_2}{2\eta^{12}u^2 - \eta^{22}u^1} & \dfrac{4u^2F_2'}{\eta^{12}} - \dfrac{4u^2F_4}{\sqrt{-2\eta^{12}u^2 + \eta^{22}u^1}} \end{pmatrix},$$

From the general theory and from relations (4.4) we know that $(\lambda_1^i)^2$ and λ_2^i are invariants. Using the invariance the proof is a straightforward computation. The case N6 with $\kappa = -2$ can be treated in a similar way. ■

Remark. The function $\Theta_{(3)}^{12}$ can also be written as

$$\Theta_{(3)}^{12} = -\frac{\eta^{12}}{2}\mathrm{Res}_{\lambda=\hat{\lambda}}\mathrm{Tr}(g_\lambda^{-1}\Lambda_\lambda)$$

where $\hat{\lambda}$ is the eigenvalue of the affinor $L = g_2 g_1^{-1}$ and $\Lambda_\lambda^{ij} = Q_\lambda^{ij} + \frac{1}{2}(g_\lambda^{-1})_{lk}P_\lambda^{li}P_\lambda^{kj}$.

5. Truncated structures

In theorems 3 and 4 we proved the invariant nature of some of the functional parameters appearing in the deformations. In this section we prove that the remaining parameters are related to truncated structures. These are Poisson pencils of the form (2.1) depending polynomially on the parameter ϵ (that is the sum in (2.1) contains finitely many terms). We show that setting the invariant parameters to zero we obtain deformations that are Miura equivalent to truncated pencils up to order three. More precisely we prove that in the cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$ the additional parameter provides a one-parameter family of truncated structures, while in the cases N4 and N6 with $\kappa = -2$ the two additional parameters provide a two-parameter family of truncated structures.

Theorem 5. *In the cases T3, N3, N5 and N6 with $\kappa \neq 0, -1, -2$, the second order deformations with $F_2 = 0$ can be reduced by a Miura transformation to the form $\Pi_\lambda = \omega_\lambda + \epsilon^2\Theta + \mathcal{O}(\epsilon^3)$ where*

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & 2f \end{pmatrix}\partial_x^3 + \begin{pmatrix} 0 & 0 \\ 0 & 3f_x \end{pmatrix}\partial_x^2 + \begin{pmatrix} 0 & 0 \\ 0 & f_{xx} \end{pmatrix}\partial_x, \tag{5.1}$$

with $f = f(u^1)$. Moreover, the truncated pencil $\omega_\lambda + \epsilon^2\Theta$ is a Poisson pencil.

Proof. The form (5.1) can easily be obtained from the results of theorem 1 rescaling the function F_1. In particular, we have to set

- $F_1(u^1) = \frac{f(u^1)}{u^1}$, for T3,
- $F_1(u^1) = -\frac{\eta^{12}f(u^1)}{u^1}$, for N5,
- $F_1(u^1) = -\frac{\eta^{12}\kappa f(u^1)}{(1+\kappa)u^1}$, for N3, N6 with $\kappa \neq 0, -1, -2$.

To prove that $\omega_\lambda + \epsilon^2\Theta$ is a Poisson pencil, we have to show that

$$\frac{1}{2}[\Theta, \Theta]^{ijk}(x, y, z)$$
$$= \frac{\partial\Theta^{ij}(x, y)}{\partial u^l_{(s)}(x)}\partial_x^s\Theta^{lk}(x, z) + \frac{\partial\Theta^{ki}(z, x)}{\partial u^l_{(s)}(z)}\partial_z^s\Theta^{lj}(z, y) + \frac{\partial\Theta^{jk}(y, z)}{\partial u^l_{(s)}(y)}\partial_y^s\Theta^{li}(y, x) = 0.$$

Taking into account that $\Theta^{11} = \Theta^{12} = \Theta^{21} = 0$ and $\frac{\partial\Theta^{22}}{\partial u^2_{(s)}} = 0$, we obtain the result. ∎

Theorem 6. *In the case N6 with $\kappa = -2$ the second order deformations with $F_2 = F_4 = 0$ can be reduced by a Miura transformation to the form $\Pi_\lambda = \omega_\lambda + \epsilon^2\Theta + \mathcal{O}(\epsilon^3)$ where*

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & 2f \end{pmatrix}\partial_x^3 + \begin{pmatrix} 0 & 0 \\ 0 & 3f_x \end{pmatrix}\partial_x^2 + \begin{pmatrix} 0 & 0 \\ 0 & f_{xx} + 2g \end{pmatrix}\partial_x + \begin{pmatrix} 0 & 0 \\ 0 & g_x \end{pmatrix}, \tag{5.2}$$

with $f = f(u^1)$ and $g = (h(u^1)u^1_x)_x + h(u^1)u^1_{xx}$. Moreover the truncated pencil $\omega_\lambda + \epsilon^2\Theta$ is a Poisson pencil.

Proof. Here we prove only the first part of the theorem. The second part can be obtained as above by straightforward computation.

By theorem 1 we have

$$\Pi_\lambda = \omega_2 - \lambda\omega_1 + \epsilon^2\mathrm{Lie}_X\omega_2 + \mathcal{O}(\epsilon^3),$$

where the components of the vector field X are given by

$$X^1 = \theta F_1(u^1_x)^2, \quad X^2 = F_3(u^1_x)^2,$$

with $\theta = (2\eta^{12}u^2 + \eta^{22}u^1)^{-1}$. The Miura transformation

$$u^i \to \exp(-\epsilon Y)u^i, \quad i = 1, 2,$$

generated by the vector field Y of components

$$Y^1 = -\eta^{12} R u^1_{xx} - \eta^{12}\partial_1 R (u^1_x)^2 - \eta^{12}\partial_2 R u^1_x u^2_x,$$
$$Y^2 = -\eta^{22} R u^1_{xx} - \eta^{22}\partial_1 R (u^1_x)^2 + (\eta^{12}\partial_1 R - \eta^{22}\partial_2 R) u^1_x u^2_x + \eta^{12}\partial_2 R (u^2_x)^2 + \eta^{12} R u^2_{xx},$$

with $R = \frac{u^1 F_1}{2\eta^{12}(2\eta^{12}u^2 + \eta^{22}u^1)}$, reduces the pencil to the form $\omega_2 - \lambda\omega_1 + \epsilon^2 \mathrm{Lie}_{\tilde{X}}\omega_2 + \mathcal{O}(\epsilon^3)$, where

$$\tilde{X}^1 = -\frac{\theta u^1 F_1 u^1_{xx}}{2} - \left(\frac{\theta u^1 F_1'}{2} - \theta^2(\eta^{12}u^2 + \eta^{22}u^1)F_1\right)(u^1_x)^2 + \theta^2\eta^{12}u^1 F_1 u^1_x u^2_x$$
$$\tilde{X}^2 = -\frac{\theta\eta^{22}u^1 F_1 u^1_{xx}}{2\eta^{12}} + \frac{\theta u^1 F_1 u^2_{xx}}{2} + \left(\frac{\theta u^1 F_1'}{2} + \theta^2(\eta^{12}u^2 + \eta^{22}u^2)F_1\right)u^1_x u^2_x$$
$$-\left(\frac{\theta\eta^{22}u^1 F_1'}{2\eta^{12}} + \theta^2\eta^{22}u^2 F_1 - F_3\right)(u^1_x)^2 - \theta^2\eta^{12}u^1 F_1 (u^2_x)^2.$$

To conclude, it is easy to check that $\mathrm{Lie}_{\tilde{X}}\omega_2$ coincides with (5.2) ($F_1 = -\frac{2\eta^{12}f}{u^1}$ and $F_3 = -\frac{h}{u^1}$). ■

Theorem 7. *In the case N4 with $F_2 = F_4 = 0$ the second order deformations can be reduced by a Miura transformation to the form $\Pi_\lambda = \omega_\lambda + \epsilon^2\Theta + \mathcal{O}(\epsilon^3)$ where*

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & q^{22}_3 \end{pmatrix}\partial^3_x + \begin{pmatrix} 0 & q^{12}_2 \\ -q^{12}_2 & q^{22}_2 \end{pmatrix}\partial^2_x + \begin{pmatrix} q^{11}_1 & q^{12}_1 \\ q^{21}_1 & q^{22}_1 \end{pmatrix}\partial_x + \begin{pmatrix} q^{11}_0 & q^{12}_0 \\ q^{21}_0 & q^{22}_0 \end{pmatrix}, \qquad (5.3)$$

with

$$q^{22}_3 = 2f,$$
$$q^{12}_2 = 4\theta\eta^{12} f u^1_x,$$
$$q^{22}_2 = 3f' u^1_x,$$
$$q^{11}_1 = -8(\theta\eta^{12})^2 f (u^1_x)^2,$$
$$q^{12}_1 = (2\theta\eta^{12}f' - 2\theta^2\eta^{12}\eta^{22}f + 2\theta^2 h)(u^1_x)^2,$$
$$q^{21}_1 = (-6\theta\eta^{12}f' - 10\theta^2\eta^{12}\eta^{22}f + 2\theta^2 h)(u^1_x)^2 + 16(\theta\eta^{12})^2 f u^1_x u^2_x - 8\theta\eta^{12} f u^1_{xx},$$
$$q^{22}_1 = (f'' + 2\theta(\eta^{12})^{-1}h' + 6\theta^2(\eta^{12})^{-1}\eta^{22}h)(u^1_x)^2 - 8\theta^2 h u^1_x u^2_x + (f' + 4\theta(\eta^{12})^{-1}h)u^1_{xx},$$
$$q^{11}_0 = -(4(\theta\eta^{12})^2 f' + 8\theta^3(\eta^{12})^2\eta^{22}f)(u^1_x)^3 + 16(\theta\eta^{12})^3 f (u^1_x)^2 u^2_x - 8(\theta\eta^{12})^2 f u^1_x u^1_{xx},$$
$$q^{12}_0 = (2\theta^2 h' + 4\theta^3\eta^{22}h)(u^1_x)^3 - 8\theta^3\eta^{12}h(u^1_x)^2 u^2_x + 4\theta^2 h u^1_x u^1_{xx},$$
$$\begin{aligned} q^{21}_0 = {} & (-2\theta\eta^{12}f'' - 8\theta^2\eta^{12}\eta^{22}f' - 12\theta^3\eta^{12}(\eta^{22})^2 f)(u^1_x)^3 \\ & + (12(\theta\eta^{12})^2 f' + 40\theta^3(\eta^{12})^2\eta^{22}f)(u^1_x)^2 u^2_x + (-8\theta\eta^{12}f' - 16\theta^2\eta^{12}\eta^{22}f)u^1_x u^1_{xx} \\ & - 32(\theta\eta^{12})^3 f u^1_x (u^2_x)^2 + 8(\theta\eta^{12})^2 f u^1_x u^2_{xx} + 16(\theta\eta^{12})^2 f u^1_{xx} u^2_x - 4\theta\eta^{12} f u^1_{xxx}, \end{aligned}$$
$$\begin{aligned} q^{22}_0 = {} & (\theta(\eta^{12})^{-1}h'' + 4\theta^2(\eta^{12})^{-1}\eta^{22}h' + 6\theta^3(\eta^{12})^{-1}(\eta^{22})^2 h)(u^1_x)^3 \\ & + (-6\theta^2 h - 20\theta^3\eta^{22}h)(u^1_x)^2 u^2_x + (4\theta(\eta^{12})^{-1}h' + 8\theta^2(\eta^{12})^{-1}\eta^{22}h)u^1_x u^1_{xx} \\ & + 16\theta^3\eta^{12}h u^1_x (u^2_x)^2 - 2\theta^2 h u^1_x u^2_{xx} - 4\theta^2 h u^1_{xx} u^2_x + \theta(\eta^{12})^{-1} h u^1_{xxx}, \end{aligned}$$

where $f=f(u^1)$, $h=h(u^1)$ and $\theta=(2\eta^{12}u^2-\eta^{22}u^1)^{-1}$. Moreover the truncated pencil $\omega_\lambda+\epsilon^2\Theta$ is a Poisson pencil.

Proof. By theorem 1 we have $\Pi_\lambda=\omega_2-\lambda\omega_1+\epsilon^2\mathrm{Lie}_X\omega_2+\mathcal{O}(\epsilon^3)$, where the components of the vector field X are given by

$$X^1=-\theta F_1(u^1_x)^2,\quad X^2=-\theta F_3(u^1_x)^2,$$

with $\theta=(2\eta^{12}u^2-\eta^{22}u^1)^{-1}$. The Miura transformation

$$u^i\to\exp(-\epsilon Y)u^i,\quad i=1,2,$$

generated by the vector field Y of components

$$\begin{aligned}Y^1&=-\eta^{12}Ru^1_{xx}-\eta^{12}\partial_1R(u^1_x)^2-\eta^{12}\partial_2Ru^1_xu^2_x,\\ Y^2&=-\eta^{22}Ru^1_{xx}-\eta^{22}\partial_1R(u^1_x)^2+(\eta^{12}\partial_1R-\eta^{22}\partial_2R)u^1_xu^2_x+\eta^{12}\partial_2R(u^2_x)^2+\eta^{12}Ru^2_{xx},\end{aligned}$$

with $R=-\frac{u^1F_1}{2\eta^{12}(2\eta^{12}u^2-\eta^{22}u^1)}$, reduces the pencil to the form

$$\omega_2-\lambda\omega_1+\epsilon^2\mathrm{Lie}_{\tilde X}\omega_2+\mathcal{O}(\epsilon^3),$$

where

$$\begin{aligned}X^1&=\frac{\theta u^1F_1u^1_{xx}}{2}+\left(\frac{\theta u^1F_1'}{2}-\theta^2(\eta^{12}u^2-\eta^{22}u^2)F_1\right)(u^1_x)^2-\theta^2\eta^{12}u^1F_1u^1_xu^2_x,\\ X^2&=\frac{\theta\eta^{22}u^1F_1u^1_{xx}}{2\eta^{12}}-\frac{\theta u^1F_1u^2_{xx}}{2}-\left(\frac{\theta u^1F_1'}{2}+\theta^2(\eta^{12}u^2+\eta^{22}u^2)F_1\right)u^1_xu^2_x\\ &\quad+\left(\frac{\theta\eta^{22}u^1F_1'}{2\eta^{12}}+\theta^2\eta^{22}u^2F_1-\theta F_3\right)(u^1_x)^2+\theta^2\eta^{12}u^1F_1(u^2_x)^2,\end{aligned}$$

To conclude the first part of the theorem we observe that it is easy to check that $\mathrm{Lie}_{\tilde X}\omega_2=\Theta$ ($F_1=\frac{2\eta^{12}f}{u^1}$ and $F_3=-\frac{h}{\eta^{12}u^1}$). The second part is a cumbersome computation. ∎

Remark. Truncated Poisson pencils of the form

$$\Pi^{ij}_\lambda=\omega^{ij}_\lambda+\epsilon\sum_{l=0}^{2}(A^{ij}_{2;1,l}-\lambda A^{ij}_{1;1,l})\partial_x^{(2-l)}+\epsilon^2\sum_{l=0}^{3}(A^{ij}_{2;2,l}-\lambda A^{ij}_{1;2,l})\partial_x^{(3-l)}\qquad(5.4)$$

where ω_λ is a Poisson pencil of hydrodynamic type associated with a Balinskiĭ–Novikov algebra appear in [38]. In this case the coefficients

$$A^{ij}_{2;1,0},\ A^{ij}_{1;1,0},\ A^{ij}_{2;2,0},\ A^{ij}_{1;2,0}$$

are related to the second and third order co-cycles of the Balinskiĭ–Novikov algebra. In order to reduce deformations of the form (5.4) to the canonical form $\Pi_\lambda=\omega_\lambda+\epsilon^2\Theta+\mathcal{O}(\epsilon^3)$ one has to perform a Miura transformation producing (in general) infinitely many terms in the right-hand side of (5.4). For this reason (in general) Strachan–Szablikowski truncated pencils correspond in our framework to non-truncated pencils.

6. Lifts of Poisson structures

Given a differentiable manifold M, there is a natural way to lift tensor fields and affine connections from M to its tangent bundle TM, viewed as a manifold itself. Such a lift is called a *complete lift* and has been extensively studied by Yano and Kobayashi [40–42]. In this section we apply this construction to Poisson tensors defined on a suitable loop space.

6.1. Complete lift

Let us recall the definition and some properties of complete lift, see the original papers mentioned above for more details.

Given local coordinates $u^1, \ldots, u^n$ on M, let $u^1, \ldots, u^n, v^1, \ldots, v^n$ be the induced bundle coordinates on TM so that any tangent vector on M has the form $v^i\frac{\partial}{\partial u^i}$. The complete lift of a function f, a one form $\alpha = \alpha_i \mathrm{d}u^i$, and a vector field $X = X^i\frac{\partial}{\partial u^i}$ are defined, respectively, by

$$\hat{f} = v^j\frac{\partial f}{\partial u^j}, \qquad \hat{\alpha} = v^j\frac{\partial \alpha_i}{\partial u^j}\mathrm{d}u^i + \alpha_i \mathrm{d}v^i, \qquad \hat{X} = X^i\frac{\partial}{\partial u^i} + v^j\frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial v^i}. \tag{6.1}$$

It follows readily from these local expressions that $\alpha(X)$ lifts to $\hat{\alpha}(\hat{X})$ and a commutator $[X, Y]$ lifts to $[\hat{X}, \hat{Y}]$.

Lifted vector fields (one-forms) span the tangent (cotangent) space of TM at any point which does not belong to the zero section $\{v = 0\}$. As a consequence, one can define the complete lift $\hat{K}$ of any given tensor field K just by imposing that any contraction with a vector field X or a one-form α on M lifts to the contraction of $\hat{K}$ with $\hat{X}$ or $\hat{\alpha}$. Then one can check that exterior derivative and Lie derivative are invariant with respect to the complete lift, meaning that $\mathrm{d}\xi$ lifts to $\mathrm{d}\hat{\xi}$ for any differential form ξ and that a Lie derivative $L_X K$ lifts to $L_{\hat{X}}\hat{K}$.

It may be useful to have at hand explicit expressions for some special classes of tensors. In particular, the complete lift of a bilinear form $g = g_{ij}\mathrm{d}u^i \otimes \mathrm{d}u^j$ turns out to be

$$\hat{g} = v^k\frac{\partial g_{ij}}{\partial u^k}\mathrm{d}u^i \otimes \mathrm{d}u^j + g_{ij}\mathrm{d}u^i \otimes \mathrm{d}v^j + g_{ij}\mathrm{d}v^i \otimes \mathrm{d}u^j, \tag{6.2}$$

and a trilinear form $T = T_{ijk}\mathrm{d}u^i \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k$ lifts to

$$\hat{T} = v^h\frac{\partial T_{ijk}}{\partial u^h}\mathrm{d}u^i \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k + T_{ijk}\mathrm{d}u^i \otimes \mathrm{d}u^j \otimes \mathrm{d}v^k + T_{ijk}\mathrm{d}u^i \otimes \mathrm{d}v^j \otimes \mathrm{d}u^k + T_{ijk}\mathrm{d}v^i \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k.$$

Moreover, an endomorphism of the tangent bundle $A = A^i_j\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j$ lifts to

$$\hat{A} = A^i_j\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j + v^k\frac{\partial A^i_j}{\partial u^k}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^j + A^i_j\frac{\partial}{\partial v^i} \otimes \mathrm{d}v^j, \tag{6.3}$$

and the lift of a bilinear product on vector fields $\cdot = c^i_{jk}\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k$ is

$$\begin{aligned}\hat{\cdot} = c^i_{jk}\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k + v^h\frac{\partial c^i_{jk}}{\partial u^h}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k \\ + c^i_{jk}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k + c^i_{jk}\frac{\partial}{\partial v^i} \otimes \mathrm{d}v^j \otimes \mathrm{d}u^k.\end{aligned} \tag{6.4}$$

Finally, any bi-vector $P = P^{ij}\frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}$ lifts to

$$\hat{P} = P^{ij}\frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial v^j} + P^{ij}\frac{\partial}{\partial v^i} \otimes \frac{\partial}{\partial u^j} + v^k\frac{\partial P^{ij}}{\partial u^k}\frac{\partial}{\partial v^i} \otimes \frac{\partial}{\partial v^j}. \tag{6.5}$$

Now, let $\nabla\frac{\partial}{\partial u^k} = \Gamma^i_{jk}\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j$ be an affine connection on M. Its complete lift $\hat{\nabla}$ is an affine connection on TM defined by requiring that for all vector fields X on M the endomorphism ∇X lifts to $\hat{\nabla}\hat{X}$. Using that $\frac{\partial}{\partial u^k}$ and $u^l\frac{\partial}{\partial u^k}$ lift to $\frac{\partial}{\partial u^k}$ and $u^l\frac{\partial}{\partial u^k} + v^l\frac{\partial}{\partial v^k}$, respectively, one can check that

$$\hat{\nabla}\frac{\partial}{\partial u^k} = \Gamma^i_{jk}\frac{\partial}{\partial u^i} \otimes \mathrm{d}u^j + v^h\frac{\partial \Gamma^i_{jk}}{\partial u^h}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^j + \Gamma^i_{jk}\frac{\partial}{\partial v^i} \otimes \mathrm{d}v^j, \tag{6.6}$$

$$\hat{\nabla}\frac{\partial}{\partial v^k} = \Gamma^i_{jk}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^j. \tag{6.7}$$

From the definition one can readily deduce that for any tensor field K on M the complete lift of ∇K equals $\hat{\nabla}\hat{K}$. In particular, any flat tensor ($\nabla K = 0$) lifts to a flat tensor ($\hat{\nabla}\hat{K} = 0$). Moreover the following holds [40, proposition 7.1]:

Proposition 8. *The torsion and the curvature of* $\hat{\nabla}$ *are the complete lift of the torsion and the curvature of* ∇.

Remark. Since the lift is well defined for tensors and connections we can apply it to the geometric structures defining Frobenius manifolds. As a result one obtains a *lifted Frobenius structure*. We discuss this construction in more detail in appendix B.

6.2. Lift of Poisson structures of hydrodynamic type

The class of structures that can be lifted to the tangent bundle by means of complete lift includes symplectic forms and more generally Poisson tensors. The latter have been studied in some detail by Mitric and Vaisman [33]. Since the Schouten bracket is defined in terms of Lie derivative, it follows that it is invariant by complete lift as well. As a consequence, the complete lift of a bi-Hamiltonian structure $P_\lambda = P + \lambda Q$, where $\lambda \in \mathbf{R}$ and P, Q are Poisson tensors on M satisfying $[P, Q] = 0$, is a bi-Hamiltonian structure $\hat{P}_\lambda = \hat{P} + \lambda\hat{Q}$.

Recall that in local coordinates u^i on M and x on S^1 the Poisson tensor P at $\gamma = u(x)$ is represented by $\frac{\partial}{\partial u^i} \otimes P^{ij}\frac{\partial}{\partial u^j}$ where

$$P^{ij} = g^{ij}\partial_x + b^{ij}_k u^k_x, \qquad i, j = 1, ..., n. \tag{6.8}$$

Here g^{ij} is the inverse of the matrix g_{ij} which represents g locally, and $b^{ij}_k = -g^{ih}\Gamma^j_{hk}$, being Γ^j_{hk} the Christoffel symbols of g. It is clear that P can be lifted to $\mathcal{L}(TM)$ defining $\hat{P}$ as

$$\hat{P}^{\alpha\beta} = \hat{g}^{\alpha\beta}\partial_x + \hat{b}^{\alpha\beta}_\gamma u^\gamma_x, \qquad \alpha, \beta = 1, ..., 2n,$$

where $\hat{g}$ is the lift of the contravariant metric, $\hat{b}^{\alpha\beta}_\gamma$ are the contravariant Christoffel symbols of the lifted Levi-Civita connection and we set $u^{n+i} = v^i$. Indeed one has only to check that $\hat{\nabla}$

is the Levi-Civita connection of the lifted metric $\hat{g}$. But this follows by the uniqueness of the Levi-Civita connection together with the fact that $\hat{\nabla}\hat{g} = 0$ for $\nabla g = 0$, and that $\hat{\nabla}$ is torsion free by proposition 8 and by the torsion-freeness of ∇. Therefore $\hat{g}$ defines a Poisson structure of hydrodynamic type $\hat{P}$ on $\mathcal{L}(TM)$.

Remark. It is easy to check that the lift $\hat{P}$ is uniquely defined by the requirement (the analogous property in the finite-dimensional case has been observed in [33])

$$\{H_\xi, H_\eta\}_{\hat{P}} = \int_{S^1} \langle v, \{\xi, \eta\}_P \rangle \, \mathrm{d}x \tag{6.9}$$

where $H_\xi = \int_{S^1} \langle \xi, v \rangle \, \mathrm{d}x$ and $\{\cdot, \cdot\}_P$ is the Poisson bracket on 1-forms [21, 32] defined by g [1]:

$$\{\xi, \eta\}_j = g^{kl} \left[\partial_x^{s+1}(\eta)_l \frac{\partial(\xi)_j}{\partial u^k_{(s)}} - \partial_x^{s+1}(\xi)_l \frac{\partial(\eta)_j}{\partial u^k_{(s)}} \right]. \tag{6.10}$$

Proposition 9. *In local coordinates u^i, v^i on TM one has*

$$\begin{aligned} \hat{P} = & \frac{\partial}{\partial v^i} \otimes (g^{ij}\partial_x + b^{ij}_k u^k_x) \frac{\partial}{\partial u^j} + \frac{\partial}{\partial u^i} \otimes (g^{ij}\partial_x + b^{ij}_k u^k_x) \frac{\partial}{\partial v^j} \\ & + \frac{\partial}{\partial v^i} \otimes \left(v^h (b^{ij}_h + b^{ji}_h)\partial_x + v^h \frac{\partial b^{ij}_k}{\partial u^h} u^k_x + b^{ij}_k v^k_x \right) \frac{\partial}{\partial v^j}. \end{aligned} \tag{6.11}$$

Proof. Thanks to (6.8) we have to determine the coefficients g^{ij} and b^{ij}_k for the lifted metric $\hat{g}$. To this end, let W^j be the metric dual of the coordinate one-form $\mathrm{d}u^j$ on M. This means that W^j is the unique vector field on M such that $g(W^j, \cdot) = \mathrm{d}u^j$, and clearly one has

$$W^j = g^{ij} \frac{\partial}{\partial u^i}. \tag{6.12}$$

Moreover, well known properties of the Christoffel symbols yield

$$\nabla W^j = b^{ij}_k \frac{\partial}{\partial u^i} \otimes \mathrm{d}u^k. \tag{6.13}$$

Therefore one can write

$$P = W^j \otimes \partial_x \frac{\partial}{\partial u^j} + \nabla_{\dot{\gamma}} W^j \otimes \frac{\partial}{\partial u^j}, \tag{6.14}$$

wehere $\dot{\gamma} = u^k_x \frac{\partial}{\partial u^k}$.

Let U^j and V^j be the metric dual of $\mathrm{d}u^j$ and $\mathrm{d}v^j$ with respect to the lifted metric $\hat{g}$ on TM. One can readily check by (6.2) that

$$U^j = g^{ij} \frac{\partial}{\partial v^i}. \tag{6.15}$$

On the other hand, by (6.1) the lift of $\mathrm{d}u^j$ turns out to be $\mathrm{d}v^j$. Therefore $V^j = \hat{W}^j$, so that

$$V^j = g^{ij}\frac{\partial}{\partial u^i} + v^k(b^{ij}_k + b^{ji}_k)\frac{\partial}{\partial v^i}, \tag{6.16}$$

where we used the identity

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k. \tag{6.17}$$

In particular $\hat{\nabla} V^j = \hat{\nabla}\hat{W}^j$, whence by definition of the lifted connection and equations (6.13) and (6.3) it follows that

$$\hat{\nabla} V^j = b^{ij}_k \frac{\partial}{\partial u^i} \otimes \mathrm{d}u^k + v^h \frac{\partial b^{ij}_k}{\partial u^h}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^k + b^{ij}_k \frac{\partial}{\partial v^i} \otimes \mathrm{d}v^k. \tag{6.18}$$

On the other hand, by (6.7) one calculates

$$\hat{\nabla} U^j = \frac{\partial g^{ij}}{\partial u^k}\frac{\partial}{\partial v^i} \otimes \mathrm{d}u^k + g^{ij}\Gamma^h_{ki}\frac{\partial}{\partial v^h} \otimes \mathrm{d}u^k, \tag{6.19}$$

whence, thanks to the identity (6.17), one concludes

$$\hat{\nabla} U^j = b^{ij}_k \frac{\partial}{\partial v^i} \otimes \mathrm{d}u^k. \tag{6.20}$$

The statement then follows by simple calculations from equations (6.15), (6.16), (6.18), (6.20) and the identity

$$\hat{P} = U^j \otimes \partial_x \frac{\partial}{\partial u^j} + \hat{\nabla}_{\dot\gamma} U^j \otimes \frac{\partial}{\partial u^j} + V^j \otimes \partial_x \frac{\partial}{\partial v^j} + \nabla_{\dot\gamma} V^j \otimes \frac{\partial}{\partial v^j}, \tag{6.21}$$

where $\dot\gamma = u^k_x \frac{\partial}{\partial u^k} + v^k_x \frac{\partial}{\partial v^k}$ for any loop $\gamma = (u(x), v(x))$ in TM. ∎

6.3. Lift of bi-vectors in the loop space

In matrix notation the lift (6.11) takes the form

$$\hat{P} = \begin{pmatrix} 0 & P^{ij} \\ P^{ij} & \sum_{k,t} v^k_{(t)} \frac{\partial P^{ij}}{\partial u^k_{(t)}} \end{pmatrix}, \tag{6.22}$$

whence it is clear that one can lift to $\mathcal{L}(TM)$ any given Poisson structure (non-necessarily of hydrodynamic type) on the loop space $\mathcal{L}(M)$. The proof of this fact is contained in [25] in the framework of the linearization of Hamiltonian objects, i.e. formal or universal linearization (see for instance [23, 26]) or tangent covering (see for instance [24]). We provide here a different direct proof which rests just on the Schouten bracket formula given in [16].

Theorem 10. *Suppose that*

$$P^{ij}_{x,y} = P^{ij}_k(x-y, u, u_x, \ldots, u_{k+1}) = \sum_{m=0}^{k+1} A^{ij}_m(u, u_x, \ldots, u_{k+1})\delta^{(k+1-m)}(x-y),$$

and

$$Q^{ij}_{x,y} = Q^{ij}_k(x-y,u,u_x,\dots,u_{k+1}) = \sum_{m=0}^{k+1} B^{ij}_m(u,u_x,\dots,u_{k+1})\delta^{(k+1-m)}(x-y),$$

have a vanishing Schouten bracket

$$\begin{aligned}[P,Q]^{ijk}_{x,y,z} &= \frac{\partial P^{ij}_{x,y}}{\partial u^l_{(s)}(x)}\partial^s_x Q^{lk}_{x,z} + \frac{\partial Q^{ij}_{x,y}}{\partial u^l_{(s)}(x)}\partial^s_x P^{lk}_{x,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^l_{(s)}(z)}\partial^s_z Q^{lj}_{z,y}\\ &+ \frac{\partial Q^{ki}_{z,x}}{\partial u^l_{(s)}(z)}\partial^s_z P^{lj}_{z,y} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_{(s)}(y)}\partial^s_y Q^{li}_{y,x} + \frac{\partial Q^{jk}_{y,z}}{\partial u^l_{(s)}(y)}\partial^s_y P^{li}_{y,x} = 0,\end{aligned}$$

then also the lifted structures

$$\hat{P} = \begin{pmatrix} 0 & P^{ij} \\ P^{ij} & \sum_{k,t} v^k_{(t)} \dfrac{\partial P^{ij}}{\partial u^k_{(t)}} \end{pmatrix}, \qquad \hat{Q} = \begin{pmatrix} 0 & Q^{ij} \\ Q^{ij} & \sum_{k,t} v^k_{(t)} \dfrac{\partial Q^{ij}}{\partial u^k_{(t)}} \end{pmatrix}$$

have a vanishing Schouten bracket.

Proof. Throughout in this proof u^{n+i} will denote v^i for all $i = 1,\dots,n$. Moreover we fix the convention that latin indices i, j, k run from 1 through n, and greek indices α, β, γ run from 1 through $2n$. By straightforward computation we obtain

- For $\alpha = i, \beta = j, \gamma = k$:

$$\begin{aligned}[\hat{P},\hat{Q}]^{\alpha\beta\gamma}_{x,y,z} &= \frac{\partial \hat{P}^{ij}_{x,y}}{\partial u^\lambda_{(s)}(x)}\partial^s_x \hat{Q}^{\lambda k}_{x,z} + \frac{\partial \hat{Q}^{ij}_{x,y}}{\partial u^\lambda_{(s)}(x)}\partial^s_x \hat{P}^{\lambda k}_{x,z} + \frac{\partial \hat{P}^{ki}_{z,x}}{\partial u^\lambda_{(s)}(z)}\partial^s_z \hat{Q}^{\lambda j}_{z,y}\\ &+ \frac{\partial \hat{Q}^{ki}_{z,x}}{\partial u^\lambda_{(s)}(z)}\partial^s_z \hat{P}^{\lambda j}_{z,y} + \frac{\partial \hat{P}^{jk}_{y,z}}{\partial u^\lambda_{(s)}(y)}\partial^s_y \hat{Q}^{\lambda i}_{y,x} + \frac{\partial \hat{Q}^{jk}_{y,z}}{\partial u^\lambda_{(s)}(y)}\partial^s_y \hat{P}^{\lambda i}_{y,x} = 0,\end{aligned}$$

  since $\hat{P}^{ij}_{x,y} = \hat{Q}^{ij}_{x,y} = \hat{P}^{ki}_{z,x} = \hat{Q}^{ki}_{z,x} = \hat{P}^{jk}_{y,z} = \hat{Q}^{jk}_{y,z} = 0$.

- For $\alpha = n+i, \beta = j, \gamma = k$:

$$\begin{aligned}[\hat{P},\hat{Q}]^{\alpha\beta\gamma}_{x,y,z} &= \frac{\partial \hat{P}^{n+i,j}_{x,y}}{\partial u^\lambda_{(s)}(x)}\partial^s_x \hat{Q}^{\lambda k}_{x,z} + \frac{\partial \hat{Q}^{n+i,j}_{x,y}}{\partial u^\lambda_{(s)}(x)}\partial^s_x \hat{P}^{\lambda k}_{x,z} + \frac{\partial \hat{P}^{k,n+i}_{z,x}}{\partial u^\lambda_{(s)}(z)}\partial^s_z \hat{Q}^{\lambda j}_{z,y}\\ &+ \frac{\partial \hat{Q}^{k,n+i}_{z,x}}{\partial u^\lambda_{(s)}(z)}\partial^s_z \hat{P}^{\lambda j}_{z,y} + \frac{\partial \hat{P}^{jk}_{y,z}}{\partial u^\lambda_{(s)}(y)}\partial^s_y \hat{Q}^{\lambda,n+i}_{y,x} + \frac{\partial \hat{Q}^{jk}_{y,z}}{\partial u^\lambda_{(s)}(y)}\partial^s_y \hat{P}^{\lambda,n+i}_{y,x}\\ &= \frac{\partial P^{ij}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x Q^{lk}_{x,z} + \frac{\partial Q^{ij}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x P^{lk}_{x,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^{n+l}_{(s)}(z)}\partial^s_z Q^{lj}_{z,y}\\ &+ \frac{\partial Q^{ki}_{z,x}}{\partial u^{n+l}_{(s)}(z)}\partial^s_z P^{lj}_{z,y} + \frac{\partial \hat{P}^{jk}_{y,z}}{\partial u^l_{(s)}(y)}\partial^s_y Q^{li}_{y,x} + \frac{\partial \hat{Q}^{jk}_{y,z}}{\partial u^l_{(s)}(y)}\partial^s_y P^{li}_{y,x} = 0,\end{aligned}$$

since $\hat{P}^{jk}_{y,z} = \hat{Q}^{jk}_{y,z} = 0$ and $P^{ij}_{x,y}$, $Q^{ij}_{x,y}$, $P^{ki}_{x,y}$, $Q^{ki}_{x,y}$ do not depend on coordinates on the fibres. Similarly one can prove the vanishing of the Schouten bracket for $\alpha = i, \beta = n + j, \gamma = k$ and $\alpha = i, \beta = j, \gamma = n + k$.

- For $\alpha = n + i, \beta = n + j, \gamma = k$:

$$
\begin{aligned}
[\hat{P},\hat{Q}]^{\alpha\beta\gamma}_{x,y,z} &= \frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{\lambda}_{(s)}(x)}\partial^s_x \hat{Q}^{\lambda k}_{x,z} + \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{\lambda}_{(s)}(x)}\partial^s_x \hat{P}^{\lambda k}_{x,z} + \frac{\partial \hat{P}^{k,n+i}_{z,x}}{\partial u^{\lambda}_{(s)}(z)}\partial^s_z \hat{Q}^{\lambda,n+j}_{z,y} \\
&\quad + \frac{\partial \hat{Q}^{k,n+i}_{z,x}}{\partial u^{\lambda}_{(s)}(z)}\partial^s_z \hat{P}^{\lambda,n+j}_{z,y} + \frac{\partial \hat{P}^{n+j,k}_{y,z}}{\partial u^{\lambda}_{(s)}(y)}\partial^s_y \hat{Q}^{\lambda,n+i}_{y,x} + \frac{\partial \hat{Q}^{n+j,k}_{y,z}}{\partial u^{\lambda}_{(s)}(y)}\partial^s_y \hat{P}^{\lambda,n+i}_{y,x} \\
&= \frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x Q^{lk}_{x,z} + \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x P^{lk}_{x,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^{l}_{(s)}(z)}\partial^s_z Q^{lj}_{z,y} \\
&\quad + \frac{\partial Q^{ki}_{z,x}}{\partial u^{l}_{(s)}(z)}\partial^s_z P^{lj}_{z,y} + \frac{\partial P^{jk}_{y,z}}{\partial u^{l}_{(s)}(y)}\partial^s_y Q^{li}_{y,x} + \frac{\partial Q^{jk}_{y,z}}{\partial u^{l}_{(s)}(y)}\partial^s_y P^{li}_{y,x}.
\end{aligned}
$$

Using the identities

$$
\frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)} = \frac{\partial P^{ij}_{x,y}}{\partial u^{l}_{(s)}(x)}, \qquad \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)} = \frac{\partial Q^{ij}_{x,y}}{\partial u^{l}_{(s)}(x)}, \tag{6.23}
$$

we finally obtain

$$
[\hat{P},\hat{Q}]^{n+i,n+j,k}_{x,y,z} = [P,Q]^{ijk}_{x,y,z} = 0.
$$

Similarly one can prove the vanishing of the Schouten bracket for $\alpha = i, \beta = n + j, \gamma = n + k$ and $\alpha = n + i, \beta = j, \gamma = n + k$.

- For $\alpha = n + i, \beta = n + j, \gamma = n + k$:

$$
\begin{aligned}
[\hat{P},\hat{Q}]^{\alpha\beta\gamma}_{x,y,z} &= \frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{\lambda}_{(s)}(x)}\partial^s_x \hat{Q}^{\lambda,n+k}_{x,z} + \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{\lambda}_{(s)}(x)}\partial^s_x \hat{P}^{\lambda,n+k}_{x,z} + \frac{\partial \hat{P}^{n+k,n+i}_{z,x}}{\partial u^{\lambda}_{(s)}(z)}\partial^s_z \hat{Q}^{\lambda,n+j}_{z,y} \\
&\quad + \frac{\partial \hat{Q}^{n+k,n+i}_{z,x}}{\partial u^{\lambda}_{(s)}(z)}\partial^s_z \hat{P}^{\lambda,n+j}_{z,y} + \frac{\partial \hat{P}^{n+j,n+k}_{y,z}}{\partial u^{\lambda}_{(s)}(y)}\partial^s_y \hat{Q}^{\lambda,n+i}_{y,x} + \frac{\partial \hat{Q}^{n+j,n+k}_{y,z}}{\partial u^{\lambda}_{(s)}(y)}\partial^s_y \hat{P}^{\lambda,n+i}_{y,x} \\
&= \frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{l}_{(s)}(x)}\partial^s_x Q^{lk}_{x,z} + \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{l}_{(s)}(x)}\partial^s_x P^{lk}_{x,z} + \frac{\partial \hat{P}^{n+k,n+i}_{z,x}}{\partial u^{l}_{(s)}(z)}\partial^s_z Q^{lj}_{z,y} \\
&\quad + \frac{\partial \hat{Q}^{n+k,n+i}_{z,x}}{\partial u^{l}_{(s)}(z)}\partial^s_z P^{lj}_{z,y} + \frac{\partial \hat{P}^{n+j,n+k}_{y,z}}{\partial u^{l}_{(s)}(y)}\partial^s_y Q^{li}_{y,x} + \frac{\partial \hat{Q}^{n+j,n+k}_{y,z}}{\partial u^{l}_{(s)}(y)}\partial^s_y P^{li}_{y,x} \\
&\quad + \frac{\partial \hat{P}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x \hat{Q}^{n+l,n+k}_{x,z} + \frac{\partial \hat{Q}^{n+i,n+j}_{x,y}}{\partial u^{n+l}_{(s)}(x)}\partial^s_x \hat{P}^{n+l,n+k}_{x,z} + \frac{\partial \hat{P}^{n+k,n+i}_{z,x}}{\partial u^{n+l}_{(s)}(z)}\partial^s_z \hat{Q}^{n+l,n+j}_{z,y} \\
&\quad + \frac{\partial \hat{Q}^{n+k,n+i}_{z,x}}{\partial u^{n+l}_{(s)}(z)}\partial^s_z \hat{P}^{n+l,n+j}_{z,y} + \frac{\partial \hat{P}^{n+j,n+k}_{y,z}}{\partial u^{n+l}_{(s)}(y)}\partial^s_y \hat{Q}^{n+l,n+i}_{y,x} + \frac{\partial \hat{Q}^{n+j,n+k}_{y,z}}{\partial u^{n+l}_{(s)}(y)}\partial^s_y \hat{P}^{n+l,n+i}_{y,x}
\end{aligned}
$$

Using the identities (6.23) and the fact that the operator ∂_x and the operator $\sum_{k,t} u^{n+k}_{(t)}\frac{\partial}{\partial u^k_{(t)}(x)}$ commute, as it is immediate to check using the identity

$$\partial_x \frac{\partial}{\partial u^k_{(t)}(x)} = \frac{\partial}{\partial u^k_{(t)}(x)}\partial_x - \frac{\partial}{\partial u^k_{(t-1)}(x)},$$

we obtain

$$[\hat{P},\hat{Q}]^{n+i,n+j,n+k}_{x,y,z} = \sum_{k,t}\left(u^{n+k}_{(t)}(x)\frac{\partial}{\partial u^k_{(t)}(x)} + u^{n+k}_{(t)}(y)\frac{\partial}{\partial u^k_{(t)}(y)} + u^{n+k}_{(t)}(z)\frac{\partial}{\partial u^k_{(t)}(z)}\right)[P,Q]^{ijk}_{x,y,z} = 0$$

since $[P,Q]^{ijk}_{x,y,z} = 0$ by the hypothesis. ■

Remark. Note that the lift of bi-vectors (6.22) is obtained from (6.5) by just replacing $\sum_j v^j \frac{\partial}{\partial u^j}$ with $\sum_{j,k} v^j_{(k)}\frac{\partial}{\partial u^j_{(k)}}$. The lift of general tensor fields can be defined in exactly the same way. For instance the lift of functionals, one forms and vector fields can be defined as

$$\hat{F} = \int v^j \frac{\delta F}{\delta u^j}\,\mathrm{d}x, \qquad \hat{\alpha} = \sum_{j,k} v^j_{(k)}\frac{\partial \alpha_i}{\partial u^j_{(k)}}\delta u^i + \alpha_i \delta v^i, \qquad \hat{X} = X^i\frac{\partial}{\partial u^i} + \sum_{j,k} v^j_{(k)}\frac{\partial X^i}{\partial u^j_{(k)}}\frac{\partial}{\partial v^i}.$$

As in the finite-dimensional case the lift $\hat{K}$ of higher order tensor fields K can be defined requiring that any contraction with a vector field X or a one-form α on the loop space lifts to the contraction of $\hat{K}$ with $\hat{X}$ or $\hat{\alpha}$. As a consequence of this general rule the lift of a Hamiltonian vector field coincides with the Hamiltonian vector field obtained lifting the Poisson bi-vector and the Hamiltonian functional: $\widehat{P\delta H} = \hat{P}\delta\hat{H}$. In appendix C we check this fact. Finally we point out that the linearization of Hamiltonian objects mentioned above is nothing but the Yano–Kobayashi complete lift in the infinite-dimensional setting.

6.4. *Lift of deformations*

We have seen in the introduction that deformations of n-component semisimple Poisson pencils of hydrodynamic type depend on n arbitrary functions of a single variable. Applying the previous construction to this case we obtain an n-parameter family of deformations of the lifted Poisson pencil of hydrodynamic type. Due to the obvious identity

$$\det\hat{\pi}^{ij} = \pm(\det\pi^{ij})^2$$

any invariant coefficient comes with double multiplicity. This example suggests that deformations of non-semisimple structures corresponding to those invariant parameters are unobstructed.

6.4.1. Example. In the scalar case all second order deformations are given by [30]

$$\Pi_\lambda = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2(2s\partial_x^3 + 3s_x\partial_x^2 + s_{xx}\partial_x) + \mathcal{O}(\epsilon^3), \tag{6.24}$$

where c is a constant and $s(u)$ is an arbitrary function of u. Applying the lift we obtain a one-parameter family of deformations of a two-component Poisson pencil of hydrodynamic type.

Here we want to show this lift is equivalent, up to Miura transformations, to the case N3 (that is, N6 with $\kappa = 1$) with $F_1(u^1) = \eta^{22} = 0$. Let us consider second order deformations of $N3$ obtained in theorem 1, and set $\eta^{22} = 0$ (otherwise g_1 would not be the lift of the scalar constant metric $\eta = 1$), $\eta^{12} = 1$, $F_1(u^1) = 0$ and $F_2(u^1) = -\frac{f(u^1)}{u^1}$.

The Miura transformation

$$u^i \to \exp(-\epsilon Y)u^i, \quad i = 1, 2,$$

generated by the vector field Y of components

$$Y^1 = \frac{f'}{3}u^1_{xx} + \frac{f''}{3}(u^1_x)^2, \quad Y^2 = -\frac{f''}{3}u^1_x u^2_x - \frac{f'}{3}u^2_{xx},$$

reduces the pencil to the form

$$\hat{\Pi}_\lambda = \begin{pmatrix} 0 & \Pi_\lambda \\ \Pi_\lambda & \sum_t v_{(t)}\frac{\partial \Pi_\lambda}{\partial u_{(t)}} \end{pmatrix},$$

where Π_λ coincides with (6.24) setting $u^1 = u$ and $f(u^1) = s(u)$.

Acknowledgments

We would like to thank Jenya Ferapontov and Raffaele Vitolo for useful discussions and Joseph Krasil'shchik and Alik Verbovetsky for pointing out the reference [25]. PL is partially supported by the Italian MIUR Research Project *Teorie geometriche e analitiche dei sistemi Hamiltoniani in dimensioni finite e infinite* and by GNFM Progetto Giovani 2014 *Aspetti geometrici e analitici dei sistemi integrabili*.

Appendix A. Computations of deformations

In this appendix we give a sketch of the proof of theorem 1, providing the computations of deformations in detail. First we observe that the pencil Π^{ij}_λ can be always reduced to the form

$$\Pi_\lambda = \omega_\lambda + \epsilon Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \dots \tag{A.1}$$

by a suitable Miura transformation. The proof is due to Getzler and it is based on the study of the Poisson–Lichnerowicz cohomology groups [22] (an alternative proof can be found in [9, 16, 28]):

$$H^j(\mathcal{L}(\mathbb{R}^n), \omega) := \frac{\ker\{\mathrm{d}_\omega : \Lambda^j_{\mathrm{loc}} \to \Lambda^{j+1}_{\mathrm{loc}}\}}{\mathrm{im}\{\mathrm{d}_\omega : \Lambda^{j-1}_{\mathrm{loc}} \to \Lambda^j_{\mathrm{loc}}\}}$$

for Poisson bi-vector of hydrodynamic type ω. The differential d_ω is defined as

$$\mathrm{d}_\omega := [\omega, \cdot\ ]$$

where the square bracket is the Schouten bracket. Getzler also proved the triviality of cohomology for any positive integer j (in particular that the triviality of deformations is related to the vanishing of the second co-homology group).

A.1. First order deformations

The pencil (A.1) is a deformation of ω_λ if it satisfies the Jacobi identity for every λ, that is

$$[Q, Q] = [\omega_1, Q] = 0.$$

where $Q = \omega_2 + \epsilon Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + ...$ This implies in particular

$$[\omega_2, Q_1] = [\omega_1, Q_1] = 0.$$

In other words Q_1 is a co-cycle for both the differentials d_{ω_1} and $d_{\omega 2}$. Using the triviality of $H^1(\mathcal{L}(\mathbb{R}^n), \omega)$ and $H^2(\mathcal{L}(\mathbb{R}^n), \omega)$ we obtain $Q_1 = d_{\omega_2} X = \mathrm{Lie}_X \omega_2$ for a suitable vector field of degree 1

$$X^i = X^i_1(u^1, u^2)u^1_x + X^i_2(u^1, u^2)u^2_x, \quad i = 1, 2,$$

satisfying

$$d_{\omega_1} d_{\omega_2} X = 0.$$

It is not difficult to prove that among the solutions of the above equation those corresponding to trivial deformations have the form $X = \omega_1 \delta H + \omega_2 \delta K$, where the Hamiltonian densities are differential polynomials of degree 0, namely $H = \int h(u^1, u^2)\, dx$ and $K = \int k(u^1, u^2)\, dx$. It turns out that in our case all first order deformations are trivial. All details are given below, case by case.

A.1.1. T3. First order deformations. Let us point out that in this case the vanishing of the coefficient η^{22} implies that the affinor L^i_j assumes a diagonal form, while for $\eta^{22} \neq 0$ it corresponds to one 2×2 Jordan block case (as well as all other cases we are dealing with). Recall that we are assuming $\eta^{12} \neq 0$. The vector field X solution of $d_{\omega_1} d_{\omega_2} X = 0$ is given in components by

$$X^1_1 = X^1_1, \quad X^1_2 = X^1_2, \quad X^2_1 = \frac{\eta^{22}}{\eta^{12}} \partial_1(X^1_1 u^1) + \int \left(\partial_1 X^1_1 - \frac{\eta^{22} u^1}{\eta^{12}} \partial^2_1 X^2_1 \right) du^2 + F,$$
$$X^2_2 = X^1_1 + \frac{\eta^{22}}{\eta^{12}} (X^1_2 + u^1(\partial_2 X^1_1 - \partial_1 X^1_2)),$$

where $F = F(u^1)$. The components Y^i of the vector field $Y = \omega_1 \delta H + \omega_2 \delta K$ are given by $Y^i = Y^i_1 u^1_x + Y^i_2 u^2_x$, where

$$\begin{aligned} Y^1_1 &= \eta^{12} \partial_1 \partial_2 H - u^1 \partial_1 \partial_2 K, & Y^1_2 &= \eta^{12} \partial^2_2 H - u^1 \partial^2_2 K, \\ Y^2_1 &= \partial_1(\eta^{12} \partial_1 H + \eta^{22} \partial_2 H - u^1 \partial_1 K), & Y^2_2 &= \partial_2(\eta^{12} \partial_1 H + \eta^{22} \partial_2 H - u^1 \partial_1 K), \end{aligned}$$

Choosing H and K such that $X^1_i = Y^1_i$ for $i = 1, 2$, one can easily see that

$$X^2_1 = Y^2_1 + F, \quad X^2_2 = Y^2_2.$$

Finally, the function F can be removed using the vector field Y such that $H = 0$ and K such that $-\partial_1(u^1 \partial_1 K) = F$. Thus, the first order deformations are trivial.

A.1.2. N5. First order deformations. Here $\eta^{12}\neq 0$. Solving $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X=0$ for $\deg(X)=1$ we obtain

$$X_2^1=\partial_1 F,\quad X_1^2=\partial_2 F,$$
$$X_1^2=\int\left(\partial_1 X_2^2+\frac{\eta^{22}\partial_2 F+\eta^{12}\partial_1 F-\eta^{12}X_2^2}{2\eta^{12}(u^1+u^2)-\eta^{22}u^1}\right)\mathrm{d}u^2+G,\quad X_2^2=X_2^2,$$

where $F=F(u^1,u^2)$ and $G=G(u^1)$.

The components Y^i of the vector field $Y=\omega_1\delta H+\omega_2\delta K$ are given by

$$\begin{aligned}
Y_1^1&=\partial_1(\eta^{12}\partial_2 H+u^1\partial_2 K),\\
Y_2^1&=\partial_2(\eta^{12}\partial_2 H+u^1\partial_2 K),\\
Y_1^2&=\eta^{12}\partial_1^2 H+\eta^{22}\partial_1\partial_2 H+u^1\partial_1^2 K+2(u^1+u^2)\partial_1\partial_2 K+\partial_2 K,\\
Y_2^2&=\eta^{12}\partial_1\partial_2 H+\eta^{22}\partial_2^2 H+u^1\partial_1\partial_2 K+2(u^1+u^2)\partial_2^2 K+\partial_2 K.
\end{aligned}$$

Choosing H and K such that $F=\eta^{12}\partial_2 H+u^1\partial_2 K,\ X_2^2=Y_2^2$, we obtain

$$X_1^1=X_2^1=X_2^2=0,\quad X_1^2=G.$$

Taking $H=0$ and K such that $\partial_2 K=0$ and $u^1\partial_1^2 K=G$, we can also remove G. Thus, deformations of degree 1 are trivial.

A.1.3. N3, N4 and N6. First order deformations. This case is more involved. Let us assume $\kappa\neq-1$, otherwise the metric g_2 would be degenerate. Here $\eta^{12}\neq 0$.
Imposing $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X=0$ for $\deg(X)=1$ we obtain

$$X_1^1=\partial_1 G+R,\quad X_2^1=\partial_2 G,\quad X_1^2=\partial_1 F,\quad X_2^2=\partial_2 F,$$
$$R=\theta^{\frac{\kappa}{2}}\int\kappa(\eta^{22}\partial_2 G+\eta^{12}\partial_1 G-\eta^{12}\partial_2 F)\theta^{-1-\frac{\kappa}{2}}\,\mathrm{d}u^2+\theta^{\frac{\kappa}{2}}S,$$

where $F=F(u^1,u^2), G=G(u^1,u^2), S=S(u^1)$ and $\theta=2\eta^{12}u^2-(1+\kappa)\eta^{22}u^1$. The components Y^i of the vector field $Y=\omega_1\delta H+\omega_2\delta K$ are given by

$$\begin{aligned}
Y_1^1&=\partial_1(\eta^{12}\partial_2 H+(1+\kappa)u^1\partial_2 K)-\kappa\partial_2 K,\\
Y_2^1&=\partial_2(\eta^{12}\partial_2 H+(1+\kappa)u^1\partial_2 K),\\
Y_1^2&=\partial_1(\eta^{12}\partial_1 H+\eta^{22}\partial_2 H+2u^2\partial_2 K+(1+\kappa)u^1\partial_1 K-K),\\
Y_2^2&=\partial_2(\eta^{12}\partial_1 H+\eta^{22}\partial_2 H+2u^2\partial_2 K+(1+\kappa)u^1\partial_1 K-K).
\end{aligned}$$

Choosing H and K such that

$$\eta^{12}\partial_2 H+(1+\kappa)u^1\partial_2 K=F,$$

$$\eta^{12}\partial_1 H+\eta^{22}\partial_2 H+2u^2\partial_2 K+(1+\kappa)u^1\partial_1 K-K=G,$$

we obtain

$$X_1^1=\theta^{\frac{\kappa}{2}}S,\quad X_2^1=X_1^2=X_2^2=0.$$

Finally, taking a suitable choice of H and K, we can also remove S. In particular, we have

- for $\kappa \neq 0, -2$

$$H = \frac{(1+\kappa)u^1\theta^{1+\frac{\kappa}{2}}S}{(\eta^{12})^2\kappa(\kappa+2)}, \quad K = -\frac{\theta^{1+\frac{\kappa}{2}}S}{\eta^{12}\kappa(\kappa+2)},$$

- for $\kappa = 0$

$$H = \frac{(2\eta^{12}u^2 - \eta^{22}u^1)(\log(2\eta^{12}u^2 - \eta^{22}u^1) - 1)u^1S}{4(\eta^{12})^2},$$

$$K = \frac{u^2\int S\,\mathrm{d}u^1}{u^1} - \frac{(2\eta^{12}u^2 - \eta^{22}u^1)(\log(2\eta^{12}u^2 - \eta^{22}u^1) - 1)S}{4\eta^{12}} - \iint \frac{\eta^{22}\partial_1(u^1S)}{2\eta^{12}u^1}\,\mathrm{d}u^1\,\mathrm{d}u^1$$

- for $\kappa = -2$

$$H = \frac{\log(2\eta^{12}u^2 + \eta^{22}u^1)u^1S}{4(\eta^{12})^2}, \quad K = \frac{\log(2\eta^{12}u^2 + \eta^{22}u^1)S}{4\eta^{12}} + \frac{\int S\,\mathrm{d}u^1}{2\eta^{12}u^1}.$$

Thus, first-order deformations are trivial.

A.2. Second order deformations

We have seen that in all cases Q_1 can be eliminated by a Miura transformation. For this reason, without loss of generality, we can assume the pencil has the form

$$\Pi_\lambda = \omega_\lambda + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \ldots$$

Using the same arguments applied to first order deformations we can easily prove that:

- general second order deformations can be always written as $Q_2 = \mathrm{d}_{\omega_2}X$ for a suitable vector field of degree 2

$$X^i = X_1^i(u^1,u^2)u^1_{xx} + X_2^i(u^1,u^2)(u^1_x)^2 + X_3^i(u^1,u^2)u^1_xu^2_x + X_4^i(u^1,u^2)(u^2_x)^2 + X_5^i(u^1,u^2)u^2_{xx},$$

satisfying

$$\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0.$$

- trivial second order deformations are those corresponding to vector fields of the form $\omega_1\delta H + \omega_2\delta K$, where the Hamiltonian functionals H and K have Hamiltonian densities of degree 1, namely

$$H = \int [h_1(u^1,u^2)u^1_x + h_2(u^1,u^2)u^2_x]\,\mathrm{d}x, \quad K = \int [k_1(u^1,u^2)u^1_x + k_2(u^1,u^2)u^2_x]\,\mathrm{d}x.$$

Before we go into the details of the computations, let us observe that

$$\delta H = \begin{pmatrix} \dfrac{\delta H}{\delta u^1} \\ \dfrac{\delta H}{\delta u^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial u^1} - \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial H}{\partial u^1_x} \\ \dfrac{\partial H}{\partial u^2} - \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial H}{\partial u^2_x} \end{pmatrix} = \begin{pmatrix} R(u^1,u^2)u^2_x \\ -R(u^1,u^2)u^1_x \end{pmatrix},$$

for $R(u^1,u^2)=\partial_1 H_2(u^1,u^2)-\partial_2 H_1(u^1,u^2)$ and similarly

$$\delta K=\begin{pmatrix}\dfrac{\delta K}{\delta u^1}\\[2mm] \dfrac{\delta K}{\delta u^2}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial K}{\partial u^1}-\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial K}{\partial u^1_x}\\[2mm] \dfrac{\partial K}{\partial u^2}-\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\partial K}{\partial u^2_x}\end{pmatrix}=\begin{pmatrix}S(u^1,u^2)u^2_x\\ -S(u^1,u^2)u^1_x\end{pmatrix},$$

for $S(u^1,u^2)=\partial_1 K_2(u^1,u^2)-\partial_2 K_1(u^1,u^2)$.

We now proceed as follows:

1. We solve the equation $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X=0$, which leads to a solution depending on two functions of two variables and at most four functions of one variable.
2. Up to Miura-type transformations, that is, using the freedom given by the functions R and S, we can eliminate the two functions of two variables.
3. In the cases T3, N3, N5 and N6 with $\kappa\neq-1,-2$, we still use a Miura-type transformation to reduce the deformation to a more suitable form (see step 4).
4. The last step is quite straightforward. We firstly take a generic Hamiltonian vector field of the form $X=\omega_1\delta H-\omega_2\delta K$ with

$$H=\int\sum_{i,j}(h_{ij}u^i_x\log u^j_x)\,\mathrm{d}x,\qquad K=\int\sum_{i,j}(k_{ij}u^i_x\log u^j_x)\,\mathrm{d}x,$$

where the coefficients h_{ij} and k_{ij} are arbitrary functions of (u^1,u^2). Then, comparing X with the vector field obtained above (step 3), we obtain the values of h_{ij} and k_{ij} which correspond to the final expression written in theorem 1.

Let us discuss in detail each case. In what follows, all the functions X^i_j,R,S, $i=1,2$, $j=1,\ldots,5$, will depend on (u^1,u^2), unless stated otherwise.

A.2.1. T3. Second order deformations. Let us assume $\eta^{22}\neq 0$. The solution of $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X=0$ for $\deg(X)=2$ is given by

$$\begin{aligned}
X^1_1&=X^1_1,\\
X^1_2&=X^1_2,\\
X^1_3&=\frac{2}{3}\partial_2X^1_1-\frac{1}{3}\partial_2X^2_5,\\
X^1_4&=0,\\
X^1_5&=0,\\
X^2_1&=\frac{\eta^{22}u^1}{\eta^{12}}\left(X^1_2-\frac{4}{3}\partial_1X^1_1\right)-\frac{\eta^{22}}{3\eta^{12}}(\partial_1(u^1X^2_5)+2X^2_5)+F_1,\\
X^2_2&=\partial_1X^2_1,\\
X^2_3&=\partial_2X^2_1+\partial_1X^2_5,\\
X^2_4&=\partial_2X^2_5,\\
X^2_5&=F_2\mathrm{e}^{\frac{-\eta^{12}u^2}{\eta^{22}u^1}}-X^1_1.
\end{aligned}$$

where F_1,F_2 depend on u^1. The components Y^i of the vector field $Y=\omega_1\delta H+\omega_2\delta K$ are

$$
\begin{aligned}
Y_1^1 &= -\eta^{12}R + u^1 S,\\
Y_2^1 &= -\eta^{12}\partial_1 R + u^1\partial_1 S,\\
Y_3^1 &= -\eta^{12}\partial_2 R + u^1\partial_2 S,\\
Y_4^1 &= 0,\\
Y_5^1 &= 0,\\
Y_1^2 &= -\eta^{22}R,\\
Y_2^2 &= -\eta^{22}\partial_1 R,\\
Y_3^2 &= \eta^{12}\partial_1 R - \eta^{22}\partial_2 R - u^1\partial_1 S - S,\\
Y_4^2 &= \eta^{12}\partial_2 R - u^1\partial_2 S,\\
Y_5^2 &= \eta^{12}R - u^1 S.
\end{aligned}
$$

Choosing R and S such that $X_i^1 = Y_i^1$ for $i = 1, 2$, we finally obtain

$$
\begin{aligned}
X_1^1 &= 0\\
X_2^1 &= 0\\
X_3^1 &= -\frac{1}{3}\partial_2 X_5^2,\\
X_4^1 &= 0,\\
X_5^1 &= 0,\\
X_1^2 &= -\frac{\eta^{22}}{3\eta^{12}}(\partial_1(u^1X_5^2) + 2X_5^2) + F_1,\\
X_2^2 &= \partial_1 X_1^2,\\
X_3^2 &= \partial_2 X_1^2 + \partial_1 X_5^2,\\
X_4^2 &= \partial_2 X_5^2,\\
X_5^2 &= F_2 \mathrm{e}^{\frac{-\eta^{12}u^2}{\eta^{22}u^1}}.
\end{aligned}
$$

Thus, these coefficients depend on two functions F_1, F_2 in the variable u^1.

In the case $\eta^{22} = 0$, the computation is easier. The condition $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0$ implies

$$
\begin{aligned}
X_1^1 &= X_1^1\\
X_2^1 &= X_2^1\\
X_3^1 &= \partial_2 X_1^1,\\
X_4^1 &= 0,\\
X_5^1 &= 0,\\
X_1^2 &= F,\\
X_2^2 &= \partial_1 F,\\
X_3^2 &= -\partial_1 X_1^1,\\
X_4^2 &= -\partial_2 X_1^1,\\
X_5^2 &= -X_1^1.
\end{aligned}
$$

where F depends on u^1. Also in this case the freedom in R and S allows us to reduce X_1^1 and X_2^1 to zero, obtaining

$$X^1 = 0, \quad X^2 = Fu^1_{xx} + \partial_1 F(u^1_x)^2 = (Fu^1_x)_x$$

The second component of the vector field can be written as

$$X_2 = \partial_x^2 \int F \, \mathrm{d}u^1,$$

and setting $f = Fu^1$ yields

$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & f_{xx}\delta' + 3f_x\delta'' + 2f\delta''' \end{pmatrix}.$$

Finally, in order to obtain the form we need to compute h_{ij} (step 3), we perform the canonical Miura transformation generated by the local Hamiltonian

$$H = -\int_{S^1} \left(\frac{\eta^{22}(u^1)^2 F_2'}{3(\eta^{12})^2} + \frac{u^2 F_2}{3\eta^{12}} \right) \mathrm{e}^{-\frac{\eta^{12}u^2}{\eta^{22}u^1}} u^1_x \, \mathrm{d}x.$$

Remark. Let us point out that this solution can be obtained from the general case in the limit $\eta^{22} \to 0$.

A.2.2. N5. Second order deformations. The condition $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2} X = 0$ for $\deg(X) = 2$ implies

$$\begin{aligned}
X_1^1 &= X_1^1, \\
X_2^1 &= \partial_1 X_1^1, \\
X_3^1 &= \partial_2 X_1^1, \\
X_4^1 &= 0, \\
X_5^1 &= 0, \\
X_1^2 &= X_1^2, \\
X_2^2 &= \partial_1 X_1^2 + \frac{2}{3}\theta^{1/2}\partial_1 F_2 + \frac{5\eta^{12} - 2\eta^{22}}{3}\theta^{3/2}F_2 + \theta(\eta^{22}X_1^1 - \eta^{12}X_1^2 + F_1), \\
X_3^2 &= \partial_2 X_1^2 - \partial_1 X_1^1 + \theta^{1/2}\partial_1 F_2 - \frac{4\eta^{12} - 3\eta^{22}}{6}\theta^{3/2}F_2, \\
X_4^2 &= -\eta^{12}\theta^{3/2}F_2 - \partial_2 X_1^1, \\
X_5^2 &= \theta^{1/2}F_2 - X_1^1,
\end{aligned}$$

where F_i, for $i = 1, 2$, are functions depending on u^1 and $\theta = (2\eta^{12}(u^1 + u^2) - \eta^{22}u^1)^{-1}$. The components Y^i of the vector field $Y = \omega_1\delta H + \omega_2\delta K$ are

$$
\begin{aligned}
Y_1^1 &= -(\eta^{12}R + u^1 S),\\
Y_2^1 &= -\partial_1(\eta^{12}R + u^1 S),\\
Y_3^1 &= -\partial_2(\eta^{12}R + u^1 S),\\
Y_4^1 &= 0,\\
Y_5^1 &= 0,\\
Y_1^2 &= -(\eta^{22}R + 2(u^1 + u^2)S),\\
Y_2^2 &= -(\eta^{22}\partial_1 R + 2(u^1 + u^2)\partial_1 S + S),\\
Y_3^2 &= \partial_1(\eta^{12}R + u^1 S) - \partial_2(\eta^{22}R + 2(u^1 + u^2)S),\\
Y_4^2 &= \partial_2(\eta^{12}R + u^1 S),\\
Y_5^2 &= \eta^{12}R + u^1 S.
\end{aligned}
$$

Choosing R, S such that $X_1^i = Y_1^i$ for $i = 1, 2$, we can reduce X^1 to zero and the coefficients of X^2, respectively, to

$$
\begin{aligned}
X_1^2 &= 0,\\
X_2^2 &= \frac{2}{3}\partial_1(\theta^{1/2}F_2) - \frac{7}{3}\partial_2(\theta^{1/2}F_2) - \eta^{22}\theta^{3/2}F_2 + \theta F_1,\\
X_3^2 &= \partial_1(\theta^{1/2}F_2) - \frac{1}{3}\partial_2(\theta^{1/2}F_2),\\
X_4^2 &= \partial_2(\theta^{1/2}F_2),\\
X_5^2 &= \theta^{1/2}F_2.
\end{aligned}
$$

Thus, the deformations of degree 2 depend on two functions of u^1.

To reduce the deformation in the form written in theorem 1 (step 3) we perform the canonical Miura transformation generated by

$$H = \int_{S^1} \frac{u^1}{(\eta^{12})^2}\left(\frac{(3\eta^{22} - 8\eta^{12})\theta^{1/2}F_2}{6} + \theta^{-1/2}F_2' + \frac{\log(\theta^{-1})F_1}{2}\right)u_x^1\,\mathrm{d}x.$$

A.2.3. N3, N4 and N6. Second order deformations. The vector fields $Y = P\delta H + Q\delta K$ are given by

$$
\begin{aligned}
Y_1^1 &= -(\eta^{12}R + (1+\kappa)u^1 S),\\
Y_2^1 &= -\partial_1(\eta^{12}R + (1+\kappa)u^1 S) + \kappa S,\\
Y_3^1 &= -\partial_2(\eta^{12}R + (1+\kappa)u^1 S),\\
Y_4^1 &= 0,\\
Y_5^1 &= 0,\\
Y_1^2 &= -(\eta^{22}R + 2u^2 S),\\
Y_2^2 &= -\partial_1(\eta^{22}R + 2u^2 S),\\
Y_3^2 &= \partial_1(\eta^{12}R + (1+\kappa)u^1 S) - \partial_2(\eta^{22}R + 2u^2 S),\\
Y_4^2 &= \partial_2(\eta^{12}R + (1+\kappa)u^1 S),\\
Y_5^2 &= \eta^{12}R + (1+\kappa)u^1 S.
\end{aligned}
$$

In studying the solutions of the equation $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0$ we have to distinguish three cases: $\kappa = 0$, $\kappa = -2$, $\kappa \neq 0, 2$. This is due to the fact that the conditions coming from this equation include the following:

$$\kappa(\kappa + 2)X^1_5(u^1, u^2) = 0.$$

Case 1*:* $\kappa = 0$. The condition $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0$ for $\deg(X) = 2$ leads to

$$\begin{aligned}
X^1_1 &= X^1_1,\\
X^1_2 &= \partial_1 X^1_1 + \theta F_1,\\
X^1_3 &= \theta\partial_1 F_2 - \eta^{22}\theta^2 F_2 + \partial_2 X^1_1,\\
X^1_4 &= 2\eta^{12}\theta^2 F_2,\\
X^1_5 &= \theta F_2,\\
X^2_1 &= X^2_1,\\
X^2_2 &= \partial_1 X^2_1 + \theta F_3,\\
X^2_3 &= -\frac{\partial_1^2 F_2}{\eta^{12}} + \theta^{\frac{1}{2}}\partial_1 F_4 - \frac{\eta^{22}}{2}\theta^{\frac{3}{2}}F_4 - \partial_1 X^1_1 + \partial_2 X^2_1,\\
X^2_4 &= \eta^{12}\theta^{\frac{3}{2}}F_4 - \partial_2 X^1_1,\\
X^2_5 &= -\frac{\partial_1 F_2}{\eta^{12}} + \theta^{\frac{1}{2}}F_4 - X^1_1,
\end{aligned}$$

where F_i for $i = 1, \ldots, 4$ are arbitrary functions depending on u^1, and $\theta = (\eta^{22}u^1 - 2\eta^{12}u^2)^{-1}$. Choosing R and S such that $X^i_1 = Y^i_1$ for $i = 1$, 2, we can reduce both X^i_1 and $i = 1$, 2 to zero, obtaining

$$\begin{aligned}
X^1_1 &= 0,\\
X^1_2 &= \theta F_1,\\
X^1_3 &= \partial_1(\theta F_2)\\
X^1_4 &= \partial_2(\theta F_2),\\
X^1_5 &= \theta F_2,\\
X^2_1 &= 0,\\
X^2_2 &= \theta F_3,\\
X^2_3 &= \partial_1\left(\theta^{\frac{1}{2}}F_4 - \frac{\partial_1 F_2}{\eta^{12}}\right),\\
X^2_4 &= \partial_2\left(\theta^{\frac{1}{2}}F_4 - \frac{\partial_1 F_2}{\eta^{12}}\right),\\
X^2_5 &= \theta^{\frac{1}{2}}F_4 - \frac{\partial_1 F_2}{\eta^{12}}.
\end{aligned}$$

In this case, the deformations of degree 2 depend on four functions on u^1.

Case 2: $\kappa = -2$. The condition $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0$ for $\deg(X) = 2$ implies

$$
\begin{aligned}
X_1^1 &= X_1^1,\\
X_2^1 &= \partial_1 X_1^1 + 2\eta^{22}\theta^{\frac{5}{2}}F_4 + \frac{4(\eta^{22})^2\theta^4F_2 - 2\eta^{22}\theta^3\partial_1F_2}{\eta^{12}}\\
&\quad + 2\eta^{12}\theta X_1^2 - 2\eta^{22}\theta X_1^1 + \theta F_1,\\
X_3^1 &= \partial_2 X_1^1 - \theta^3\partial_1F_2 + 3\eta^{22}\theta^4F_2 + 2\eta^{12}\theta^{\frac{5}{2}}F_4,\\
X_4^1 &= -4\eta^{12}\theta^4F_2,\\
X_5^1 &= \theta^3F_2,\\
X_1^2 &= X_1^2,\\
X_2^2 &= \partial X_1^2 + F_3,\\
X_3^2 &= \partial_2X_1^2 - \partial_1X_1^1 + \frac{4\eta^{22}\theta^3\partial_1F_2 - \theta^2\partial_1^2F_2 - 6(\eta^{22})^2\theta^4F_2}{\eta^{12}}\\
&\quad + \theta^{\frac{3}{2}}\partial_1F_4 - \frac{3}{2}\eta^{22}\theta^{\frac{5}{2}}F_4,\\
X_4^2 &= 4\theta^3\partial_1F_2 - 12\eta^{22}\theta^4F_2 - 3\eta^{12}\theta^{\frac{5}{2}}F_4 - \partial_2X_1^1,\\
X_5^2 &= \frac{2\eta^{22}\theta^3F_2 - \theta^2\partial_1F_2}{\eta^{12}} - X_1^1 + \theta^{\frac{3}{2}}F_4,
\end{aligned}
$$

here $\theta = (2\eta^{12}u^2 + \eta^{22}u^1)^{-1}$ and $F_i = F_i(u^1)$, for $i = 1, \ldots, 4$. Choosing R, S such that $X_1^i = Y_1^i$ for $i = 1, 2$, we can reduce X_1^i to zero, obtaining

$$
\begin{aligned}
X_1^1 &= 0,\\
X_2^1 &= 2\eta^{22}\theta\left(\theta^{\frac{3}{2}}F_4 - \frac{\partial_1(\theta^2F_2)}{\eta^{12}}\right) + \theta F_1,\\
X_3^1 &= 2\eta^{12}\theta^{\frac{5}{2}}F_4 - \partial_1(\theta^3F_2),\\
X_4^1 &= -4\eta^{12}\theta^4F_2,\\
X_5^1 &= \theta^3F_2,\\
X_1^2 &= 0,\\
X_2^2 &= F_3,\\
X_3^2 &= \partial_1(\theta^{\frac{3}{2}}F_4) - \frac{\partial_1^2(\theta^2F_2)}{\eta^{12}},\\
X_4^2 &= 4\partial_1(\theta^3F_2) + \partial_2(\theta^{\frac{3}{2}}F_4),\\
X_5^2 &= \theta^{\frac{3}{2}}F_4 - \frac{\partial_1(\theta^2F_2)}{\eta^{12}}.
\end{aligned}
$$

Also in this case, the deformations depend on four functions on u^1.

Case 3*:* $\kappa \neq 0, -1, -2$. The condition $\mathrm{d}_{\omega_1}\mathrm{d}_{\omega_2}X = 0$ for $\deg(X) = 2$ implies

$$\begin{aligned}
X_1^1 &= X_1^1,\\
X_2^1 &= \partial_1 X_1^1 + \frac{\kappa(\kappa+2)}{3(\kappa+1)^2}\theta^{\frac{\kappa-1}{2}}\partial_1 F_2 - \frac{\kappa(\kappa^2+7\kappa+4)\eta^{22}}{6(\kappa+1)}\theta^{\frac{\kappa-3}{2}}F_2\\
&\quad + \theta^{-1}(\kappa(\eta^{22}X_1^1 - \eta^{12}X_1^2) + F_1),\\
X_3^1 &= \partial_2 X_1^1 - \frac{\kappa(\kappa-1)\eta^{12}}{3(\kappa+1)}\theta^{\frac{\kappa-3}{2}}F_2,\\
X_4^1 &= 0,\\
X_5^1 &= 0,\\
X_1^2 &= X_1^2,\\
X_2^2 &= \partial_2 X_1^2,\\
X_3^2 &= \partial_2 X_1^2 - \partial_1 X_1^1 + \theta^{\frac{\kappa-1}{2}}\partial_1 F_2 - \frac{1}{2}\eta^{22}(\kappa-1)(\kappa+1)\theta^{\frac{\kappa-3}{2}}F_2,\\
X_4^2 &= (\kappa-1)\eta^{12}\theta^{\frac{\kappa-3}{2}}F_2 - \partial_1 X_1^1,\\
X_5^2 &= \theta^{\frac{\kappa-1}{2}}F_2 - X_1^1,
\end{aligned}$$

here $\theta = 2\eta^{12}u^2 - (\kappa+1)\eta^{22}u^1$ and F_i for $i = 1, 2$ are arbitrary functions depending on u^1. Choosing R, S such that $X_1^i = Y_1^i$ for $i = 1, 2$ we can remove X_1^i, obtaining

$$\begin{aligned}
X_1^1 &= 0,\\
X_2^1 &= \frac{\kappa(\kappa+2)}{3(\kappa+1)^2}\theta^{\frac{\kappa-1}{2}}\partial_1 F_2 - \frac{\kappa(\kappa^2+7\kappa+4)\eta^{22}}{6(\kappa+1)}\theta^{\frac{\kappa-3}{2}}F_2 + \theta^{-1}F_1,\\
X_3^1 &= -\frac{\kappa}{3(\kappa+1)}\partial_2(\theta^{\frac{\kappa-1}{2}}F_2),\\
X_4^1 &= 0,\\
X_5^1 &= 0,\\
X_1^2 &= 0,\\
X_2^2 &= 0,\\
X_3^2 &= \partial_1(\theta^{\frac{\kappa-1}{2}}F_2),\\
X_4^2 &= \partial_2(\theta^{\frac{\kappa-1}{2}}F_2),\\
X_5^2 &= \theta^{\frac{\kappa-1}{2}}F_2.
\end{aligned}$$

In this last case, the deformations depend on two functions of u^1. The canonical Miura transformation reducing the pencil to the form described in the step 3 is generated by the Hamiltonian functional

$$H = \int_{S^1}\left(-\frac{\theta^{\frac{\kappa-1}{2}}(4\kappa\eta^{12}(\kappa-1)u^2 + \eta^{22}(\kappa+1)(2\kappa^3+7\kappa^2+12\kappa+3)u^1)F_2}{6(\eta^{11})^2(\kappa+1)^2(\kappa-1)}\right.$$
$$\left. + \frac{\theta^{\frac{\kappa+1}{2}}(2\kappa+3)u^1 F_2'}{3(\eta^{11})^2(\kappa+1)^2} + \frac{\log\theta(\kappa+1)u^1 F_1}{2(\eta^{11})^2\kappa}\right)u_x^1\,\mathrm{d}x.$$

Appendix B. Lift of Frobenius structures

Recall that a Frobenius manifold is a smooth manifold M equipped with a pseudo-metric g with Levi-Civita connection ∇, a symmetric bilinear tensorial product on vector fields $\cdot$ and two vector fields e, E such that

- $\nabla^{\lambda}_X Y = \nabla_X Y + \lambda X \cdot Y$ defines a flat affine connection ∇^{λ} for all $\lambda \in \mathbf{R}$,
- $\nabla e = 0$, $[e, E] = e$ and $e \cdot X = X$ for all vector fields X,
- $\nabla(\nabla E) = 0$, $L_E \cdot = \cdot$ and $L_E g = k\, g$ for some constant k.

Theorem 11. *Let $(M, g, \cdot, e, E)$ be a Frobenius manifold. Then the lifted tensors $\hat{g}, \hat{\cdot}, \hat{e}, \hat{E}$ define a structure of Frobenius manifold on TM. The Frobenius potential of the lifted structure is given by the lift of the Frobenius potential $\hat{F} = v^i \frac{\partial F}{\partial u^i}$.*

Proof. From (6.2) one readily sees that $\hat{g}$ is symmetric and non-degenerate as soon as g is. If ∇ is the Levi-Civita connection of g, then the lift $\hat{\nabla}$ is the Levi-Civita connection of $\hat{g}$. This follows by the uniqueness of the Levi-Civita connection once one notes that $\hat{\nabla}\hat{g} = 0$ and that $\hat{\nabla}$ is torsion free. To see this note that $\hat{\nabla}\hat{g} = 0$ for $\nabla g = 0$ and that $\hat{\nabla}$ is torsion free by proposition 8 and by the torsion-freeness of ∇.

From (6.4) is clear that $\hat{\cdot}$ is symmetric if $\cdot$ is. Moreover, by the definition of complete lift for connections it follows that $\hat{\nabla}^{\lambda}_X Y = \hat{\nabla}_X Y + \lambda X \mathbin{\hat{\cdot}} Y$ for all $\lambda \in \mathbf{R}$, where now X, Y are arbitrary tensor fields on TM. Thanks to proposition 8, then $\hat{\nabla}^{\lambda}$ is flat. All other conditions follows directly from definition of complete lift and invariance of the Lie derivative under complete lift. ■

At this point recall that a Frobenius manifold is said to be *massive* if the algebra structure induced by the product $\cdot$ on any tangent space to M is semisimple. More explicitly this means that there is no tangent vector X on M such that $X \cdot \ldots \cdot X = 0$ for some finite product. One may wonder whether the semisimplicity assumption is preserved by complete lift or not. In fact it is not, nor is possible to obtain a massive Frobenius manifold by complete lift of any Frobenius structure on M. The reason is that any vector Y which is tangent to the fibres of TM is an idempotent for the algebra structure induced by $\hat{\cdot}$. Indeed any such vector has the local expression $Y^i \frac{\partial}{\partial y^i}$, whence it follows that $Y \mathbin{\hat{\cdot}} Y = 0$ thanks to (6.4).

Remark. Given a Frobenius manifold $(M, g, \cdot, e, E)$ one can define a hierarchy of quasilinear systems of PDEs of the form

$$u^i_{t_{p,\alpha}} = P^{ij} \frac{\delta H_{p,\alpha}}{\delta u^j}, \qquad i = 1, \ldots, n,\ \ p = 1, \ldots, n,\ \ \alpha = 0, 1, 2, 3, \ldots$$

where P^{ij} is Hamiltonian operator of hydrodynamic type associated with the invariant metric g and $H_{p,\alpha}$ are suitable local functionals in involution

$$\{H_{p,\alpha}, H_{q,\beta}\}_P = \int_{S^1} \frac{\delta H_{p,\alpha}}{\delta u^i} (g^{ij}\partial_x + b^{ij}_k u^k_x) \frac{\delta H_{q,\beta}}{\delta u^j}\, \mathrm{d}x = 0$$

with respect to the associated Poisson bracket $\{,\}_P$. It is easy to check that the flows of the lifted hierarchy

$$u^i_{t_{p,\alpha}} = \hat{P}^{ij} \frac{\delta \hat{H}_{p,\alpha}}{\delta u^j}, \qquad i = 1, \ldots, 2n,\ \ p = 1, \ldots, n,\ \ \alpha = 0, 1, 2, 3, \ldots$$

coincide with 'half' of the flows of the principal hierarchy of the lifted Frobenius structure. The involutivity of the lifted Hamiltonian functionals

$$\hat{H}_{p,\alpha} = \int_{S^1} v^s \partial_s h_{p,\alpha}\, \mathrm{d}x$$

follows from the identity (6.9). Indeed, due to this identity any family of 1-forms in involution with respect to $\{\cdot,\cdot\}_P$ defines a family of Hamiltonians in involution with respect to $\{\cdot,\cdot\}_{\hat{P}}$. If the 1-forms are exact the Hamiltonians on the tangent bundle are the lift of the Hamiltonians on the base manifold.

Appendix C. Lift of Hamiltonian vector fields

Given a Hamiltonian vector field $P\delta H$ with $\int_{S^1} h(u, u_x, ...)\, \mathrm{d}x$, we want to compare its complete lift

$$\widehat{P\delta H} = P\frac{\delta H}{\delta u}\frac{\partial}{\partial u} + \sum_k v_{(k)} \frac{\partial (P\frac{\delta H}{\delta u})}{\partial u_{(k)}} \frac{\partial}{\partial v}$$

with the vector field

$$\hat{P}\delta\hat{H} = P\frac{\delta H}{\delta u}\frac{\partial}{\partial u} + \left(P\frac{\delta \hat{H}}{\delta u} + \sum_t v_{(t)} \frac{\partial P}{\partial u_{(t)}} \frac{\delta \hat{H}}{\delta v}\right)\frac{\partial}{\partial v}$$

where $\hat{H}[u, v] = \int_{S^1} v \frac{\delta H}{\delta u}\, \mathrm{d}x$. Since the components along $\frac{\partial}{\partial u}$ coincide we have to show that

$$P\frac{\delta \hat{H}}{\delta u} + \sum_t v_{(t)} \frac{\partial P}{\partial u_{(t)}} \frac{\delta \hat{H}}{\delta v} = \sum_k v_{(k)} \frac{\partial (P\frac{\delta H}{\delta u})}{\partial u_{(k)}}.$$

We observe that

$$\frac{\delta \hat{H}}{\delta v} = \frac{\delta H}{\delta u}, \qquad \frac{\delta \hat{H}}{\delta u} = \frac{\delta}{\delta u}\left(\sum_k \int_{S^1} v_{(k)} \frac{\partial h}{\partial u_{(k)}}\, \mathrm{d}x\right),$$

where the second identity has been obtained integrating by parts. Using these facts and taking into account that the operators ∂_x and $\sum_k v_{(k)} \frac{\partial}{\partial u_{(k)}}$ commute, we obtain

$$\begin{aligned}
&P\frac{\delta \hat{H}}{\delta u} + \sum_k v_{(k)} \frac{\partial P}{\partial u_{(k)}} \frac{\delta \hat{H}}{\delta v} \\
&= P\frac{\delta}{\delta u}\left(\sum_k \int_{S^1} v_{(k)} \frac{\partial h}{\partial u_{(k)}}\, \mathrm{d}x\right) + \sum_k v_{(k)} \frac{\partial P}{\partial u_{(k)}} \frac{\delta H}{\delta u} \\
&= P\sum_{h,k} (-1)^h \partial_x^h \left(v_{(k)} \frac{\partial^2 h}{\partial u_{(k)} \partial u_{(h)}}\right) + \sum_k v_{(k)} \frac{\partial P}{\partial u_{(k)}} \frac{\delta H}{\delta u} \\
&= P\sum_k v_{(k)} \frac{\partial}{\partial u_{(k)}}\left[\sum_h (-1)^h \partial_x^h \left(\frac{\partial h}{\partial u_{(k)}}\right)\right] + \sum_k v_{(k)} \frac{\partial P}{\partial u_{(k)}} \frac{\delta H}{\delta u} \\
&= \sum_k v_{(k)} \frac{\partial (P\frac{\delta H}{\delta u})}{\partial u_{(k)}}.
\end{aligned}$$

In the non-scalar case the proof works in exactly the same way.

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