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A mathematical study of meteo and landslide tsunamis: the Proudman resonance

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Abstract

In this paper, we want to understand the Proudman resonance. It is a resonant respond in shallow waters of a water body on a traveling atmospheric disturbance when the speed of the disturbance is close to the typical water wave velocity. We show here that the same kind of resonance exists for landslide tsunamis and we propose a mathematical approach to investigate these phenomena based on the derivation, justification and analysis of relevant asymptotic models. This approach allows us to investigate more complex phenomena that are not dealt with in the physics literature such as the influence of a variable bottom or the generalization of the Proudman resonance in deeper waters. First, we prove a local well-posedness of the water waves equations with a moving bottom and a non constant pressure at the surface taking into account the dependence of small physical parameters and we show that these equations are a Hamiltonian system (which extends the result of Zakharov (1968 *J. Appl. Mech. Tech. Phys.* **9** 190–4)). Then, we justify some linear asymptotic models in order to study the Proudman resonance and submarine landslide tsunamis; we study the linear water waves equations and dispersion estimates allow us to investigate the amplitude of the sea level. To complete these asymptotic models, we add some numerical simulations.

Keywords: water waves equations, quasilinear hyperbolic system, asymptotic models, dispersion estimates
Mathematics Subject Classification numbers: 35Q35, 76B15, 76B03, 35L80, 35J65

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Presentation of the problem

A tsunami is popularly an elevation of the sea level due to an earthquake. However, tsunamis induced by seismic sources represent only 80% of the tsunamis. 6% are due to landslides and 3% to meteorological effects (see the book of Levin and Nosov [21]). Big traveling storms for instance can give energy to the sea and lead to an elevation of the surface. In some cases, this amplification is important and this phenomenon is called the Proudman resonance in the physics literature. Similarly, submarine landslides can significantly increase the level of the sea and we talk about landslide tsunamis. In this paper, we study mathematically these two phenomena. We model the sea by an irrotational and incompressible ideal fluid bounded from below by the seabed and from above by a free surface. We suppose that the seabed and the surface are graphs above the still water level. We model an underwater landslide by a moving seabed (moving bottom) and the meteorological effects by a non constant pressure at the surface (air-pressure disturbance). Therefore, we suppose that $b(t,X) = b_0(X) + b_m(t,X)$, where b_0 represents a fixed bottom and b_m the variation of the bottom because of the landslide. Similarly, the pressure at the surface is of the form $P + P_{\text{ref}}$, where P_{ref} is a constant which represents the pressure far from the meteorological disturbance, and $P(t,X)$ models the meteorological disturbance (we assume that the pressure at the surface is known). We denote by d the horizontal dimension, which is equal to 1 or 2. $X \in \mathbb{R}^d$ stands for the horizontal variable and $z \in \mathbb{R}$ is the vertical variable. H is the typical water depth. The water occupies a moving domain $\Omega_t := \{(X,z) \in \mathbb{R}^{d+1}, -H + b(t,X) < z < \zeta(t,X)\}$. The water is homogeneous (constant density ρ), inviscid, irrotational with no surface tension. We denote by $\mathbf{U}$ the velocity and Φ the velocity potential. We have $\mathbf{U} = \nabla_{\mathbf{X,z}}\boldsymbol{\Phi}$. The law governing the irrotational fluids is the Bernoulli law

$$\partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = \frac{1}{\rho}(P_{\text{ref}} - \mathcal{P}) \text{ in } \Omega_t, \tag{1}$$

where $\mathcal{P}$ is the pressure in the fluid domain. Changing Φ if necessary, it is possible to assume that $P_{\text{ref}} = 0$. Furthermore, the incompressibility of the fluid implies that

$$\Delta_{X,z}\Phi = 0 \text{ in } \Omega_t. \tag{2}$$

We suppose also that the fluid particles do not cross the bottom or the surface. We denote by $\mathbf{n}$ the unit normal vector, pointing upward and $\partial_{\mathbf{n}}$ the upward normal derivative. Then, the boundary conditions are

$$\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi = 0 \text{ on } \{z = \zeta(t,X)\}, \tag{3}$$

and

$$\partial_t b - \sqrt{1+|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi = 0 \text{ on } \{z = -H + b(t,X)\}. \tag{4}$$

In 1968, Zakharov (see [33]) showed that the water waves problem is a Hamiltonian system and that ψ, the trace of the velocity potential at the surface ($\psi = \Phi_{|z=\zeta}$), and the surface ζ are canonical variables. Then, Craig, Sulem and Sulem (see [11] and [12]) formulate this remark into a system of two non local equations. We follow their construction to formulate our problem. Using the fact that Φ satisfies (2) and (4), we can characterize Φ thanks to ζ and $\psi = \Phi_{|z=\zeta}$

$$\begin{cases} \Delta_{X,z}\Phi = 0 \text{ in } \Omega_t\,, \\ \Phi_{|z=\zeta} = \psi, \sqrt{1+|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi_{|z=-H+b} = \partial_t b. \end{cases} \tag{5}$$

We decompose this equation in two parts, the surface contribution and the bottom contribution

$$\Phi = \Phi^S + \Phi^B,$$

such that

$$\begin{cases} \Delta_{X,z}\Phi^S = 0 \text{ in } \Omega_t\,, \\ \Phi^S{}_{|z=\zeta} = \psi, \sqrt{1+|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi^S{}_{|z=-H+b} = 0, \end{cases} \tag{6}$$

and

$$\begin{cases} \Delta_{X,z}\Phi^B = 0 \text{ in } \Omega_t\,, \\ \Phi^B{}_{|z=\zeta} = 0, \sqrt{1+|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi^B{}_{|z=-H+b} = \partial_t b. \end{cases} \tag{7}$$

In the purpose of expressing (3) with ζ and ψ, we introduce two operators. The first one is the Dirichlet–Neumann operator

$$G[\zeta,b] : \psi \mapsto \sqrt{1+|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi^S{}_{|z=\zeta}, \tag{8}$$

where Φ^S satisfies (6). The second one is the Neumann–Neumann operator

$$G^{\mathrm{NN}}[\zeta,b] : \partial_t b \mapsto \sqrt{1+|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi^B{}_{|z=\zeta}, \tag{9}$$

where Φ^B satisfies (7). Then, we can reformulate (3) as

$$\partial_t\zeta - G[\zeta,b](\psi) = G^{\mathrm{NN}}[\zeta,b](\partial_t b). \tag{10}$$

Furthermore thanks to the chain rule, we can express $(\partial_t\Phi)_{|z=\zeta}$, $(\nabla_{X,z}\Phi)_{|z=\zeta}$ and $(\partial_z\Phi)_{|z=\zeta}$ in terms of ψ, ζ, $G[\zeta,b](\psi)$ and $G^{\mathrm{NN}}[\zeta,b](\partial_t b)$. Then, we take the trace at the surface of (1) (since there is no surface tension we have $\mathcal{P}_{|z=\zeta} = P$) and we obtain a system of two scalar equations that reduces to the standard Zakharov/Craig-Sulem formulation when $\partial_t b = 0$ and $P = 0$,

$$\begin{cases} \partial_t\zeta - G[\zeta,b](\psi) = G^{\mathrm{NN}}[\zeta,b](\partial_t b), \\ \partial_t\psi + g\zeta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(G[\zeta,b](\psi) + G^{\mathrm{NN}}[\zeta,b](\partial_t b) + \nabla\zeta\cdot\nabla\psi)^2}{(1+|\nabla\zeta|^2)} = -\dfrac{P}{\rho}. \end{cases} \tag{11}$$

In the following, we work with a nondimensionalized version of the water waves equations with small parameters ε, β and μ (see section 2.1). The wellposedness of the water waves problem with a constant pressure and a fixed bottom was studied by many people. Wu proved it in the case of an infinite depth without nondimensionalization ([31] and [32]). Then, Lannes treated the case of a finite bottom without nondimensionalization ([18]), Iguchi proved a local wellposedness result for μ small enough in order to justify shallow water approximations for water waves ([16]), and Lannes and Alvarez-Samaniego showed, in the case of the nondimensionalized equations, that we can find an existence time $T = \frac{T_0}{\max(\varepsilon,\beta)}$ where T_0 does not depend on ε, β and μ ([7]). More recently, Mésognon-Gireau improved the result of Lannes and Alvarez-Samaniego and proved that if we add enough surface tension we can find an existence time $T = \frac{T_0}{\varepsilon}$ where T_0 does not depend on ε and μ ([24]). Iguchi studied the case of a moving bottom in order to justify asymptotic models for tsunamis ([17]). Finally, Alazard, Burq and Zuily study the optimal regularity for the initial data ([2]) and more recently, Alazard, Baldi and Han-Kwan show that a well-chosen non constant external pressure can create any small amplitude two-dimensional gravity-capillary water waves ([3]). We organize this paper in two part. Firstly in section 2, we prove two local existence theorems for the water waves problem with a moving bottom and a non constant pressure at the surface by differentiating and 'quasi-linearizing' the water waves equations and we pay attention to the dependence of the time of existence and the size of the solution with respect to the parameters ε, β, λ and μ. This theorem extends the result of Iguchi ([17]) and Lannes (chapter 4 in [20]). We also prove that the water waves problem can be viewed as a Hamiltonian system. Secondly in section 3, we justify some linear asymptotic models and study the Proudman resonance. First, in section 3.1 we study the case of small topography variations in shallow waters, approximation used in the Physics literature to investigate the Proudman resonance; then in section 3.2 we derive a model when the topography is not small in the shallow water approximation; and in section 3.3 we study the linear water waves equations in order to extend the Proudman resonance in deep water with a small fixed topography. Finally, appendix A contains results about the elliptic problem (17) and appendix B contains results about the Dirichlet–Neumann and the Neumann–Neumann operators. Appendix C comprises standard estimates that we use in this paper.

1.2. Notations

A good framework for the velocity in the Euler equations is the Sobolev spaces H^s. But we do not work with $\mathbf{U}$ but with ψ the trace of Φ, and $\mathbf{U} = \nabla_{\mathbf{X,z}}\mathbf{\Phi}$. It will be too restrictive to take ψ in a Sobolev space. A good idea is to work with the Beppo Levi spaces (see [14]). For $s \geqslant 0$, the Beppo Levi spaces are

$$\dot{H}^s(\mathbb{R}^d) := \{\psi \in L^2_{\text{loc}}(\mathbb{R}^d), \nabla\psi \in H^{s-1}(\mathbb{R}^d)\}.$$

In this paper, C is a constant and for a function f in a normed space $(X, |\cdot|)$ or a parameter γ, $C(|f|, \gamma)$ is a constant depending on $|f|$ and γ whose exact value has non importance. The norm $|\cdot|_{L^2}$ is the L^2-norm and $|\cdot|_\infty$ is the L^∞-norm in $\mathbb{R}^d$. Let $f \in \mathcal{C}^0(\mathbb{R}^d)$ and $m \in \mathbb{N}$ such that $\frac{f}{1+|x|^m} \in L^\infty(\mathbb{R}^d)$. We define the Fourier multiplier $f(D) : H^m(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as

$$\forall\, u \in H^m(\mathbb{R}^d), \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

In $\mathbb{R}^d$ we denote the gradient operator by ∇ and in Ω or $S = \mathbb{R}^d \times (-1, 0)$ the gradient operator is denoted $\nabla_{X,z}$. Finally, we denote by $\Lambda := \sqrt{1+|D|^2}$ with $D = -\mathrm{i}\nabla$.

2. Local existence of the water waves equations

This part is devoted to the wellposedness of the water waves equations (theorems 2.3 and 2.4). We carefully study the dependence on the parameters ε, β, λ and μ of the existence time and of the size of the solution. Contrary to [20] and [17], we exhibit the nonlinearities of the water waves equations in order to obtain a better existence time.

2.1. The model

In this part, we present a nondimensionalized version of the water waves equations. In order to derive some asymptotic models to the water waves equations we introduce some dimensionless parameters linked to the physical scales of the system. The first one is the ratio between the typical free surface amplitude a and the water depth H. We define $\varepsilon := \frac{a}{H}$, called the nonlinearity parameter. The second one is the ratio between H and the characteristic horizontal scale L. We define $\mu := \frac{H^2}{L^2}$, called the shallowness parameter. The third one is the ratio between the order of bottom bathymetry amplitude a_{bott} and H. We define $\beta := \frac{a_{\text{bott}}}{H}$, called the bathymetric parameter. Finally, we denote by λ the ratio of the typical landslide amplitude $a_{\text{bott},m}$ and a_{bott}. We also nondimensionalize the variables and the unknowns. We introduce

$$\begin{cases} X' = \dfrac{X}{L}, z' = \dfrac{z}{H}, \zeta' = \dfrac{\zeta}{a}, b' = \dfrac{b}{a_{\text{bott}}}, b_0' = \dfrac{b_0}{a_{\text{bott}}}, b_m' = \dfrac{b_m}{a_{\text{bott},m}}, t' = \dfrac{\sqrt{gH}}{L}t, \\ (\Phi^S)' = \dfrac{H}{aL\sqrt{gH}}\Phi^S, (\Phi^B)' = \dfrac{L}{Ha_{\text{bott},m}\sqrt{gH}}\Phi^B, \psi' = \dfrac{H}{aL\sqrt{gH}}\psi, P' = \dfrac{P}{a\rho g}, \end{cases} \tag{12}$$

where

$$\Omega_t' = \{(X', z') \in \mathbb{R}^{d+1}, -1 + \beta b'(t', X') < z' < \varepsilon\zeta'(t', X')\}.$$

Remark 2.1. It is worth noting that the nondimensionalization of Φ^S, ψ and t comes from the linear wave theory (in shallow water regime, the characteristic speed is $\sqrt{gH}$). See section 1.3.2 in [20]. Let us explain the nondimensionalization of Φ^B. Consider the linear case

$$\begin{cases} \Delta_{X,z}\Phi^B = 0, -H < z < 0, \\ {\Phi^B}_{|z=0} = 0, \partial_z {\Phi^B}_{|z=-H} = \partial_t b. \end{cases}$$

A straightforward computation gives $\Phi^B = \frac{\sinh(z\mid D\mid)}{|D\mid\cosh(H\mid D\mid)}\partial_t b$. If the typical wavelength is L, the typical wave number is $\frac{2\pi}{L}$. Furthermore, the typical order of magnitude of $\partial_t b$ is $\frac{a_{\text{bott},m}\sqrt{gH}}{L}$. Then, the order of magnitude of Φ^B in the shallow water case is

$$\frac{L}{2\pi}\frac{\sqrt{gH}\,a_{\text{bott},m}}{L}\frac{\sinh\left(2\pi\frac{H}{L}\right)}{\cosh\left(2\pi\frac{H}{L}\right)} \sim \frac{\sqrt{gH}\,a_{\text{bott},m}H}{L}.$$

For the sake of clarity, we omit the primes. We can now nondimensionalize the water waves problem. Using the notation

$$\nabla^\mu_{X,z} := (\sqrt{\mu}\,\nabla_X\ , \partial_z\ )^t \text{ and } \Delta^\mu_{X,z}\ := \mu\Delta_X\ + \partial_z^2,$$

the water waves equations (11) become in dimensionless form

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b](\psi) = \dfrac{\beta\lambda}{\varepsilon}G_\mu^{\text{NN}}[\varepsilon\zeta, \beta b](\partial_t b), \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \dfrac{\varepsilon}{2\mu}\dfrac{\left(G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta\mu}{\varepsilon}G_\mu^{\text{NN}}[\varepsilon\zeta, \beta b](\partial_t b) + \mu\nabla(\varepsilon\zeta)\cdot\nabla\psi\right)^2}{(1 + \varepsilon^2\mu|\nabla\zeta|^2)} = -P. \end{cases} \tag{13}$$

In the following $\partial_{\mathbf{n}}$ is the upward *conormal* derivative

$$\partial_{\mathbf{n}}\Phi^S = \mathbf{n}\cdot\begin{pmatrix}\sqrt{\mu}\,\mathbf{I_d} & \mathbf{0} \\ 0 & 1\end{pmatrix}\nabla^\mu_{\mathbf{X,z}}\mathbf{\Phi^S}_{|\partial\Omega}.$$

Then, The Dirichlet–Neumann operator $G_\mu[\varepsilon\zeta, \beta b]$ is

$$G_\mu[\varepsilon\zeta, \beta b](\psi) := \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi^S{}_{|z=\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta\cdot\nabla_X\Phi^S{}_{|z=\varepsilon\zeta} + \partial_z\Phi^S{}_{|z=\varepsilon\zeta}, \tag{14}$$

where Φ^S satisfies

$$\begin{cases}\Delta^\mu_{X,z}\Phi^S = 0 \text{ in } \Omega_t\,,\\ \Phi^S{}_{|z=\varepsilon\zeta} = \psi, \partial_{\mathbf{n}}\Phi^S{}_{|z=-1+\beta b} = 0,\end{cases} \tag{15}$$

while the Neumann–Neumann operator $G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b]$ is

$$G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](\partial_t b) := \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi^B{}_{|z=\varepsilon\zeta} = -\mu\nabla(\varepsilon\zeta)\cdot\nabla_X\Phi^B{}_{|z=\varepsilon\zeta} + \partial_z\Phi^B{}_{|z=\varepsilon\zeta}, \tag{16}$$

where Φ^B satisfies

$$\begin{cases}\Delta^\mu_{X,z}\Phi^B = 0 \text{ in } \Omega_t\,,\\ \Phi^B{}_{|z=\varepsilon\zeta} = 0, \sqrt{1+\beta^2|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi^B{}_{|z=-1+\beta b} = \partial_t b.\end{cases} \tag{17}$$

Remark 2.2. We have nondimensionalized the Dirichlet–Neumann and the Neumann–Neumann operators as follows

$$G[\zeta, b](\psi) = \frac{aL\sqrt{gH}}{H^2}G_\mu[\varepsilon\zeta', \beta b'](\psi'), G^{\mathrm{NN}}[\zeta, b](\partial_t b) = \frac{a_{\mathrm{bott},m}\sqrt{gH}}{L}G^{\mathrm{NN}}_\mu[\varepsilon\zeta', \beta b'](\partial_{t'}b').$$

We add two classical assumptions. First, we assume some constraints on the nondimensionalized parameters and we suppose there exist $\rho_{\max} > 0$ and $\mu_{\max} > 0$, such that

$$0 < \varepsilon, \beta, \beta\lambda \leqslant 1, \frac{\beta\lambda}{\varepsilon} \leqslant \rho_{\max} \text{ and } \mu \leqslant \mu_{\max}. \tag{18}$$

Furthermore, we assume that the water depth is bounded from below by a positive constant

$$\exists\ h_{\min} > 0, \varepsilon\zeta + 1 - \beta b \geqslant h_{\min}. \tag{19}$$

In order to quasilinearize the water waves equations, we have to introduce the vertical speed at the surface $\underline{w}$ and horizontal speed at the surface $\underline{V}$. We define

$$\underline{w} := \underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) = \frac{G_\mu[\varepsilon\zeta, \beta b](\psi) + \mu\frac{\beta\lambda}{\varepsilon}G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](\partial_t b) + \varepsilon\mu\nabla\zeta\cdot\nabla\psi}{1+\varepsilon^2\mu|\nabla\zeta|^2}, \tag{20}$$

and

$$\underline{V} := \underline{V}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) = \nabla\psi - \varepsilon\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\nabla\zeta. \tag{21}$$

2.2. Notations and statement of the main results

In this paper, $d = 1$ or 2, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ and $s \geqslant 0$. The constant $T \geqslant 0$ represents a final time. The pressure P and the bottom b are given functions. We suppose that $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. We denote by M_N a constant of the form

$$M_N = C\left(\frac{1}{h_{\min}}, \mu_{\max}, \varepsilon|\zeta|_{H^{\max(t_0+2,N)}}, \beta|b|_{L^\infty_t H_X^{\max(t_0+2,N)}}\right). \tag{22}$$

We denote by $U := (\zeta, \psi)^t$ the unknowns of our problem. We want to express (11) as a quasilinear system. It is well-known that the good energy for the water waves problem is

$$\mathcal{E}^N(U) = |\mathfrak{P}\psi|^2_{H^{\frac{3}{2}}} + \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leqslant N} (|\zeta_{(\alpha)}|^2_{L^2} + |\mathfrak{P}\psi_{(\alpha)}|^2_{L^2}), \tag{23}$$

where $\zeta_{(\alpha)} := \partial^\alpha \zeta$, $\psi_{(\alpha)} := \partial^\alpha \psi - \varepsilon \underline{w} \partial^\alpha \zeta$ and $\mathfrak{P} := \frac{|D|}{\sqrt{1+\sqrt{\mu}\,|D|}}$. This energy is motivated by the linearization of the system around the rest state (see 4.1 in [20]). $\mathfrak{P}$ acts as the square root of the Dirichlet–Neumann operator (see [20]). Here, $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$ are the *Alinhac's good unknowns of the system* (see [5] and [4] in the case of the standard water waves problem). We define $U_{(\alpha)} := (\zeta_{(\alpha)}, \psi_{(\alpha)})^t$. We can introduce an associated energy space. Considering a $T \geqslant 0$,

$$E^N_T := \{U \in \mathcal{C}([0, T]; H^{t_0+2}(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d)), \mathcal{E}^N(U) \in L^\infty([0, T])\}. \tag{24}$$

Our main results are the following theorems. We give two existence results. The first theorem extends the result of Iguchi (theorem 2.4 in [17]) since we give a control of the dependence of the solution with respect to the parameters ε, β and μ and we add a non constant pressure at the surface and also extends the result of Lannes (theorem 4.16 in [20]), since we improve the regularity of the initial data and add a non constant pressure pressure at the surface and a moving bottom. Notice that we explain later what is condition (29) (it corresponds to the positivity of the so called Rayleigh-Taylor coefficient).

Theorem 2.3. *Let $A > 0$, $t_0 > \frac{d}{2}$, $N \geqslant \max(1, t_0) + 3$, $U^0 \in E^N_0$, $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ such that*

$$\mathcal{E}^N(U^0) + \frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L^\infty_t H^N_X} + |\nabla P|_{L^\infty_t H^N_X} \leqslant A.$$

We suppose that the parameters $\varepsilon, \beta, \mu, \lambda$ satisfy (18) and that (19) and (29) are satisfied initially. Then, there exists $T > 0$ and a unique solution $U \in E^N_T$ to (13) with initial data U^0. Moreover, we have

$$T = \min\left(\frac{T_0}{\max(\varepsilon, \beta)}, \frac{T_0}{\frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L^\infty_t H^N_X} + |\nabla P|_{L^\infty_t H^N_X}}\right), \frac{1}{T_0} = c^1 \text{ and } \sup_{t\in[0,T]} \mathcal{E}^N(U) = c^2,$$

with $c^j = C\left(A, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, \rho_{\max}, |b|_{W^{3,\infty}_t H^N_X}, |\nabla P|_{W^{1,\infty}_t H^N_X}\right)$.

Notice that if $\partial_t b$ and P are of size $\max(\varepsilon, \beta)$, we find the same existence time that in theorem 4.16 in [20]. The second result shows that it is possible to go beyond the time scale of the previous theorem; although the norm of the solution is not uniformly bounded in terms of ε and β, we are able to make this dependence precise. This theorem will be used to justify some of the asymptotic models derived in section 3 over large time scales when the pressure at the surface and the moving bottom are not supposed small. We introduce $\delta := \max(\varepsilon, \beta^2)$.

Theorem 2.4. *Under the assumptions of the previous theorem, there exists $T_0 > 0$ such that $U \in E^N_{\frac{T_0}{\sqrt{\delta}}}$. Moreover, for all $\alpha \in [0, \frac{1}{2}]$, we have*

$$\frac{1}{T_0} = c^1, \sup_{t\in\left[0,\frac{T_0}{\delta^\alpha}\right]} \mathcal{E}^N(U) \leqslant \frac{c^3}{\delta^{2\alpha}}, c^j = C\left(A, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, \rho_{\max}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right).$$

Notice that when $\partial_t b$ and P are of size $\max(\varepsilon, \beta)$, the existence time of theorem 2.3 is better than the one of theorem 2.4. Theorem 2.4 is only useful when $\partial_t b$ and P are not small. Notice finally, that condition (29) is satisfied if ε is small enough. Hence, since in the following, ε is small, it is reasonable to assume it.

2.3. Quasilinearization

Firstly, we give some controls of $|\mathfrak{P}\psi|_{H^s}$ and $|\mathfrak{P}\psi_{(\alpha)}|_{H^s}$ with respect to the energy $\mathcal{E}^N(U)$.

Proposition 2.5. *Let $T > 0$, $t_0 > \frac{d}{2}$ and $N \geqslant 2 + \max(1, t_0)$. Consider $U \in E_T^N$, $b \in W^{1,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$, such that ζ and b satisfy condition (19) for all $0 \leqslant t \leqslant T$. We assume also that μ satisfies (18). Then, for $0 \leqslant t \leqslant T$, for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant N - 1$ and for $0 \leqslant s \leqslant N - \frac{1}{2}$,*

$$|\partial^\alpha \mathfrak{P}\psi|_{L^2} + |\mathfrak{P}\psi_{(\alpha)}|_{H^1} + |\mathfrak{P}\psi|_{H^s} \leqslant M_N \mathcal{E}^N(U)^{\frac{1}{2}} + \frac{\beta\lambda}{\varepsilon} M_N \, |\partial_t b|_{L_t^\infty H_X^N}.$$

Proof. For the first inequality, we have thanks to Proposition C.1,

$$\begin{aligned} |\partial^\alpha \mathfrak{P}\psi|_{L^2} &\leqslant |\mathfrak{P}\psi_{(\alpha)}|_{L^2} + \varepsilon |\mathfrak{P}(\underline{w}\partial^\alpha\zeta)|_{L^2}, \\ &\leqslant |\mathfrak{P}\psi_{(\alpha)}|_{L^2} + \frac{\varepsilon}{\mu^{\frac{1}{4}}} |\underline{w}\partial^\alpha\zeta|_{H^{\frac{1}{2}}}. \end{aligned}$$

But $\psi \in \dot{H}^2(\mathbb{R}^d)$. Then by Proposition B.8, $\underline{w} \in H^1(\mathbb{R}^d)$ and $\partial^\alpha\zeta \in H^1(\mathbb{R}^d)$. Using Proposition C.2, we obtain

$$|\partial^\alpha \mathfrak{P}\psi|_{L^2} \leqslant |\mathfrak{P}\psi_{(\alpha)}|_{L^2} + C\varepsilon \left|\frac{\underline{w}}{\mu^{\frac{1}{4}}}\right|_{H^1} |\zeta|_{H^N} \leqslant |\mathfrak{P}\psi_{(\alpha)}|_{L^2} + M_N \varepsilon |\zeta|_{H^N} \left(|\mathfrak{P}\psi|_{H^{\frac{3}{2}}} + \frac{\beta\lambda}{\varepsilon} |\partial_t b|_{H^1} \right).$$

The other inequalities follow with the same arguments, see for instance lemma 4.6 in [20]. $\square$

The following statement is a first step to the quasilinearization of the water waves equations. It is essentially proposition 4.5 in [20] and lemma 6.2 in [17]. However, we improve the minimal regularity of U (we decrease the minimal value of N to 4 in dimension 1) and we provide the dependence in $\partial_t b$ which does not given in [17]. For those reasons, we give a proof of this proposition.

Proposition 2.6. *Let $t_0 > \frac{d}{2}$, $T > 0$, $N \geqslant \max(t_0, 1) + 3$, $b \in W^{1,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $U \in E_T^N$, such that ζ and b satisfy condition (19) for all $0 \leqslant t \leqslant T$. We assume also that μ satisfies (18). Then, for all $\alpha \in \mathbb{N}^d$, $1 \leqslant |\alpha| \leqslant N$, we have,*

$$\begin{aligned} \partial^\alpha \left(\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta}{\varepsilon} G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b) \right) &= \frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi_{(\alpha)}) + \frac{\beta\lambda}{\varepsilon} G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^\alpha \partial_t b) \\ &\quad - \varepsilon \mathbf{1}_{\{|\alpha| = N\}} \nabla \cdot (\zeta_{(\alpha)} \underline{V}) + R_\alpha. \end{aligned}$$

Furthermore R_α is controlled

$$|R_\alpha|_{L^2} \leqslant M_N |(\varepsilon\zeta, \beta b)|_{H^N} \mathcal{E}^N(U)^{\frac{1}{2}} + \frac{\beta\lambda}{\varepsilon} M_N \, |\partial_t b|_{L^\infty_t H^N_X}.$$

Proof. We adapt and follow the proof of proposition 4.5 in [20]. See also Proposition C.4 in [17]. Using Proposition B.13, we obtain

$$\partial^\alpha\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta}{\varepsilon}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b)\right) = \frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b](\psi_{(\alpha)}) + \frac{\beta\lambda}{\varepsilon}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^\alpha\partial_t b)$$
$$-\varepsilon\mathbf{1}_{\{|\alpha|=N\}}\nabla\cdot(\zeta_{(\alpha)}\underline{V}) + R_\alpha,$$

where R_α is a sum of terms of the form (we adopt the notation of Remark B.12 in appendix B.3)

$$A_{j,\iota,\nu} := d^j\left(\frac{1}{\mu}G_\mu(\partial^\nu\psi) + \frac{\beta\lambda}{\varepsilon}G_\mu^{\mathrm{NN}}(\partial^\nu\partial_t b)\right).(\partial^{\iota^1}\zeta, ..., \partial^{\iota^j}\zeta; \partial^{\iota^1}b, ..., \partial^{\iota^j}b),$$

where j is an integer and $\iota^1, ..., \iota^j$ and ν are multi-index, and

$$\sum_{1\leqslant l\leqslant j}|\iota^l| + |\nu| = N,$$

with $(j, |\iota^{l_0}|, |\nu|) \neq (1, N, 0)$ and $(0, 0, N)$. Here ι^{l_0} is such that $\max_{1\leqslant l\leqslant j}|\iota^l| = |\iota^{l_0}|$. In particular, $1 \leqslant |\iota^{l_0}| \leqslant N$. We distinguish several cases.

(a) $|\iota^{l_0}| + |\nu| \leqslant N-2$ *and* $|\iota^{l_0}| \leqslant N-3$ *or* $|\iota^{l_0}| + |\nu| \leqslant N$, $|\iota^{l_0}| \leqslant N-3$ *and* $|\nu| \leqslant N-2$ *:*

Applying the second point of theorem 3.28 in [20] and the first point of Proposition B.15 with $s = \frac{1}{2}$ and $t_0 = \min\left(t_0, \frac{3}{2}\right)$, we get that

$$|A_{j,\iota,\nu}|_{L^2} \leqslant M_N \prod_l |(\varepsilon\partial^{\iota^l}\zeta, \beta\partial^{\iota^l}b)|_{H^3}\left[|\mathfrak{P}\partial^\nu\psi|_{H^1} + \frac{\beta\lambda}{\varepsilon}|\partial^\nu\partial_t b|_{L^2}\right],$$

and the result follows by proposition 2.5.

(b) $|\iota^{l_0}| = N-2$ *and* $|\nu| = 0, 1$ *or* 2 *:*

We apply the fourth point of theorem 3.28 in [20] and the second point of Proposition B.15 with $s = \frac{1}{2}$ and $t_0 = \max(t_0, 1)$,

$$|A_{j,\iota,\nu}|_{L^2} \leqslant M_N |(\varepsilon\partial^{\iota^{l_0}}\zeta, \beta\partial^{\iota^{l_0}}b)|_{H^{\frac{3}{2}}} \prod_{l\neq l_0} |(\varepsilon\partial^{\iota^l}\zeta, \beta\partial^{\iota^l}b)|_{H^{N-2}}\left[|\mathfrak{P}\partial^\nu\psi|_{H^{N-2}} + \frac{\beta\lambda}{\varepsilon}|\partial^\nu\partial_t b|_{H^{N-2}}\right].$$

Then, we get the result thanks to proposition 2.5.

(c) $\iota^1 = \iota$ *with* $|\iota| = N-1$, $|\nu| = j = 1$ *:*

We proceed as in proposition 4.5 in [20], using theorem 3.15 in [20] and Propositions B.7, B.8 and B.13.

(d) $|\iota^{l_0}| = N-1$ *and* $|\nu| = 0$ *:*

Here $j = 2$ and $|\iota^2| = 1$. For instance we consider that $l_0 = 1$ and $|\iota^2| = 1$. Using the second inequality of Proposition B.15 we have

$$\left|d^2 G_\mu^{\mathrm{NN}}(\partial_t b).(\partial^{\iota^1}\zeta, \partial^{\iota^2}\zeta; \partial^{\iota^1}b, \partial^{\iota^2}b)\right|_{L^2} \leqslant M_N \left|(\varepsilon\partial^{\iota^1}\zeta, \beta\partial^{\iota^1}b)\right|_{H^1} \left|(\varepsilon\partial^{\iota^2}\zeta, \beta\partial^{\iota^2}b)\right|_{H^2} |\partial_t b|_{H^2}.$$

Furthermore, using two times Proposition B.13, we get

$$\begin{aligned}\frac{1}{\mu}d^2 G_\mu(\psi).(\partial^{\iota^1}\zeta, \partial^{\iota^2}\zeta; \partial^{\iota^1}b, \partial^{\iota^2}b) = &-\frac{\varepsilon}{\sqrt{\mu}}dG_\mu[\varepsilon\zeta, \beta b]\left(\partial^{\iota^1}\zeta\frac{1}{\sqrt{\mu}}\underline{w}(\psi, 0)\right).(\partial^{\iota^2}\zeta, 0)\\ &-\frac{\varepsilon}{\sqrt{\mu}}G_\mu[\varepsilon\zeta, \beta b]\left(\partial^{\iota^1}\zeta\frac{1}{\sqrt{\mu}}d\underline{w}(\psi, 0).(\partial^{\iota^2}\zeta, 0)\right)\\ &-\varepsilon\nabla\cdot(\partial^{\iota^1}\zeta d\underline{V}(\psi, 0).(\partial^{\iota^2}\zeta, 0))\\ &+\beta dG_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^{\iota^1}b\ \underline{\widetilde{V}}(\psi, 0)).(0, \partial^{\iota^2}b)\\ &+\beta G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^{\iota^1}b\ d\underline{\widetilde{V}}(\psi, 0).(0, \partial^{\iota^2}b)).\end{aligned}$$

The control follows from the first inequality of theorem 3.15 and proposition 4.4 in [20], and Propositions B.8, B.11 and B.14.

(e) $|\nu| = N-1$ *and* $|\iota^{l_0}| = 1$*:*

Here, $j=1$. It is clear that

$$\left|\frac{\beta\lambda}{\varepsilon}dG_\mu^{\mathrm{NN}}(\partial^\nu\partial_t b).(\partial^{\iota^1}\zeta; \partial^{\iota^1}b)\right|_{L^2} \leqslant \frac{\beta\lambda}{\varepsilon}M_N|\partial_t b|_{H^N}.$$

Furthermore,

$$\frac{1}{\mu}dG_\mu(\partial^\nu\psi).(\partial^{\iota^1}\zeta; \partial^{\iota^1}b) = \frac{1}{\mu}dG_\mu(\psi_{(\nu)}).(\partial^{\iota^1}\zeta; \partial^{\iota^1}b) + \frac{1}{\sqrt{\mu}}dG_\mu\left(\frac{\varepsilon}{\sqrt{\mu}}\underline{w}\partial^\nu\zeta\right).(\partial^{\iota^1}\zeta; \partial^{\iota^1}b).$$

Then, using theorem 3.15 in [20] and proposition 2.5, we get the result. □

This proposition enables to quasilinearize the first equation of the water waves equations. For the second equation, it is the purpose of the following proposition.

Proposition 2.7. *Let $T>0$, $N \geqslant \max(t_0, 1)+3$, $b \in W^{1,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $U \in E_T^N$, such that ζ and b satisfy (19) for all $0 \leqslant t \leqslant T$. We assume also that μ satisfies (18). Then, for all $\alpha \in \mathbb{N}^d$, $1 \leqslant |\alpha| \leqslant N$, we have,*

$$\begin{aligned}\partial^\alpha\left[\frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\varepsilon}{2\mu}(1+\varepsilon^2\mu|\nabla\zeta|^2)\underline{w}^2\right] = \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) - \frac{\varepsilon}{\mu}\underline{w}\ \partial^\alpha G_\mu(\psi)\\ -\beta\lambda\underline{w}\ \partial^\alpha G_\mu^{\mathrm{NN}}(\partial_t b) + S_\alpha.\end{aligned}$$

Furthermore S_α is controlled

$$|\mathfrak{P}S_\alpha|_{L^2} \leqslant \varepsilon M_N \mathcal{E}^N(U) + C\left(M_N, \frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L_t^\infty H_X^N}\right)\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} + M_N\left(\frac{\beta\lambda}{\sqrt{\varepsilon}}|\partial_t b|_{L_t^\infty H_X^N}\right)^2.$$

Proof. The proof of this proposition is similar to the proof of proposition 4.10 in [20] expect we use Propositions B.8 and B.13. See also Proposition C.4 in [17]. □

Thanks to this linearization, we can 'quasilinarize' equations (13). It is the purpose of the next proposition. Let us introduce, the Rayleigh-Taylor coefficient

$$\underline{\mathfrak{a}} := \underline{\mathfrak{a}}(U, \beta b) = 1 + \varepsilon \partial_t \left(\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right) + \varepsilon^2 \underline{V}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) \cdot \nabla\left(\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right). \tag{25}$$

This quantity plays an important role. We also introduce two new operators,

$$\mathcal{A}[U, \beta b] := \begin{pmatrix} 0 & -\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b] \\ \underline{\mathfrak{a}}(U, \beta b) & 0 \end{pmatrix} \tag{26}$$

and

$$\mathcal{B}[U, \beta b] := \begin{pmatrix} \varepsilon \nabla \cdot (\bullet \underline{V}) & 0 \\ 0 & \varepsilon \underline{V} \cdot \nabla \end{pmatrix}. \tag{27}$$

We can now quasilinearize the water waves equations. We use the same arguments as in proposition 4.10 in [20] and part 6 in [17]. Notice that we give here a precise estimate with respect to $\partial_t b$ and P of the residuals R_α and S_α and that the minimal value of N, regularity of U, is smaller than in proposition 4.10 in [20].

Proposition 2.8. *Let $T > 0$, $N \geqslant \max(t_0, 1) + 3$, $b \in W^{2,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$, $P \in L^\infty(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ and $U \in E_T^N$ satisfies (19) for all $0 \leqslant t \leqslant T$ and solving (13). We assume also that μ satisfies (18). Then, for all $\alpha \in \mathbb{N}^d$, $1 \leqslant |\alpha| \leqslant N$, we have,*

$$\partial_t U_{(\alpha)} + \mathcal{A}[U, \beta b](U_{(\alpha)}) + \mathbf{1}_{\{|\alpha|=N\}} \mathcal{B}[U, \beta b](U_{(\alpha)}) = \left(\frac{\lambda\beta}{\varepsilon} G_\mu^{\mathrm{NN}}[0,0](\partial^\alpha \partial_t b), -\partial^\alpha P\right)^t + (\widetilde{R_\alpha}, S_\alpha)^t. \tag{28}$$

Furthermore, $\widetilde{R_\alpha}$ and S_α satisfy

$$\begin{cases} |\widetilde{R_\alpha}|_{L^2} \leqslant M_N |(\varepsilon\zeta, \beta b)|_{H^N} \mathcal{E}^N(U)^{\frac{1}{2}} + \dfrac{\beta\lambda}{\varepsilon} M_N \, |\partial_t b|_{L_t^\infty H_X^N}, \\ |\mathfrak{P} S_\alpha|_{L^2} \leqslant \varepsilon M_N \mathcal{E}^N(U) + C\left(M_N, \dfrac{\beta\lambda}{\varepsilon}\, |\partial_t b|_{L_t^\infty H_X^N}\right) \varepsilon \mathcal{E}^N(U)^{\frac{1}{2}} + M_N \left(\dfrac{\beta\lambda}{\sqrt{\varepsilon}}\, |\partial_t b|_{L_t^\infty H_X^N}\right)^2. \end{cases}$$

Proof. Thanks to Proposition B.15, we get

$$\begin{aligned} \left| G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^\alpha \partial_t b) - G_\mu^{\mathrm{NN}}[0,0](\partial^\alpha \partial_t b) \right|_{L^2} &\leqslant \int_0^1 \left| dG_\mu^{\mathrm{NN}}[z\varepsilon\zeta, z\beta b](\partial^\alpha \partial_t b).(\zeta, b) \right|_{L^2} \mathrm{d}z \\ &\leqslant M_N |(\varepsilon\zeta, \beta b)|_{H^N} \, |\partial^\alpha \partial_t b|_{L_t^\infty H_X^N}. \end{aligned}$$

Then, denoting $\widetilde{R_\alpha} = R_\alpha + G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial^\alpha \partial_t b) - G_\mu^{\mathrm{NN}}[0,0](\partial^\alpha \partial_t b)$, we obtain the first equation thanks to proposition 2.6. For the second equation, using proposition 2.7 and the first equation of the water waves problem, we have

$$
\begin{aligned}
\partial_t\partial^\alpha\psi &= -\partial^\alpha\zeta - \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) + \frac{\varepsilon}{\mu}\underline{w}\,\partial^\alpha G_\mu(\psi) + \beta\underline{w}\,\partial^\alpha G_\mu^{\mathrm{NN}}(\partial_t b) - \partial^\alpha P + S_\alpha \\
&= -\partial^\alpha\zeta - \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) + \varepsilon\underline{w}\partial_t\partial^\alpha\zeta - \partial^\alpha P + S_\alpha \\
&= -\partial^\alpha\zeta(1 + \varepsilon\partial_t\underline{w} + \varepsilon^2\underline{V}\cdot\nabla\underline{w}) - \varepsilon\underline{V}\cdot\nabla\psi_{(\alpha)} + \varepsilon\partial_t(\underline{w}\partial^\alpha\zeta) - \partial^\alpha P + S_\alpha \\
&= -\underline{\mathfrak{a}}\partial^\alpha\zeta - \varepsilon\underline{V}\cdot\nabla\psi_{(\alpha)} + \varepsilon\partial_t(\underline{w}\partial^\alpha\zeta) - \partial^\alpha P + S_\alpha,
\end{aligned}
$$

and the result follows. $\square$

In the case of a constant pressure at the surface and a fixed bottom, it is well-known that system (28) is symmetrizable if

$$\exists\ \mathfrak{a}_{\min} > 0,\ \underline{\mathfrak{a}}(U,\beta b) \geqslant \mathfrak{a}_{\min}. \tag{29}$$

Then, we introduce the symmetrizer

$$\mathcal{S}[U,\beta b] := \begin{pmatrix} \underline{\mathfrak{a}}(U,\beta b) & 0 \\ 0 & \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b] \end{pmatrix}. \tag{30}$$

This symmetrization has an associated energy

$$
\begin{aligned}
\mathcal{F}^\alpha(U) &= \frac{1}{2}(\mathcal{S}[U,\beta b](U_{(\alpha)}), U_{(\alpha)})_{L^2},\ \text{if } \alpha \neq 0, \\
\mathcal{F}^0(U) &= \frac{1}{2}|\zeta|^2_{H^{\frac{3}{2}}} + \frac{1}{2}\left(\Lambda^{\frac{3}{2}}\psi, \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b]\left(\Lambda^{\frac{3}{2}}\psi\right)\right)_{L^2}, \\
\mathcal{F}^{[N]}(U) &= \sum_{|\alpha|\leqslant N}\mathcal{F}^\alpha(U).
\end{aligned} \tag{31}
$$

As in lemma 4.27 in [20], it can be shown that $\mathcal{F}^{[N]}$ and $\mathcal{E}^{[N]}$ are equivalent in the following sense.

Proposition 2.9. *Let $T > 0$, $N \in \mathbb{N}$, $U \in E_T^N$ satisfying (19) and (29) for all $0 \leqslant t \leqslant T$. Then, for all $0 \leqslant k \leqslant N$, $\mathcal{F}^{[k]}$ is comparable to $\mathcal{E}^k$*

$$\frac{1}{|\underline{\mathfrak{a}}(U,\beta b)|_{L^\infty} + M_N}\mathcal{F}^{[k]}[U,b] \leqslant \mathcal{E}^k(U) \leqslant \left(M_N + \frac{1}{\mathfrak{a}_{\min}}\right)\mathcal{F}^{[k]}[U,b]. \tag{32}$$

2.4. Local existence

The water water equations can be written as follow:

$$\partial_t U + \mathcal{N}(U) = (0, -P)^t, \tag{33}$$

with $\mathcal{N}(U) = (\mathcal{N}_1(U), \mathcal{N}_2(U))^t$ and

$$
\begin{aligned}
\mathcal{N}_1(U) &:= -\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi) - \frac{\beta\lambda}{\varepsilon}G_\mu^{\mathrm{NN}}[\varepsilon\zeta,\beta b](\partial_t b), \\
\mathcal{N}_2(U) &:= \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\varepsilon}{2\mu}(1 + \varepsilon^2\mu|\nabla\zeta|^2)\left(\underline{w}[\varepsilon\zeta,\beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right)^2.
\end{aligned} \tag{34}
$$

According to our quasilinearization, we need that $\underline{\mathfrak{a}}$ be a positive real number. Therefore, we have to express $\underline{\mathfrak{a}}$ without partial derivative with respect to t, particularly when $t = 0$. It is easy to check that (we adopt the notation of Remark B.12 in appendix B.3)

$$\begin{aligned}\underline{\mathfrak{a}}(U, \beta b) = 1 + \varepsilon^2 \underline{V}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) \cdot \nabla\left[\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right] \\ + \varepsilon d\underline{w}\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right).(-\mathcal{N}_1(U), \partial_t b) + \varepsilon \underline{w}[\varepsilon\zeta, \beta b]\left(-P - \mathcal{N}_2(U), \frac{\beta\lambda}{\varepsilon}\partial_t^2 b\right).\end{aligned} \tag{35}$$

The following proposition gives estimates for $\underline{\mathfrak{a}}(U, \beta b)$. It is adapted from Proposition C.6 in [17].

Proposition 2.10. *Let $T > 0$, $t_0 > \frac{d}{2}$, $N \geqslant \max(t_0, 1) + 3$, $(\zeta, \psi) \in E_T^N$ is a solution of the water waves equations (13), $P \in L^\infty(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ and $b \in W^{2,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$, such that condition (19) is satisfied. We assume also that μ satisfies (18). Then, for all $0 \leqslant t \leqslant T$,*

$$\begin{aligned}|\underline{\mathfrak{a}}(U, \beta b) - 1|_{H^{t_0}} \leqslant C\left(M_N, \max(\beta\lambda, \beta)\, |\partial_t b|_{L_t^\infty H_X^N}, \varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\right)\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} \\ + \varepsilon M_N\left(|\nabla P|_{L_t^\infty H_X^N} + \frac{\beta\lambda}{\varepsilon}\,|\partial_t^2 b|_{L_t^\infty H_X^N}\right).\end{aligned}$$

Furthermore, if $\partial_t^3 b \in L^\infty(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $\partial_t P \in L^\infty(\mathbb{R}^+; \dot{H}^N(\mathbb{R}^d))$, then,

$$\begin{aligned}|\partial_t(\underline{\mathfrak{a}}(U, \beta b))|_{H^{t_0}} \leqslant C\left(M_N, \max(\beta\lambda, \beta)\, |\partial_t b|_{W_t^{1,\infty} H_X^N}, |\nabla P|_{L_t^\infty H_X^N}, \varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\right)\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} \\ + \varepsilon C\left(M_N, \max(\beta\lambda, \beta)\, |\partial_t b|_{L_t^\infty H_X^N}\right)\left(|\nabla P|_{W_t^{1,\infty} H_X^N} + \frac{\beta\lambda}{\varepsilon}\left|\partial_t^2 b\right|_{W_t^{1,\infty} H_X^N}\right).\end{aligned}$$

Proof. Using the first point of Proposition B.8 and Product estimate Proposition C.2 we have

$$|\underline{V}[\varepsilon\zeta, \beta b](\varepsilon\psi, \beta\lambda\partial_t b) \cdot \nabla\, [\underline{w}[\varepsilon\zeta, \beta b](\varepsilon\psi, \beta\lambda\partial_t b)]\,|_{H^{t_0}} \leqslant M_N\left(|\mathfrak{P}\varepsilon\psi|_{H^{t_0+\frac{1}{2}}} + \beta\lambda\, |\partial_t b|_{L_t^\infty H_X^{t_0}}\right)^2.$$

Furthermore, thanks to the first point of Proposition B.15 and the first point of theorem 3.28 in [20] we obtain

$$\left|\varepsilon d\underline{w}\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right).(-\mathcal{N}_1(U), \partial_t b)\right|_{H^{t_0}} \leqslant M_N\, |(\varepsilon\mathcal{N}_1(U), \beta\partial_t b)|_{H^{t_0+1}}\left(|\mathfrak{P}\varepsilon\psi|_{H^{t_0+\frac{1}{2}}} + \beta\lambda\, |\partial_t b|_{L_t^\infty H_X^{t_0}}\right).$$

Then, the first inequality follows easily from Propositions B.8 and C.2 and Product estimate Proposition C.2. The second inequality can be proved similarly. □

We can now prove theorems 2.3 and 2.4. We recall that $\delta := \max(\varepsilon, \beta^2)$.

Proof. We slice up this proof in three parts. First we regularize and symmetrize the equations, then we find some energy estimates and finally we conclude by convergence. We only give the energy estimates in this paper and a carefully study of the nonlinearities of the water waves equations is done. We refer to the proof of theorem 4.16 in [20] for the regularization, the convergence and the uniqueness (see also part 7 in [17]). For theorem 2.3 (respectively theorem 2.4), we assume that U solves (13) on $[0, T]$ $\left(\text{respectively on } \left[0, \frac{T}{\sqrt{\delta}}\right]\right)$ and that (19) and (29) are satisfied for $\frac{h_{\min}}{2}$ and $\frac{\mathfrak{a}_{\min}}{2}$ on $[0, T]$ $\left(\text{respectively on } \left[0, \frac{T}{\sqrt{\delta}}\right]\right)$ for some $T > 0$.

(a) $|\alpha| = 0$, *The* 0—*energy*

We proceed as in section 4.3.4.3 in [20] and part 6 in [17]. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^0(U) = \frac{1}{2\mu}\left(dG_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right).(\partial_t\zeta, \partial_t b), \Lambda^{\frac{3}{2}}\psi\right)_{L^2} + \frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right)_{L^2}$$
$$-\left(\Lambda^{\frac{3}{2}}(\mathcal{N}_2(U) - \zeta), \frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right)\right)_{L^2} - \left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right), \Lambda^{\frac{3}{2}}P\right)_{L^2}. \quad (36)$$

We have to control all the term in the rhs.

- Control of $\frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right)_{L^2}$.
  Using Proposition B.7, we get

$$\left|\frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right)_{L^2}\right| \leqslant M_N\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\mathcal{E}^N(U)^{\frac{1}{2}}.$$

- Control of $\left(\Lambda^{\frac{3}{2}}(\mathcal{N}_2(U) - \zeta), \frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right)\right)_{L^2}$.
  Using propositions 2.5 and B.8, we get

$$\left|\left(\Lambda^{\frac{3}{2}}(\mathcal{N}_2(U) - \zeta), \frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right)\right)_{L^2}\right| \leqslant \left|\mathcal{N}_2(U) - \zeta\right|_{H^{\frac{3}{2}}}\left|\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right)\right|_{L^2},$$
$$\leqslant \varepsilon M_N\mathcal{E}^N(U)^{\frac{3}{2}} + M_N\left(\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\right)^2\varepsilon\mathcal{E}^N(U).$$

- Control of $\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right), \Lambda^{\frac{3}{2}}P\right)_{L^2}$.
  We get, using remark 3.13 in [20],

$$\left|\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right), \Lambda^{\frac{3}{2}}P\right)_{L^2}\right| \leqslant M_N E^N(U)^{\frac{1}{2}}|\nabla P|_{L_t^\infty H_X^N}.$$

- Control of $\frac{1}{2\mu}\left(dG_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right).(\partial_t\zeta, \partial_t b), \Lambda^{\frac{3}{2}}\psi\right)_{L^2}$.
  Using proposition 3.29 in [20], the second point of theorem 3.15 in [20], Propositions B.7 and C.2 , we get

$$\left|\frac{1}{\mu}\left(dG_\mu[\varepsilon\zeta, \beta b]\left(\Lambda^{\frac{3}{2}}\psi\right).(\partial_t\zeta, \partial_t b), \Lambda^{\frac{3}{2}}\psi\right)_{L^2}\right| \leqslant M_N\left|(\varepsilon\mathcal{N}_1(U), \beta\partial_t b)\right|_{H^{N-2}}|\mathfrak{P}\psi|^2_{H^{\frac{3}{2}}}$$
$$\leqslant M_N\varepsilon\mathcal{E}^N(U)^{\frac{3}{2}} + \max(\beta, \beta\lambda)\left|\partial_t b\right|_{L_t^\infty H_X^N}\mathcal{E}^N(U).$$

Finally, gathering all the previous estimates, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^0(U) \leqslant \varepsilon M_N\mathcal{E}^N(U)^{\frac{3}{2}} + M_N C\left(\rho_{\max}, |\partial_t b|_{L_t^\infty H_X^N}\right)\max(\varepsilon, \beta)\mathcal{E}^N(U)$$
$$+ M_N\sqrt{E^N(U)}\left(|\nabla P|_{L_t^\infty H_X^N} + \frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L_t^\infty H_X^N}\right). \quad (37)$$

(b) $|\alpha| > 0$, *the higher orders energies*

We proceed as in section 4.3.4.3 in [20] and part 6 in [17]. A simple computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}^{\alpha}(U)\right) = &-\varepsilon \mathbf{1}_{\{|\alpha|=N\}}\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \nabla \cdot (\zeta_{(\alpha)}\underline{V})\right)_{L^2} + \left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \frac{\beta\lambda}{\varepsilon} G_{\mu}^{\mathrm{NN}}[0,0](\partial_t \partial^{\alpha} b) + \widetilde{R_{\alpha}}\right)_{L^2} \\ &- \varepsilon \mathbf{1}_{\{|\alpha|=N\}}\left(\frac{1}{\mu} G_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}), [\underline{V} \cdot \nabla \psi_{(\alpha)}]\right)_{L^2} + \left(\frac{1}{\mu} G_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}), S_{\alpha} - \partial^{\alpha} P\right)_{L^2} \\ &+ \frac{1}{2}\left(\partial_t \underline{\mathfrak{a}}\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)_{L^2} + \frac{1}{2}\left(\frac{1}{\mu} dG_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}) \,.\, (\partial_t \zeta, \partial_t b), \psi_{(\alpha)}\right)_{L^2}. \end{aligned} \tag{38}$$

We have to control all the term in the rhs.

- Control of $\left(\partial_t \underline{\mathfrak{a}}\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)_{L^2}$.

  Using the second point of proposition 2.10 we get

$$\begin{aligned}\left|\left(\partial_t \underline{\mathfrak{a}}\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)_{L^2}\right| \leqslant\, & M_N C\left(\rho_{\max}, |\partial_t b|_{W_t^{1,\infty}H_X^N}, |\nabla P|_{L_t^{\infty}H_X^N}, \varepsilon \mathcal{E}^N(U)^{\frac{1}{2}}\right) \varepsilon \mathcal{E}^N(U)^{\frac{3}{2}} \\ &+ C\left(M_N, \beta\lambda |\partial_t b|_{L_t^{\infty}H_X^N}\right)\left(|\nabla P|_{W_t^{1,\infty}H_X^N} + \frac{\beta\lambda}{\varepsilon} |\partial_t^2 b|_{W_t^{1,\infty}H_X^N}\right) \varepsilon \mathcal{E}^N(U).\end{aligned}$$

- Control of $\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \frac{\beta\lambda}{\varepsilon} G_{\mu}^{\mathrm{NN}}[0,0](\partial_t \partial^{\alpha} b)\right)_{L^2}$.

  We get, thanks to propositions 2.10 and B.7,

$$\begin{aligned}\left|\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \frac{\beta\lambda}{\varepsilon} G_{\mu}^{\mathrm{NN}}[0,0](\partial_t \partial^{\alpha} b)\right)_{L^2}\right| \leqslant\, & C\left(\rho_{\max}, \mu_{\max}, |b|_{W_t^{2,\infty}H_X^N}, |\nabla P|_{L_t^{\infty}H_X^N}, \varepsilon \mathcal{E}^N(U)^{\frac{1}{2}}\right) \\ &\times \frac{\beta\lambda}{\varepsilon} |\partial_t b|_{L_t^{\infty}H_X^N} \mathcal{E}^N(U)^{\frac{1}{2}}.\end{aligned}$$

- Controls of $\varepsilon \mathbf{1}_{\{|\alpha|=N\}}\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \nabla \cdot (\zeta_{(\alpha)}\underline{V})\right)_{L^2}$.

  Inspired by section 4.3.4.3 in [20], a simple computation gives

$$\begin{aligned}\left|\varepsilon\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \nabla \cdot (\zeta_{(\alpha)}\underline{V})\right)_{L^2}\right| &= \left|\varepsilon\left(\underline{\mathfrak{a}}\zeta_{(\alpha)} \nabla \cdot \underline{V}, \zeta_{(\alpha)}\right)_{L^2}\right| \\ &\leqslant C\left(\rho_{\max}, \mu_{\max}, |b|_{W_t^{2,\infty}H_X^N}, |\nabla P|_{L_t^{\infty}H_X^N}, \delta \mathcal{E}^N(U)\right) \varepsilon \left[\mathcal{E}^N(U)^{\frac{3}{2}} + \mathcal{E}^N(U)\right].\end{aligned}$$

  using propositions 2.10 and B.8.

- Controls of $\left(\frac{1}{\mu} G_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}), S_{\alpha} - \partial^{\alpha} P\right)_{L^2}$, $\left(\frac{1}{\mu} dG_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}).(\partial_t \zeta, \partial_t b), \psi_{(\alpha)}\right)_{L^2}$ and $\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}, \widetilde{R_{\alpha}}\right)_{L^2}$.

  We can use the same arguments as in the third and the fourth point of part (a) using propositions 2.8 and 2.10.

- Control of $\varepsilon\left(\frac{1}{\mu} G_{\mu}[\varepsilon\zeta, \beta b](\psi_{(\alpha)}), [\underline{V} \cdot \nabla \psi_{(\alpha)}]\right)_{L^2}$.

  We refer to the section 4.3.4.3 and proposition 3.30 in [20] for this control.

  Gathering the previous estimates and using proposition 2.9, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^N(U) \leqslant C\left(\rho_{\max}, \frac{1}{h_{\min}}, \mu_{\max}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}, \varepsilon\mathcal{F}^N(U)^{\frac{1}{2}}\right)$$
$$\times\left(\varepsilon\mathcal{F}^N(U)^{\frac{3}{2}} + \max(\varepsilon,\beta)\mathcal{F}^N(U) + \mathcal{F}^N(U)^{\frac{1}{2}}\left[\frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L_t^\infty H_X^N} + |\nabla P|_{L_t^\infty H_X^N}\right]\right). \tag{39}$$

Then, we easily prove theorem 2.3, using the same arguments as section 4.3.4.4 in [20]. Furthermore, for $\alpha \in \left[0, \frac{1}{2}\right]$, defining $\widetilde{\mathcal{F}^N}(U)(\tau) = \delta^{2\alpha}\mathcal{F}^N(U)\left(\frac{\tau}{\delta^\alpha}\right)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathcal{F}^N}(U) \leqslant C\left(\rho_{\max}, \mu_{\max}, \frac{1}{\mathfrak{a}_{\min}}, \frac{1}{h_{\min}}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}, \widetilde{\mathcal{F}^N}(U)\right).$$

We can also apply the same arguments as section 4.3.4.4 in [20] and theorem 2.4 follows. □

2.5. Hamiltonian system

In this section we prove that the water waves problem (13) is a Hamiltonian system in the Sobolev framework. This extends the classical result of Zakharov ([33]) to the case where the bottom is moving and the atmospheric pressure is not constant (see also [13]). In the case of a moving bottom, Guyenne and Nicholls already pointed out it in [15][1]. We have to introduce the Dirichlet–Dirichlet and the Neumann–Dirichlet operators

$$\begin{cases} G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](\psi) = (\Phi^S)_{|z=-1+\beta b}, \\ G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b](\partial_t b) = (\Phi^B)_{|z=-1+\beta b}, \end{cases} \tag{40}$$

where Φ^S is defined in (15) and Φ^B is defined in (17). We postpone the study of these operators to appendix B.

Remark 2.11. If we denote $\Phi := \Phi^S + \frac{\beta\lambda\mu}{\varepsilon}\Phi^B$, Φ satisfies

$$\begin{cases} \Delta_{X,z}^\mu \Phi = 0 \text{ in } \Omega_t, \\ \Phi_{|z=\varepsilon\zeta} = \psi, \sqrt{1+\beta^2|\nabla b|^2}\,\partial_{\mathbf{n}}\Phi_{|z=-1+\beta b} = \dfrac{\beta\lambda\mu}{\varepsilon}\partial_t b. \end{cases}$$

Then

$$\sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,\partial_{\mathbf{n}}\Phi_{|z=\varepsilon\zeta} = G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\beta\mu\lambda}{\varepsilon}G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b), \tag{41}$$

and

$$\Phi_{|z=-1+\beta b} = G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](\psi) + \frac{\beta\mu\lambda}{\varepsilon}G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b](\partial_t b). \tag{42}$$

Theorem 2.12. *Let* $T > 0$, $t_0 > \frac{d}{2}$, $\zeta, b \in \mathcal{C}^0([0,T]; H^{t_0+1}(\mathbb{R}^d))$, $\psi \in \mathcal{C}^0([0,T]; H^2(\mathbb{R}^d))$, $\partial_t b \in \mathcal{C}^0([0,T]; H^1(\mathbb{R}^d))$, $P \in \mathcal{C}^0([0,T]; L^2(\mathbb{R}^d))$ *such that* (ζ, ψ) *is a solution of* (13)*. Define* $H = H(\zeta,\psi) = \mathcal{T}(\zeta,\psi) + \mathcal{U}(\zeta,\psi)$*, where* $\mathcal{T}(\zeta,\psi) = \mathcal{T}$ *is*

[1] It seems that there is a typo in their hamiltonian; '$-\zeta v$' should read '$+\zeta v$'.

$$\mathcal{T} = \frac{1}{2\mu}\int_{\Omega_t}\left|\nabla^{\mu}_{X,z}\left(\Phi^{S} + \frac{\beta\lambda\mu}{\varepsilon}\Phi^{B}\right)\right|^{2} + \int_{\mathbb{R}^d}\frac{\beta\lambda}{\varepsilon}\partial_t b\left(G^{\mathrm{DD}}_{\mu}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda\mu}{\varepsilon}G^{\mathrm{ND}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b)\right), \quad (43)$$

and $\mathcal{U}(\zeta,\psi) = \mathcal{U}$ *is*

$$\mathcal{U} = \frac{1}{2}\int_{\mathbb{R}^d}\zeta^2 \mathrm{d}X + \int_{\mathbb{R}^d}\zeta P \mathrm{d}X. \quad (44)$$

Then, the water waves equations (13) take the form

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\zeta H\\ \partial_\psi H\end{pmatrix}.$$

Remark 2.13. $\mathcal{T}$ is the sum of the kinetic energy and the moving bottom contribution and $\mathcal{U}$ the sum of the potential energy and the pressure contribution. Using Green's formula and remark 2.11 we obtain that

$$\mathcal{T} = \frac{1}{2}\int_{\mathbb{R}^d}\psi\left(\frac{1}{\mu}G_{\mu}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda}{\varepsilon}G^{\mathrm{NN}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b)\right)\mathrm{d}X$$
$$+\frac{1}{2}\int_{\mathbb{R}^d}\frac{\beta\lambda}{\varepsilon}\partial_t b\left(G^{\mathrm{DD}}_{\mu}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda\mu}{\varepsilon}G^{\mathrm{ND}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b)\right)\mathrm{d}X,$$

Proof. Using the linearity of the Dirichlet–Neumann and the Dirichlet–Dirichlet operators with respect to ψ and the fact that the adjoint of $G^{\mathrm{NN}}_{\mu}[\varepsilon\zeta,\beta b]$ is $G^{\mathrm{DD}}_{\mu}[\varepsilon\zeta,\beta b]$ (see Proposition B.5), we get that

$$\partial_\psi H = \frac{1}{\mu}G_{\mu}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda}{\varepsilon}G^{\mathrm{NN}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b).$$

Applying Proposition B.13 (which provides explicit expressions for shape derivatives) and remark 2.13, we obtain that

$$\begin{aligned}2\ \partial_\zeta H &= -\frac{\varepsilon}{\mu}G_{\mu}[\varepsilon\zeta,\beta b](\psi)\underline{w} + \varepsilon\nabla\psi\cdot\underline{V} - \varepsilon\frac{\beta\lambda}{\varepsilon}G^{\mathrm{NN}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b)\underline{w} + 2P + 2\zeta,\\ &= -\frac{\varepsilon}{\mu}G_{\mu}[\varepsilon\zeta,\beta b](\psi)\underline{w} + \varepsilon\nabla\psi\cdot\nabla\psi - \varepsilon^2\underline{w}\nabla\psi\cdot\nabla\zeta - \varepsilon\frac{\beta\lambda}{\varepsilon}G^{\mathrm{NN}}_{\mu}[\varepsilon\zeta,\beta b](\partial_t b)\underline{w} + 2P + 2\zeta,\\ &= \varepsilon|\nabla\psi|^2 - \frac{\varepsilon}{\mu}\underline{w}^2(1 + \varepsilon^2\mu|\nabla\zeta|^2) + 2P + 2\zeta,\end{aligned}$$

which ends the proof. □

In fact, working in the Beppo Levi framework for ψ requires that $\frac{1}{|D|}\partial_t b \in L^2(\mathbb{R}^d)$ and results that are not dealing with this paper.

3. Asymptotic models

In this part, we derive some asymptotic models in order to model two different types of tsunamis. The most important phenomenon that we want to catch is the Proudman resonance (see for instance [25] or [30] for an explanation of the Proudman resonance) and the submarine landslide tsunami phenomenon (see [21, 27] or [28]). These resonances occur in a linear case. The duration of the resonance depends on the phenomenon. For a meteotsunami, the duration of the resonance corresponds to the time the meteorological disturbance takes to reach the coast (see [25]). However, for a landslide tsunami, the duration of the resonance corresponds to the duration of the landslide (which depends on the size of the slope, see [21] or [27]). If the landslide is offshore, it is unreasonable to assume that the duration of the landslide is the time the water waves take to reach the coast. A variation of the pressure of 1 hPa creates a water wave of 1 cm whereas a moving bottom of 1 cm tends to create a water wave of 1 cm. Therefore we assume in the following that $a_{\mathrm{bott},m} = a$ (and hence $\beta\lambda = \varepsilon$). However, it is important to notice that even if for storms, a variation of the pressure of 100 hPa is very huge, it is quite ordinary that a submarine landslide have a thickness of 1 m. Typically, a storm makes a variation of few Hpa, and the thickness of a submarine landslide is few dm (we refer to [21]). In this part, we only study the propagation of such phenomena. Therefore, we take $d = 1$. In the following, we give three linear asymptotic models of the water waves equations and we give examples of pressures and moving bottoms that create a resonance. The pressure at the surface P and the moving bottom b_m move from the left to the right. We consider that the system is initially at rest. We start this part by giving an asymptotic expansion with respect to μ and $\max(\varepsilon, \beta)$ of $G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b]$.

Proposition 3.1. *Let $t_0 > \frac{d}{2}$, ζ and $b \in H^{t_0+2}(\mathbb{R}^d)$ such that condition (19) is satisfied. We suppose that the parameters ε, β and μ satisfy (18). Then, for all $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$ with $0 \leqslant s \leqslant t_0 + \frac{3}{2}$, we have*

$$\left|G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](B) - G_\mu^{\mathrm{NN}}[0, 0](B)\right|_{H^{s-\frac{1}{2}}} \leqslant M_0 |(\varepsilon\zeta, \beta b)|_{H^{t_0+2}} |B|_{H^{s-\frac{1}{2}}}$$

and

$$\left|G_\mu^{\mathrm{NN}}[0, 0](B) - B\right|_{H^{s-\frac{1}{2}}} \leqslant C\mu\, |B|_{H^{s+\frac{3}{2}}}.$$

Proof. The first inequality follows from Proposition B.15 and the second from Remark B.1. □

Remark 3.2. In the same way and under the assumptions of the previous proposition, we can prove that (see proposition 3.28 in [20]), for $0 \leqslant s \leqslant t_0 + \frac{3}{2}$,

$$\left|G_\mu[\varepsilon\zeta, \beta b](\psi) - G_\mu[0, 0](\psi)\right|_{H^{s-\frac{1}{2}}} \leqslant \mu M_0 |(\varepsilon\zeta, \beta b)|_{H^{t_0+2}} |\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}}$$

and

$$\left|\frac{1}{\mu} G_\mu[0, 0](\psi) + \Delta\psi\right|_{H^{s-\frac{1}{2}}} \leqslant \mu C\, |\nabla\psi|_{H^{s+\frac{5}{2}}}.$$

We denote by $\overline{V}$ the vertically averaged horizontal component,

$$\overline{V} = \overline{V}[\varepsilon\zeta, \beta b](\psi, \partial_t b) = \frac{1}{1 + \varepsilon\zeta - \beta b} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla_X(\Phi[\varepsilon\zeta, \beta b](\psi, \partial_t b)(\cdot, z)) \mathrm{d}z, \tag{45}$$

where $\Phi = \Phi[\varepsilon\zeta, \beta b](\psi, \partial_t b)$ satisfies

$$\begin{cases} \Delta_{X,z}^{\mu}\Phi = 0, -1 + \beta b \leqslant z \leqslant \varepsilon\zeta, \\ \Phi_{|z=\varepsilon\zeta} = \psi, \sqrt{1 + \beta^2 |\nabla b|^2}\, \partial_{\mathbf{n}}\Phi_{|z=-1+\beta b} = \mu\partial_t b. \end{cases}$$

The following proposition is remark 3.36 and a small adaptation of proposition 3.37 and lemma 5.4 in [20] (see also section A.5.5 in [20]).

Proposition 3.3. *$T > 0$, $t_0 > \frac{d}{2}$, $0 \leqslant s \leqslant t_0$ and $\zeta, b \in W^{1,\infty}([0,T]; H^{t_0+2}(\mathbb{R}^d))$ such that condition (19) is satisfied on $[0, T]$. We suppose that the parameters ε, β and μ satisfy (18). We also assume that $\psi \in W^{1,\infty}([0,T]; \dot{H}^{s+3}(\mathbb{R}^d))$. Then,*

$$G_\mu[\varepsilon\zeta, \beta b](\psi) + \mu G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\partial_t b) = -\mu\nabla\cdot((1+\varepsilon\zeta - \beta b)\overline{V}) + \mu\partial_t b,$$

and

$$\begin{cases} \left|\overline{V} - \nabla\psi\right|_{H^s} \leqslant \mu C\left(\dfrac{1}{h_{\min}}, \mu_{\max}, \varepsilon|\zeta|_{H^{t_0+2}}, \beta|b|_{L_t^\infty H_X^{t_0+2}}\right)\max(|\nabla\psi|_{H^{s+2}}, |\partial_t b|_{L_t^\infty H_X^{s+1}}), \\ \left|\partial_t\overline{V} - \nabla\partial_t\psi\right|_{H^s} \leqslant \mu C\left(\dfrac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+2}}, |\partial_t\zeta|_{H^{t_0+2}}, |b|_{W_t^{2,\infty}H_X^{t_0+2}}, |\nabla\psi|_{H^{s+2}}, |\partial_t\nabla\psi|_{H^{s+2}}\right). \end{cases}$$

In this part, we will consider symmetrizable linear hyperbolic systems of the first order. We refer to [8] for more details about the wellposedness. In the following, we will only give the energy associated to the symmetrization.

3.1. A shallow water model when β is small

3.1.1. Linear asymptotic. We consider the case that ε, β, μ are small. Physically, this means that we consider small amplitudes for the surface and the bottom (compared to the mean depth) and waves with large wavelengths (compared to the mean depth). The asymptotic regime (in the sense of Definition A.19 in [20]) is

$$\mathcal{A}_{\mathrm{LW}} = \{(\varepsilon, \beta, \lambda, \mu), 0 < \mu, \varepsilon, \beta \leqslant \delta_0, \beta\lambda = \varepsilon\}, \tag{46}$$

with $\delta_0 \ll 1$.

Proposition 3.4. *Let $t_0 > \frac{d}{2}$, $N \geqslant \max(1, t_0) + 3$, $U^0 \in E_0^N$, $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ and $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$. We suppose (19) and (29) are satisfied initially. Then, there exists $T > 0$, such that for all $(\varepsilon, \beta, \lambda, \mu) \in \mathcal{A}_{\mathrm{LW}}$, there exists a solution $U = (\zeta, \psi) \in E^N_{\frac{T}{\sqrt{\delta_0}}}$ to the water waves equations with initial data U^0 and this solution is unique. Furthermore, for all $\alpha \in [0, \frac{1}{3})$,*

$$\left|\zeta - \widetilde{\zeta}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-4}(\mathbb{R}^d)\right)} + \left|\nabla\psi - \nabla\widetilde{\psi}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} \leqslant T\delta_0^{1-3\alpha}\widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N(U^0), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right),$$

and with, $(\widetilde{\zeta}, \widetilde{\psi})$ solution of the waves equation

$$\begin{cases} \partial_t\widetilde{\zeta} + \Delta_X\widetilde{\psi} = \partial_t b, \\ \partial_t\widetilde{\psi} + \widetilde{\zeta} = -P, \end{cases} \tag{47}$$

with initial data U^0.

Proof. First, the system (47) is wellposed since it can be symmetrized thanks to the energy

$$\mathcal{E}(t) = \left|\widetilde{\zeta}\right|_{L^2}^2 + \left|\nabla\widetilde{\psi}\right|_{L^2}^2.$$

Using theorem 2.4 we get a uniform time of existence $\frac{T}{\sqrt{\delta_0}} > 0$ for the water waves equation and for all parameters in $\mathcal{A}_{\mathrm{LW}}$. Then, using proposition 3.1, remark 3.2, Propositions B.8 and C.1 and standard controls we get that

$$\begin{cases} \partial_t\zeta + \Delta_X\psi = \partial_t b + R_1, \\ \partial_t\psi + \zeta = -P + R_2, \end{cases} \tag{48}$$

with

$$\begin{cases} |R_1|_{H^{N-4}} \leqslant C(\varepsilon|\zeta|_{H^N}, |b|_{L_t^\infty H_X^N})(|(\varepsilon\zeta, \beta b)|_{H^N} + \mu) \max\left(|\mathfrak{P}\psi|_{H^{N-\frac{1}{2}}}, |\partial_t b|_{H^N}\right), \\ |R_2|_{H^{N-1}} \leqslant \varepsilon C(\varepsilon|\zeta|_{H^N}, |b|_{L_t^\infty H_X^N}) \max\left(|\mathfrak{P}\psi|_{H^{N-\frac{1}{2}}}^2, |\partial_t b|_{H^N}^2\right). \end{cases}$$

If we denote $\zeta_1 = \zeta - \widetilde{\zeta}$ and $\psi_1 = \psi - \widetilde{\psi}$, we see that (ζ_1, ψ_1) satisfies

$$\begin{cases} \partial_t\zeta_1 + \Delta_X\psi_1 = R_1, \\ \partial_t\psi_1 + \zeta_1 = R_2. \end{cases}$$

Differentiating the energy

$$\mathcal{E}^N(t) = \frac{1}{2}|\zeta_1|_{H^{N-4}}^2 + \frac{1}{2}|\nabla\psi_1|_{H^{N-2}}^2,$$

we get the estimate thanks to proposition 2.5 and energy estimate in theorem 2.4. □

This model is well-known in the physics literature (see [26]).

3.1.2. Resonance in shallow waters when β *is small.* We consider the equation (47) for $d = 1$. We transform it in order to have a unique equation for $h := \widetilde{\zeta} - b$,

$$\begin{cases} \partial_t^2 h - \partial_X^2 h = \partial_X^2(P + b), \\ h_{|t=0} = -b(0, .), \\ \partial_t h_{|t=0} = 0. \end{cases} \tag{49}$$

We denote $f(t, X) := (P + b)(t, X)$, which represents a disturbance. We want to understand the resonance for landslide and meteo tsunamis. In both cases, it is a linear respond, in the shallow water case, of a body of water due to a moving pressure or a moving bottom, when the speed of the storm or the landslide is close to the typical wave celerity (here 1). We can compute h thanks to the d'Alembert's formula

$$h(t,X) = \underbrace{-\frac{1}{2}(b(0,X-t) + b(0,X+t))}_{h_T(t,X)} + \underbrace{\frac{1}{2}\int_0^t \partial_X f(\tau, X+t-\tau)\mathrm{d}\tau}_{:=h_L(t,X)}$$
$$\underbrace{-\frac{1}{2}\int_0^t \partial_X f(\tau, X-t+\tau)\mathrm{d}\tau}_{:=h_R(t,X)}.$$

We are interesting in disturbances f moving from the left to the right (propagation to a coast). Therefore, we study only h_R. The following proposition shows that a disturbance moving with a speed equal to 1 makes appear a resonance.

Proposition 3.5. *Let $f \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^d))$ and $\partial_X f \in L^\infty_{t\times X}(\mathbb{R}\times\mathbb{R}^d)$. Then, for all $X \in \mathbb{R}$, $t > 0$,*

$$|h_R(t,X)| \leqslant \frac{t}{2}\,|\partial_X f|_\infty .$$

Furthermore, if $f(t, X) = f_0(X - t)$, $f_0 \in H^1(\mathbb{R}^d)$ and $\left|f_0'(X_0 - t_0)\right| = |f'|_\infty$ the equality holds for (t_0, X_0). If $f(t, X) = f_0(X - Ut)$ with $f_0 \in H^1(\mathbb{R}^d)$ and $U \neq 1$,

$$|h_R|_\infty \leqslant \min\left(\frac{|f_0|_\infty}{|1-U|}, \frac{t}{2}\left|f_0'\right|_\infty\right).$$

Proof. If $f(t, X) = f_0(X - Ut)$,

$$h_R(t,X) = -\frac{1}{2}\int_0^t f_0'(X - t + (1-U)\tau)\mathrm{d}\tau,$$

and the result follows. □

This proposition corresponds to the historical work of Proudman ([26]). We rediscover the fact that the resonance occurs if the speed of the disturbance is 1. For a disturbance with a speed different from 1, we notice a saturation effect (also pointed out in [27]). The graph in figure 1, gives the typical evolution of $|h(t,\cdot)|_\infty$ with respect to the time t for different values of the speed. We can see the saturation effect. We compute h with a finite difference method and we take $f(t,X) = \mathrm{e}^{-\frac{1}{2}(X-Ut)^2}$. We see also that the landslide resonance and the Proudman resonance have the same effects. There are however two important differences that we exposed in the introduction of this part. The first one is the duration of the resonance. A landslide is quicker than a meteorological effect. The second one, is the fact that the typical size of the landslide (few dm) is bigger than the size of a storm (few hPa). For instance, for a moving storm which creates a variation of the pressure of 3 hPa during 15 t_0, the final wave can reach a amplitude of 13 cm (it is for example the case of the meteotsunami in Nagasaki in 1979, see [25]). Conversely, an offshore landslide with a thickness of 1 m that lasts t_0, can create a wave of 50 cm (which corresponds to the results in [27]). Therefore, we see that the principal difference between an offshore landslide and a moving storm is the size.

3.2. A shallow water model when β is large

3.2.1. Linear asymptotic. In this case, we suppose only that ε and μ are small. We recall that $\beta b(t,X) = \beta b_0(X) + \beta\lambda b_m(t,X)$. Then, we assume also that $1 - b_0 \geqslant h_{\min} > 0$. In the following, we denote $h_0 := 1 - \beta b_0$. The asymptotic regime is

$$\mathcal{A}_{\mathrm{LVW}} = \{(\varepsilon, \beta, \lambda, \mu), 0 < \varepsilon, \mu \leqslant \delta_0, 0 < \beta \leqslant 1, \beta\lambda = \varepsilon\}, \tag{50}$$

with $\delta_0 \ll 1$. We can now give a asymptotic model.

Proposition 3.6. *Let $t_0 > \frac{d}{2}$, $N \geqslant \max(1, t_0) + 4$, $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$, $U^0 = (\zeta_0, \psi_0) \in E_0^N$, and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. We suppose that (19) and (29) are satisfied initially. We suppose*

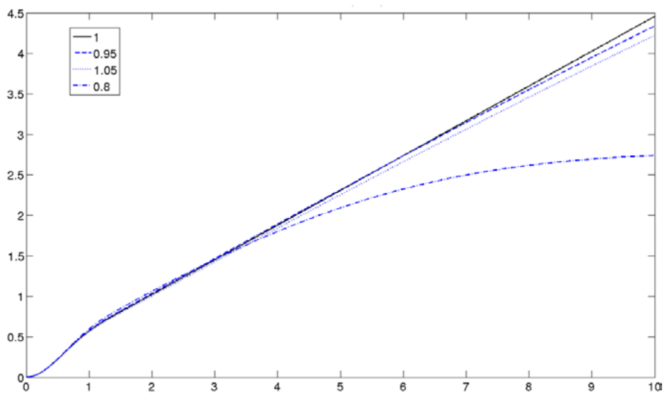

Figure 1. Evolution of the maximum of h, solution of equation (49), with different values of the speed U.

also that $b_0 \in H^N(\mathbb{R}^d)$ *and that* $h_0 = 1 - \beta b_0 \geqslant h_{\min}$. *Then, there exists* $T > 0$, *such that for all* $(\varepsilon, \beta, \lambda, \mu) \in \mathcal{A}_{\mathrm{LVW}}$, *there exists a unique solution* $U = (\zeta, \psi) \in E_T^N$ *to the water waves equations with initial data* U^0. *Furthermore, for* $\overline{V}$ *as in* (45),

$$|\zeta - \zeta_1|_{L^\infty([0,T];H^{N-4}(\mathbb{R}^d))} + \left|\overline{V} - \overline{V}_1\right|_{L^\infty([0,T];H^{N-4}(\mathbb{R}^d))} \leqslant T\delta_0 \widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N(U^0), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right),$$

and $(\zeta_1, \overline{V}_1)$ *solution of the waves equation*

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot (h_0 \overline{V}_1) = \partial_t b_m, \\ \partial_t \overline{V}_1 + \nabla \zeta_1 = -\nabla P, \\ (\zeta_1)_{|t=0} = \zeta_0, (V_1)_{|t=0} = \overline{V}\,[\varepsilon\zeta_0, \beta b_{|t=0}]\,(\psi_0, (\partial_t b)_{|t=0}). \end{cases} \tag{51}$$

Proof. The system (51) is wellposed since it can be symmetrized thanks to the energy

$$\mathcal{E}(t) = \frac{1}{2}|\zeta_1|_{L^2}^2 + \frac{1}{2}(h_0 \overline{V}_1, \overline{V}_1)_{L^2}.$$

For the inequality, we proceed as in proposition 3.4, differentiating the energy

$$\mathcal{E}^N(t) = \frac{1}{2}|\zeta_2|_{H^{N-4}}^2 + \frac{1}{2}(h_0 \Lambda^{N-4}\overline{V}_2, \Lambda^{N-4}\overline{V}_2)_{L^2},$$

with $\zeta_2 = \zeta - \zeta_1$ and $\overline{V}_2 = \overline{V} - \overline{V}_1$. Using Gronwall's lemma, proposition 3.3 and standard controls, we get result. $\square$

This model is well-known in the physics literature to investigate the landslide tsunami phenomenon (see [27]).

3.2.2. Amplification in shallow waters when β is large. In this part, $d = 1$ and we suppose that $P = 0$. The same study can be done for a non constant pressure. For the sake of simplicity, we assume also that initially the velocity of the landslide is zero and hence that $(\partial_t b_m)_{|t=0} = 0$ (the bottom does not move at the beginning). We transform the system (51) in order to get an equation for ζ_1 only. We obtain that ζ_1 satisfies

$$\partial_t^2 \zeta_1 - \partial_X (h_0 \partial_X \zeta_1) = \partial_t^2 b_m, \tag{52}$$

with $(\zeta_1)_{|t=0} = 0$ and $(\partial_t \zeta_1)_{|t=0} = 0$. We wonder now if we can catch an elevation of the sea level with this asymptotic model. Therefore, we are looking for solutions of the form

$$\zeta_2(t, X) = t\zeta_3(t, X). \tag{53}$$

The following proposition gives example of such solutions for bounded moving bottoms (with finite energy).

Proposition 3.7. *Suppose that $h_0 \geqslant h_{\min} > 0$ with $h_0 \in H^1(\mathbb{R})$. Let $(\zeta_3, \overline{V}_3)$ be a solution of*

$$\begin{cases} \partial_t \zeta_3 + \partial_X (h_0 \overline{V}_3) = 0, \\ \partial_t \overline{V}_3 + \partial_X \zeta_3 = 0, \end{cases}$$

with $(\zeta_3, \overline{V}_3)_{|t=0} = (0, f')$ with $f \in H^1(\mathbb{R})$. Then, $\zeta_1(t, X) = t\zeta_3(t, X)$ is a non trivial solution of (52) with

$$b_m(t, X) = 2 \int_0^t \zeta_3(s, X) \mathrm{d}s, \tag{54}$$

and $b_m(t, \cdot)$ is bounded in $L^2(\mathbb{R}^d)$ and in $L^\infty(\mathbb{R}^d)$ uniformly with respect to t

$$|b_m(t, \cdot)|_{L^2} + |b_m(t, \cdot)|_{L^\infty} \leqslant C,$$

where C is independent on t.

Proof. Plugging the expression of ζ and b_m in (52), we get the first result. We have to show that $\zeta_3 \in L^1(\mathbb{R}^+; L^2(\mathbb{R}^d))$. Consider the linear hyperbolic equation

$$\begin{cases} \partial_t \eta + \partial_X (h_0 W) = 0, \\ \partial_t W + \partial_X \eta = 0, \end{cases}$$

with $(\eta, W)_{|t=0} = (-f, 0)$. This system has a unique solution $(\eta, W) \in \mathcal{C}^0(\mathbb{R}; H^1(\mathbb{R}))$. Furthermore, $(\partial_t \eta, \partial_t W) \in \mathcal{C}^0(\mathbb{R}; L^2(\mathbb{R}))$, and $(\partial_t \eta, \partial_t W)$ satisfies the same linear hyperbolic system as $(\zeta_3, \overline{V}_3)$. By uniqueness, $\zeta_3 = \partial_t \eta$ and

$$b_m(t, X) = 2\eta(t, X) + 2f(X).$$

Since, for all t,

$$\int_{\mathbb{R}} \eta(t, X)^2 + h_0(X) W(t, X)^2 \mathrm{d}X = \int_{\mathbb{R}} f(X)^2 \mathrm{d}X,$$

and $h_0 \geqslant h_{\min} > 0$, we get the control of $|b_m(t, \cdot)|_{L^2}$. Finally, η satisfies the waves equation

$$\partial_t^2 \eta - \partial_X (h_0 \partial_X \eta) = 0,$$

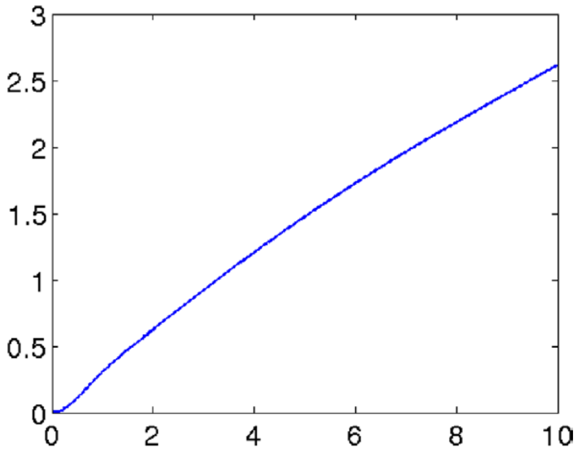

Figure 2. Evolution of the maximum of ζ_1, solution of (52), for a non flat bottom b_0.

with $(\eta, \partial_t \eta)_{|t=0} = (-f, 0) \in H^1(\mathbb{R}^d)$. Then, for all t,

$$\int_{\mathbb{R}} |\partial_t \eta(t, X)|^2 + h_0(X) \, |\partial_X \eta(t, X)|^2 \mathrm{d}X = \int_{\mathbb{R}} h_0(X) f'(X)^2 \mathrm{d}X.$$

Therefore, $|\eta|_{H^1}$ (and $|\eta|_{L^\infty}$ by Sobolev embedding) is controlled uniformly with respect to t. □

In the following, we compute numerically some solutions of equations (52) of the form (53) with a finite difference method. We take $b_0(X) = -\tanh(X)$, $\beta = \frac{1}{2}$ and $(\partial_t \zeta_3)_{|t=0} = (4X^2 - 2)\mathrm{e}^{-X^2}$. The figure 2 is the evolution of the maximum of ζ_1. The figure 3 is the graph at different times of the waves and the landslide. The dashed curves are the landslide, the solid curves are the waves and the dotted curve is the slope. Therefore, we see that an important elevation of the sea level is possible even if we do not consider that the seabed is flat.

Remark 3.8. In order to simplify, we consider that the system is initially at rest. But our study can easily be extended to waves with non trivial initial data. In particular, we can study a wave amplified by a landslide. This is what happened during the tsunami in Fukushima in 2011 (see [1]). We compute numerically this amplification. We consider a wave moving with a speed equal to 1 (typical speed in the sea after nondimensionalization) that is amplified by a landslide. Figure 4 represents the evolution of the maximum of this wave. We can see an amplification.

3.3. Linear asymptotic and resonance in intermediate depths

In this case, we consider only that ε, β are small. Physically, this means that we consider small amplitudes for the surface and the bottom (compared to the mean depth) and that the depth is not small compared to wavelength of the waves. In this part, we generalize the Proudman resonance in deeper waters. The asymptotic regime is

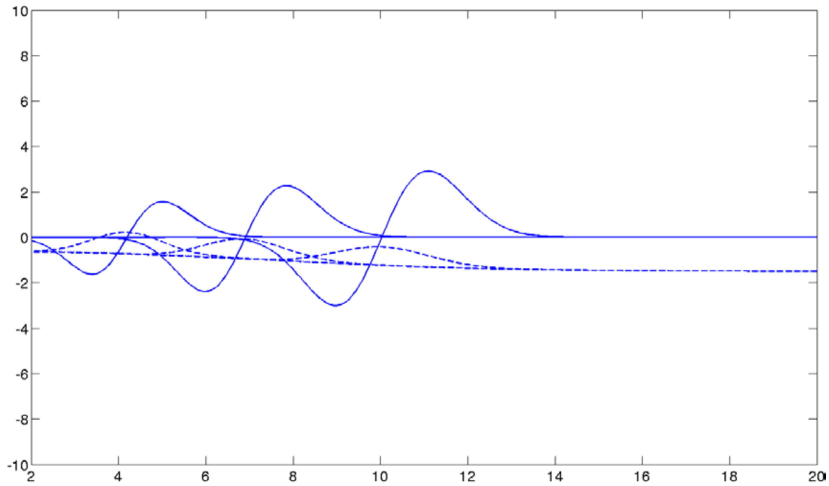

Figure 3. Evolution of the surface ζ_1 (solid line), solution of (52), and the landslide b_m (dashed line).

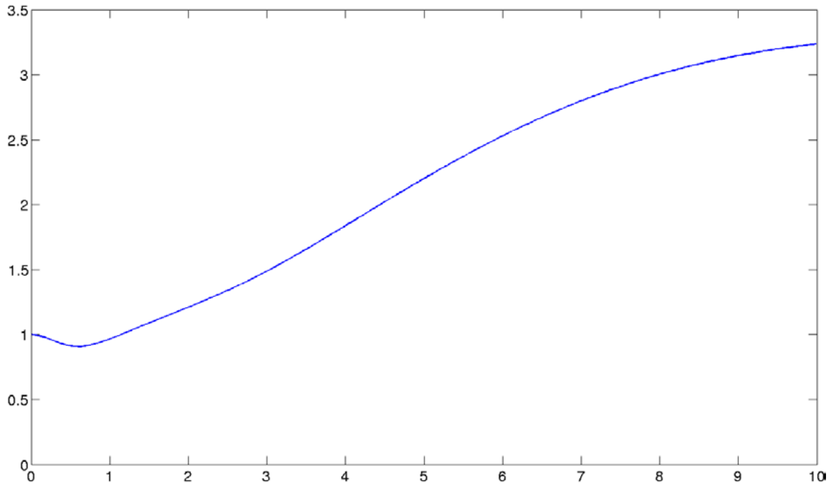

Figure 4. Evolution of the maximum of h, solution of (52), with non trivial initial data and with b_m like in figure 3.

$$\mathcal{A}_{\mathrm{LWW}} = \left\{(\varepsilon, \beta, \lambda, \mu), 0 < \varepsilon, \beta \leqslant \delta_0, \beta\lambda = \varepsilon \text{ and } 0 < \mu \leqslant \mu_{\max}\right\}, \tag{55}$$

with $\delta_0 \ll 1$ and $0 < \mu_{\max}$. Using the energy

$$\mathcal{E}(t) = \frac{1}{2}|\zeta|_{L^2}^2 + \frac{1}{2}\left(\frac{1}{\mu}G_\mu[0,0](\psi), \psi\right)_{L^2},$$

and proceeding as in proposition 3.4 (we need also proposition 3.12 in [20]), we get a new asymptotic model.

Proposition 3.9. *Let* $t_0 > \frac{d}{2}$, $N \geqslant \max(1, t_0) + 3$, $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$, $U^0 = (\zeta_0, \psi_0) \in E_0^N$ *and* $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. *We suppose that (19) and (29) are satisfied initially. Then,* *there exists* $T > 0$*, such that for all* $(\varepsilon, \beta, \lambda, \mu) \in \mathcal{A}_{\mathrm{LWW}}$*, there exists a unique solution* $U = (\zeta, \psi) \in E^{N}_{\frac{T}{\sqrt{\delta_0}}}$ *to the water waves equations with initial data* U^0*. Furthermore, for all* $\alpha \in \left[0, \frac{1}{3}\right)$*,*

$$\left|\zeta - \widetilde{\zeta}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} + \left|\frac{|D|}{\sqrt{1+|D|}}(\psi - \widetilde{\psi})\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} \leqslant T\delta_0^{1-3\alpha}\widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N(U^0), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right),$$

where $(\widetilde{\zeta}, \widetilde{\psi})$ *is a solution of the waves equation*

$$\begin{cases} \partial_t\widetilde{\zeta} - \dfrac{1}{\mu}G_\mu[0,0](\widetilde{\psi}) = G_\mu^{\mathrm{NN}}[0,0](\partial_t b), \\ \partial_t\widetilde{\psi} + \widetilde{\zeta} = -P, \end{cases} \tag{56}$$

with initial data U^0*.*

The Proudman resonance is a phenomenon which occurs in shallow water regime. We wonder if there is also a resonance in deeper waters. In this part, we only work with a non constant pressure and hence $\partial_t b = 0$. The same study can be done for a moving bottom. We consider the equation (56) for $d = 1$. Since, the initial data does not affect the possible resonance, we suppose in the following that $U^0 = 0$. We transform the system (56) in order to have a unique equation for $\widetilde{\zeta}$ (in the following we denote $\widetilde{\zeta}$ by ζ to simplify the notation)

$$\begin{cases} \partial_t^2\zeta + \dfrac{1}{\mu}G_\mu[0,0](\zeta) = -\dfrac{1}{\mu}G_\mu[0,0](P), \\ \zeta_{|t=0} = 0, \partial_t\zeta_{|t=0} = 0. \end{cases}$$

We can solve explicitly the previous equation, we get that

$$\widehat{\zeta}(t,\xi) = \underbrace{\frac{\mathrm{i}}{2}\int_0^t \xi\sqrt{\frac{\tanh\left(\sqrt{\mu}\,|\xi|\right)}{\sqrt{\mu}\,|\xi|}}\,\widehat{P}(\tau,\xi)\mathrm{e}^{\mathrm{i}(t-\tau)\xi\sqrt{\frac{\tanh\left(\sqrt{\mu}\,|\xi|\right)}{\sqrt{\mu}\,|\xi|}}}\,\mathrm{d}\tau}_{:=\widehat{\zeta}_L(t,\xi)}$$
$$\underbrace{-\frac{\mathrm{i}}{2}\int_0^t \xi\sqrt{\frac{\tanh\left(\sqrt{\mu}\,|\xi|\right)}{\sqrt{\mu}\,|\xi|}}\,\widehat{P}(\tau,\xi)\mathrm{e}^{\mathrm{i}(\tau-t)\xi\sqrt{\frac{\tanh\left(\sqrt{\mu}\,|\xi|\right)}{\sqrt{\mu}\,|\xi|}}}\,\mathrm{d}\tau}_{:=\widehat{\zeta}_R(t,\xi)}.$$

In order to find a resonant pressure, we suppose that P has the form $\mathrm{e}^{-\mathrm{i}ta(D)}P_0$, where a is a real smooth odd function which is sublinear, there exists $C > 0$ such that $|a(\xi)| \leqslant C|\xi|$. We also suppose that the phase velocity of the disturbance is positive, $\frac{a(\xi)}{\xi} \geqslant 0$. P_0 is a smooth function in a Sobolev space with $\widehat{P_0}(0) \neq 0$. We denote $\omega(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$. A simple computation gives that

$$|\zeta_L(t,\cdot)| \leqslant |\widehat{\zeta_L}(t,\cdot)|_{L^1} \leqslant \left|\widehat{P_0}\right|_{L^1}.$$

Furthermore, we have

$$\begin{aligned} |\widehat{\zeta_R}(t,\xi)| &= \frac{1}{2}\left|\int_0^t \xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)\mathrm{e}^{\mathrm{i}\tau(\xi\omega(\sqrt{\mu}\,\xi)-a(\xi))}\mathrm{d}\tau\right| \\ &\leqslant \frac{t}{2}\left|\xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)\right|, \end{aligned}$$

with an equality if and only if $a(\xi) = \xi\omega(\sqrt{\mu}\,\xi)$. Hence, it is natural to consider that

$$\widehat{P}(t,\xi) = \mathrm{e}^{-\mathrm{i}t\xi\omega(\sqrt{\mu}\,\xi)}P_0(\xi). \tag{57}$$

A simple computation gives

$$\zeta_R(t,X) = -\frac{\mathrm{i}t}{2}\int_{\mathbb{R}} \xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)\mathrm{e}^{-\mathrm{i}t\xi\omega(\sqrt{\mu}\,\xi)}\mathrm{e}^{\mathrm{i}X\xi}\mathrm{d}\xi. \tag{58}$$

We wonder now if a resonance occurs. We need a dispersion estimate for the linear water waves equation.

Proposition 3.10. *Let $f \in W^{1,1}(\mathbb{R})$ such that $\widehat{f}\,(0) = 0$. Then,*

$$\left|\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\xi\omega(\sqrt{\mu}\,\xi)}\mathrm{e}^{\mathrm{i}X\xi}\widehat{f}\,(\xi)\mathrm{d}\xi\right| \leqslant \frac{C}{\sqrt{t}}\left(\frac{1}{\sqrt{\mu}}\left|\frac{1}{\sqrt{|\xi|}}(\widehat{f}\,)'\right|_{L^1(\mathbb{R})} + \mu^{\frac{1}{8}}\left||\xi|^{\frac{3}{4}}(\widehat{f}\,)'\right|_{L^1(\mathbb{R})}\right).$$

Proof. We denote $I(t)$,

$$\begin{aligned} I(t) &:= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\xi\omega(\sqrt{\mu}\,\xi)}\mathrm{e}^{\mathrm{i}X\xi}\widehat{f}\,(\xi)\mathrm{d}\xi \\ &= \frac{1}{\sqrt{\mu}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\left(y\omega(y)-\frac{X}{t}y\right)}\widehat{f}\left(\frac{y}{\sqrt{\mu}}\right)\mathrm{d}y. \end{aligned}$$

We denote ϕ,

$$\phi(y) = y\omega(y) - \frac{X}{t}y,$$

and y_0 the unique minimum of ϕ''. Figure 5 represents ϕ'' on $[0,+\infty[$.

To estimate $I(t)$ we decompose $I(t)$ into four parts.

$$\begin{aligned} I_1(t) &= \frac{1}{\sqrt{\mu}}\int_0^{y_0} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(y)}\widehat{f}\left(\frac{y}{\sqrt{\mu}}\right)\mathrm{d}y \\ &= \frac{1}{\sqrt{\mu}}\int_0^{y_0} -\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_y^{y_0}\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(z)}\mathrm{d}z\right)\widehat{f}\left(\frac{y}{\sqrt{\mu}}\right)\mathrm{d}y \\ &= \frac{1}{\mu}\int_0^{y_0}\int_y^{y_0} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(z)}\mathrm{d}z(\widehat{f}\,)'\left(\frac{y}{\sqrt{\mu}}\right)\mathrm{d}y. \end{aligned}$$

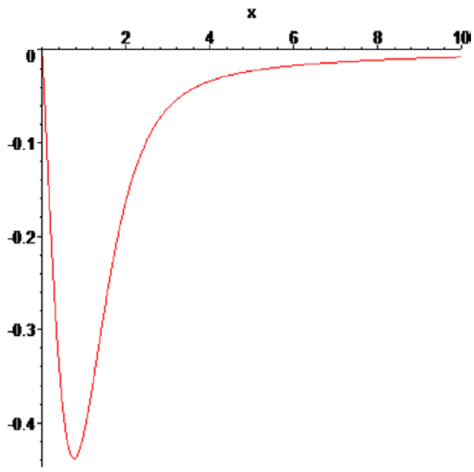

Figure 5. Profile of ϕ'' .

Then, using Van der Corput's lemma (see [29]) and the fact that for $z \in [y, y_0]$,

$|\phi''(z)| \geqslant |\phi''(y)|$ and $|\phi''(z)| \geqslant Cz$,

$$
\begin{aligned}
|I_1(t)| &\leqslant \frac{C}{\mu^{\frac{3}{4}}\sqrt{t}} \int_0^{y_0} \left| \frac{1}{\sqrt{y}} (\widehat{f}\,)'\left(\frac{y}{\sqrt{\mu}}\right) \right| \mathrm{d}y \\
&\leqslant \frac{C}{\sqrt{\mu}\,\sqrt{t}} \int_0^{+\infty} \left| \frac{1}{\sqrt{\xi}} (\widehat{f}\,)'(\xi) \right| \mathrm{d}\xi.
\end{aligned}
$$

Furthermore, for $M > y_0$ large enough,

$$
\begin{aligned}
I_2(t) &= \frac{1}{\sqrt{\mu}} \int_{y_0}^{M} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(y)} \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) \mathrm{d}y \\
&= \frac{1}{\sqrt{\mu}} \int_{y_0}^{M} \frac{\mathrm{d}}{\mathrm{d}y}\left(\int_{y_0}^{y} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(z)} \mathrm{d}z \right) \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) \mathrm{d}y \\
&= \int_{y_0}^{M} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(z)} \frac{\mathrm{d}z}{\sqrt{\mu}} \widehat{f}\left(\frac{M}{\sqrt{\mu}}\right) - \frac{1}{\mu} \int_{y_0}^{M} \int_{y_0}^{y} \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\phi(z)} \mathrm{d}z (\widehat{f}\,)'\left(\frac{y}{\sqrt{\mu}}\right) \mathrm{d}y.
\end{aligned}
$$

Then, using Van der Corput's lemma and the fact that for $z \in [y_0, y]$,

$|\phi''(z)| \geqslant |\phi''(y)|$ and $|\phi''(z)| \geqslant Cz^{-\frac{3}{2}}$,

$$
\begin{aligned}
|I_2(t)| &\leqslant \left| \frac{M}{\sqrt{\mu}} \widehat{f}\left(\frac{M}{\sqrt{\mu}}\right) \right| + \frac{C}{\mu^{\frac{3}{4}}\sqrt{t}} \int_{y_0}^{M} \left| y^{\frac{3}{4}} (\widehat{f}\,)'\left(\frac{y}{\sqrt{\mu}}\right) \right| \mathrm{d}y \\
&\leqslant \left| \widehat{f'}\left(\frac{M}{\sqrt{\mu}}\right) \right| + \frac{C\mu^{\frac{1}{8}}}{\sqrt{t}} \int_0^{+\infty} \left| \xi^{\frac{3}{4}} (\widehat{f}\,)'(\xi) \right|.
\end{aligned}
$$

Tending M to $+\infty$ we get the result. The control for $\xi < 0$ is similar. □

Therefore, in the linear case, we have also a resonance.

Corollary 3.11. *Let $P_0 \in H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ such that $XP_0 \in H^3(\mathbb{R})$ and let $0 < \mu \leqslant \mu_{\max}$. Consider,*

$$
\zeta_R(t, X) = -\frac{\mathrm{i}t}{2} \int_{\mathbb{R}} \xi \omega(\sqrt{\mu}\,\xi) \widehat{P_0}(\xi) \mathrm{e}^{-\mathrm{i}t\xi\omega(\sqrt{\mu}\,\xi)} \mathrm{e}^{\mathrm{i}X\xi} \mathrm{d}\xi.
$$

Then,

$$
\left|\zeta_R(t, \cdot)\right|_\infty \leqslant C(\mu_{\max}) \sqrt{\frac{t}{\mu}} (|P_0|_{H^3} + |P_0|_{L^1} + |XP_0|_{H^3}),
$$

and

$$
\varliminf_{t \to +\infty} \left| \frac{1}{\sqrt{t}} \zeta_R(t, \cdot) \right|_\infty \geqslant C(P_0) > 0.
$$

Proof. We take $\widehat{f}\,(\xi) = \xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)$. Then,

$$
|(\widehat{f}\,)'(\xi)| \leqslant (1 + \sqrt{\mu}\,|\xi|)|\widehat{P_0}(\xi)| + |\xi||(\widehat{P_0})'(\xi)|,
$$

and the first inequality follows from the previous proposition. For the second inequality, we use a stationary phase approximation. We denote $\phi(\xi) = \xi\omega(\xi)$. Let $\xi_0 > 0$, such that $\left|\xi_0 \widehat{P_0}(\xi_0)\right| = \left|\xi \widehat{P_0}\right|_{L^\infty}$, and $X_\mu < 0$, such that $\phi'(\sqrt{\mu}\,\xi_0) = X_\mu$. Then, we have,

$$
\begin{aligned}
\lim_{t \to +\infty} \left| \frac{1}{\sqrt{t}} \zeta_R(t, tX_\mu) \right| &= \lim_{t \to +\infty} \frac{\sqrt{t}}{2\mu} \left| \int_{\mathbb{R}} \xi\omega(\xi) \widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right) \mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi(\omega(\xi) - X_\mu)} \mathrm{d}\xi \right| \\
&= \frac{\sqrt{2\pi}}{2\mu^{\frac{1}{4}}} \left| \frac{\omega\left(\xi_0\sqrt{\mu}\right)\xi_0 \widehat{P_0}(\xi_0)}{\sqrt{\left|\phi''\left(\xi_0\sqrt{\mu}\right)\right|}} \right|.
\end{aligned}
$$

Since $|\phi''(\xi)| \leqslant C|\xi|$ and $\omega(\xi_0\sqrt{\mu}\,) \geqslant C(\xi_0)\sqrt{\mu}$, we get the result. □

Remark 3.12. Notice that for all $s \in \mathbb{R}$,

$$
\left| \zeta_R(t, \cdot) + \frac{t}{2} P_0'(\cdot - t) \right|_{H^s} \leqslant \sqrt{\mu}\, t^2 \left|\nabla P_0\right|_{H^{s+2}}.
$$

Hence, by tending formally μ to 0, we rediscover the result we get in the shallow water case (section 3.1).

Remark 3.13. Notice that for a general pressure term $P(t, X)$ we can show that the amplitude ζ satisfying

$$\widehat{\zeta}(t,\xi) = \frac{\mathrm{i}}{2}\int_0^t \xi\sqrt{\frac{\tanh(\sqrt{\mu}\,|\xi|)}{\sqrt{\mu}\,|\xi|}}\,\widehat{P}(\tau,\xi)\mathrm{e}^{\mathrm{i}(t-\tau)\xi\sqrt{\frac{\tanh(\sqrt{\mu}\,|\xi|)}{\sqrt{\mu}\,|\xi|}}}\mathrm{d}\tau$$
$$-\frac{\mathrm{i}}{2}\int_0^t \xi\sqrt{\frac{\tanh(\sqrt{\mu}\,|\xi|)}{\sqrt{\mu}\,|\xi|}}\,\widehat{P}(\tau,\xi)\mathrm{e}^{\mathrm{i}(\tau-t)\xi\sqrt{\frac{\tanh(\sqrt{\mu}\,|\xi|)}{\sqrt{\mu}\,|\xi|}}}\mathrm{d}\tau,$$

satisfies also

$$|\zeta(t,\cdot)|_\infty \leqslant C(\mu_{\max})\sqrt{\frac{t}{\mu}}\left(|P|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d))} + |P|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}^d)} + |XP|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}^d)}\right).$$

Hence, contrary to the shallow water case, we can not hope a linear amplification with respect to the time t. Corollary 3.11 also shows that the factor of amplification of $\sqrt{t}$ is optimal.

Hence, we observe that in intermediate water depths, a resonance can occur but with a factor of amplification of $\sqrt{t}$ and not t. But we saw that in the shallow water case, the resonance occurs for a moving pressure with a speed equal to 1, $P(t, X) = P_0(X - t)$. We wonder if this pressure can create a resonance. The following proposition shows that the previous pressure can create a resonance with a factor of amplification of $t^{\frac{1}{3}}$.

Proposition 3.14. *Let $0 < \mu \leqslant \mu_{\max}$. Let $P_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ such that $\widehat{P_0}(0) \neq 0$. Consider, the amplitude ζ_R created by $P(t, X) = P_0(X - t)$,*

$$\widehat{\zeta_R}(t,\xi) = -\frac{\mathrm{i}}{2}\xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)\mathrm{e}^{-\mathrm{i}t\xi}\int_{-t}^0 \mathrm{e}^{\mathrm{i}s\xi(\omega(\sqrt{\mu}\,\xi)-1)}\mathrm{d}s. \tag{59}$$

Then,

$$\left|\zeta_R(t,\cdot)\right|_\infty \leqslant C(\mu_{\max})\left(\frac{t^{\frac{1}{3}}}{\mu}|P_0|_{L^1} + \mu^{\frac{1}{4}}|P_0|_{H^1}\right).$$

Furthermore, if $XP_0 \in H^1(\mathbb{R})$,

$$\underline{\lim}_{t\to+\infty}\left|\frac{1}{t^{\frac{1}{3}}}\zeta_R(t,\cdot)\right|_\infty \geqslant \frac{C}{\mu^{\frac{2}{3}}}\left|\widehat{P_0}(0)\right|.$$

Proof. We have

$$\zeta_R(t,X) = -\frac{\mathrm{i}}{2}\int_{\mathbb{R}}\xi\omega(\sqrt{\mu}\,\xi)\widehat{P_0}(\xi)\mathrm{e}^{-\mathrm{i}t\xi}\int_{-t}^0 \mathrm{e}^{\mathrm{i}s\xi(\omega(\sqrt{\mu}\,\xi)-1)}\mathrm{e}^{\mathrm{i}X\xi}\mathrm{d}s\mathrm{d}\xi$$
$$= -\frac{\mathrm{i}}{2}\frac{1}{\mu}\int_{\mathbb{R}}\xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^0 \mathrm{e}^{\mathrm{i}\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\mathrm{d}s\mathrm{d}\xi.$$

We decompose this integral into 3 parts.

$$|I_1(t)| = \left|\frac{1}{\mu}\int_{|\xi|\leqslant t^{-\frac{1}{3}}} \xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}\mathrm{e}^{\mathrm{i}\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\mathrm{d}\xi\mathrm{d}s\right|$$
$$\leqslant \frac{t^{\frac{1}{3}}}{\mu}\left|\widehat{P_0}\right|_\infty.$$

Furthermore, since $|\omega(\xi)-1| \geqslant C\xi^2$ for $0 \leqslant |\xi| \leqslant 1$, we have

$$|I_2(t)| = \left|\frac{1}{\mu}\int_{t^{-\frac{1}{3}}\leqslant|\xi|\leqslant 1} \xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}\mathrm{e}^{\mathrm{i}\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\mathrm{d}\xi\mathrm{d}s\right|$$
$$= \left|\frac{1}{\sqrt{\mu}}\int_{t^{-\frac{1}{3}}\leqslant|\xi|\leqslant 1}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\frac{\omega(\xi)}{\omega(\xi)-1}\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\left(\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}-\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi\omega(\xi)}\right)\mathrm{d}\xi\right|$$
$$\leqslant C\frac{t^{\frac{1}{3}}}{\sqrt{\mu}}\left|\widehat{P_0}\right|_\infty.$$

Finally,

$$|I_3(t)| = \left|\frac{1}{\mu}\int_{|\xi|\geqslant 1} \xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}\mathrm{e}^{\mathrm{i}\frac{s}{\sqrt{\mu}}\xi(\omega(\sqrt{\mu}\xi)-1)}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\mathrm{d}\xi\mathrm{d}s\right|$$
$$= \left|\frac{1}{\sqrt{\mu}}\int_{|\xi|\geqslant 1}\mathrm{e}^{\mathrm{i}\frac{X}{\sqrt{\mu}}\xi}\frac{\omega(\xi)}{\omega(\xi)-1}\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\left(\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi}-\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{\mu}}\xi\omega(\xi)}\right)\mathrm{d}\xi\right|$$
$$\leqslant C\int_{|\xi|\geqslant\frac{1}{\sqrt{\mu}}}\left|\widehat{P_0}(\xi)\right|\mathrm{d}\xi,$$
$$\leqslant C\mu^{\frac{1}{4}}\,|P_0|_{H^1},$$

and the first inequality follows. For the second inequality, we use a stationary phase approximation. We denote $\phi(\xi) := \xi(\omega(\xi)-1)$. We recall that $\phi(\xi) = -\frac{1}{6}\xi^3 + o(\xi^3)$. Using a generalization of Morse lemma at the order 3, there exists $a > 0$ and $\psi \in \mathcal{C}^\infty([-a,a])$, such that for all $|y| \leqslant a$,

$$\phi(\psi(y)) = \frac{1}{6}\phi'''(0)y^3, \psi(0) = 0 \text{ and } \psi'(0) = 1.$$

Then,

$$I(s) := \int_{\mathbb{R}}\omega(\xi)\xi\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\mathrm{e}^{\mathrm{i}\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}\mathrm{d}\xi$$
$$= \int_{-a}^{a}\psi'(y)\omega(\psi(y))\psi(y)\widehat{P_0}\left(\frac{\psi(y)}{\sqrt{\mu}}\right)\mathrm{e}^{\mathrm{i}\frac{s}{6\sqrt{\mu}}y^3}\mathrm{d}y + o(s^{-\frac{2}{3}})$$
$$= \left(\frac{6\sqrt{\mu}}{s}\right)^{\frac{2}{3}}\widehat{P_0}(0)\int_{z\in\mathbb{R}} z\mathrm{e}^{\mathrm{i}z^3}\mathrm{d}z + o(s^{-\frac{2}{3}}).$$

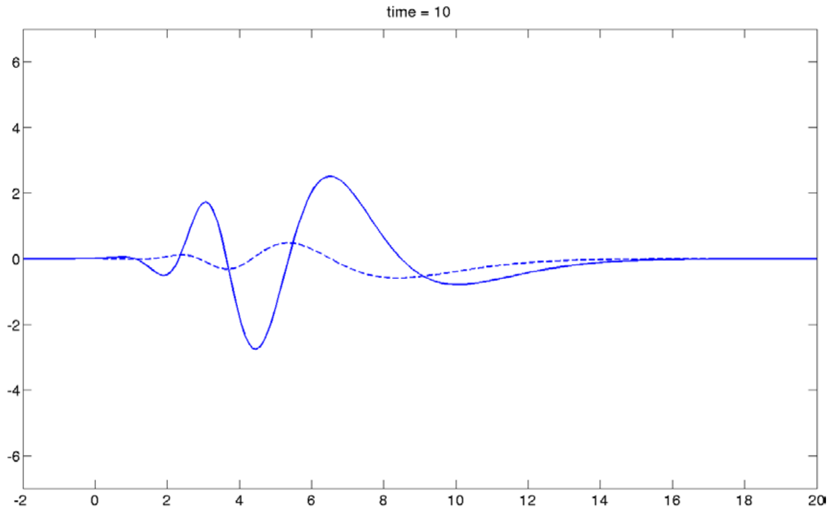

Figure 6. Evolution of the surface elevation ζ_R in (58) (solid line) because of a resonant moving pressure P in (57) (dashed line).

Therefore,

$$\lim_{t\to+\infty}\left|\frac{1}{t^{\frac{1}{3}}}\zeta_R(t,t)\right| = \frac{C}{\mu^{\frac{2}{3}}}\left|\widehat{P_0}(0)\right|.$$

□

Then, in intermediate water depths, a traveling pressure with a constant speed equal to 1 is also resonant, but it takes more time to obtain a significant elevation of the level of the sea. In the following, we compute numerically some solutions. We take $P_0(X) = -\mathrm{e}^{-X^2}$ and $\mu = 1$. The figure 6 is the evolution of a water wave because of a pressure of the form (57). The solid curve is the wave and the dashed curve is the moving pressure. The figure 7 is the evolution is a water wave when the pressure moves with a speed 1. The figure 8 compares the evolution of the maximum of the resonant case and the case when the speed is equal to 1.

Remark 3.15. In our work, we neglect the Coriolis effect. However, in view of the duration of the meteotsunami phenomenon, it would be more realistic to consider it. It will be studied in a future work ([22]) based on the work of Castro and Lannes ([9] and [10]).

Acknowledgments

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Appendix A. The Laplace problem

A.1. Formulation of the problems

In this part, we recall some results of chapter 2 in [20] and section 4 of [17] and study the Laplace problem (17) in the Beppo Levi spaces. We suppose that the parameters ε, μ and β satisfy condition (18). The Laplace problem (17) is

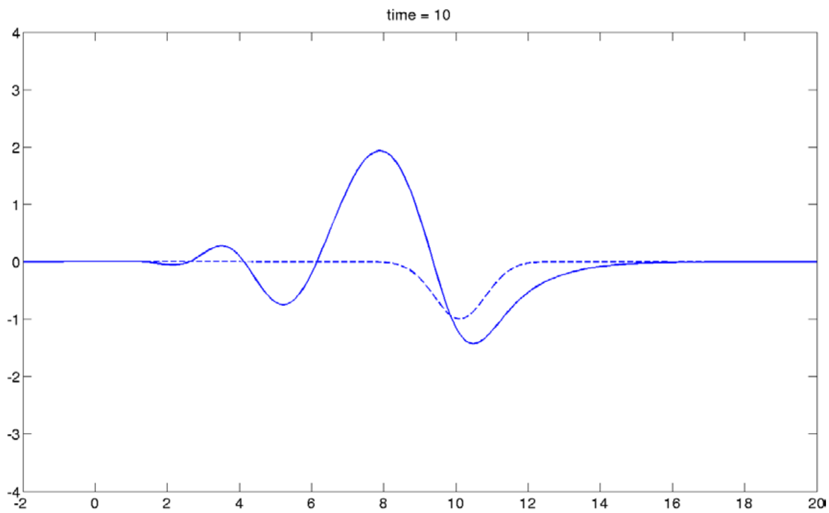

Figure 7. Evolution of the surface elevation ζ_R in (59) (solid line) because of a moving pressure P with a speed of 1 (dashed line).

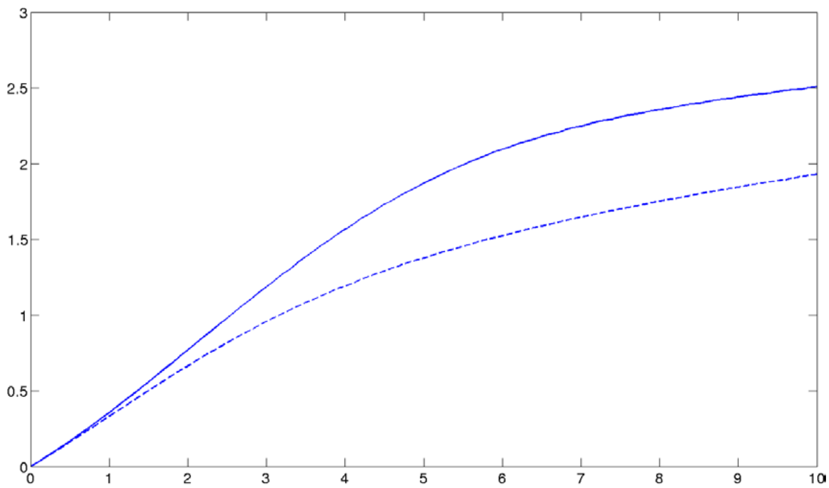

Figure 8. Evolution of the maximum of ζ_R in the resonant case (solid line) and the moving pressure with a speed of 1 (dashed line).

$$\begin{cases} \Delta^{\mu}_{X,z}\Phi^B = 0 \text{ in } \Omega_t\,, \\ {\Phi^B}_{|z=\varepsilon\zeta} = 0, \sqrt{1+\beta^2|\nabla b|^2}\,\partial_{\mathbf{n}}{\Phi^B}_{|z=-1+\beta b} = B, \end{cases}$$

where $B = \partial_t b$. Notice that $\partial_{\mathbf{n}}$ is here the upward *conormal* derivative

$$\partial_{\mathbf{n}}\Phi^B = \mathbf{n}\cdot\begin{pmatrix}\sqrt{\mu}\,\mathbf{I_d} & 0\\ 0 & 1\end{pmatrix}\nabla^{\mu}_{\mathbf{X,z}}{\mathbf{\Phi^B}}_{\mathbf{|z=-1+\beta b}}\,.$$

For the study of (15) we refer to [20]. We work with Beppo Levi spaces. We refer to [14] and proposition 2.3 in [20] for general results about these spaces. We recall that, for $s \geqslant 0$,

$$\dot{H}^s(\mathbb{R}^d) := \{\psi \in L^2_{\text{loc}}(\mathbb{R}^d), \nabla\psi \in H^{s-1}(\mathbb{R}^d)\},$$

that $\dot{H}^1(\mathbb{R}^d \times (-1,0))/\mathbb{R}$ is a Hilbert space for the norm $|\nabla_{X,z}\cdot|_{L^2}$ and that $H^s(\mathbb{R}^d)$ is dense in $\dot{H}^s(\mathbb{R}^d)$. In order to fix the domain, we transform the problem into variable coefficients elliptic problem on $S := \mathbb{R}^d \times (-1,0)$ (the flat strip). We introduce a regularizing diffeomorphism. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a positive, compactly supported, smooth, even function equal to one near 0. For $\delta > 0$ we define

$$\Sigma := \begin{array}{ccc} S & \longrightarrow & \Omega \\ (X,z) & \mapsto & (X, z+\sigma(X,z)), \end{array}$$

and

$$\sigma(X,z) := [\theta(\delta z|D|)\varepsilon\zeta(X) - \theta(\delta(z+1)|D|)\beta b(X)]\, z + \varepsilon\theta(\delta z|D|)\zeta(X).$$

We omit the dependence on t here. In the following, we denote by M a constant of the form

$$M = C\left(\frac{1}{h_{\min}}, \mu_{\max}, \varepsilon|\zeta|_{H^{t_0+1}(\mathbb{R}^d)}, \beta|b|_{H^{t_0+1}(\mathbb{R}^d)}\right).$$

In order to study the Laplace problems in S, we have to treat the regularity in the direction X and in the direction z one at a time. We introduce the following spaces.

Definition A.1. Let $s \in \mathbb{R}$. We define $(H^{s,1}(S), |\cdot|_{s,1})$ and $(H^{s,0}(S), |\cdot|_{0,1})$

$$H^{s,1}(S) := L^2_z H^s_X(S) \cap H^1_z H^{s-1}_X(S) \text{ , and } |u|^2_{H^{s,1}} = |\Lambda^s u|^2_{L^2} + |\Lambda^{s-1}\partial_z u|^2_{L^2},$$

and

$$H^{s,0}(S) := L^2_z H^s_X(S) \text{ , and } |u|^2_{H^{s,0}} = |\Lambda^s u|^2_{L^2} \text{ .}$$

Remark A.2. We have the following embedding (see proposition 2.10 in [20]) for $s \in \mathbb{R}$

$$H^{s+\frac{1}{2},1}(S) \subset L^\infty_z H^s_X(S).$$

In the following, we fix $\delta > 0$ small enough. Then, we can transform our equations. We denote $\phi^B := \Phi^B \circ \Sigma$ and we get that

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z}\phi^B = 0 \text{ in } S, \\ \phi^B_{\;|z=0} = 0, \partial_{\mathbf{n}}\phi^B_{\;|z=-1} = B, \end{cases} \tag{A.1}$$

with $P(\Sigma) = I_{d+1\times d+1} + Q(\Sigma)$ and

$$Q(\Sigma) := \begin{pmatrix} \partial_z\sigma I_{d\times d} & -\sqrt{\mu}\,\nabla_X\sigma \\ -\sqrt{\mu}\,\nabla_X\sigma^t & \dfrac{-\partial_z\sigma + \mu\,|\nabla_X\sigma|^2}{1+\partial_z\sigma} \end{pmatrix}. \tag{A.2}$$

Notice that $P(\Sigma)$ is well defined if δ is small enough and that $\partial_{\mathbf{n}} := \mathbf{e_z} \cdot (\mathbf{P}(\boldsymbol{\Sigma})\nabla^\mu_{\mathbf{X,z}}\,\cdot\,)$. We have to know the regularity of $P(\Sigma)$. It is the subject of the next proposition (see proposition 2.18 and lemma 2.26 in [20]).

Proposition A.3. *Let $t_0 > \frac{d}{2}$, $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then,*

$$|Q(\Sigma)|_{H^{t_0+\frac{1}{2},1}}, |\Lambda^{t_0}Q(\Sigma)|_{L^\infty_z L^2_X(S)}, \left|\Lambda^{t_0-1}\partial_z Q(\Sigma)\right|_{L^\infty_z L^2_X(S)} \leqslant M.$$

Furthermore, $P(\Sigma)$ is coercive. There exist a constant $k(\Sigma) > 0$ such that $\frac{1}{k(\Sigma)} \leqslant M$ and

$$\forall \Theta \in \mathbb{R}^{d+1}, \forall (X,z) \in S, P(\Sigma)(X,z)\Theta \cdot \Theta \geqslant k(\Sigma)|\Theta|^2.$$

We have a variational formulation of the Laplace problem (A.1). We introduce

$$H^1_{0,\mathrm{surf}}(S) := \overline{\mathcal{D}(S \cup \{z=-1\})}^{|\,|_{H^1(S)}} = \overline{\mathcal{D}(S \cup \{z=-1\})}^{|\,|_{\dot{H}^1(S)}}.$$

See proposition 2.3 (3) in [20] for a proof of the second equality.

Definition A.4. Let $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$. We say that $\phi \in H^1_{0,\mathrm{surf}}(S)$ is a variational solution of (A.1) if for all $\varphi \in H^1_{0,\mathrm{surf}}(S)$,

$$\int_S \nabla^\mu_{X,z}\phi \cdot P(\Sigma)\nabla^\mu_{X,z}\varphi = -\langle B, \varphi_{|z=-1}\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}}.$$

We have also the following trace result that we can prove easily using a density argument.

Lemma A.5. *For all $\varphi \in H^1_{0,\mathrm{surf}}(S)$ we have*

$$\left|\sqrt{1+\sqrt{\mu}\,|D|}\ \varphi_{|z=-1}\right|_{L^2(\mathbb{R}^d)} \leqslant 2\left|\nabla^\mu_{X,z}\varphi\right|_{L^2(S)}.$$

We can now establish existence and uniqueness results.

Proposition A.6. *Let $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ satisfying (19). Then, the problem (A.1) has a unique variational solution named $B^{\mathfrak{d}} \in H^1_{0,\mathrm{surf}}(S)$.*

Proof. Because S is bounded in the direction z and that $P(\Sigma)$ is uniformly coercive, the results follow from the Lax–Milgram theorem and Poincaré inequality in $H^1_{0,\mathrm{surf}}(S)$. □

In this part, we study the Laplace problem (17), but the same work can be done for (15) (see chapter 2 in [20]) and we can transform (17) as follows

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z}\phi^S = 0 \text{ in } S, \\ \phi^S{}_{|z=0} = \psi, \partial_{\mathbf{n}}\phi^S{}_{|z=-1} = 0. \end{cases} \tag{A.3}$$

In the following, we denote by $\psi^{\mathfrak{h}}$, the unique solution of (A.3).

A.2. Regularity estimates of the solutions

In this part, we give some regularity estimates.

Theorem A.7. *Let $t_0 > \frac{d}{2}$ and $0 \leqslant s \leqslant t_0 + \frac{1}{2}$. Let $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that condition (19) is satisfied. Then, for all $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left|\Lambda^s \nabla^\mu_{X,z} B^{\mathfrak{d}}\right|_{L^2(S)} \leqslant M \left|\frac{1}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{H^s}.$$

Futhermore, if $s \geqslant \max(0, 1-t_0)$, we have

$$\left|\Lambda^{s-1}\partial_z \nabla^\mu_{X,z} B^{\mathfrak{d}}\right|_{L^2(S)} \leqslant M \left|\frac{1}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{H^s}.$$

Proof. Let $\delta > 0$ and χ be a smooth compactly supported real function that is equal to 1 near 0. We introduce the smoothing operator $\Lambda_\delta^s := \chi(\delta\Lambda)\Lambda^s$. We know that $B^\mathfrak{d} \in H^1_{0,\mathrm{surf}}(S)$. Therefore, using $\Lambda_\delta^{2s} B^\mathfrak{d}$ a test function, we have

$$\int_S \nabla^\mu_{X,z} B^\mathfrak{d} \cdot P(\Sigma) \nabla^\mu_{X,z} \Lambda_\delta^{2s} B^\mathfrak{d} = -\int_{\mathbb{R}^d} B(\Lambda_\delta^{2s} B^\mathfrak{d})_{|z=-1}.$$

Since $P(\Sigma)$ is symmetric, Λ_δ^s commutes with $\nabla^\mu_{X,z}$ and is independent of z we obtain that

$$\begin{aligned}\int_S P(\Sigma)\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d} \cdot \nabla^\mu_{X,z} \Lambda_\delta^s B^\mathfrak{d} = &-\int_S [\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu_{X,z} B^\mathfrak{d} \cdot \nabla^\mu_{X,z} \Lambda_\delta^s B^\mathfrak{d} \\ &- \int_{\mathbb{R}^d} \frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}\,|D|}} B\left(\sqrt{1+\sqrt{\mu}\,|D|}\,\Lambda_\delta^s B^\mathfrak{d}\right)_{|z=-1}.\end{aligned}$$

Then by coercivity of $P(\Sigma)$ and trace inequality Lemma A.5

$$\begin{aligned}k(\Sigma)\left|\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d}\right|^2_{L^2(S)} \leqslant &\left|[\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2} \left|\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} \\ &+2\left|\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} \left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{L^2},\end{aligned}$$

and

$$k(\Sigma)\left|\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} \leqslant \left|[\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{L^2}.$$

We have to distinguish two cases.

(a) $0 \leqslant s \leqslant t_0$ *:*

The commutator estimate Proposition C.6 (with $T_0 = t_0$) and Proposition A.3 give

$$\begin{aligned}k(\Sigma)\left|\Lambda_\delta^s \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} &\leqslant C\left|Q(\Sigma)\right|_{L^\infty_z H^{t_0}_X(S)} \left|\Lambda_\delta^{s-\varepsilon} \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{L^2} \\ &\leqslant M\left|\Lambda_\delta^{s-\varepsilon} \nabla^\mu_{X,z} B^\mathfrak{d}\right|_{L^2(S)} + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}\,|D|}} B\right|_{L^2}\end{aligned}$$

for some $\varepsilon > 0$ small enough ($\varepsilon < t_0 - \frac{d}{2}$). Using a finite induction on s and taking the limit when δ goes to 0, the first inequality follows. For the second estimate, we only need to give a control of $\partial_z^2 B^\mathfrak{d}$. We use equation (A.1) satisfied by $B^\mathfrak{d}$. We express $P(\Sigma)$ as

$$P(\Sigma) := \begin{pmatrix}(1+a(X,z))I_{d\times d} & \mathbf{q}(X,z)\\ \mathbf{q}^t(X,z) & 1+q_{d+1}(X,z)\end{pmatrix}.$$

A simple computation gives

$$\begin{aligned}(1+q_{d+1})\partial_z^2 B^\mathfrak{d} = &-\sqrt{\mu}\,\nabla_X \cdot \left((1+a)\sqrt{\mu}\,\nabla_X B^\mathfrak{d}\right) - \sqrt{\mu}\,\nabla_X \cdot (\partial_z B^\mathfrak{d}\mathbf{q}) \\ &-\sqrt{\mu}\,\partial_z \mathbf{q} \cdot \nabla_X B^\mathfrak{d} - \sqrt{\mu}\,\partial_z \nabla_X B^\mathfrak{d} \cdot \mathbf{q} - \partial_z q_{d+1} \partial_z B^\mathfrak{d}.\end{aligned}$$

We have $a, \mathbf{q}, q_{d+1} \in L^\infty_z H^{t_0}_X(S)$, $\partial_z \mathbf{q}, \partial_z q_{d+1} \in L^\infty_z H^{t_0-1}_X(S)$ and $1 + q_{d+1} \geqslant k(\Sigma)$. Then, since $s \geqslant 1 - t_0$ and $\nabla_X B^{\mathfrak{d}} \in H^{s,1}(S)$, by the product estimates Propositions C.3 and C.4 (with $T_0 = t_0$), we obtain the result.

(b) $t_0 \leqslant s \leqslant t_0 + \frac{1}{2}$:

The commutator estimate Proposition C.7 (with $T_0 = t_0 + \frac{1}{2}$ and $t_1 > \frac{1}{2}$) and Proposition A.3 give

$$k(\Sigma) \left| \Lambda^s_\delta \nabla^\mu_{X,z} B^{\mathfrak{d}} \right|_{L^2(S)} \leqslant M \left[\left| \Lambda^{s+\frac{1}{2}-t_1}_\delta \nabla^\mu_{X,z} B^{\mathfrak{d}} \right|_{L^2(S)} \right.$$
$$\left. + \left| \Lambda^{s-1+\frac{1}{2}-t_1}_\delta \partial_z \nabla^\mu_{X,z} B^{\mathfrak{d}} \right|_{L^2(S)} + 2 \left| \frac{\Lambda^s_\delta}{\sqrt{1+\sqrt{\mu}\,|D|}} B \right|_{L^2} \right].$$

We denote $\varepsilon := \frac{1}{2} - t_1$. We obtain the first inequality for $t_0 \leqslant s \leqslant t_0 + \varepsilon$ thanks to the previous case. Furthermore, we saw that

$$(1 + q_{d+1}) \partial_z^2 B^{\mathfrak{d}} = -\sqrt{\mu}\, \nabla^\mu_X \cdot \left((1+a) \sqrt{\mu}\, \nabla_X B^{\mathfrak{d}} \right) - \sqrt{\mu}\, \nabla_X \cdot (\partial_z B^{\mathfrak{d}} \mathbf{q})$$
$$- \sqrt{\mu}\, \partial_z \mathbf{q} \cdot \nabla_X B^{\mathfrak{d}} - \sqrt{\mu}\, \partial_z \nabla_X B^{\mathfrak{d}} \cdot \mathbf{q} - \partial_z q_{d+1} \partial_z B^{\mathfrak{d}}.$$

We have $a, \mathbf{q}, q_{d+1} \in L^2_z H^{t_0+\frac{1}{2}}_X(S)$, $\partial_z \mathbf{q}, \partial_z q_{d+1} \in L^2_z H^{t_0-\frac{1}{2}}_X(S)$ and $1 + q_{d+1} \geqslant k(\Sigma)$. Then, since $s \geqslant 1 - t_0$ and $\nabla_X B^{\mathfrak{d}} \in L^\infty_z H^{s-\frac{1}{2}}_X(S)$, by the product estimates Propositions C.3 and C.5 (with $T_0 = t_0$), and we obtain the second inequality for $t_0 \leqslant s \leqslant t_0 + \varepsilon$. Using a finite induction, we obtain the first and the second inequality. □

Appendix B. The Dirichlet–Neumann and the Neumann–Neumann operators

We refers to chapter 3 of [20] for more details about the Dirichlet–Neumann operator and section 3 in [17] for the study of these operators.

B.1. Main properties

We can express the Neumann–Neumann operator with the formalism of the previous section. For $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$ we have

$$G_\mu[\varepsilon\zeta, \beta b](\psi) = (\mathbf{e_z} \cdot \mathbf{P}(\mathbf{\Sigma}) \nabla^\mu_{\mathbf{X,z}} \psi^{\mathfrak{h}})_{|z=0}, \tag{B.1}$$

and

$$G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](B) = (\mathbf{e_z} \cdot \mathbf{P}(\mathbf{\Sigma}) \nabla^\mu_{\mathbf{X,z}} \mathbf{B}^{\mathfrak{d}})_{|z=0}. \tag{B.2}$$

Remark B.1. Notice that (see proposition 3.2 in [17])

$$\frac{1}{\mu} G_\mu[0,0](\psi) = |D|^2 \frac{\tanh(\sqrt{\mu}\,|D|)}{\sqrt{\mu}\,|D|} \psi \text{ and } G^{\mathrm{NN}}_\mu[0,0](B) = \frac{1}{\cosh(\sqrt{\mu}\,|D|)} B.$$

We recall that $G_\mu[\varepsilon\zeta, \beta b]$ is symmetric and maps continuously $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ into $\left(\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/\mathbb{R}\right)'$ (see section 3.1. in [20]). We need an extension result in $H^{-\frac{1}{2}}(\mathbb{R}^d)$ in order to give a dual formulation of the Neumann–Neumann operator.

Definition B.2. Let $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^d)$. We define $\varphi^\#$ as

$$\varphi^\# = \frac{\sinh([z+1]\sqrt{\mu}\,|D|)}{\sinh(\sqrt{\mu}\,|D|)}\varphi.$$

Remark B.3. $\varphi^\#$ satisfies weakly

$$\begin{cases} \Delta^\mu_{X,z}\varphi^\# = 0 \text{ in } S, \\ {\varphi^\#}_{|z=0} = \varphi, {\varphi^\#}_{|z=-1} = 0. \end{cases}$$

We can prove easily regularity results for $\varphi^\#$ similar to $\varphi^\natural$.

Proposition B.4. *Let $s \geqslant 0$ and $\varphi \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$\left|\Lambda^{s-1}\nabla^\mu_{X,z}\varphi^\#\right|_{L^2(S)} + \frac{1}{\sqrt{\mu}}\left|\Lambda^{s-2}\partial_z\nabla^\mu_{X,z}\varphi^\#\right|_{L^2(S)} \leqslant C\left|\sqrt{1+\sqrt{\mu}\,|D|}\,\varphi\right|_{H^{s-1}}.$$

We can now give a dual formulation of the Neumann–Neumann operator. We introduce the Dirichlet–Dirichlet operator, for $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$,

$$G^{\mathrm{DD}}_\mu[\varepsilon\zeta, \beta b](\psi) := (\psi^\natural)_{|z=-1}. \tag{B.3}$$

The following result is proposition 3.3 in [17].

Proposition B.5. *Let $t_0 > \frac{d}{2}$, $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that (19) is satisfied. $G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](\cdot)$ can be extended to $H^{-\frac{1}{2}}(\mathbb{R}^d)$ with the dual formulation*

$$G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](B) = \begin{cases} H^{\frac{1}{2}}(\mathbb{R}^d) & \longrightarrow \quad \mathbb{R} \\ \quad\varphi & \longmapsto \displaystyle\int_S P(\Sigma)\nabla^\mu_{X,z}B^\mathfrak{d} \cdot \nabla^\mu_{X,z}\varphi^\#. \end{cases} \tag{B.4}$$

Furthermore, the adjoint of $G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b]$ is $G^{\mathrm{DD}}_\mu[\varepsilon\zeta, \beta b]$. For all $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$,

$$(G^{\mathrm{NN}}_\mu[\varepsilon\zeta, \beta b](B), \varphi)_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}} = (B, G^{\mathrm{DD}}_\mu[\varepsilon\zeta, \beta b](\varphi))_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}}.$$

In order to study shape derivatives of the Dirichlet–Neumann and the Neumann–Neumann operators, we have to introduce the Neumann–Dirichlet operator. For $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$, we define

$$G^{\mathrm{ND}}_\mu[\varepsilon\zeta, \beta b](B) := (B^\mathfrak{d})_{|z=-1}. \tag{B.5}$$

The following result is a symmetry property and a dual formulation of the Neumann–Dirichlet operator.

Proposition B.6. *Let $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that (19) is satisfied. $G^{\mathrm{ND}}_\mu[\varepsilon\zeta, \beta b](B)$ can be view as*

$$G_\mu^{\mathrm{ND}}[\varepsilon\zeta,\beta b](B)=\begin{cases} H^{-\frac{1}{2}}(\mathbb{R}^d) & \longrightarrow \quad \mathbb{R} \\ \quad C & \longmapsto -\displaystyle\int_S P(\Sigma)\nabla^\mu_{X,z}B^{\mathfrak{d}}\cdot\nabla^\mu_{X,z}C^{\mathfrak{d}}. \end{cases} \tag{B.6}$$

Furthermore, $G_\mu^{\mathrm{ND}}[\varepsilon\zeta,\beta b](\cdot)$ is a negative symmetric operator and, for all B_1, B_2 in $H^{-\frac{1}{2}}(\mathbb{R}^d)$,

$$(G_\mu^{\mathrm{ND}}[\varepsilon\zeta,\beta b](B_1),B_2)_{(H^{-1/2})'-H^{-1/2}}=(G_\mu^{\mathrm{ND}}[\varepsilon\zeta,\beta b](B_2),B_1)_{(H^{-1/2})'-H^{-1/2}}.$$

We refer to proposition 3.3 in [17] for a proof of this result.

B.2. Regularity estimates

In this part we give some controls the Neumann–Neumann operators.

Proposition B.7. *Let $t_0>\frac{d}{2}$, $0\leqslant s\leqslant t_0+\frac{1}{2}$ and $\zeta, b\in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, $G_\mu^{\mathrm{NN}}[\varepsilon\zeta,\beta b]$ maps continuously $H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into itself*

$$|G_\mu^{\mathrm{NN}}[\varepsilon\zeta,\beta b](B)|_{H^{s-\frac{1}{2}}}\leqslant M\,|B|_{H^{s-\frac{1}{2}}}.$$

Proof. This proposition follows by Theorem A.7 and by using the same arguments that theorem 3.15 in [20]. □

We can extend these estimates to $\underline{w}[\varepsilon\zeta,\beta b]$, the vertical velocity at the surface and to $\underline{V}[\varepsilon\zeta,\beta b]$ the horizontal velocity at the surface. These operators appear naturally when we differentiate the Dirichlet–Neumann and the Neumann–Neumann operator with respect to the surface ζ. We define

$$\underline{w}[\varepsilon\zeta,\beta b]:=\begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d)\to & H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi,B) & \mapsto \dfrac{G_\mu[\varepsilon\zeta,\beta b](\psi)+\mu G_\mu^{\mathrm{NN}}[\varepsilon\zeta,\beta b](B)+\varepsilon\mu\nabla\zeta\cdot\nabla\psi}{1+\varepsilon^2\mu|\nabla\zeta|^2}, \end{cases} \tag{B.7}$$

and

$$\underline{V}[\varepsilon\zeta,\beta b]:=\begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d)\to & H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi,B) & \mapsto \nabla\psi-\varepsilon\underline{w}[\varepsilon\zeta,\beta b](\psi,B)\nabla\zeta. \end{cases} \tag{B.8}$$

Proposition B.8. *Let $t_0>\frac{d}{2}$, $0\leqslant s\leqslant t_0+\frac{1}{2}$ and $\zeta, b\in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, $\underline{w}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$*

$$|\underline{w}[\varepsilon\zeta,\beta b](\psi,B)|_{H^{s-\frac{1}{2}}}\leqslant M\Big(\mu^{\frac{3}{4}}|\mathfrak{P}\psi|_{H^s}+\mu|B|_{H^{s-\frac{1}{2}}}\Big).$$

Furthermore, if $1\leqslant s\leqslant t_0$, $\underline{w}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+1}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$

$$|\underline{w}[\varepsilon\zeta,\beta b](\psi,B)|_{H^{s-\frac{1}{2}}}\leqslant M\mu\Big(|\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}}+|B|_{H^{s-\frac{1}{2}}}\Big).$$

Finally, we have the same continuity result for $\underline{V}[\varepsilon\zeta,\beta b]$.

We can also give some regularity estimates for $G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b]$ since it is the adjoint of $G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b]$.

Proposition B.9. *Let $t_0 > \frac{d}{2}$, $0 \leqslant s \leqslant t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, $G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ into itself*

$$\left|\nabla G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](\psi)\right|_{H^{s-\frac{1}{2}}} \leqslant M \left|\nabla\psi\right|_{H^{s-\frac{1}{2}}}.$$

Finally, we can give some regularity estimates for $G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b]$.

Proposition B.10. *Let $t_0 > \frac{d}{2}$, $0 \leqslant s \leqslant t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, $G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b]$ maps continuously $H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s+\frac{1}{2}}(\mathbb{R}^d)$*

$$\left|G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b](B)\right|_{H^{s+\frac{1}{2}}} \leqslant M \left|B\right|_{H^{s-\frac{1}{2}}}.$$

In the same way, we can extend also these estimates to $\underline{\widetilde{w}}[\varepsilon\zeta, \beta b]$, the vertical velocity at the bottom and to $\underline{\widetilde{V}}[\varepsilon\zeta, \beta b]$ the horizontal velocity at the bottom. These operators appear naturally when we differentiate the Dirichlet–Neumann and the Neumann–Neumann operator with respect to the bottom b

$$\underline{\widetilde{w}}[\varepsilon\zeta, \beta b](\psi, B) = \frac{\mu B + \beta\mu\nabla b \cdot \nabla(G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](\psi) + \mu G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b](B))}{1 + \beta^2\mu|\nabla b|^2}, \tag{B.9}$$

and

$$\underline{\widetilde{V}}[\varepsilon\zeta, \beta b](\psi, B) = \nabla(G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](\psi) + \mu G_\mu^{\mathrm{ND}}[\varepsilon\zeta, \beta b](B)) - \beta\underline{\widetilde{w}}[\varepsilon\zeta, \beta b](\psi, B)\nabla b. \tag{B.10}$$

Proposition B.11. *Let $t_0 > \frac{d}{2}$, $0 \leqslant s \leqslant t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, $\underline{\widetilde{w}}[\varepsilon\zeta, \beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$*

$$|\underline{\widetilde{w}}[\varepsilon\zeta, \beta b](\psi, B)|_{H^{s-\frac{1}{2}}} \leqslant M\Big(|\nabla\psi|_{H^{s-\frac{1}{2}}} + \mu|B|_{H^{s-\frac{1}{2}}}\Big).$$

Finally, we have the same continuity result for $\underline{\widetilde{V}}[\varepsilon\zeta, \beta b]$.

B.3. Shape derivatives

Let $t_0 > \frac{d}{2}$. Given $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$. We denote by Γ the set of functions (ζ, b) in $H^{t_0+1}(\mathbb{R}^d)$ satisfying (19). We introduce the map

$$G_\mu^{\mathrm{NN}}(B) := \begin{cases} \quad\Gamma \quad \to H^{\frac{1}{2}}(\mathbb{R}^d) \\ (\zeta, b) \mapsto G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](B), \end{cases} \tag{B.11}$$

which is the Neumann–Neumann operator. We can also define $G_\mu(\psi)$, $\underline{w}(\psi, B)$ and $\underline{V}(\psi, B)$.

Remark B.12. When no confusion is possible and to the sake of simplicity, we write $G_\mu(\psi)$, $G_\mu^{\mathrm{NN}}(B)$, $\underline{w}(\psi, B)$ and $\underline{V}(\psi, B)$ instead of $G_\mu[\varepsilon\zeta, \beta b](\psi)$, $G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](B)$, $\underline{w}[\varepsilon\zeta, \beta b](\psi, B)$ and $\underline{V}[\varepsilon\zeta, \beta b](\psi, B)$.

In order to linearize the water waves equations, we need a shape derivative formula for the Dirichlet–Neumann and the Neumann–Neumann operators. The following proposition is a summarize of theorems 3.5 and 3.6 in [17] and theorem 3.21 in [20].

Proposition B.13. *Let $t_0 > \frac{d}{2}$, $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then, $G_\mu(\psi)$ and $G_\mu^{\mathrm{NN}}(B)$ are Fréchet differentiable. For $(h, k) \in H^{t_0+1}(\mathbb{R}^d)$, we have*

$$\begin{aligned} dG_\mu(\psi).(h,0) + \mu dG_\mu^{\mathrm{NN}}(B).(h,0) = &-\varepsilon G_\mu[\varepsilon\zeta, \beta b](h\, \underline{w}[\varepsilon\zeta, \beta b](\psi, B)) \\ &-\varepsilon\mu\nabla\cdot(h\, \underline{V}[\varepsilon\zeta, \beta b](\psi, B)), \end{aligned}$$

and

$$dG_\mu(\psi).(0,k) + \mu dG_\mu^{\mathrm{NN}}(B).(0,k) = \beta\mu G_\mu^{\mathrm{NN}}[\varepsilon\zeta, \beta b](\nabla\cdot(k\, \underline{\widetilde{V}}[\varepsilon\zeta, \beta b](\psi, B))).$$

Furthermore,

$$dG_\mu^{\mathrm{DD}}(\psi).(h,0) + \mu dG_\mu^{\mathrm{ND}}(B).(h,0) = -\varepsilon G_\mu^{\mathrm{DD}}[\varepsilon\zeta, \beta b](h\underline{w}[\varepsilon\zeta, \beta b](\psi, B)).$$

Thanks to these formulae we can give some controls to the first shape derivatives of the operators. For instance, we give an estimate for $d\underline{\widetilde{w}}$ and $d\underline{\widetilde{V}}$.

Proposition B.14. *Let $t_0 > \frac{d}{2}$ and $(\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then, for $0 \leqslant s \leqslant t_0 + \frac{1}{2}$, for $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ and $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left| d\underline{\widetilde{V}}(\psi, B).(h,k) \right|_{H^{s-\frac{1}{2}}}, \left| d\underline{\widetilde{w}}(\psi, B).(h,k) \right|_{H^{s-\frac{1}{2}}} \leqslant M \left|(h,k)\right|_{H^{t_0+1}} \left(|\nabla\psi|_{H^{s-\frac{1}{2}}} + |B|_{H^{s-\frac{1}{2}}} \right).$$

Proof. This result follows from Propositions B.13 and B.7. □

We end this part by giving some controls of the shape derivatives of G_μ and G_μ^{NN}. We do not use the previous method, we differentiate j times directly the dual formulation of both operators. We refer to proposition 3.28 in [20] for a control of $d^j G_\mu.(\mathbf{h}, \mathbf{k})(\psi)$.

Proposition B.15. *Let $t_0 > \frac{d}{2}$ and $(\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)$ such that condition (19) is satisfied. Then for all $0 \leqslant s \leqslant t_0 + \frac{1}{2}$ and $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left| d^j G_\mu^{\mathrm{NN}}.(\mathbf{h}, \mathbf{k})(\mathbf{B}) \right|_{H^{s-\frac{1}{2}}} \leqslant M \prod_{i \geqslant 1} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} |B|_{H^{s-\frac{1}{2}}}.$$

Furthermore, if $0 \leqslant s \leqslant t_0$ and $B \in H^{t_0}(\mathbb{R}^d)$,

$$\left| d^j G_\mu^{\mathrm{NN}}.(\mathbf{h}, \mathbf{k})(\mathbf{B}) \right|_{H^{s-\frac{1}{2}}} \leqslant M \left|(\varepsilon h_1, \beta k_1)\right|_{H^{s+\frac{1}{2}}} \prod_{i \geqslant 2} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} |B|_{H^{t_0}}.$$

We do not prove this proposition here (which is based on a shape derivative of $B^{\mathfrak{d}}$). We refer to [23].

Appendix C. Useful estimates

In this part, we give some useful estimates, product and commutator estimates. We refer to appendix B in [19, 20] and chapter II in [6] for the proofs. The first estimates are useful to control $\mathfrak{P}f$. We recall that $\mathfrak{P} = \frac{|D|}{\sqrt{1+\sqrt{\mu}\,|D|}}$.

Proposition C.1. *Let $f \in H^1(\mathbb{R}^d)$ and $g \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$|\mathfrak{P}g|_{L^2} \leqslant \mu^{-\frac{1}{4}}|g|_{H^{\frac{1}{2}}}, |\mathfrak{P}f|_{H^{\frac{1}{2}}} \leqslant \max\left(1, \mu^{-\frac{1}{4}}\right)|\nabla f|_{L^2} \text{ and } |\nabla f|_{L^2} \leqslant \max\left(1, \mu^{\frac{1}{4}}\right)|\mathfrak{P}f|_{H^{\frac{1}{2}}}.$$

Proof. The first inequality follows from the fact that $1 + \sqrt{\mu}\,|\xi| \geqslant \sqrt{\mu}\,|\xi|$, the second inequality from $\frac{(1+|\xi|^2)^{\frac{1}{4}}}{\sqrt{1+\sqrt{\mu}\,|\xi|}} \leqslant \max\left(1, \frac{1}{\mu^{\frac{1}{4}}}\right)$ and the third from $\frac{\sqrt{1+\sqrt{\mu}\,|\xi|}}{\sqrt{1+|\xi|}} \leqslant \max\left(1, \mu^{\frac{1}{4}}\right)$. $\square$

We need some product estimates in $\mathbb{R}^d$. The following proposition is proposition 2.1.2 in [6].

Proposition C.2. *Let $s, s_1, s_2 \in \mathbb{R}$ such that $s \leqslant s_1$, $s \leqslant s_2$, $s_1 + s_2 \geqslant 0$ and $s < s_1 + s_2 - \frac{d}{2}$. Then, there exists a constant $C > 0$ such that for all $f \in H^{s_1}(\mathbb{R}^d)$ and for all $g \in H^{s_2}(\mathbb{R}^d)$, we have $fg \in H^s(\mathbb{R}^d)$ and*

$$|fg|_{H^s} \leqslant C|f|_{H^{s_1}}|g|_{H^{s_2}}.$$

We also need some product estimates in $S := \mathbb{R}^d \times (-1, 0)$. The following proposition is the corollary B.5 in [20].

Proposition C.3. *Let $s, s_1, s_2 \in \mathbb{R}$ such that $s \leqslant s_1$, $s \leqslant s_2$, $s_1 + s_2 \geqslant 0$, $s < s_1 + s_2 - \frac{d}{2}$ and $p \in \{2, +\infty\}$. Then, there exists a constant $C > 0$ such that for all $f \in L^\infty_z H^{s_1}_X(S)$ and for all $g \in L^p_z H^{s_2}_X(S)$, we have $fg \in L^p_z H^s_X(S)$ and*

$$|\Lambda^s(fg)|_{L^p_z L^2_X(S)} \leqslant C|\Lambda^{s_1}f|_{L^\infty_z L^2_X(S)}|\Lambda^{s_2}g|_{L^p_z L^2_X(S)}.$$

The following propositions gives estimates for $1/(1+g)$ in the flat strip S. We refer to corollary B.6 in [20].

Proposition C.4. *Let $T_0 > \frac{d}{2}$, $-T_0 \leqslant s \leqslant T_0$, $k_0 > 0$ and $p \in \{2, +\infty\}$. Then, for all $f \in L^p_z H^s_X(S)$ and $g \in L^\infty_z H^{T_0}_X(S)$ with $1 + g \geqslant k_0$, we have*

$$\left|\Lambda^s \frac{f}{1+g}\right|_{L^p_z L^2(S)} \leqslant C\left(\frac{1}{k_0}, |g|_{L^\infty_z H^{T_0}_X}\right)|\Lambda^s f|_{L^p_z L^2(S)}.$$

Proposition C.5. *Let $T_0 > \frac{d}{2}$, $s \geqslant -T_0$ and $k_0 > 0$. Then, for all $f, g \in L^\infty_z H^{T_0}_X(S) \cap H^{s,0}(S)$ with $1 + g \geqslant k_0$, we have*

$$\left|\frac{f}{1+g}\right|_{H^{s,0}} \leqslant C\left(\frac{1}{k_0}, |g|_{L^\infty_z H^{T_0}_X}\right)\left(|f|_{H^{s,0}} + \mathbf{1}_{\{s > T_0\}}|f|_{L^\infty_z H^{T_0}_X}|g|_{H^{s,0}}\right).$$

Notice that if $s \leqslant T_0$, $f \in H^{s,0}(S)$ is enough.

We need some commutator estimates in S. The following propositions are corollary B.17 in [20].

Proposition C.6. *Let $T_0 > \frac{d}{2}$, $\delta \geqslant 0$, $0 < t_1 \leqslant 1$ with $t_1 < T_0 - \frac{d}{2}$ and $-\frac{d}{2} < s \leqslant T_0 + t_1$. Then for all $u \in L^\infty_z H^{T_0}_X$ and $v \in H^{s-t_1,0}(S)$ we have*

$$\left|[\Lambda^s_\delta, u]\,v\right|_{L^2(S)} \leqslant C\left|\Lambda^{T_0}_\delta u\right|_{L^\infty_z L^2_X(S)}\left|\Lambda^{s-t_1}_\delta v\right|_{L^2(S)}.$$

Proposition C.7. *Let $T_0 > \frac{d}{2}$, $\delta \geqslant 0$, $0 < t_1 \leqslant 1$ with $t_1 < T_0 - \frac{d}{2}$ and $-\frac{d}{2} < s \leqslant T_0 + t_1$. Then for all $u \in H^{T_0,0}$ and $v \in L_z^\infty H_X^{s-t_1}$ we have*

$$\left|[\Lambda_\delta^s, u]\, v\right|_{L^2(S)} \leqslant C \left|\Lambda_\delta^{T_0} u\right|_{L^2(S)} \left|\Lambda_\delta^{s-t_1} v\right|_{L_z^\infty L_X^2(S)}.$$

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