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Smooth solutions for the dyadic model

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Abstract
We consider the dyadic model, which is a toy model to test issues of well-posedness and blow-up for the Navier–Stokes and Euler equations. We prove well-posedness of positive solutions of the viscous problem in the relevant scaling range which corresponds to Navier–Stokes. Likewise we prove well-posedness for the inviscid problem (in a suitable regularity class) when the parameter corresponds to the strongest transport effect of the nonlinearity.

Mathematics Subject Classification: 76D03, 76B03, 35Q35, 35Q30, 76D05, 35Q31

1. Introduction

We consider the dyadic model introduced in [9, 11] and lately extensively studied in several variants (viscous [5, 6, 10], inviscid [1, 3, 7, 12, 14] and stochastically forced [2]).

The dyadic model has been derived in [11] as an approximation of the ordinary differential equations describing the evolution of the coefficients of the wavelet expansion of a viscous fluid. Other authors have obtained the same model from the Fourier expansion of the solutions of the Navier–Stokes equations [8] or by the Burgers equation [14]. A remarkable fact is that the dyadic model enjoys the main features of the equations for fluid dynamics, such as the formal energy conservation, while having a much simpler mathematical structure.

Here we focus on regularity and well-posedness for positive solutions to the viscous (1.1) and to the inviscid problem (1.2).
1.1. The viscous problem

Let \( \nu > 0 \), \( \beta > 0 \) and consider

\[
\begin{align*}
\dot{X}_n &= -\nu \lambda^2 n X_n + \lambda^\beta n-1 X_{n-1}^2 - \lambda^\beta n X_n X_{n+1}, \\
X_n(0) &= x_n,
\end{align*}
\]

where \( X_0 = 0, \lambda_0 = 0, \lambda_n = \lambda^n \) for \( n \geq 1 \) and \( \lambda = 2 \). We assume that \( x_n \geq 0 \) and this implies (see [5]) that the solution remains positive at all times. The parameter \( \beta \) measures the relative strength of the nonlinearity versus the dissipation. The range of values \( \beta \in [1, \frac{5}{2}] \) is essentially that corresponding, within the simplification of the model, to the three dimensional Navier–Stokes equations. The range arises from scaling arguments applied to the nonlinear term, we refer to [6] for further details.

If \( \beta \leq 2 \) the nonlinear term is dominated by the dissipative one, in this case Cheskidov [5] proved existence of regular global solutions using classical techniques, while if \( \beta > 3 \) the nonlinearity is too strong and all solutions with large enough initial condition develop a blow-up [5].

The two results above are based on ‘energy methods’ and do not cover the range \( \beta \in (2, \frac{5}{2}] \), where it becomes crucial to understand how the structure of the nonlinearity drives the dynamics. The method proposed here (which is reminiscent of a technique used in the context of fluid mechanics in [13]) is based on purely dynamical systems techniques.

In order to prove well-posedness of the viscous problem, we identify a minimal condition that implies smoothness of solutions (proposition 3.4). The main idea then is to show the existence of an invariant region for the vector \((X_n, X_{n+1})\) by a dynamical argument (lemma 2.1) which provides the minimal condition. We are led to the following result.

**Theorem A.** Let \( \beta \in (2, \frac{5}{2}] \), then for every initial condition \((x_n)_{n \geq 1}\) such that

\[
x_n \geq 0 \quad \text{for all } n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} x_n^2 < \infty,
\]

there exists a unique weak solution in \( H \) to problem (1.1). Moreover, the solution is smooth, that is

\[
\sup_{n \geq 1} (\lambda^\gamma n X_n(t)) < \infty
\]

for all \( \gamma > 0 \) and \( t > 0 \).

1.2. The inviscid problem

It turns out that the invariant region provided by lemma 2.1 is independent of the viscosity. This allows us to consider the inviscid problem

\[
\begin{align*}
\dot{X}_n &= \lambda^\beta n X_n^{\beta} n-1 - \lambda^\beta n X_n X_{n+1}, \\
X_n(0) &= x_n,
\end{align*}
\]

It is known that there are local in time regular solutions (namely, with strong enough decay in \( n \)) and that there is a finite time blow-up, that is the quantity

\[
\sum_{n=1}^{\infty} (\lambda^\beta n X_n(t))^2 \not\to \infty
\]

when \( t \) approaches a finite time [9, 11]. Our result gives a different picture, as we prove that the dynamics generated by (1.2) is well-posed in a larger space. The correct interpretation of
both results is that the condition above involving the blowing up quantity does not provide the natural space for the solutions of the inviscid problem. Indeed, a $\lambda_n^{-\beta/3}$ decay is borderline for the conservation of energy (which does not hold rigorously for weaker decay, a proof for $\beta \leq 1$ is given in [3]).

To support the physical validity of the solutions we consider, we also prove that the global solution we have found is the unique vanishing viscosity limit. The main result for (1.2) is given in detail as follows.

**Theorem B.** Let $\beta = \frac{5}{2}$. There exists $\gamma_0 > \beta - 2$ such that for every $\gamma \in (\beta - 2, \gamma_0]$ the following statement holds. Let $x = (x_n)_{n \geq 1}$ with $x_n \geq 0$ for all $n \geq 1$ and

$$\sup_{n \geq 1}(\lambda_n^r x_n) < \infty,$$

then there is a global in time weak solution $X = (X_n)_{n \geq 1}$ to (1.2) with initial condition $x$ such that

$$\sup_{t \geq 0}(\sup_{n \geq 1}(\lambda_n^r X_n(t))) < \infty,$$

which is unique in the class of solutions satisfying the bound (1.3) above.

Moreover, $X$ is the unique vanishing viscosity limit. More precisely, if $X^{[\nu]}$ is the solution to the viscous problem (1.1) with viscosity $\nu$ and with initial condition $x$, then

$$X^{[\nu]}_n \to X_n, \quad n \geq 1,$$

as $\nu \to 0$, uniformly in time on compact sets.

The paper is organized as follows. In section 2 we prove the fundamental invariant region lemma with a dynamical systems technique. The well-posedness of the viscous problem is established in section 3, while the vanishing viscosity limit and the inviscid problem are analysed in section 4.

### 2. The invariant region lemma

In this section we prove the key result of the paper. Let $(X_n)_{n \geq 1}$ be a solution to problem (1.1) on a time interval $[0, T]$. In view of proposition 3.4, it is natural to apply the following change of variables:

$$Y_n = \lambda_n^{\beta-2\epsilon} X_n,$$

where $\epsilon > 0$ will be chosen suitably in the proof of the lemma below. A straightforward computation shows that $(Y_n)_{n \geq 1}$ solves

$$\dot{Y}_n = -\nu \lambda_n^2 Y_n + \lambda_n^{2-\epsilon} \lambda_n^{\beta-2\epsilon} Y_{n-1} - \lambda_n^{2-\epsilon} \lambda_n^{2-\beta-\epsilon} Y_n Y_{n+1},$$

$$Y_n(0) = y_n,$$

for $n \geq 1$ and $t \in [0, T]$, where clearly $y_n = \lambda_n^{\beta-2\epsilon} X_n(0)$ for all $n \geq 1$.

For technical reasons we consider a finite dimensional (truncated) version of the equations for $Y$. For every $N \geq 1$ let $(Y^{(N)}_n)_{1 \leq n \leq N}$ be the solution to

$$\dot{Y}^{(N)}_n = -\nu \lambda_n^2 Y^{(N)}_n + \lambda_n^{2-\epsilon} \lambda_n^{\beta-2\epsilon} Y^{(N)}_{n-1} - \lambda_n^{2-\epsilon} \lambda_n^{2-\beta-\epsilon} Y^{(N)}_n Y^{(N)}_{n+1},$$

$$Y^{(N)}_n(0) = y_n,$$

for $n = 1, \ldots, N$, where for the sake of simplicity we have set $Y^{(N)}_0 = 0$ and $Y^{(N)}_{N+1} = Y^{(N)}_N$, so to avoid writing the border equations in a different form. Let us now introduce the region $A$ of $\mathbb{R}^2$ that will be invariant for the vectors $(Y^{(N)}_n, Y^{(N)}_{n+1})$,

$$A := \{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, h(x) < y < g(x) \},$$

where $h$ and $g$ are the solutions to the inviscid problem with initial condition $x$. This region is defined in such a way that it is a subset of $A$. Moreover, it is clear that $A$ is invariant under the flow of the vector field $(\lambda_n^r, \lambda_n^s)$, where $r$ and $s$ are constants. This follows from the fact that the solutions of the inviscid problem are unique and the flow of the vector field is well-defined for all $x$ in $A$. Therefore, the region $A$ is invariant for the solutions of the inviscid problem.

To prove that $A$ is a fundamental invariant region, we need to show that the solutions of the viscous problem remain in $A$ for all time. This is done by using a dynamical systems technique. The well-posedness of the viscous problem is established in section 3, while the vanishing viscosity limit and the inviscid problem are analysed in section 4.
where the functions $h$ and $g$ that provide the lower and upper bound of $A$ are defined as

\[
g(x) = \min\{mx + \theta, 1\}, \quad h(x) = \begin{cases} 
0 & x \leq \delta, \\
\lambda^2 \frac{(x - \delta)}{1 - \delta} & x > \delta,
\end{cases}
\]

and $\theta \geq \delta$ so that the little square $[0, \delta]^2$ is all inside $A$.

**Lemma 2.1.** There exist $\delta \in (0, 1), c \in (0, 1), \theta \in (\delta, 1), m > 1 - \theta$ and $\epsilon_0 > 0$ such that for every $\beta \in (2, \frac{5}{2}], \nu > 0$ and $\epsilon \in (0, \epsilon_0)$ the following statement holds true: if $N \geq 1$ and if $(Y_n, Y_{n+1}) \in A$ for all $n \leq N$, then $(Y_n^{(N)}(t), Y_{n+1}^{(N)}(t)) \in A$ for all $n = 1, \ldots, N$ and $t \geq 0$, where $(Y_n^{(N)})_{1 \leq n \leq N}$ is the solution to (2.2) with initial condition $(y_n)_{1 \leq n \leq N}$.

**Proof.** For simplicity we drop the superscript $(N)$ along this proof. Since the pairs $(Y_n, Y_{n+1})_{1 \leq n \leq N}$ satisfy a finite dimensional system of differential equation, it is sufficient to show that the derivative in time of $(Y_n, Y_{n+1})$ points inward on the border of $A$ when $(Y_n, Y_{n+1}) \in A$ for each $n = 1, \ldots, N$ or, equivalently, that the scalar product with the inward normal of the border of $A$ with the vector field

\[
\mathfrak{B} = (\dot{Y}_n, Y_{n+1}) = \nu \lambda^2_n \begin{pmatrix} -Y_n \\
-\lambda^2 Y_{n+1} \end{pmatrix} + \frac{\lambda^2 - 4 + 2\epsilon}{\lambda^2} \begin{pmatrix} \frac{Y^2_{n-1} - \lambda^2 - 2\beta - 3\epsilon Y_n}{\lambda^2} Y_{n+1} \\
\lambda^2 - (\frac{Y^2_n - \lambda^2 - 2\beta - 3\epsilon Y_{n+1} Y_{n+2}}{\lambda^2}) \end{pmatrix}
\]

is positive when $(Y_n, Y_{n+1}) \in A$ for all $n = 1, \ldots, N$. The set $A$ is convex, hence we can consider separately the viscous and the inviscid contribution to $\mathfrak{B}$.

We start with the viscous part, which we denote by $\mathfrak{B}_v$ (we neglect the multiplicative constant $\nu \lambda^2_n$) and we denote the inward normals as in figure 1. The scalar product of $\mathfrak{B}_v$ with each $\mathfrak{n}_1, \mathfrak{n}_3, \mathfrak{n}_4, \mathfrak{n}_6$ on the respective pieces of the border of $A$ with the vector field

\[
\mathfrak{B}_v = -Y_n = 0, \quad \mathfrak{B}_v \cdot \mathfrak{n}_3 = \lambda^2 Y_{n+1} = \lambda^2, \quad \mathfrak{B}_v \cdot \mathfrak{n}_4 = Y_n = 1, \quad \mathfrak{B}_v \cdot \mathfrak{n}_6 = -\lambda^2 Y_{n+1} = 0,
\]

so we are left with the last two cases, in which, for simplicity, we set $x = Y_n$. First,

\[
\mathfrak{B}_v \cdot \mathfrak{n}_2 = -xg'(x) + \lambda^2 Y_{n+1} = -xg'(x) + \lambda^2 g(x) = m(\lambda^2 - 1)x + \theta \lambda^2 > 0,
\]

then

\[
\mathfrak{B}_v \cdot \mathfrak{n}_5 = xh'(x) - \lambda^2 Y_{n+1} = xh'(x) - \lambda^2 h(x) = \frac{c\delta \lambda^2 (x - \delta)}{1 - \delta} \frac{x - \delta}{1 - \delta} \geq 0.
\]
We consider now the inviscid term, that we denote by $\mathcal{B}_1$ (and again we neglect the irrelevant multiplicative factor). Again we set $x = Y_n$ and, for simplicity, $\gamma' = 6 - 2\beta - 3\epsilon$. We consider first the easy terms,

$$
\mathcal{B}_1 \cdot \tilde{n}_1 = Y_{n-1}^2 - \lambda\gamma' x Y_{n+1} = Y_{n-1}^2 \geq 0.
$$

$$
\mathcal{B}_1 \cdot \tilde{n}_2 = \lambda^{2-\gamma}(x^2 - \lambda\gamma' Y_{n+1} Y_{n+2}) = \lambda^{2-\gamma} x^2 \geq 0.
$$

Next, we consider the piece of the border of $A$ corresponding to $\tilde{n}_3$. Here $Y_{n+1} = 1$ and $x \leq 1$, moreover since $(Y_{n+1}, Y_{n+2}) \in A$, it follows that $Y_{n+2} \geq c$, hence

$$
\mathcal{B}_1 \cdot \tilde{n}_3 = \lambda^{2-\gamma}(\lambda\gamma' Y_{n+1} Y_{n+2} - x^2) \geq \lambda^{2-\gamma}(\lambda^{\gamma} c - 1).
$$

The term on the right-hand side in the formula above is positive if we choose $\lambda^{\gamma} c = 1$. Likewise on the piece corresponding to $\tilde{n}_4$ we have $x = Y_n = 1$, $Y_{n+1} \geq c$ and $Y_{n-1} \leq 1$, hence

$$
\mathcal{B}_1 \cdot \tilde{n}_4 = \lambda^{\gamma} x Y_{n+1} - Y_{n-1}^2 \geq \lambda^{\gamma} c - 1 \geq 0.
$$

We are left with the two challenging inequalities, that we are going to analyse. The first is on the piece of boundary corresponding to $\tilde{n}_2$, where we have $Y_{n+1} = g(x)$ and, since $(Y_{n+1}, Y_{n+2}) \in A$, $Y_{n+2} \geq h(Y_{n+1}) = h(g(x)) = c \left( \frac{m x^{\gamma} \lambda}{1-m} \right)^{\frac{1}{2}}$, since $\theta \geq \delta$. Hence, using the fact that $\gamma^{\gamma} c = 1$ and that $\gamma \leq 2 - 3\epsilon$,

$$
\mathcal{B}_1 \cdot \tilde{n}_2 = g'(x)(Y_{n-1}^2 - \lambda\gamma' x Y_{n+1}) - \lambda^{2-\gamma}(x^2 - \lambda\gamma' Y_{n+1} Y_{n+2})
$$

$$
\geq -\lambda\gamma' x g'(x) g(x) - \lambda^{2-\gamma}(x^2 - \lambda\gamma' g(x) h(g(x)))
$$

$$
= \lambda^{2-\gamma}(mx + \theta) \left( \frac{mx + \theta - \delta}{1 - \delta} \right)^{\frac{1}{2}} - \lambda^{2-\gamma} x^2 - \lambda^{\gamma} m x (mx + \theta)
$$

$$
\geq \lambda^{2-\gamma}(mx + \theta) \left( \frac{mx + \theta - \delta}{1 - \delta} \right)^{\frac{1}{2}} - \lambda^{2-\gamma} x^2 - \lambda^{\gamma} m x (mx + \theta).
$$

This last expression depends on $x$ but not on $\beta$ and it is sufficient to show that it is non-negative for $x \in [0, \frac{1-\delta}{m}]$. This will be done later by a suitable choice of the parameters.

Prior to this, we consider the second inequality, on the piece corresponding to $\tilde{n}_5$. Here we have that $Y_{n+1} = h(x)$ and $Y_{n-1} \leq 1$, and, since $(Y_{n+1}, Y_{n+2}) \in A$, $Y_{n+2} \leq g(Y_{n+1}) = g(h(x)) \leq mh(x) + \theta$. Therefore, since $x(\frac{x}{1-\delta})^2 \leq 1$ and $\gamma \geq 1 - 3\epsilon$, hence $\lambda^{\gamma} \leq \lambda^{3\epsilon - 1}$,

$$
\mathcal{B}_1 \cdot \tilde{n}_5 = \lambda^{2-\gamma}(x^2 - \lambda\gamma' Y_{n+1} Y_{n+2}) - h'(x)(Y_{n-1}^2 - \lambda\gamma' x Y_{n+1})
$$

$$
\geq \lambda^{2-\gamma}(x^2 - \lambda\gamma' h(x) g(h(x))) - h'(x)(1 - \lambda\gamma' x h(x))
$$

$$
\geq \lambda^{2-\gamma} \left[ x^2 - \theta \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}} \right] - m \lambda^{2-\gamma} \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}}
$$

$$
= \lambda^{2-\gamma} \left[ x - \theta \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}} \right] - m \lambda^{2-\gamma} \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}}
$$

$$
\geq \lambda^{2-\gamma} \left[ x - \theta \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}} \right] - m \lambda^{2+2\epsilon} \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}}
$$

$$
\geq \lambda^{2-\gamma} \left[ x - \theta \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}} \right] - m \lambda^{2+2\epsilon} \left( \frac{x - \delta}{1 - \delta} \right)^{\frac{1}{2}}
$$

for $x \in [\delta, 1]$. Also this lower bound does not depend on $\beta$. 

\[ \text{Smooth solutions for the dyadic model} \]
Let $\psi_1$ and $\psi_2$ be the right-hand sides of (2.3) and (2.4), respectively, when $\epsilon = 0$, namely,

$$\psi_1(x) = \lambda_2^2 \left(\frac{mx + \theta - \delta}{1 - \delta}\right)^{\lambda^2} - \lambda_2^2 x^2 - m \lambda_2^2 x (mx + \theta),$$

$$\psi_2(x) = \lambda_2^2 \left[\frac{x^2 - \theta}{1 - \delta}\right]^{\lambda^2} - m \lambda_2 \left(\frac{x - \delta}{1 - \delta}\right)^{2\lambda^2} - \frac{\lambda_2}{1 - \delta} \left(\frac{x - \delta}{1 - \delta}\right)^{\lambda^2 - 1} \left[1 - x \left(\frac{x - \delta}{1 - \delta}\right)^{\lambda^2}\right].$$

It is sufficient to show that both functions have positive minimal values. Continuity then ensures that the same is true for small $\epsilon$. A direct computation shows that both $\psi_1$ and $\psi_2$ are positive with the choice $\delta = \frac{1}{10}$, $\theta = \frac{3}{5}$, $m = \frac{3}{4}$ (we recall that $\lambda = 2$). Figure 2 shows a plot of the two functions.

**Remark 2.2.** A cleverer choice of the parameters $\delta$, $\theta$ and $m$ allows us to extend the above result, and in turn the main results of the paper, to values of $\beta$ larger than $\frac{5}{2}$ (although smaller than 3, due to the blow-up results in [5, 9]).

3. Uniqueness and regularity in the viscous case

Define

$$H = \left\{ x = (x_n)_{n \geq 1} \subset \mathbb{R} : \|x\|_H^2 := \sum_{n=1}^{\infty} x_n^2 < \infty \right\}. \quad (3.1)$$

Following Cheskidov [5] we introduce weak and Leray–Hopf solutions for (1.1).

**Definition 3.1.** A weak solution to (1.1) on $[0, T]$ is a sequence of functions $X = (X_n)_{n \geq 1}$ such that $X_n \in C^1([0, T]; \mathbb{R})$ for every $n \geq 1$ and (1.1) is satisfied.

A positive solution is a weak solution having all components positive.

A Leray–Hopf solution is a weak solution $X$ with values in $H$ and such that the energy inequality

$$\|X(t)\|_H^2 + 2v \int_s^t \sum_{n=1}^{\infty} (\lambda_n X_n(r))^2 \, dr \leq \|X(s)\|_H^2,$$

holds for a.e. $s \geq 0$ and all $t > s$. 
The following facts are proved in [5],

- existence of global in time Leray–Hopf solutions for all initial conditions in $H$,
- if the initial condition $(x_n)_{n \geq 1}$ is positive, namely $x_n \geq 0$ for all $n \geq 1$, then every weak solution is a Leray–Hopf solution, stays positive for all times and the energy inequality holds for all times $s, t$ (not only for a.e. $s$),
- if $\beta \leq 2$, there is a unique Leray–Hopf solution which is smooth, for every initial condition in $H$,
- if $\beta > 3$, then every positive solution (starting from a large enough initial condition) cannot be smooth for all times.

Our first result is a criterion for uniqueness of positive solutions.

**Proposition 3.2 (Uniqueness).** Let $X = (X_n)_{n \geq 1}$ be a positive solution to (1.1) on $[0, T]$ with initial condition $X(0) \in H$ and such that the quantity

$$\sup_{t \in [0,T], n \geq 1} (\lambda_n^{\beta-3} X_n(t))$$

is finite. Then $X$ is the unique weak solution with initial condition $(X_n(0))_{n \geq 1}$.

In particular, if $\beta \leq 3$, there is a unique weak solution for any positive initial condition in $H$.

**Proof.** The proof is a minor variation of the idea in [1]. Let $Y = (Y_n)_{n \geq 1}$ be another solution with the same initial condition of $X$ and set $Z_n = Y_n - X_n$, $W_n = X_n + Y_n$, then

$$\dot{Z}_n = -\nu \lambda_n^2 Z_n + \lambda_{n-1}^\beta Z_{n-1} W_{n-1} - \frac{1}{2} \lambda_n^\beta (Z_n W_{n+1} + Z_{n+1} W_n).$$

Fix $N \geq 1$ and set $\psi_N(t) = \sum_{n=1}^N \frac{1}{2n} Z_n^2$, then $\psi_N(0) = 0$ and it is elementary to verify that

$$\frac{d}{dt} \psi_N(t) + 2\nu \sum_{n=1}^N \frac{\lambda_n^\beta}{2n} Z_n^2 = - \sum_{n=1}^N \frac{\lambda_n^\beta}{2n} Z_n^2 W_{n+1} - \frac{\lambda_N^\beta}{2N} Z_N Z_{N+1} W_N.$$

In particular (we recall that $\lambda = 2$ and $\lambda_n = \lambda^n$),

$$\frac{d}{dt} \psi_N(t) \leq -\lambda_N^{-\beta-1} Z_N Z_{N+1} W_N$$

$$= -\lambda_N^{-\beta-1} (Y_N^2 Y_{N+1} + X_N^2 X_{N+1} - X_N Y_N) Y_N^2 - X_N Y_{N+1})$$

$$\leq \lambda_N^{-\beta-1} (X_{N+1}^2 Y_N^2 + X_N^2 Y_{N+1}).$$

By assumption (3.2) there exists a constant $c_0 > 0$ such that

$$\frac{d}{dt} \psi_N(t) \leq c_0 \lambda_N^2 (X_N^2 + Y_N^2 + Y_{N+1}^2),$$

and so by integrating in time,

$$\psi_N(t) \leq c_0 \int_0^t \lambda_N^2 (X_s^2 + Y_s^2 + Y_{s+1}^2) \, ds.$$

Since $X$ and $Y$ are both Leray–Hopf solutions, the right-hand side in the above inequality converges to 0 as $N \to \infty$ and in conclusion $\psi_n(t) = 0$ for all $t \geq 0$ and all $n \geq 1$. \qed
3.1. Proof of theorem A

Having the key lemma 2.1 in hand, we have now all ingredients for the proof of the main theorem concerning the viscous case. Set
\[ D^\infty = \left\{ (x_n)_{n \geq 1} \subset \mathbb{R} : \sup_{n \geq 1} (\lambda_n^{\nu} |x_n|) < \infty \text{ for all } \nu > 0 \right\}, \]
then theorem A can be rephrased in the following way: given a positive initial condition \( x \in \mathcal{H} \), there is a unique weak solution \( X \), moreover \( X(t) \in D^\infty \) for every \( t > 0 \).

**Proof of theorem A.** Let \( x \in \mathcal{H} \) be positive and let \( X = (X_n)_{n \geq 1} \) be the weak solution starting at \( x \) which is unique due to proposition 3.2. The proof is divided for the sake of clarity into three steps.

**Step 1: Reduction to smoother initial conditions.** We show that we can work with smoother initial conditions. Indeed, by the energy inequality,
\[ \sum_{n=1}^{\infty} \int_{\mathbb{R}} (\lambda_n X_n(r))^2 \, dr < \infty, \]
for a.e. \( s > 0 \) and all \( t > s \), hence \( \sup_{n \geq 1} (\lambda_n X_n(t)) < \infty \) for a.e. \( t > 0 \). In particular we can find a sequence \( t_k \downarrow 0 \) such that
\[ \sup_{n \geq 1} (\lambda_n X_n(t_k)) < \infty. \]
To prove the theorem, it is sufficient to prove that the solution is smooth for all \( t \geq t_k \), for each \( k \). Therefore there is no loss of generality if we assume that the initial condition satisfies \( \sup_{n \geq 1} \lambda_n x_n < \infty \). We will do so in the rest of the proof.

**Step 2: The invariant area argument.** Assume that \( \sup_{n \geq 1} \lambda_n x_n < \infty \). Let \( \epsilon \in (0, 3 - \beta] \) be such that \( \epsilon \leq \epsilon_0 \), where \( \epsilon_0 \) is in the statement of lemma 2.1, and set \( K_0 = \sup_{n \geq 1} \lambda_n^{\beta - 2 \epsilon} x_n \). Set
\[ \tilde{Y}_n(t) = \frac{\delta}{K_0} \lambda_n^{\beta - 2 \epsilon} X_n \left( \frac{\delta}{K_0} t \right), \quad n \geq 1, \ t \geq 0, \]
where \( \delta \) is the constant from lemma 2.1. It turns out that \( (\tilde{Y}_n)_{n \geq 1} \) is a solution to (2.1) but with viscosity \( \tilde{\nu} = \frac{\lambda_n}{K_0} \nu \). Uniqueness of \( (X_n)_{n \geq 1} \) clearly ensures uniqueness of \( (\tilde{Y}_n)_{n \geq 1} \) for equation (2.1) and so it is standard to show that the solutions \( (\bar{Y}_n(N))_{n \geq 1} \) of (2.2) (with viscosity \( \bar{\nu} \)) converge to \( (\tilde{Y}_n)_{n \geq 1} \). Clearly \( \sup_{n \leq N} \bar{Y}_n(0) \leq \delta \) for all \( N \geq 1 \), therefore lemma 2.1 ensures that \( Y_n(N)(t) \leq 1 \) and in turn
\[ \sup_{n \geq 1} \lambda_n^{\beta - 2 \epsilon} X_n(t) \leq \frac{K_0}{\delta}. \quad (3.3) \]

**Step 3: Smoothness by a local solution argument.** It is sufficient now to show that there is \( \eta > 0 \) such that for every \( t_0 > 0 \), \( X(t) \in D^\infty \) for all \( t \in (t_0, t_0 + \eta) \).

Let \( V_n(t) = X_n(t)e^{\lambda_n(t-t_0)} \), a direct computation shows that
\[ \dot{V}_n = -\nu(\lambda_n^2 - \lambda_n) V_n + \lambda_n^{\beta - 2 \epsilon} V_{n-1}^2 - \lambda_n^{\beta - 2 \epsilon} e^{-\nu(t-t_0)} V_n V_{n+1} \]
with \( V_n(t_0) = X_n(t_0) \). To prove smoothness of \( X \) in a small interval, it is sufficient to show that \( V \) is bounded (uniformly in \( n \)) in the same interval. This follows by a standard Banach’s fixed point argument, which we sketch below. Note that we will be able to find a common size \( \eta \) for the small interval due to the uniform estimate (3.3). Let
\[ V_\epsilon = \left\{ x = (x_n)_{n \geq 1} : \|x\|_{V_\epsilon} := \sup_{n \geq 1} (\lambda_n^{\beta - 2 \epsilon} |x_n|) < \infty \right\}. \]
and define component-wise the map $\mathcal{F}$ on $L^\infty(t_0, t_0 + \eta; V_\epsilon)$ as

$$
\mathcal{F}_n(V)(t) = e^{-\nu(\lambda^2 - \lambda_n) t} X_n(t_0) + \int_{t_0}^t e^{-\nu(\lambda^2 - \lambda_n) (s-t)} \left( \lambda^2_{n-1} V_n(t) - \lambda_n^2 e^{-\nu(\lambda^2 - \lambda_n) (t-s)} V_{n+1}(s) \right) \, ds
$$

for $t \geq t_0$. Note that $\lambda^2_\nu - \lambda_\nu \geq \frac{1}{2}\lambda^2_\nu$. To apply the fixed point theorem we need to show that there are suitable $\eta > 0$ and $R > 0$ such that $\mathcal{F}$ maps a ball of $L^\infty(t_0, t_0 + \eta; V_\epsilon)$ centred in the initial condition (considered as a constant function) and of radius $R$ into itself and is a contraction in that ball.

It is easy to see that there is $c > 0$ (depending only on $\lambda$, $\nu$, $\beta$ and $\epsilon$) such that

$$
\|\mathcal{F}(V) - X(t_0)\|_{\infty, V_\epsilon} \leq \|X(t_0)\|_{V_\epsilon} + c\eta^{\frac{1}{2}} \|V\|_{\infty, V_\epsilon},
$$

$$
\|\mathcal{F}(V) - \mathcal{F}(\tilde{V})\|_{\infty, V_\epsilon} \leq c\eta^{\frac{1}{2}} \|V - \tilde{V}\|_{\infty, V_\epsilon} (\|V\|_{\infty, V_\epsilon} + \|\tilde{V}\|_{\infty, V_\epsilon}),
$$

for instance for $t \in [t_0, t_0 + \eta]$,

$$
\lambda_n^{\beta - 2\epsilon} |\mathcal{F}(V)(t) - \mathcal{F}(\tilde{V})(t)| \\
\leq \lambda_n^{\beta - 2\epsilon} \int_{t_0}^t e^{-\frac{\nu}{2}(\lambda^2 - \lambda_n) (s-t)} (\lambda^2_{n-1} - \lambda^2_n) V_n(t) - \lambda^2_n e^{-\nu(\lambda^2 - \lambda_n) (t-s)} V_{n+1}(s) \, ds \\
\leq c \|V - \tilde{V}\|_{\infty, V_\epsilon} \|V\|_{\infty, V_\epsilon} (\|V\|_{\infty, V_\epsilon} + \|\tilde{V}\|_{\infty, V_\epsilon}) \lambda_n^{\beta - 2\epsilon} \int_{t_0}^t e^{-\frac{\nu}{2}(\lambda^2 - \lambda_n) (t-s)} \, ds \\
\leq c\eta^{\frac{1}{2}} \|V - \tilde{V}\|_{\infty, V_\epsilon} (\|V\|_{\infty, V_\epsilon} + \|\tilde{V}\|_{\infty, V_\epsilon}),
$$

and the other inequality follows by similar computations.

Hence it is sufficient to choose $R > \|X(t_0)\|_{V_\epsilon}$ and $\eta$ small enough so that the required properties are satisfied. We stress again that the size of $\eta$ depends only on $\|X(t_0)\|_{V_\epsilon}$ and this quantity is uniformly bounded by (3.3). In other words, the size of $\eta$ depends only on $\frac{K}{\nu}$.

We state explicitly a consequence of one of the steps of the proof above for later use in the inviscid case. Its proof follows directly from the second step of the proof above (see formula (3.3)).

**Corollary 3.3.** If $x \in H$ is positive and satisfies

$$
\sup_{n \geq 1} \lambda_n^{\beta - 2\epsilon} x_n < \infty
$$

for some $\epsilon > 0$, then

$$
\sup_{\nu > 0} \sup_{t \geq 0} \sup_{n \geq 1} (\lambda_n^{\beta - 2\epsilon} X^\nu_n(t)) \leq \frac{1}{\delta} \sup_{n \geq 1} \lambda_n^{\beta - 2\epsilon} x_n,
$$

where $\delta$ is the constant of lemma 2.1 and $X^\nu$ is the weak solution of (1.1) with viscosity $\nu$ and with initial condition $x$.

### 3.2. Stationary solutions and critical regularity

In this section we elaborate with more details on the conditions for regularity for solutions of (1.1). The problem is completely solved in theorem A whenever the solution is positive. For this reason in this section (and only in this section) we shall consider solutions to (1.1) which may have also non-positive components.

The following proposition gives an optimal condition of regularity for solutions with no sign. The condition is quite similar to the one given in [4, theorem 3.1] if one interprets the components of $X$ as the Littlewood–Paley decomposition of the solution to the Navier–Stokes equations. The main interest in the case discussed here is that there exist weak solutions that are not smooth, although they are very close to satisfy the regularity criterion (3.4).
Proposition 3.4. Let $\beta > 2$. There exists $c_0 > 0$ such that if $T > 0$ and $X$ is a weak solution of (1.1) on $[0, T]$ with $X(0) \in D^{s\infty}$ and

$$\limsup_{n \to \infty} \left( \sup_{t \in [0, T]} (\lambda_n^{\beta-2} |X_n(t)|) \right) < c_0 v,$$

then $X(t) \in D^{s\infty}$ for all $t \in [0, T]$.

**Proof.** Choose $c_0 > 0$ so that $c_0(\lambda_0^{\beta-4} \vee \lambda^{2-\beta}) < \frac{1}{2}$ and fix $\epsilon > 0$ such that also $c_0(\lambda_0^{\beta-4+\epsilon} \vee \lambda^{2-\beta}) < \frac{1}{2}$. By the assumption there is an integer $n_0 \geq 1$ such that

$$\sup_{n \geq n_0} \left( \sup_{t \in [0, T]} (\lambda_n^{\beta-2} |X_n(t)|) \right) < c_0 v.$$

Since

$$X_n(t) = e^{-\nu \lambda_n^2 t} X_n(0) + \int_0^t e^{-\nu \lambda_n^2 (t-s)} (\lambda_n^\beta X_n^2 - \lambda_n^\beta X_{n+1}) ds,$$

we have that for $n \geq n_0 + 1$,

$$|X_n(t)| \leq |X_n(0)| + c_0 v 2^\epsilon \int_0^t e^{-\nu \lambda_n^2 (t-s)} (\lambda_n^\beta |X_n-1| + \lambda^{2-\beta} |X_n|) ds.$$

Set $G_n = \sup_{t \in [0, T]} (\lambda_n^{\beta-2+\epsilon} |X_n(t)|)$, then

$$G_n \leq \lambda_n^{\beta-2+\epsilon} |X_n(0)| + c_0(\lambda_n^{\beta-4+\epsilon} \vee \lambda^{2-\beta})(G_{n-1} + G_n),$$

Set $c_1 = c_0(\lambda_n^{\beta-4+\epsilon} \vee \lambda^{2-\beta})$, then $c_1 < \frac{1}{2}$, $c_1 < 1$ and

$$G_n \leq \frac{1}{1 - c_1} \sup_{t \in [0, T]} (\lambda_n^{\beta-2+\epsilon} |X_n(t)|) + \frac{c_1}{1 - c_1} G_{n-1},$$

therefore $\sup_{n \geq n_0} G_n < \infty$. In conclusion,

$$G := \sup_{n \geq 1} \left( \sup_{t \in [0, T]} (\lambda_n^{\beta-2+\epsilon} X_n(t)) \right) < \infty.$$

The fact that the above quantity is finite allows us to show that $X$ is actually a Leray–Hopf solution, hence smoothness follows as in the third step of the proof of theorem A. It remains to show that $X$ is a Leray–Hopf, that is that the energy inequality holds true. Indeed, for $n \geq 1$,

$$\frac{d}{dt} \sum_{k=1}^{n} X_k^2 + 2v \sum_{k=1}^{n} (\lambda_k X_k^2) \leq -2\lambda_n^\beta X_n^2 X_{n+1} \leq v \sum_{k=1}^{n} (\lambda_k X_k^2) + cG^{2\epsilon},$$

since by Young’s inequality,

$$|\lambda_n^\beta X_n^2 X_{n+1}| \leq \lambda_n^{-\epsilon} |\lambda_n^{\beta-2+\epsilon} X_n| \leq c G^{2\epsilon} \leq v \sum_{k=1}^{n} (\lambda_k X_k)^2 + cG^{2\epsilon},$$

hence $X$ satisfies

$$\|X(t)\|^2 + v \sum_{n=1}^{\infty} \int_0^t (\lambda_n X_n(r))^2 dr \leq \|X(0)\|^2 + cG^{2\epsilon} t,$$

and with similar computations as above and the Hölder inequality, for every $s, t, s \leq t$,

$$\left| \int_s^t \lambda_n X_n^2 X_{n+1} dr \right| \leq G^{1+\epsilon} (t-s)^{\frac{1}{2}} \left( \int_s^t (\lambda_n X_n)^2 dr \right)^{\frac{1}{2}} \to 0,$$

as $n \to \infty$, which actually implies the energy equality. 

$\square$
Remark 3.5. It is clear by the proof that it is sufficient to assume that
\[ \sup_{n \geq 1} (\lambda_n^{\beta - 2\gamma} |X_n(0)|) < \infty \]
for some \( \gamma > 0 \) to conclude that \( X(t) \in D^\infty \) for \( t > 0 \).

Remark 3.6. The \( \lambda_n^{\beta - 2} \) decay can be interpreted in terms of local existence and uniqueness of smooth solutions. Indeed, this decay is critical in the sense that only exponents larger or equal than \( \beta - 2 \) allow for local smooth solutions (for any general quadratic finite-range interaction nonlinearity with growth of order \( \lambda_n^{\beta} \), without taking the geometry into account). This can be seen easily as in the third step of the proof of theorem A.

The critical exponent \( \beta - 2 \) is also related to a scale invariance property of the equations given by
\[ X_n(t) \rightarrow \lambda_k^{\beta - 2} X_{n+k}(\lambda_k^{\beta} t) , \]
which is somewhat reminiscent of the scale invariance \( u(t, x) \rightarrow cu(c^2 t, cx) \) for the Navier–Stokes equations. The above scale invariance is not exact, due to the ‘boundary’ term \( X_0 = 0 \).

3.2.1. Existence of stationary solutions. We show that the condition given in proposition 3.4 is optimal by showing that there is a weak solution to (1.1) such that the quantity \( \sup_{t,n} (\lambda_n^{\beta - 2} |X_n(t)|) \) is bounded but the solution is not smooth. The example is provided by a time-stationary solution.

It is clear that any stationary solution cannot be a Leray–Hopf solution, hence it cannot be positive [5]. The proposition below will show that a stationary solution must have all components negative.

We shall call stationary solution any sequence \( \gamma = (\gamma_n)_{n \geq 1} \) such that
\[ \nu \lambda_n^{2\gamma_n} + \lambda_n^{\beta} \gamma_n \gamma_{n+1} - \lambda_{n-1}^{\beta} \gamma_{n-1}^2 = 0, \quad n \geq 1, \tag{3.5} \]

Proposition 3.7. Let \( \gamma = (\gamma_n)_{n \geq 1} \) be a non-zero stationary solution.

• If there is \( n_0 \geq 1 \) such that \( \gamma_{n_0} = 0 \), then \( \gamma_n = 0 \) for all \( n \leq n_0 \).
• Let \( n_0 \) be the first index such that \( \gamma_{n_0} \neq 0 \). Then \( \gamma_n < 0 \) for all \( n > n_0 \).
• Let \( n_0 \) be the first index such that \( \gamma_{n_0} \neq 0 \). Then there is \( c > 0 \) such that \( \lambda_n^{\beta - 2} |\gamma_n| \geq c \), for all \( n \geq n_0 \).

Proof. Multiply (3.5) by \( \gamma_n \) and sum up to \( N \) to obtain
\[ \nu \sum_{n=1}^{N} \lambda_n^{2\gamma_n} + \lambda_n^{\beta} \gamma_n \gamma_{n+1} = 0, \quad N \geq 1. \]

The first two properties follow from this equality. For the third property, (3.5) implies that
\[ \gamma_{n+1} = \frac{\gamma_{n+1}^{\beta - 2}}{\lambda_n^{\beta}} - \nu \lambda_n^{2 - \beta} \leq -\nu \lambda_n^{2 - \beta} , \]
since all \( \gamma_n \) are negative. \( \square \)

Hence a stationary solution can decay at most as the critical profile which is borderline in proposition 3.4. So the existence of a stationary solutions shows that the condition of proposition 3.4 is optimal. Moreover, if the stationary solution is in \( H \), this provides an example of two weak solutions with the same initial condition (the stationary solution and the Leray–Hopf solution).
Proposition 3.8. If \( \lambda^{2\beta-6} < \frac{1}{u} \) (that is, \( \beta \leq 2.2 \)), then there are infinitely many stationary solutions which decay exactly as \( \lambda^{2-\beta} \). More precisely, there are infinitely many stationary solutions \((\gamma_n)_{n \geq 1}\) such that

- \( \gamma_n < 0 \) for every \( n \geq 1 \),
- there are \( c_1, c_2 > 0 \) such that \( c_1 \leq \lambda^{\beta-2} |\gamma_n| \leq c_2 \).

Proof. We look for a stationary solution \((\gamma_n)_{n \geq 1}\). To this end, set \( u = \lambda^{2\beta-6} \) and consider the change of unknowns \( \gamma_n = -\nu \lambda^{2\beta-6} \), so that \( \gamma \) is a stationary solution if and only if

\[
\begin{align*}
  a_1(a_2 - 1) &= 0, \\
  a_n a_{n+1} &= a_n + u a_{n-1}^2, \quad n \geq 2.
\end{align*}
\]  (3.6)

Consider \( a_1 \geq 0 \) (to be determined), set \( a_2 = 1 \), so that all others \( a_n \) can be computed explicitly in terms of \( a_1 \) and let

\[
\begin{align*}
  A_u &= \frac{1}{2u} \left( 1 - \sqrt{\frac{1 - 3u}{1 + u}} \right), \\
  B_u &= \frac{1}{2u} \left( 1 + \sqrt{\frac{1 - 3u}{1 + u}} \right).
\end{align*}
\]

It is easy to verify that the recursion in (3.6) has the fixed point \( \frac{1}{1-u} \) and an order 2 cycle exchanging \( A_u \) and \( B_u \). Moreover if \( a_{n-1}, a_n \in [A_u, B_u] \), then \( a_{n+1} \in [A_u, B_u] \).

So to prove the existence of a stationary solution, it is sufficient to find values of \( a_1 \) such that the sequence \((a_n)_{n \geq 1}\) ends up in \([A_u, B_u]\). For instance, if we ask that \( a_3, a_4 \in [A_u, B_u] \), after some obvious computations we find the condition

\[
\frac{A_u - 1}{u} < a_1^2 < \frac{1}{A_u - 1} - \frac{1}{u}.
\]

which ensures an infinite number of suitable \( a_1 \) when \( u \) is smaller than about 0.31. A picture of the region, in \( u \) and \( a_1 \), is sketched in figure 3. Larger values of \( u \) may be achieved with similar computations. \( \Box \)

4. The inviscid limit

Following [1], we give the following definitions of solution.

Definition 4.1. A weak solution on \([0, T)\) (global if \( T = \infty \)) of (1.2) is a sequence \( X = (X_n)_{n \geq 1} \) of functions such that \( X_n \in C^1([0, T]; \mathbb{R}) \) for all \( n \geq 1 \) and (1.2) is satisfied.

A positive solution is a weak solution having all components positive.

A Leray–Hopf solution is a weak solution such that \( X(t) \in H \) (where \( H \) is defined in (3.1)) and the energy inequality

\[
\|X(t)\|_H \leq \|X(s)\|_H
\]

holds for a.e. \( s \geq 0 \) and all \( t \geq s \).
We give a short summary of known facts on solutions to (1.2).

• There is at least one global in time Leray–Hopf solutions for all initial conditions in $H$ (see [7], the proof is given for $\beta = \frac{5}{2}$ but the extension to all $\beta$ is straightforward).

• There is a unique local in time solution for ‘regular’ enough initial conditions [9].

• There is a unique local in time solution for ‘regular’ enough initial conditions [9].

• If the initial condition $(x_n)_{n \geq 1}$ is positive, then every weak solution stays positive for all times, is Leray–Hopf and the energy inequality holds for all times [1].

• If $\beta \leq 1$, there is a unique Leray–Hopf solution for every positive initial condition [1].

• No positive solution can be smooth for all times [7, 12]: if $\beta = \frac{5}{2}$, then the quantity $\lambda_n^{5/6}X_n(t)$ cannot be bounded for all times.

We first start by giving a uniqueness criterion, based again on the idea in [1].

Lemma 4.2 (Uniqueness). Given $T > 0$, let $X = (X_n)_{n \geq 1}$ be a positive solution to (1.2) on $[0, T]$.

• If the quantity
  \[ \sup_{t \in [0, T]} \sup_{n \geq 1} (\beta - 1) X_n(t) \]  (4.1)
is finite, then $X$ is the unique solution with initial condition $(X_n(0))_{n \geq 1}$ in the class of Leray–Hopf solutions.

• If for some $\epsilon > 0$ the quantity
  \[ \sup_{t \in [0, T]} \sup_{n \geq 1} \left( \frac{1}{\lambda_n} \left( \beta - 1 + \epsilon \right) X_n(t) \right) \]  (4.2)
is finite, then $X$ is the unique weak solution with initial condition $(X_n(0))_{n \geq 1}$ in the class of weak solutions satisfying (4.2).

Proof. We follow the same lines (with the same notation) of the proof of proposition 3.2. Denote by $c_0$ the quantity (4.1). Let $Y = (Y_n)_{n \geq 1}$ be another weak solution with the same initial condition of $X$. Then for $N \geq 1$,

\[
\frac{d}{dt} \psi_N(t) \leq -\lambda_N^{\beta - 1} Z_N Z_{N+1} W_N \\
\quad \leq \lambda_N^{\beta - 1} (X_{N+1} Y_N^2 + X_N^2 Y_{N+1}) \\
\quad \leq c_0 (X_N^2 + Y_N^2 + Y_{N+1}^2),
\]

and so by integrating in time,

\[
\psi_N(t) \leq c_0 \int_0^t (X_n^2 + Y_n^2 + Y_{n+1}^2) \, ds.
\]

Since $X$ and $Y$ are both Leray–Hopf solutions, the right–hand side in the above inequality converges to 0 as $N \to \infty$ and in conclusion $\psi_n(t) = 0$ for all $t \geq 0$ and all $n \geq 1$.

For the second statement, let $X, Y$ be two solutions in the class, that is with (4.2) finite for both $X$ and $Y$. As in the proof of the previous claim,

\[
\frac{d}{dt} \psi_N(t) \leq \lambda_N^{\beta - 1} (X_{N+1} Y_N^2 + X_N^2 Y_{N+1}) \leq c \lambda_N^{-3N},
\]

and so $\psi_N(t) \leq \lambda_N^{-3N} t$, which implies that $X = Y$.

Proof of theorem B. Assume that $\beta = \frac{5}{2}$ and take $\gamma_0 = \beta - 2 + \epsilon_0$, where $\epsilon_0$ is given in lemma 2.1.
Let $x \in H$ be such that $\sup_n \lambda_n^\gamma x_n < \infty$ for some $\gamma \in (\beta - 2, \gamma_0]$. First, note that by the choice of $\beta$ it turns out that $\beta - 2 = \frac{1}{4}(\beta - 1)$, hence the second statement of the previous lemma ensures that there is at most one weak solution satisfying (1.3) as long as one can show that there is a solution which is bounded in the scale $\lambda_n^\gamma$. This will turn out to be true by corollary 3.3.

Given $\nu > 0$, let $(X^{[\nu]}_n)_{n \geq 1}$ be the solution to the viscous problem (1.1) with viscosity $\nu$. For every $n \geq 1$ and $\nu \leq 1$, 

$$|\dot{X}^{[\nu]}_n| \leq \nu^2 X^{[\nu]}_n + \lambda_n^\beta (X^{[\nu]}_{n-1})^2 + \lambda_n^\beta X^{[\nu]}_n X^{[\nu]}_{n+1} \leq c_n$$

where $c_n$ is a number independent of $\nu \leq 1$ (although it does depend on $n$). Hence by the Ascoli–Arzelà theorem for each $n$ the family $\{X^{[\nu]}_n : \nu \in (0, 1]\}$ is compact in $C([0, \infty); \mathbb{R})$. By a diagonal procedure, we can find a common sequence $(\nu_k)_{k \in \mathbb{N}}$ and a limit point $(X^{[0]}_n)_{n \geq 1}$ such that $X^{[\nu_k]}_n \to X^{[0]}_n$ uniformly on compact intervals of $[0, \infty)$ for every $n \geq 1$.

Clearly any limit point is positive and satisfies equations (1.2). Moreover, we know by corollary 3.3 that

$$\sup_{n \geq 1} \sup_{t \geq 0} (\lambda_n^\gamma X^{[\nu]}_n(t)) < \infty,$$

hence any limit point satisfies (1.3), and in particular (4.2). By the previous lemma there is only one limit point and $X^{[\nu]}_n \to X^{[0]}_n$ uniformly as $\nu \downarrow 0$. □

**Remark 4.3.** Clearly the family $(X^{[\nu]}_{n \geq 1})_{\nu \leq 1}$ has limit points, which are solutions of (1.2), also when $\beta \neq \frac{5}{2}$. Moreover, all limit points are bounded in the scaling $\lambda_n^{\beta-2}$ by virtue of lemma 2.1.

For $\beta < \frac{5}{2}$ we cannot prove uniqueness because $\beta - 2 < \frac{1}{4}(\beta - 1)$. On the other hand, if $\beta > \frac{5}{2}$ the result can be extended if one extends the range of values for which lemma 2.1 holds (see remark 2.2).

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**References**


