The recognition problem for equivariant singularities

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The recognition problem for equivariant singularities

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Abstract. Singularity theory involves the classification of singularities up to some equivalence relation. The solution to a particular recognition problem is the characterisation of an equivalence class in terms of a finite number of polynomial equalities and inequalities to be satisfied by the Taylor coefficients of a singularity.

The recognition problem can be simplified by decomposing the group of equivalences into a unipotent group and a group of matrices. Building upon results of Bruce and co-workers, we show for contact equivalence that in many cases the unipotent problem can be solved by just using linear algebra. We give a necessary and sufficient condition for this, namely that the tangent space be invariant under unipotent equivalence. We then develop efficient methods for checking whether the tangent space is invariant, and give several examples drawn from equivariant bifurcation theory.

1. Introduction

Golubitsky and Schaeffer (1979a,b) introduced the idea of applying singularity-theoretic methods to the study of equivariant bifurcation problems. Subsequently, many authors have produced classifications up to some codimension in a given context. These classifications include the following three components.

(i) A list of normal forms, with the property that all bifurcation problems up to the given codimension are equivalent to precisely one normal form.

(ii) The universal unfolding of each normal form.

(iii) The solution to the recognition problem for each normal form.

The *recognition problem* is one of the least explored facets of singularity theory and it is with this third component that we deal in this paper. We are interested in knowing precisely when a bifurcation problem is equivalent to a given normal form. Hence we must find a characterisation of the orbit of the normal form under the group of equivalences $\mathscr{D}$. This problem can often be reduced to one of finite dimensions via a key idea from singularity theory, that of *finite determinacy*. Many smooth map germs are determined up to $\mathscr{D}$ equivalence by finitely many coefficients in their Taylor expansion. Modulo other *high-order terms* $\mathscr{D}$ acts as a Lie group. It is well known that the orbits under the resulting Lie group are semi-algebraic sets, so

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we can characterise the orbit as comprising those germs whose Taylor coefficients satisfy a finite number of polynomial constraints in the form of equalities and inequalities. This characterisation is the solution to the recognition problem.

We will always assume that the bifurcation problems under discussion are finitely determined. Indeed, finite codimension implies finite determinacy, see lemma 2.1, and so for the purpose of classifying bifurcation problems up to low codimension, this assumption is always valid. The next step is to discover precisely which terms are high-order terms. Gaffney (1986) uses results from Bruce *et al* (1985) in providing the answer to this problem. However an additional assumption is required, namely that $\mathscr{D}$ acts linearly. The group of (contact) equivalences used in studying bifurcation problems does indeed act linearly and the results in this paper require the same assumption. In fact, the linearity of the group action is the key hypothesis in our results which hold equally well for the recognition problem under the right equivalence and contact equivalence in classical singularity theory.

Because of the Lie group structure of $\mathscr{D}$, we can speak of the *tangent space* to the orbit of a bifurcation problem f, or the *Lie algebra* at f

$$T(f, \mathscr{D}) = L\mathscr{D}f = \{\mathrm{d}(\delta_t f)/\mathrm{d}t|_{t=0} \mid \delta_t \in \mathscr{D}, \delta_0 = 1\}. \tag{1.1}$$

Most of the low-codimension classifications in the literature have been performed in the presence of a group of symmetries Γ acting *absolutely irreducibly* (the only linear maps commuting with the group action are real multiples of the identity). Such classifications include bifurcation problems in one state variable with no symmetry up to codimension seven (Keyfitz 1986) and with $\mathbb{Z}_2$ symmetry up to codimension three (Golubitsky and Schaeffer 1984), in two state variables with D_4 symmetry up to topological codimension two (Golubitsky and Roberts 1986), and in three state variables with $\mathbb{O}$ symmetry up to topological codimension one (Melbourne 1988). Apart from these, the most exhaustive classification in the literature is that performed by Dangelmayr and Armbruster (1983) who consider an action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ which is not irreducible. They go up to codimension four.

It is shown in §3 that provided Γ acts absolutely irreducibly, then the group of equivalences $\mathscr{D}(\Gamma)$ can be decomposed into a group $\mathrm{U}(\Gamma)$ of equivalences whose linear parts are the identity and a group $\mathrm{S}(\Gamma)$ of linear equivalences (which hence must be scalar multiples of the identity). We refer to these as the group of *unipotent* equivalences and the group of *scalings* and define the *unipotent tangent space* $T(f, \mathrm{U}(\Gamma))$ in an analogous way to $T(f, \mathscr{D}(\Gamma))$.

Examination of the solutions of the recognition problem in the aforementioned classifications leads to the following observations.

(i) Calculating the effect of the scalings alone is easy, although the results look complicated and are often very nonlinear.

(ii) If we consider the recognition problems with respect to unipotent equivalences alone, the solutions consist only of equalities.

(iii) In many cases, these equalities are linear.

(iv) The linearity of these equalities is usually disguised when the effect of the scalings is included.

The following remarks on these observations are in order.

(i) If Γ does not act absolutely irreducibly then it is possible for the effect of the linear equivalences to be rather complicated (for example, two-state variable problems with no symmetry, Golubitsky and Schaeffer 1984). This complexity does not occur provided linear equivalences are forced by the action of Γ to be diagonal matrices; see §7. In this paper we study only such examples.

(ii) This property is in fact always true and is stated algebraically in proposition 3.3 of Bruce *et al* (1985) and theorem 3.2(a) of this paper.

(iii) The main result of this paper, theorem 4.4, gives a necessary and sufficient condition for this property of *linear determinacy* to hold. The condition is that $T(f, \mathrm{U}(\Gamma))$ should be invariant under $\mathrm{U}(\Gamma)$. In this case the orbit of f under $\mathrm{U}(\Gamma)$ is simply the affine space $f + T(f, \mathrm{U}(\Gamma))$.

(iv) A graphic example is given in example 6.4. In the light of this and other examples, it seems reasonable to solve the unipotent part of a recognition problem separately, whether or not the bifurcation problem is linearly determined.

The organisation of this paper is as follows. Section 2 sets up the necessary background. In §3 we show that $\mathscr{D}(\Gamma)$ can be decomposed into $\mathrm{U}(\Gamma)$ and $\mathrm{S}(\Gamma)$, and that the recognition problem can be similarly decomposed. We then give a theory for $\mathrm{U}(\Gamma)$ equivalence that is almost identical to that developed by Gaffney (1986) for $\mathscr{D}(\Gamma)$ equivalence. In particular, results by Bruce *et al* (1985) lead to a characterisation of a module of high-order terms. Section 4 contains our main result which gives a criterion for a bifurcation problem to be linearly determined. In §5 we give results which make it easier to check whether or not this criterion holds. Even if the bifurcation problem in question is not linearly determined, the calculations discussed in §5 are still necessary in order to determine the module of high-order terms.

In §6 we solve the recognition problem for many linearly determined bifurcation problems. A common link between these examples is that Γ acts absolutely irreducibly. We conclude by discussing briefly in §7 the complications that can be introduced into both the $\mathrm{U}(\Gamma)$ and the $\mathrm{S}(\Gamma)$ recognition problems when Γ does not act absolutely irreducibly.

2. Background

We summarise the main concepts that will be needed, and establish the notation. The notation is the same as that used in Golubitsky and Schaeffer (1984), Golubitsky *et al* (1988), Melbourne (1987), Golubitsky and Roberts (1986) and Stewart (1987). Let Γ be a compact Lie group acting on $\mathbb{R}^n$. A smooth map germ at 0, $g:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is said to be Γ *equivariant* if

$$g(\gamma x, \lambda) = \gamma g(x, \lambda) \qquad \text{for all } \gamma \in \Gamma, x \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

We denote the space of all such mappings by $\vec{\mathscr{E}}_{x,\lambda}(\Gamma)$. The variable $x = (x_1, \ldots, x_n)$ is called the *state variable* and λ is the *bifurcation parameter*. Let $\mathscr{E}_{x,\lambda}(\Gamma)$ be the ring of all Γ-*invariant* smooth function germs at $0, f:\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$; that is, those f satisfying

$$f(\gamma x, \lambda) = f(x, \lambda) \qquad \text{for all } \gamma \in \Gamma, x \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

Then $\vec{\mathscr{E}}_{x,\lambda}(\Gamma)$ is a module over $\mathscr{E}_{x,\lambda}(\Gamma)$. We must also consider the $\mathscr{E}_{x,\lambda}(\Gamma)$ module $\overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma)$, which consists of the germs at 0 of all smooth-matrix-valued maps $S:\mathbb{R}^n \times \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the condition

$$\gamma^{-1}S(\gamma x, \lambda)\gamma = S(x, \lambda) \qquad \text{for all } \gamma \in \Gamma, x \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

A result of Schwarz (1975) ensures that there exists a finite set of invariant generators $u_1, \ldots, u_r \in \mathscr{E}_{x,\lambda}(\Gamma)$ such that any element $f \in \mathscr{E}_{x,\lambda}(\Gamma)$ can be written as a function of $u_1, \ldots, u_r$. In other words $\mathscr{E}_{x,\lambda}(\Gamma) = \mathscr{E}_{u,\lambda}$. The ring $\mathscr{E}_{u,\lambda}$ has a unique maximal ideal $\mathscr{M}_{u,\lambda} = \langle u_1, \ldots, u_r, \lambda \rangle$ comprising all invariant functions that vanish at the origin. The kth power of the maximal ideal $\mathscr{M}^k_{u,\lambda}$ consists of all invariant functions whose derivatives in u and λ up to any degree less than k vanish at the origin. Similarly we can define $\vec{\mathscr{M}}^k_{x,\lambda}(\Gamma)$ to be the space of equivariant maps whose derivatives in x and λ of degree less than k vanish at the origin. A *bifurcation problem with Γ symmetry* is an equation $g(x, \lambda) = 0$ where $g \in \vec{\mathscr{M}}_{x,\lambda}(\Gamma)$ and $(d_x g)_0 = 0$.

The group of Γ equivalences acting on $\vec{\mathscr{M}}_{x,\lambda}(\Gamma)$ is defined in the following way. Let $\mathscr{L}(\Gamma)^\circ$ denote the connected component of $\mathrm{Hom}_\Gamma(\mathbb{R}^n) \cap GL(\mathbb{R}^n)$ containing the identity, where $\mathrm{Hom}_\Gamma(\mathbb{R}^n)$ is the vector space of all Γ-equivariant linear mappings on $\mathbb{R}^n$. Then $g, h \in \vec{\mathscr{M}}_{x,\lambda}(\Gamma)$ are Γ *equivalent* if there exists a triple $(S, X, \Lambda) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma) \times \vec{\mathscr{M}}_{x,\lambda}(\Gamma) = \mathscr{M}_\lambda$ such that

$$h(x, \lambda) = S(x, \lambda) g(X(x, \lambda), \Lambda(\lambda)) \qquad S(0), (d_x X)_0 \in \mathscr{L}(\Gamma)^\circ, \Lambda'(0) > 0.$$

Let

$$\mathscr{D}(\Gamma) = \{(S, X, \Lambda) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma) \times \vec{\mathscr{M}}_{x,\lambda}(\Gamma) \times \mathscr{M}_\lambda \mid S(0), (d_x X)_0 \in \mathscr{L}(\Gamma)^\circ, \Lambda'(0) > 0\}.$$

Then, under a suitable multiplication, the group action of $\mathscr{D}(\Gamma)$ on $\vec{\mathscr{M}}_{x,\lambda}(\Gamma)$ induces the required equivalence relation. If we write $\varphi_i = (X_i, \Lambda_i)$ $i = 1, 2,$ then the multiplication is given by

$$(S_2, \varphi_2) \circ (S_1, \varphi_1) = (S_2 \cdot (S_1 \circ \varphi_2), \varphi_1 \circ \varphi_2)$$

where

$$S_2 \cdot (S_1 \circ \varphi_2)(x, \lambda) = S_2(x, \lambda) \cdot S_1(\varphi_2(x, \lambda))$$
$$\varphi_1 \circ \varphi_2(x, \lambda) = (X_1 \circ \varphi_2(x, \lambda), \Lambda_1 \circ \Lambda_2(\lambda)).$$

A calculation using (1.1) shows that the tangent space is given by

$$T(f, \mathscr{D}(\Gamma)) = \tilde{T}(f, \mathscr{D}(\Gamma)) + \mathscr{E}_\lambda\{\lambda f_\lambda\} \tag{2.1a}$$

where

$$\tilde{T}(f, \mathscr{D}(\Gamma)) = \{Sf + (df)X \mid (S, X) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma) \times \vec{\mathscr{M}}_{x,\lambda}(\Gamma)\}. \tag{2.1b}$$

Note that $\tilde{T}(f, \mathscr{D}(\Gamma))$ is an $\mathscr{E}_{x,\lambda}(\Gamma)$ module, but this is not necessarily so for $T(f, \mathscr{D}(\Gamma))$. Equation (2.1) gives an alternative 'formal' definition for $T(f, \mathscr{D}(\Gamma))$. Unlike in (1.1) we do not require $\mathscr{D}(\Gamma)$ to be a Lie group. The following result is a fundamental lemma from singularity theory relating the concepts of finite determinacy and finite codimension.

Lemma 2.1. The following are equivalent.

(a) $T(f, \mathscr{D}(\Gamma))$ has finite codimension in $\vec{\mathscr{E}}_{x,\lambda}(\Gamma)$, that is

$$T(f, \mathscr{D}(\Gamma)) \oplus V = \vec{\mathscr{E}}_{x,\lambda}(\Gamma)$$

for some finite-dimensional vector space V.

(b) f is finitely determined, that is there is some $k > 0$ such that

$$f + p \in \mathscr{D}(\Gamma) \cdot f \qquad \text{for all } p \in \vec{\mathscr{M}}^k_{x,\lambda}(\Gamma).$$

If (a) and (b) hold then $\mathscr{D}(\Gamma)$ can be considered as acting modulo $\vec{\mathscr{M}}^k_{x,\lambda}(\Gamma)$. The induced action is that of a Lie group acting algebraically. The Lie algebra defined in (1.1) coincides with the tangent space defined in (2.1).

Proof of (b) is implied by (a) by theorem 10.2 of Damon (1984). The converse is proved by an easy calculation and is not in any case used in this paper. The consequences of (a) and (b) are well known (see Thom and Levine 1971).

Definition 2.2. A bifurcation problem f has finite Γ codimension if $T(f, \mathscr{D}(\Gamma)$ has finite codimension in $\vec{\mathscr{E}}_{x,\lambda}(\Gamma)$.

3. Unipotent actions and the recognition problem

Let $\mathscr{D}(\Gamma)$ be the following group of Γ equivalences acting on $\vec{\mathscr{M}}_{x,\lambda}(\Gamma)$:

$\mathscr{D}(\Gamma)=\{(S, X, \Lambda) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma)\times\vec{\mathscr{M}}_{x,\lambda}(\Gamma)\times\mathscr{M}_\lambda \mid \mathrm{S}(0), (d_xX)_0 \in \mathscr{L}(\Gamma)^\circ, \Lambda'(0)>0\}.$

Consider the map projecting equivalences onto their linear parts

$$\pi : \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma)\times\vec{\mathscr{M}}_{x,\lambda}(\Gamma)\times\mathscr{M}_\lambda \to \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma)\times\vec{\mathscr{M}}_{x,\lambda}(\Gamma)\times\mathscr{M}_\lambda$$

$$\pi(S, X, \Lambda)=(\mathrm{S}(0), (d_xX)_0, \Lambda'(0)).$$

Let $\mathrm{S}(\Gamma)=\mathscr{L}(\Gamma)^\circ\times\mathscr{L}(\Gamma)^\circ\times\mathbb{R}^{>0}$ where $\mathbb{R}^{>0}$ is the set of positive real numbers. It is easy to check that

$$\pi|_{\mathscr{D}(\Gamma)} : \mathscr{D}(\Gamma)\to \mathrm{S}(\Gamma)$$

is a group epimorphism. Its kernel

$$\mathrm{U}(\Gamma)=\{(S, X, \Lambda)\in\mathscr{D}(\Gamma) \mid \mathrm{S}(0)=1, (d_xX)_0=1, \Lambda'(0)=1\} \tag{3.1}$$

is therefore a normal subgroup of $\mathscr{D}(\Gamma)$. We can decompose $\delta\in\mathscr{D}(\Gamma)$ as

$$\delta = su_1 = u_2s$$

where $s\in \mathrm{S}(\Gamma)$, u_1, $u_2\in\mathrm{U}(\Gamma)$. To do this we set

$$s=\pi(\delta) \qquad u_1=\pi(\delta)^{-1}\delta \qquad u_2=\delta\pi(\delta)^{-1}.$$

Furthermore the decomposition is unique since

$$\pi(\delta)=\pi(s)\pi(u_1)=s.$$

Note however that in general $u_1\neq u_2$.

The group $\mathrm{U}(\Gamma)$ consists of unipotent diffeomorphisms, whose linear parts are unipotent matrices. (In general, a unipotent matrix is one that in some coordinate system can be written as an upper triangular matrix with ones on the diagonal. We have the special case where there are no non-zero superdiagonal entries.) In consequence we can use the methods of Bruce *et al* (1985), from algebraic geometry.

Remark 3.1.

(a) The decomposition described above allows us to solve a $\mathscr{D}(\Gamma)$ recognition problem by combining the solutions of the corresponding $\mathrm{U}(\Gamma)$ and $\mathrm{S}(\Gamma)$ recognition problems in the following way. Our method is to compute $\mathrm{S}(\Gamma)\cdot n$ for a given normal form n, and then to calculate $U(\Gamma) \cdot f$ for all $f \in \mathrm{S}(\Gamma) \cdot n$. Since

$$\mathscr{D}(\Gamma) \cdot n = \mathrm{U}(\Gamma) \cdot \mathrm{S}(\Gamma) \cdot n$$

we have $g \in \mathscr{D}(\Gamma) \cdot n$ if and only if $g \in \mathrm{U}(\Gamma) \cdot f$ for some $f \in \mathrm{S}(\Gamma) \cdot n$.

The elements of $\mathrm{S}(\Gamma)$ are linear, hence we might hope to solve the $\mathrm{S}(\Gamma)$ recognition problem without too much difficulty. This hope is not always realised; see ch IX of Golubitsky and Schaeffer (1984) for the case of two state variables without symmetry. However, in the examples which we consider in this paper, Γ acts in such a way that $\mathrm{S}(\Gamma)$ is *scalar,* that is $\mathscr{L}(\Gamma)^{\circ}$ contains only diagonal matrices (in some coordinate system). In §7 we give a criterion for $\mathrm{S}(\Gamma)$ to be scalar in terms of the action of Γ. In these cases solving $\mathrm{S}(\Gamma)$ recognition problems is a trivial matter. In the remainder of this section we concentrate on the $\mathrm{U}(\Gamma)$ recognition problem. From now on we usually suppress the Γ dependence.

(*b*) Our results require bifurcation problems $f \in \vec{\mathscr{M}}_{x,\lambda}$ to have finite codimension. It is not necessary to specify whether this is finite codimension with respect to $\mathscr{D}$ or U. A calculation shows that

$$T(f, U) = \tilde{T}(f, U) + \mathscr{E}_{\lambda}\{\lambda^2 f_{\lambda}\} \tag{3.2a}$$

where

$$\tilde{T}(f, U) = \{Sf + (df)X \mid (S, X) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda} \times \vec{\mathscr{M}}_{x,\lambda},\ \mathrm{S}(0) = (dX)_0 = 0\}. \tag{3.2b}$$

By (2.1) and (3.2)

$$T(f, \mathscr{D}) = T(f, U) + W \tag{3.3}$$

where

$$W = \mathbb{R}\{Sf + (d_x f)X + \lambda f_{\lambda} \mid S,\ d_x X \in \mathscr{L}\}.$$

Now $\vec{\mathscr{M}}_{x,\lambda}$ and $\overleftrightarrow{\mathscr{E}}_{x,\lambda}$ are finitely generated as modules over $\overleftrightarrow{\mathscr{E}}_{x,\lambda}$, say by

$$X_1, \ldots, X_r, S_1, \ldots, S_s$$

(theorems XII,5.2 and XII,5.3, and exercise XIV,1.3 of Golubitsky *et al* (1988)) and so $\mathscr{L}$ is spanned by

$$(d_x X_1)_0, \ldots, (d_x X_r)_0; S_1(0), \ldots, S_s(0).$$

Therefore W is a finite-dimensional vector space and hence, by (3.3), it follows that the two tangent spaces have finite or infinite codimension in $\overleftrightarrow{\mathscr{E}}_{x,\lambda}$ together.

(*c*) The results in §§3 and 4, in particular corollary 3.9 and theorem 4.4, hold in a more general setting. Here U and S can be any subgroups of $\mathscr{D}$ satisfying the following three properties:

for all $\delta \in \mathscr{D}$, $\delta = su$ for some $u \in U$, $s \in S$,
U acts unipotently,
the codimension property (3.3) holds with W finite dimensional.

We will require the following two results from algebraic geometry. They deal with actions of unipotent groups and are proposition 3.3 and corollary 3.5 respectively of Bruce *et al* (1985).

Theorem 3.2. Let U be a unipotent affine algebraic group over $\mathbb{R}$ acting algebraically on an affine variety V. Then

(a) The orbits of U are Zariski closed in V;

(b) If $x \in V$ and W is a U-invariant subspace of V then $x + W$ is contained in an orbit of U if and only if $LUx \supset W$.

Theorem 3.2 is restated in our particular context in corollary 3.6.

Definition 3.3. For $f \in \vec{\mathcal{M}}_{x,\lambda}$,

$$\begin{aligned} M(f, U) &= \{p \in \vec{\mathcal{M}}_{x,\lambda} \mid f + p \in Uf\} \\ &= \{uf - f \mid u \in U\}. \end{aligned}$$

Remark 3.4. Notice that $g \in Uf$ if and only if $g - f \in M(f, U)$. Hence, solving the U-recognition problem amounts to computing $M(f, U)$.

Definition 3.5. A subspace of $\vec{\mathcal{M}}_{x,\lambda}$ is U *intrinsic* if it is invariant under the action of U. If a subset M of $\vec{\mathcal{M}}_{x,\lambda}$ contains a unique maximal U-intrinsic subspace, then this subspace is called the U-*intrinsic* part of M and is denoted $\mathrm{Itr}_U\, M$.

Note that a U-intrinsic subspace of $\vec{\mathcal{M}}_{x,\lambda}$ is automatically an $\mathscr{E}_{x,\lambda}$ submodule of $\vec{\mathcal{M}}_{x,\lambda}$ since it is closed under multiplication on the left by $S = h\mathrm{I}$ for any $h \in \mathscr{E}_{x,\lambda}$.

Clearly $\mathrm{Itr}_U\, M$ exists for any subspace M. In proposition 3.8 we see that $\mathrm{Itr}_U\, M(f, U)$ always exists provided f has finite codimension.

Corollary 3.6. Suppose $f \in \vec{\mathcal{M}}_{x,\lambda}$ is of finite codimension. Then

(a) The orbit Uf is determined by a finite system of polynomial equations.

(b) Suppose M is a U-intrinsic subspace of $\vec{\mathcal{M}}_{x,\lambda}$. Then

$$M \subset M(f, U) \qquad \text{if and only if } M \subset T(f, U).$$

Proof. By lemma 2.1 we can work modulo $\vec{\mathcal{M}}^k_{x,\lambda}$, some $k > 0$, and so regard U as an algebraic group acting algebraically. Now (a) and (b) are then just rewordings of theorem 3.2(a) and (b) respectively.

We now define a module of high-order terms $\mathscr{P}(f, U)$ which is analogous to the module $\mathscr{P}$ of high-order terms in the $\mathscr{D}$ context (see Gaffney 1986).

Definition 3.7. $\mathscr{P}(f, U) = \{p \in \vec{\mathcal{M}}_{x,\lambda} \mid g + p \in Uf \text{ for all } g \in Uf\}$.

Proposition 3.8. If f has finite codimension then

$$\mathscr{P}(f, U) = \mathrm{Itr}_U\, M(f, U).$$

Proof. We have to show that $\mathscr{P}(f, U)$ is the unique maximal U-intrinsic subspace contained in $M(f, U)$. The proof is identical to that of proposition 1.7 in Gaffney (1986) with one exception. Closure under addition is still straightforward: if $p_1, p_2 \in \mathscr{P}(f, U)$ and $g \in U \cdot f$ then $g + p_1 \in U \cdot f$ and so $(g + p_1) + p_2 \in U \cdot f$ by definition. The problem is closure under scalar multiplication. However, consider the set

$$T = \{t \in \mathbb{R} \mid g + tp \in Uf\}$$

where $p \in \mathcal{P}(f, U)$, $g \in Uf$. By the property of closure under addition, we have $\mathbb{N} \subset T$. However by corollary 3.6(a), $U \cdot f$ is determined by finitely many polynomials. Therefore $t \in T$ if and only if t is a simultaneous zero of a finite set of polynomials. However, T contains $\mathbb{N}$, an infinite set, and so $T = \mathbb{R}$ as required. Therefore $\mathcal{P}(f, U)$ is a subspace.

The rest of the proof proceeds as expected. Suppose $g \in \mathcal{P}(f, U)$, $u \in U$. Then $g + up = u(u^{-1}g + p) \in Uf$, so $up \in \mathcal{P}(f, U)$. Therefore $\mathcal{P}(f, U)$ is a U-intrinsic subspace. Clearly $\mathcal{P}(f, U) \subset M(f, U)$. Suppose $P \subset M(f, U)$ where P is U intrinsic. Let $p \in P$ and $g = uf$, $u \in U$. Then

$$g + p = uf + p = u(f + u^{-1}p) \in U \cdot f.$$

Thus $P \in \mathcal{P}(f, U)$ and $\mathcal{P}(f, U)$ is maximal and unique.

Corollary 3.9. If f has finite codimension then

$$\mathcal{P}(f, U) = \mathrm{Itr}_U\, T(f, U).$$

Proof. Taking U-intrinsic parts in corollary 3.6(b) and applying proposition 3.8 yields

$$M \subset \mathcal{P}(f, U) \qquad \text{if and only if } M \subset \mathrm{Itr}_U\, T(f, U)$$

for any U-intrinsic subspace M. Setting $M = \mathcal{P}(f, U)$ and $M = \mathrm{Itr}_U\, T(f, U)$ in turn gives the result.

4. Linearly determined bifurcation problems

In remark 3.4, we observed that the computation of $M(f, U)$ would solve the U-recognition problem. By corollary 3.6(a), $M(f, U)$ is determined by a finite set of polynomial equations. We concentrate on the simplest case when these equations are linear, so that $M(f, U)$ is a vector subspace of finite codimension. Note that this codimension is the same as that of $T(f, U)$, because

$$\begin{aligned} \operatorname{codim} T(f, U) &= \text{number of defining equations for } Uf \\ &= \operatorname{codim} M(f, U). \end{aligned}$$

Definition 4.1. A bifurcation problem $f \in \vec{\mathcal{M}}_{x,\lambda}$ of finite codimension is *linearly determined* if $M(f, U)$ is a vector subspace of $\vec{\mathcal{M}}_{x,\lambda}$.

Remark 4.2. Linearly determined bifurcation problems are by no means rare. Indeed in examples that have been studied up to now, the majority of bifurcation problems are linearly determined. In the context of one state variable with no symmetry, nine out of the thirteen bifurcation problems of codimension 4 or less are linearly determined, whilst if $\Gamma = \mathbb{Z}_2$ all problems up to at least codimension 3 are linearly determined. In this section we give a simple criterion for linear determinacy. If this is satisfied, then $M(f, U)$ is immediately known.

Proposition 4.3. f is linearly determined if and only if $M(f, U) = \mathcal{P}(f, U)$.

Proof. We have to show that $M(f, U)$ is a subspace if and only if it is a U-intrinsic subspace. One implication is trivial. To prove the converse suppose $p \in M(f, U)$ and $g \in Uf$, so that there exist $u, u' \in U$ such that

$$f + p = uf \qquad g = u'f.$$

Then

$$(g + p) - f = (u'f - f) + (uf - f) \in M(f, U).$$

Therefore $g + p \in Uf$ and so $p \in \mathcal{P}(f, U)$.

Theorem 4.4. f is linearly determined if and only if $T(f, U)$ is U intrinsic, in which case

$$M(f, U) = T(f, U).$$

Proof. Suppose that f is linearly determined. Then by proposition 4.3,

$$M(f, U) = \mathcal{P}(f, U) \subset T(f, U).$$

But $M(f, U)$ is a subspace with the same codimension as $T(f, U)$. Therefore

$$T(f, U) = M(f, U) = \mathcal{P}(f, U)$$

the latter being a U-intrinsic subspace. The converse can be proved directly in the case when $U(\Gamma)$ is defined as in (3.1). However the proof is quite unwieldy. Wall found a more natural setting for the result in lemma 4.5. The upshot of this lemma is that $T(f, U) = M(f, U)$. However, $T(f, U) = \mathcal{P}(f, U)$ and so f is linearly determined by proposition 4.3.

In the remainder of this section we revert to the notation of theorem 3.2. We recall that the Lie algebra LU at f and the tangent space $T(f, U)$ are the same space.

Lemma 4.5. Let U be a unipotent group acting linearly on a vector space V, and let $v \in V$ be such that $LU \cdot v$ is a U-invariant subspace of V. Then $U \cdot v$ is the affine subspace $v + LU \cdot v$.

Proof. (Wall 1986). Let $N_1, \ldots, N_k$ be a basis of the Lie algebra LU. Since this is nilpotent, there is an integer r such that any product of more than r of the N_i is zero. The tangent space $LU \cdot v$ is spanned by the $N_i v$. Since it is invariant, any $N_i N_j v$ also belongs to $LU \cdot v$ (see proposition 5.1).

It suffices to show that $U \cdot v \subset v + LU \cdot v$ for these have the same dimension. As $U \cdot v$ is closed, it follows that it is the whole space. Because the exponential map for U is surjective, it is enough to show that for any $N = \sum \lambda_i N_i$ in LU, $e^N v$ belongs to $v + LU \cdot v$. But

$$e^N v = v + \sum \lambda_i N_i v + \frac{1}{2}\left(\sum \lambda_i N_i\right)^2 v + \ldots + \frac{1}{n!}\left(\sum \lambda_i N_i\right)^r v$$

and since any $N_i N_j v$ is a linear combination of the $N_i v$ it follows by induction that each term except the first lies in $LU \cdot v$.

Corollary 4.6. Let U be a unipotent group acting linearly on a vector space V and let $v \in V$. Then $LU \cdot v$ is a U-invariant subspace of V if and only if

$$U \cdot v = v + LU \cdot v.$$

Proof. It remains to prove that if $U \cdot v = v + LU \cdot v$ then $LU \cdot v$ is U invariant. Suppose that $M \in LU$, $u \in U$. We must show that $uMv \in LU \cdot v$. The hypothesis implies that $v + LU \cdot v$ is invariant under U and so

$$u(v + Mv) \in v + LU \cdot v.$$

Therefore

$$uv + uMv - v \in LU \cdot v.$$

But $uv \in U \cdot v$ and so $uv - v \in LU \cdot v$. Hence we have

$$uMv \in LU \cdot v$$

as required.

5. Tools for calculating maximal U-intrinsic subspaces

In order to calculate $\mathscr{P}(f, U)$ we need an efficient method for calculating the U-intrinsic part of a subspace. The first result gives a necessary and sufficient condition for a subspace to be U intrinsic.

Proposition 5.1. If $M \subset \vec{\mathscr{M}}_{x,\lambda}$ is a subspace of finite codimension then M is U intrinsic ($\mathscr{D}$ intrinsic) if and only if $LU \cdot M \subset M$ ($L\mathscr{D} \cdot M \subset M$).

Proof. By the finite codimension of M we can work modulo $\vec{\mathscr{M}}^k_{x,\lambda}$, $k > 0$, and so regard U as a Lie group or as an algebraic group acting algebraically. For a unipotent group U, the exponential map

$$\exp : LU \to U$$

is continuous and surjective (lemma 3.1 of Bruce *et al* (1985)), so U is the continuous image of a connected space. Therefore U is a connected Lie group acting smoothly on $\vec{\mathscr{M}}_{x,\lambda}$. Hence by lemma 2.2 of Bruce *et al* (1985) we obtain the required result for U. The result holds also for $\mathscr{D}$ since $\mathscr{D}$ is a connected Lie group by lemma 2.3, Melbourne (1987).

In general verifying the condition in proposition 5.1 is a laborious task. A better method is to recognise that a 'large part' of a subspace is U intrinsic and then apply proposition 5.1 as a last resort on whatever is remaining.

It is clear that applying a Γ equivalence to a monomial $p \in \vec{\mathscr{M}}_{x,\lambda}(\Gamma)$ cannot reduce the overall degree of p. Furthermore, because the Λ part of a Γ equivalence is only allowed to depend on λ, the degree of p in λ alone can also not be reduced. Hence for all $k, l > 0$, the subspace

$$\vec{\mathscr{M}}^k_{x,\lambda}(\Gamma)\langle \lambda^l \rangle \tag{5.1}$$

is both $\mathscr{D}$ intrinsic and U intrinsic. By the linearity of the action of $\mathscr{D}$, sums of subspaces such as in (5.1) are also intrinsic.

In the examples considered in §6, the action of Γ is irreducible. Suppose further that the action is non-trivial. The fixed point subspace

$$V^\Gamma = \{v \in \mathbb{R}^n \mid \gamma v = v \text{ for all } \gamma \in \Gamma\}$$

is a Γ-invariant subspace of $\mathbb{R}^n$ and so is just $\{0\}$. Now suppose $X \in \vec{\mathscr{E}}_{x,\lambda}(\Gamma)$. Then

$$\gamma X(0, \lambda) = X(\gamma \cdot 0, \lambda) = X(0, \lambda) \qquad \text{for all } \gamma = \Gamma.$$

Hence $X(0, \lambda) \in V^\Gamma$ and so $X(0, \lambda) \equiv 0$. Thus the following useful hypothesis is often satisfied:

$$X(0, \lambda) \equiv 0 \qquad \text{for all } X \in \vec{\mathscr{E}}_{x,\lambda}(\Gamma). \tag{5.2}$$

Condition (5.2) implies that the degree in x is preserved by Γ equivalence in the same way as the degree in λ is preserved. Therefore it is useful to define a space of germs vanishing up to some specified degree in x. For $k \geq 1$, we define

$$\vec{\mathscr{M}}_k(\Gamma) = \left\{ f \in \vec{\mathscr{E}}_x(\Gamma) \;\middle|\; \frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(0) = 0 \quad \begin{array}{l}\text{for all multi-indices}\\ |\alpha| < k\end{array} \right\}.$$

The following result is elementary.

Proposition 5.2. Suppose condition (5.2) holds. Then sums of subspaces of the form

$$\vec{\mathscr{M}}_k(\Gamma)\langle\lambda^l\rangle \qquad k \geq 1, l \geq 0$$

are $\mathscr{D}$ intrinsic and U intrinsic.

Note that

$$\vec{\mathscr{E}}_x(\Gamma) = \;= \vec{\mathscr{M}}_1(\Gamma) \supset \vec{\mathscr{M}}_2(\Gamma) \supset \vec{\mathscr{M}}_3(\Gamma) \supset \ldots.$$

These inclusions need not be strict. For example, consider $\Gamma = \mathbb{Z}_2$ acting on $\mathbb{R}$. Then $\vec{\mathscr{E}}_x(\mathbb{Z}_2)$ consists only of odd functions and so

$$\vec{\mathscr{M}}_{2k}(\mathbb{Z}_2) = \vec{\mathscr{M}}_{2k+1}(\mathbb{Z}_2) \qquad \text{for all } k \geq 1.$$

For $k > 1$, let k^- denote the largest integer less than k such that $\vec{\mathscr{M}}_k(\Gamma)$ is strictly contained in $\vec{\mathscr{M}}_{k^-}(\Gamma)$.

Remark 5.3.

(*a*) k^- is either $k-1$ or $k-2$. This is due to the fact that Γ is a compact Lie group acting on $\mathbb{R}^n$ and so is a subgroup of $\mathrm{O}(n)$. Hence there is always an invariant of degree two, the norm $\|x\|$. In consequence, there is an equivariant of degree r for any odd number r. Furthermore, the existence of an equivariant of degree two would guarantee the existence of an equivariant of any given degree. Hence we have the following:

$$\text{either } k^- = k-1 \qquad \text{for all } k > 1 \qquad \text{or } 3^- = 1. \tag{5.3}$$

(*b*) Both cases in (5.3) can obtain for $V^\Gamma = \{0\}$. The examples in §6 all satisfy $3^- = 1$, but if $\Gamma = S_3$ acting on $\mathbb{C}$ as the symmetries of an equilateral triangle, then $\bar{z}^2$ is an equivariant of degree two (see Golubitsky and Schaeffer 1983).

Theorem 5.4. Suppose (5.2) holds. Let V be a subspace of

$\vec{\mathcal{M}}_{k_1^-}(\Gamma)\{\lambda^{l_1}-1\rangle + \ldots + \vec{\mathcal{M}}_{k_s^-}(\Gamma)\langle\lambda^{l_s}-1\rangle, \qquad k_i > 1,\ l_i > 0,\ i = 1, \ldots, s.$

Then

$$\vec{\mathcal{M}}_{k_1}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{\mathcal{M}}_{k_1^-}(\Gamma)\langle\lambda^{l_1}\rangle + \ldots + \vec{\mathcal{M}}_{k_s}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{\mathcal{M}}_{k_s^-}(\Gamma)\langle\lambda^{l_s}\rangle + V$$

is U intrinsic.

Proof. By proposition 5.2

$$H = \vec{\mathcal{M}}_{k_1}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{\mathcal{M}}_{k_1^-}(\Gamma)\langle\lambda^{l_1}\rangle + \ldots + \vec{\mathcal{M}}_{k_s}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{\mathcal{M}}_{k_s^-}(\Gamma)\langle\lambda^{l_s}\rangle$$

is U intrinsic. Hence by proposition 5.1 it suffices to show that

$$LU \cdot V \subset H.$$

We show that if $p \in \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^{l-1}\rangle$ then

$$T(p, U) \subset H_0 = \vec{\mathcal{M}}_k(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle.$$

The result follows by linearity of the $\mathcal{D}$ action. Now

$$T = (p, U) = \left\{ \begin{array}{l} Sp + (\mathrm{d}p)X + \Lambda p_\lambda \mid (S, X, \Lambda) \in \overleftrightarrow{\mathscr{E}}_{x,\lambda}(\Gamma) \times \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \times \mathcal{M}_\lambda \\ S(0) = 0,\ (\mathrm{d}X)_0 = 0,\ \Lambda'(0) = 0 \end{array} \right\}.$$

It is easy enough to see that

$$Sp \in H_0,\ \Lambda p_\lambda \in \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle \subset H_0.$$

To show that $(\mathrm{d}p)X \in H_0$ we have to use remark 5.3(a). By (5.3) we have two cases to consider.

Case 1. $k^- = k - 1$ for all $k > 1$. Now p is of degree at least $k-1$ in x and at least $l-1$ in λ, and so $\mathrm{d}p$ is of degree at least $k-2$ in x and at least $l-1$ in λ. Also we have $X \in \vec{\mathcal{M}}_2(\Gamma)\mathscr{E}_\lambda + \vec{\mathcal{M}}_1(\Gamma)\langle\lambda\rangle$ since $X(0, \lambda) \equiv 0$ and $(\mathrm{d}X)_0 = 0$. Thus

$$(\mathrm{d}p)X \in \vec{\mathcal{M}}_k(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k-1}(\Gamma)\langle\lambda^l\rangle = H_0$$

as required.

Case 2. $3^- = 1$. This time $X \in \vec{\mathcal{M}}_3(\Gamma)\mathscr{E}_\lambda + \vec{\mathcal{M}}_1(\Gamma)\langle\lambda\rangle$. Hence

$$(\mathrm{d}p)X \in \vec{\mathcal{M}}_{k^-+2}(\Gamma)\langle\lambda^{l-1}\rangle + \vec{\mathcal{M}}_{k^-}(\Gamma)\langle\lambda^l\rangle.$$

By remark 5.2(a), $k^- + 2 \geqslant k$ and so the result is proved.

If (5.2) does not hold then the property of 'preservation of degree in x' does not stand. However we can prove a weak analogue of theorem 5.4 which holds true for all compact Lie group actions. Note that

$$\vec{\mathscr{E}}_{x,\lambda}(\Gamma) = \vec{\mathcal{M}}^0_{x,\lambda}(\Gamma) \supset \vec{\mathcal{M}}^1_{x,\lambda}(\Gamma) \supset \vec{\mathcal{M}}^2_{x,\lambda}(\Gamma) \supset \ldots.$$

This time each inclusion is strict.

Theorem 5.5. Let W be a subspace of

$\vec{\mathscr{E}}_{x,\lambda}(\Gamma)\langle\lambda^{l_0}\rangle + \vec{\mathcal{M}}^{k_1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1-1}\rangle + \ldots + \vec{\mathcal{M}}^{k_s}_{x,\lambda}(\Gamma)\langle\lambda^{l_s-1}\rangle \qquad k_i > 0,\ l_i > 0.$

Then

$$\vec{\mathcal{M}}_{x,\lambda}(\Gamma)\langle\lambda^{l_0}\rangle + \vec{\mathcal{M}}^{k_1+1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1-1}\rangle + \vec{\mathcal{M}}^{k_1-1}_{x,\lambda}(\Gamma)\langle\lambda^{l_1}\rangle + \dots$$
$$+ \vec{\mathcal{M}}^{k_s-1}_{x,\lambda}(\Gamma)\langle\lambda^{l_s-1}\rangle + \vec{\mathcal{M}}^{k_s-1}_{x,\lambda}(\Gamma)\langle\lambda^{l_s}\rangle + W$$

is U intrinsic.

Proof. This is similar to that of theorem 5.4. However we have only

$$X \in \vec{\mathcal{M}}_{x,\lambda}(\Gamma) + \vec{\mathscr{E}}_{x,\lambda}(\Gamma)\langle\lambda\rangle$$

rather than $\vec{\mathcal{M}}_{x,\lambda}(\Gamma) + \vec{\mathcal{M}}_{x,\lambda}(\Gamma)\langle\lambda\rangle$ as in case 1 of the proof of theorem 5.4. In particular

$$X(x, \lambda) = a\lambda \qquad a \in \mathbb{R}^n$$

is a possibility now that the restriction $X(0, \lambda) \equiv 0$ no longer holds in general. This accounts for the slightly weaker result.

6. Examples with Γ acting absolutely irreducibly

6.1. One state variable. No symmetry (Keyfitz 1986)

Up to codimension 4, all bifurcation problems fall into one of the following families:

$\varepsilon x^k + \delta\lambda$	$k \geqslant 2$	codim $= k - 2$
$\varepsilon x^k + \delta x\lambda$	$k \geqslant 3$	codim $= k - 1$
$\varepsilon x^2 + \delta\lambda^k$	$k \geqslant 2$	codim $= k - 1$
$\varepsilon x^3 + \delta\lambda^2$		codim $= 3$.

(See table IV2.2 and exercise IV2.1 of Golubitsky and Schaeffer (1984).) Our methods apply to all the above germs except those in the third family. Indeed even the solutions to the full recognition problems consist of linear defining and non-degeneracy conditions. Furthermore, in these cases the unipotent tangent spaces are invariant not only under unipotent equivalences but under the full group of equivalences. For this reason, the solution of these recognition problems is almost trivial even without making use of the results in this paper. Therefore it is necessary to go up to higher codimension to find instructive examples. First, however, we must calculate the unipotent tangent space $T(f, U)$. By definition

$$\begin{aligned} T(f, U) &= \{(\mathrm{d}/\mathrm{d}t)U_t|_{t=0} \,|u_t \in U, u_0 = 1\} \\ &= \{Sf + (d_x f)X + \Lambda f_\lambda |\, (S, X, \Lambda) \\ &\quad \in \mathscr{E}_{x,\lambda} \times \vec{\mathcal{M}}_{x,\lambda} \times \mathcal{M}_\lambda,\ \mathrm{S}(0) = (d_x X)_0 = \Lambda'(0) = 0\}. \end{aligned}$$

Therefore

$$T(f, U) = \tilde{T}(f, U) + \mathscr{E}_\lambda(\lambda^2 f_\lambda\} \tag{6.1a}$$

where

$$\tilde{T}(f, U) = \mathscr{E}_{x,\lambda}\{xf, \lambda f, x^2 f_x, \lambda f_x\}. \tag{6.1b}$$

Tangent space $T(f, U)$ is the same as L_{MAX} in corollary 1.9 of Gaffney (1986).

Example 6.1

(i) $\varepsilon x^k + \delta x\lambda^2$, $k \geq 4$; codim $= 2k - 1$. This is the family II.5,2 in table 1 of Keyfitz (1986). The lowest codimension in the family is seven. First we calculate the orbit of $\varepsilon x^k + \delta x\lambda^2$ under scaling equivalences (S, X, Λ) where

$$S(x, \lambda) \equiv \mu \qquad X(x, \lambda) \equiv \nu x \qquad \Lambda(\lambda) \equiv l\lambda \qquad \mu, \nu, l > 0.$$

It is easy to ascertain that the orbit is

$$\{\mu\nu^k\varepsilon x^k + \mu\nu l^2\delta x\lambda^2 \mid \mu, \nu, l > 0\}$$

and that f is contained in this orbit if and only if

$$f = ax^k + bx\lambda^2 \qquad \operatorname{sign} a = \varepsilon \qquad \operatorname{sign} b = \delta. \tag{6.2}$$

Now consider the unscaled germ

$$f(x, \lambda) = ax^k + bx\lambda^2 \qquad a, b \neq 0 \qquad k \geq 4.$$

By (6.1) we have

$$T(f, U) = \tilde{T}(f, U) + \mathscr{E}_\lambda\{\lambda^2 f_\lambda\}$$

where

$$\begin{aligned}\tilde{T}(f, U) &= \mathscr{E}_{x,\lambda}\{x^2 f_x, \lambda f_x, xf, \lambda f\}\\ &= \mathscr{E}_{x,\lambda}\{kax^{k+1} + bx^2\lambda^2, kax^{k-1}\lambda + b\lambda^3, ax^{k+1} + bx^2\lambda^2, ax^k\lambda + bx\lambda^3\}.\end{aligned}$$

The first and third generators simplify to x^{k+1} and $x^2\lambda^2$ and then it is easy to obtain

$$T(f, U) = \mathscr{M}^{k+1} + \mathscr{M}^2\langle\lambda^2\rangle + \mathbb{R}\{kax^{k-1}\lambda + b\lambda^3\}$$

where $\mathscr{M} = \vec{\mathscr{M}}_{x,\lambda} = \langle x, \lambda\rangle$ is the maximal ideal in $\mathscr{E}_{x,\lambda}$. Note that

$$\mathscr{P}(f, \mathscr{D}) = \mathscr{M}^{k+1} + \mathscr{M}^2\langle\lambda^2\rangle.$$

Now $kax^{k-1}\lambda + b\lambda^3 \notin \mathscr{P}(f, \mathscr{D})$, for if we apply the scaling

$$\lambda \mapsto 2\lambda$$

then

$$kax^{k-1}\lambda + b\lambda^3 \mapsto 2(kax^{k-1}\lambda + 4b\lambda^3) \notin T(f, U).$$

Hence $T(f, U)$ is not $\mathscr{D}$ intrinsic. However

$$\mathbb{R}\{kax^{k-1}\lambda + b\lambda^3\} \subset \mathscr{M}^{k-1}\langle\lambda\rangle + \langle\lambda^3\rangle$$

and

$$T(f, U) \supset \mathscr{M}^k\langle\lambda\rangle + \mathscr{M}^{k-2}\langle\lambda^2\rangle + \mathscr{M}\langle\lambda^3\rangle$$

and so by theorem 5.5

$$\mathscr{P}(f, U) = T(f, U).$$

Hence by theorem 4.4, f is linearly determined and

$$\begin{aligned}Uf &= f + T(f, U)\\ &= ax^k + bx\lambda^2 + A(kax^{k-1}\lambda + b\lambda^3) + \mathscr{M}^{k+1} + \mathscr{M}^2\langle\lambda^2\rangle.\end{aligned}$$

Further, $g \in U \cdot f$ if and only if

$$g = g_x = \dots = g_{x^{k-1}} = 0 \qquad g_\lambda = g_{x\lambda} = \dots = g_{x^{k-2}\lambda} = 0 \qquad g_{\lambda\lambda} = 0 \tag{6.3a}$$

$$g_{x^k} = k!a \qquad g_{x\lambda\lambda} = 2b \tag{6.3b}$$

$$g_{x^{k-1}\lambda} = k!Aa \qquad g_{\lambda\lambda\lambda} = 6Ab. \tag{6.3c}$$

The equations in (6.3c) are equivalent to the condition

$$k!ag_{\lambda\lambda\lambda} - 6bg_{x^{k-1}\lambda} = 0 \tag{6.3d}$$

We have now solved both the unipotent recognition problem (6.3a,b,d) and the scaling recognition problem (6.2). Combining the two solutions gives the solution to the full recognition problem. Hence we see that g is $\mathscr{D}$ equivalent to $\varepsilon x^k + \delta x\lambda^2$ if and only if

$$g = g_x = \dots = g_{x^{k-1}} = 0 \qquad g_\lambda = g_{x\lambda} = \dots = g_{x^{k-2}\lambda} = 0 \qquad g_{\lambda\lambda} = 0$$
$$\operatorname{sign} g_{x^k} = \varepsilon \qquad \operatorname{sign} g_{x\lambda\lambda} = \delta$$

and

$$g_{x^k\lambda} g_{\lambda\lambda\lambda} - 3g_{x^{k-1}\lambda} g_{x\lambda\lambda} = 0.$$

Although the defining conditions for the unipotent problem are linear, the defining and non-degeneracy conditions for the corresponding full problem are not linear.

(ii) $\varepsilon(x^2 + \delta\lambda)^2 + \sigma x^5$, codim $= 5$. (See table 3.5 of Keyfitz (1986) and example 1.13 of Gaffney (1986).) It is easy to check that f is equivalent by scalings to $\varepsilon(x^2 + \delta\lambda)^2 + \sigma x^5$ if and only if

$$f = a(x^2 + b\lambda)^2 + cx^5 \qquad \operatorname{sign} a = \varepsilon \qquad \operatorname{sign} b = \delta \qquad \operatorname{sign} c = \sigma. \tag{6.4}$$

Consider

$$f(x, \lambda) = a(x^2 + b\lambda)^2 + cx^5 \qquad a, b, c \neq 0.$$

Computations show that

$$T(f, U) = \tilde{T}(f, U) = H + \mathbb{R}\{x^5 + bx^3\lambda, x^3\lambda + bx\lambda^2\}$$

where

$$H = \mathcal{M}^6 + \mathcal{M}^4\langle\lambda\rangle + \mathcal{M}^2\langle\lambda^2\rangle + \langle\lambda^3\rangle$$

and that

$$\mathscr{P}(f, U) = T(f, U).$$

However $\mathscr{P}(f, \mathscr{D})$ is only H. Gaffney shows that in this case a sufficient condition for g to be $\mathscr{D}$ equivalent to f is that $g \equiv f \bmod T(f, U)$. In fact theorem 4.4 shows that this condition is necessary and sufficient for U equivalence. Hence

$$U \cdot f = ax^4 + 2abx^2\lambda + ab^2\lambda^2 + cx^5 + A(x^5 + bx^3\lambda) + B(x^3\lambda + bx\lambda^2) + H$$

and $g \in U \cdot f$ if and only if

$$
\begin{aligned}
&g = g_x = g_{xx} = g_{xxx} = 0 \qquad g_\lambda = g_{x\lambda} = 0 \\
&g_{xxxx} = 24a \qquad g_{xx\lambda} = 4ab \qquad g_{\lambda\lambda} = 2ab^2 \\
&g_{xxxxx} = 120(c + A) \qquad g_{xxx\lambda} = 6(Ab + B) \qquad g_{x\lambda\lambda} = 2Bb.
\end{aligned}
\tag{6.5}
$$

Conditions (6.5) are equivalent to

$$
\begin{aligned}
&g = g_x = g_{xx} = g_{xxx} = 0 \qquad g_\lambda = g_{x\lambda} = 0 \\
&g_{xxxx} = 24a \qquad 6g_{xx\lambda} = bg_{xxxx} \qquad g_{xxxx}g_{\lambda\lambda} - 3g_{xx\lambda}^2 = 0
\end{aligned}
$$

and (6.6)

$$
\frac{g_{xxxxx}}{120} - \frac{g_{xxx\lambda}}{6b} + \frac{g_{x\lambda\lambda}}{2b^2} = c.
$$

Combining this with (6.4) yields the required result: $g \in \mathscr{D}f$ if and only if

$$
\begin{aligned}
&g = g_x = g_{xx} = g_{xxx} = 0 \qquad g_\lambda = g_{x\lambda} = 0 \\
&\operatorname{sign} g_{xxxx} = \varepsilon \qquad \operatorname{sign} g_{xx\lambda} = \varepsilon\delta \qquad g_{xxxx}g_{\lambda\lambda} - 3g_{xx\lambda}^2 = 0 \\
&\operatorname{sign}\left(g_{xxxxx} - 10\frac{g_{xxx\lambda}g_{xx\lambda}}{g_{\lambda\lambda}} + 15\frac{g_{x\lambda\lambda}g_{xx\lambda}^2}{g_{\lambda\lambda}^2}\right) = \sigma.
\end{aligned}
$$

Note that example 6.1(ii) is the first member of the infinite family

$$\varepsilon(x^2 + \delta\lambda)^2 + \sigma x^l \qquad l \geqslant 5, \text{ codim} = l$$

in Keyfitz (1986). In fact it is the only member of the family that is linearly determined.

6.2. One state variable. $\Gamma = \mathbb{Z}_2$ (Golubitsky and Schaeffer (1984) ch VI)

Here Γ acts on $\mathbb{R}$ as multiplication by -1. The ring of Γ-invariant *polynomials* in x is merely the ring of even polynomials, while the module of Γ-equivariant polynomials just consists of odd polynomials. Every odd polynomial can be written as an even polynomial multiplied by x, and so the module of Γ-equivariant polynomials is generated over the ring of Γ-invariant polynomials by the single element x. Results of Schwarz (1975) and Poénaru (1976) state that these properties are shared by smooth germs. Thus if we let $u = x^2$, then

$$
\begin{aligned}
&\mathscr{E}_{x,\lambda}(\mathbb{Z}_2) = \mathscr{E}_{u,\lambda} \\
&\vec{\mathscr{E}}_{x,\lambda}(\mathbb{Z}_2) = \mathscr{E}_{u,\lambda} \cdot x.
\end{aligned}
$$

Suppose $f \in \vec{\mathscr{E}}_{x,\lambda}(\mathbb{Z}_2)$, $f(x, \lambda) = r(u, \lambda) \cdot x$, $r \in \mathscr{E}_{u,\lambda}$. The unipotent tangent space is given by

$$T(f, U, \mathbb{Z}_2) = \tilde{T}(f, U, \mathbb{Z}_2) + \mathscr{E}_\lambda\{\lambda^2 r_\lambda \cdot x\} \tag{6.7a}$$

where

$$\tilde{T}(f, U, \mathbb{Z}_2) = \mathscr{E}_{u,\lambda}\{ur, \lambda r, u^2 r_u, u\lambda r_u\} \cdot x. \tag{6.7b}$$

A list of $\mathbb{Z}_2$-equivariant germs up to codimension 3 is given in table VI,5.1 of

Golubitsky and Schaeffer (1984). It turns out that all but one of the eleven bifurcation problems satisfy

$$\mathscr{P}(f, \mathscr{D}, \mathbb{Z}_2) = \mathscr{P}(f, U, \mathbb{Z}_2) = T(f, U, \mathbb{Z}_2).$$

The missing problem is linearly determined but $\mathscr{P}(f, \mathscr{D}, \mathbb{Z}_2)$ is strictly contained in $T(f, U, \mathbb{Z}_2)$. This means that there is a distinct advantage in considering the unipotent recognition problem separately and we choose this as our next example.

Example 6.2. $(\varepsilon(u+\delta\lambda)^2+\sigma u^3)x$, $\operatorname{codim}_{\mathbb{Z}_2}=3$. Now f is equivalent by scalings to $[\varepsilon(u+\delta\lambda)^2+\sigma u^3]x$ if and only if

$$f=[a(u+b\lambda)^3]x \qquad \operatorname{sign} a=\varepsilon \qquad \operatorname{sign} b=\delta \qquad \operatorname{sign} c=\sigma. \tag{6.8}$$

Consider the germ

$$f(x, \lambda)=r(u, \lambda)x$$

where

$$r(u, \lambda)=a(u+b\lambda)^2+cu^3 \qquad a, b, c \neq 0.$$

A computation using (6.7) shows that

$$T(f, U, \mathbb{Z}_2)=\tilde{T}(f, U, \mathbb{Z}_2)=H+V$$

where

$$H=\mathscr{E}_{u,\lambda}\{u^4, u^3\lambda, u^2\lambda^2, u\lambda^3, \lambda^4\}\cdot x$$

and

$$V=\mathbb{R}\{u^3+bu^2\lambda, u^2\lambda+bu\lambda^2, u\lambda^2+b\lambda^3\}\cdot x.$$

Notice that $u^4\cdot x \in H$ and hence H contains any monomial of order 9 or more in x. Therefore $H \supset \vec{\mathscr{M}}_9(\mathbb{Z}_2)\mathscr{E}_\lambda$. In this way we see that

$$H=\vec{\mathscr{M}}_9(\mathbb{Z}_2)\mathscr{E}_\lambda+\vec{\mathscr{M}}_7(\mathbb{Z}_2)\langle\lambda\rangle+\vec{\mathscr{M}}_5(\mathbb{Z}_2)\langle\lambda^2\rangle+\vec{\mathscr{M}}_3(\mathbb{Z}_2)\langle\lambda^3\rangle+\vec{\mathscr{M}}_1(\mathbb{Z}_2)\langle\lambda^4\rangle$$

and so by proposition 5.2

$$\mathscr{P}(f, \mathscr{D}, \mathbb{Z}_2)=H.$$

Now

$$V \subset \vec{\mathscr{M}}_7(\mathbb{Z}_2)\mathscr{E}_\lambda+\vec{\mathscr{M}}_5(\mathbb{Z}_2)\langle\lambda\rangle+\vec{\mathscr{M}}_3(\mathbb{Z}_2)\langle\lambda^2\rangle+\vec{\mathscr{M}}_1(\mathbb{Z}_2)\langle\lambda^3\rangle$$

and since there are no equivariants of even order $9^-=7$, $7^-=5$, $5^-=3$ and $3^-=1$. Thus, by theorem 5.4

$$T(f, U, \mathbb{Z}_2)=H+V$$

is U intrinsic and so f is linearly determined. Therefore

$$\begin{aligned} U\cdot f &= (au^2+2abu\lambda+ab^2\lambda^2+cu^3)\cdot x \\ &\quad +[A(u^3+bu^2\lambda)+B(u^2\lambda+bu\lambda^2)+C(u\lambda^2+b\lambda^3)]\cdot x+H. \end{aligned}$$

Hence $g(x, \lambda) = s(u, \lambda)x$ is U equivalent to f if and only if

$$
\begin{aligned}
&s = s_u = s_\lambda = 0\\
&s_{uu} = 2a \qquad s_{u\lambda} = 2ab \qquad s_{\lambda\lambda} = 2ab^2\\
&s_{uuu} = 6(c + A) \qquad s_{uu\lambda} = 2(Ab + B) \qquad s_{u\lambda\lambda} = 2(Bb + C) \qquad s_{\lambda\lambda\lambda} = 6Cb.
\end{aligned}
\tag{6.9}
$$

Equations (6.9) can be replaced by

$$
\begin{aligned}
&s = s_u = s_\lambda = 0\\
&s_{uu} = 2a \qquad s_{u\lambda} = bs_{uu} \qquad s_{uu}s_{\lambda\lambda} - s_{u\lambda}^2 = 0\\
&s_{uuu} - 3\frac{s_{uu\lambda}}{b} + 3\frac{s_{u\lambda\lambda}}{b^2} - \frac{s_{\lambda\lambda\lambda}}{b^3} = 6c.
\end{aligned}
\tag{6.10}
$$

Together with (6.8) this gives the necessary and sufficient conditions for g to be $\mathbb{Z}_2$ equivalent to $\varepsilon(u + \delta\lambda)^2 + \sigma u^3$, namely

$$
\begin{aligned}
&s = s_u = s_\lambda = 0\\
&\operatorname{sign} s_{uu} = \varepsilon \qquad \operatorname{sign} s_{u\lambda} = \varepsilon\delta \qquad s_{uu}s_{\lambda\lambda} - s_{u\lambda}^2 = 0\\
&\operatorname{sign}\left(s_{uuu} - 3\frac{s_{uu\lambda}s_{uu}}{s_{u\lambda}} + 3\frac{s_{u\lambda\lambda}s_{uu}^2}{s_{u\lambda}^2} - \frac{s_{\lambda\lambda\lambda}s_{uu}^3}{s_{u\lambda}^3}\right) = \sigma.
\end{aligned}
$$

6.3. Two state variables. $\Gamma = D_4$ (Golubitsky and Roberts 1986)

Here D_4 is taken to be acting on $\mathbb{R}^2$ as the symmetry group of the square and is generated by the symmetries

$$(x, y) \mapsto (x, -y), \ (x, y) \mapsto (y, x).$$

The ring of D_4-invariant germs is given by

$$\mathscr{E}_{x,y,\lambda}(D_4) = \mathscr{E}_{N,\Delta,\lambda}$$

where

$$N = x^2 + y^2 \qquad \text{and} \qquad \Delta = (x^2 - y^2)^2.$$

$\vec{\mathscr{E}}_{x,y,\lambda}(D_4)$ is generated as a module over $\mathscr{E}_{x,y,\lambda}(D_4)$ by

$$\begin{pmatrix} x \\ y \end{pmatrix}, \ (y^2 - x^2)\begin{pmatrix} x \\ -y \end{pmatrix}.$$

Hence every D_4-equivariant map germ can be written as

$$f(x, y, \lambda) = p(N, \Delta, \lambda)\begin{pmatrix} x \\ y \end{pmatrix} + r(N, \Delta, \lambda)(y^2 - x^2)\begin{pmatrix} x \\ -y \end{pmatrix}.$$

We adopt the 'invariant coordinate' notation

$$f = [p, r].$$

Table 2.1 of Golubitsky and Roberts (1986) gives a list of the fifteen bifurcation problems with D_4 symmetry of topological codimension 2 or less. Of these, ten are linearly determined. We remark that these are precisely those bifurcation problems satisfying the non-degeneracy condition $r(0) \neq 0$. An analogous situation exists in the $\mathbb{O}$-symmetric context; see §6.4. Of the linearly determined germs, $\mathscr{P}(f, U, D_4)$ is strictly larger than $\mathscr{P}(f, \mathscr{D}, D_4)$ for all but cases I and II. We treat problem XII.

Example 6.3. $[\varepsilon N + \delta\lambda^2 + \sigma\Delta + mN\lambda, \varepsilon]$, $m^2 \neq 4\delta\sigma$, top.codim$_{D_4} = 2$. The scaling problem is not quite as trivial as in the previous examples. f is equivalent by scalings to $[\varepsilon N + \delta\lambda^2 + \sigma\Delta + mN\lambda, \varepsilon]$ if and only if

$$f = [aN + b\lambda^2 + c\Delta + dN\lambda, a] \tag{6.11a}$$

and there are positive numbers μ, v, l such that

$$\varepsilon\mu v^3 = a \qquad \delta\mu v l^2 = b \qquad \sigma\mu v^5 = c \qquad m\mu v^3 l = d.$$

Clearly we require

$$\operatorname{sign} a = \varepsilon \qquad \operatorname{sign} b = \delta \qquad \operatorname{sign} c = \sigma. \tag{6.11b}$$

A short computation shows that in addition we require

$$m = \frac{d}{|bc|^{1/2}}. \tag{6.11c}$$

As usual we now consider the unscaled germ

$$f = [aN + b\lambda^2 + c\Delta + dN\lambda, a] \qquad d^2 \neq 4bc.$$

In example 9.2 of Golubitsky and Roberts (1986) it is shown that

$$T(f, U, D_4) \supset H$$

where

$$H = [\mathcal{M}^3 + \mathcal{M}\langle\Delta\rangle, \mathcal{M}^2 + \langle\Delta\rangle]$$

$\mathcal{M}$ being the maximal ideal $\langle N, \Delta, \lambda\rangle$ in $\mathscr{E}_{N,\Delta,\lambda}$. In fact

$$T(f, U, D_4) = H + \mathbb{R}\{[N^2, N], [\Delta, N], [N\lambda, \lambda]\}. \tag{6.12}$$

In order to translate (6.12) into the notation of §5, we first note that H is generated as an $\mathscr{E}_{N,\Delta,\lambda}$ module by

$$[N^3, 0], [\Delta^2, 0], [\lambda^3, 0], [N^2\lambda, 0], [N\Delta, 0], [\Delta\lambda, 0], [0, N^2], [0, \Delta], [0, \lambda^2], [0, N\lambda]. \tag{6.13}$$

Ignoring factors of λ we start to list monomials in $\vec{\mathscr{E}}_{x,u,\lambda}(D_4)$ in order of degree in (x, y). Note that N and Δ have degrees 2 and 4 and that $[1, 0]$ and $[0, 1]$ have degrees 1 and 3.

Order	$[*, 0]$	$[0, *]$	
1	1		
3	N	1	
5	N^2, Δ	N	
7	N^3, $N\Delta$	N^2, Δ	etc

Glancing at (6.13) we note that the only monomials in (x, y) which are missing are

$$[1, 0], [N, 0], [N^2, 0], [\Delta, 0], [0, 1], [0, N].$$

These are all terms of degree 5 or less in (x, y) and hence

$$H \supset \vec{\mathcal{M}}_7(D_4)\mathcal{E}_\lambda.$$

In this way it is easily seen that

$$H = \vec{\mathcal{M}}_7(D_4)\mathcal{E}_\lambda + \vec{\mathcal{M}}_5(D_4)\langle\lambda\rangle + \vec{\mathcal{M}}_3(D_4)\langle\lambda^2\rangle + \vec{\mathcal{M}}_1(D_4)\langle\lambda^3\rangle.$$

Thus by proposition 5.2 H is U intrinsic and so is contained in $\mathcal{P}(f, U, D_4)$. Furthermore

$$\mathbb{R}\{[N^2, N], [\Delta, N], [N\lambda, \lambda]\} \subset \vec{\mathcal{M}}_5(D_4)\mathcal{E}_\lambda + \vec{\mathcal{M}}_3(D_4)\langle\lambda\rangle$$

and so by theorem 5.4, $\mathcal{P}(f, U, D_4) = T(f, U, D_4)$. Therefore by theorem 4.4 we have

$$U \cdot f = [aN + b\lambda^2 + c\Delta + dN\lambda, a] + A[N^2, N] + B[\Delta, N] + C[N\lambda, \lambda] + H.$$

Hence $[p, r] \in U \cdot f$ if and only if

$$\begin{array}{lllll} p = p_\lambda = 0 & p_N = a & p_{\lambda\lambda} = 2b & p_\Delta = c + B & \\ p_{N\lambda} = d + C & p_{NN} = 2A & r = a & r_N = A + B & r_\lambda = C \end{array}$$

that is if and only if

$$\begin{array}{llll} p = p_\lambda = 0 & p_N = a & p_{\lambda\lambda} = 2b & p_N - r = 0 \\ p_{N\lambda} - r_\lambda = d & p_{NN} + 2p_\Delta - 2r_N = 2c. & & \end{array} \tag{6.14}$$

Combining (6.14) with (6.11) we see that $[p, r]$ is D_4 equivalent to $[\varepsilon N + \delta\lambda^2 + \sigma\Delta + mN\lambda, \varepsilon]$ if and only if

$$\begin{array}{llll} p = p_\lambda = 0 & \operatorname{sign} p_N = \varepsilon & \operatorname{sign} p_{\lambda\lambda} = \delta & p_N - r = 0 \end{array}$$
$$\operatorname{sign}(p_{NN} + 2p_\Delta - 2r_N) = \sigma$$
$$m = \frac{2(p_{N\lambda} - r_\lambda)}{|p_{\lambda\lambda}(p_{NN} + 2p_\Delta - 2r_N)|^{1/2}}.$$

6.4. Three state variables. $\Gamma = \mathbb{O}$ (Melbourne 1988)

We take $\mathbb{O}$ to be acting as the symmetry group of the cube. The action is generated by

$$(x, y, z) \mapsto (-x, y, z) \qquad (x, y, z) \mapsto (y, x, z) \qquad (x, y, z) \mapsto (x, z, y).$$

We have

$$\mathcal{E}_{x,y,z\lambda}(\mathbb{O}) = \mathcal{E}_{u,v,w,\lambda}$$

where

$$u = x^2 + y^2 + z^2 \qquad v = x^2y^2 + y^2z^2 + z^2x^2 \qquad w = x^2y^2z^2$$

and $\vec{\mathcal{E}}_{x,y,z,\lambda}(\mathbb{O})$ is generated as a $\mathcal{E}_{u,y,w,\lambda}$ module by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}, \begin{pmatrix} y^2z^2x \\ z^2x^2y \\ x^2y^2z \end{pmatrix}.$$

It turns out that there are seven bifurcation problems with $\mathbb{O}$ symmetry of topological codimension 1 or less, and all but one of these are linearly determined. As in the D_4-symmetric context, linear determinacy holds for these low-codimension problems if and only if the coefficient of the third-order equivariant generator is non-zero at the origin. We include an example which gives a graphic illustration of the complications which can be introduced at the scalings stage. The equations for the unipotent recognition problem are not that pleasant but they are at least linear.

Example 6.4. $[-\varepsilon u + \delta\lambda + \sigma u^k + pu^{k+1}, \varepsilon, 0]$, $k \geqslant 2$, top. $\mathrm{codim}_{\mathbb{O}} = k - 1$. This is family 3 in Melbourne (1988). Now f is equivalent by scalings to $[-\varepsilon u + \delta\lambda + \sigma u^k - pu^{k+1}, \varepsilon, 0]$ if and only if

$$f = [-au + b\lambda + cu^k + du^{k+1}, a, 0] \tag{6.15}$$

and there exist positive numbers μ, v and l such that

$$\varepsilon\mu v^3 = a \tag{6.16a}$$

$$\delta\mu vl = b \tag{6.16b}$$

$$\sigma\mu v^{2k+1} = c \tag{6.16c}$$

$$p\mu v^{2k+3} = d. \tag{6.16d}$$

Conditions (6.16*a,b,c*) yield

$$\operatorname{sign} a = \varepsilon \qquad \operatorname{sign} b = \delta \qquad \operatorname{sign} c = \sigma. \tag{6.17}$$

In addition (6.16*a,c*) can be solved for μ and v in terms of $|a|$ and $|c|$ and substituting in (6.16*d*) yields

$$p = \frac{d}{|c|}\left(\frac{|a|}{|c|}\right)^{1/(k-1)}. \tag{6.18}$$

The calculations for this bifurcation problem in Melbourne (1988) yield the following:

$$[P, Q, R] \in U([-au + b\lambda + cu^k + du^{k+1}, a, 0])$$

if and only if

$$P = P_u + Q = \ldots = P_{u^{k-1}} + (k-1)Q_{u^{k-2}} = 0$$

$$Q = a,\ P_\lambda = b,\ P_{u^k} + kQ_{u^{k-1}} = k!c$$

$$\frac{P_{u^{k+1}} + (k+1)Q_{u^k}}{(k+1)!} + c\left(\frac{k}{2}\frac{(P_v - 2Q_u - R)}{a} + (k-1)\frac{(P_{u\lambda} + Q_\lambda)}{b}\right) = d.$$

Combining this result with (6.15), (6.17) and (6.18) we have that $[P, Q, R]$ is $\mathbb{O}$ equivalent to $[-\varepsilon u + \delta\lambda + \sigma u^k + pu^{k+1}, \varepsilon, 0]$ if and only if, letting $T_r = P_{u^r} + rQ_{u^{r-1}}$, we have

$$T_0 = T_1 = \ldots = T_{k-1} = 0$$

$$\operatorname{sign} Q = \varepsilon \qquad \operatorname{sign} P_\lambda = \delta \qquad \operatorname{sign} T_k = \sigma$$

$$\sigma\left(\frac{k!\,|Q|}{T_k}\right)^{1/(k-1)}\left(\frac{T_{k+1}}{(k+1)T_k} + \frac{k}{2}\frac{(P_v - 2Q_u - R)}{Q} + (k-1)\frac{(P_{u\lambda} + Q_\lambda)}{P_\lambda}\right) = d.$$

7. Examples with Γ not acting absolutely irreducibly

In §6 we considered examples where Γ acts irreducibly. Using theorem 5.4 or theorem 5.5, we were able to show that the unipotent tangent spaces of certain bifurcation problems are U intrinsic. Then by theorem 4.4 the unipotent recognition problems can be solved using only linear algebra. Furthermore it is then trivial to recover the solution to the full recognition problem because the group $S(\Gamma)$ of linear Γ equivalences just consists of scalar multiples of the identity. In other words, the triviality of the $S(\Gamma)$-recognition problems in §6 relies on the absolute irreducibility rather than the irreducibility of the Γ action.

Schur's lemma (theorem 2, p 119 of Kirillov 1976) states that if Γ acts irreducibly on V and $\mathrm{Hom}_\Gamma(V)$ denotes the space of linear maps on V that commute with Γ, then

$$\mathrm{Hom}_\Gamma(V) \simeq \mathbb{R},\ \mathbb{C} \text{ or } \mathbb{H}.$$

If $\mathrm{Hom}_\Gamma(V) \simeq \mathbb{R}$, then Γ acts absolutely irreducibly, whereas if $\mathrm{Hom}_\Gamma(V) \cong \mathbb{C}$ or $\mathbb{H}$, then there is no coordinate system in which $\mathrm{Hom}_\Gamma(V)$ consists only of diagonal matrices.

Definition 7.1. Suppose Γ is a compact Lie group acting on $\mathbb{R}^n$. We say that $S(\Gamma)$ is *scalar* if in some coordinate system

$$\mathrm{Hom}_\Gamma(\mathbb{R}^n) \subset \{\text{diagonal matrices}\}.$$

Proposition 7.2. Suppose Γ acts irreducibly on $\mathbb{R}^n$. Then $S(\Gamma)$ is scalar if and only if Γ acts absolutely irreducibly.

Suppose now that Γ does not act irreducibly. By theorem 3.20 of Adams (1969), $\mathbb{R}^n$ can be decomposed into irreducible subspaces

$$\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k.$$

Lemma 7.3. $S(\Gamma)$ is scalar if

(i) the actions of Γ on V_i and V_j are not isomorphic for $i \neq j$;
(ii) $\mathrm{Hom}_\Gamma(V_i) \simeq \mathbb{R} \qquad i = 1, \ldots, k.$

Proof. Suppose $L \in \mathscr{L}(\Gamma)^\circ \subset \mathrm{Hom}_\Gamma(\mathbb{R}^n)$. Then, as in proposition 4.2 of Stewart (1987),

$$L(V_i) \subset V_i \qquad i = 1, \ldots, k$$

and so L has the block matrix structure

$$\begin{array}{c} V_1 \\ \vdots \\ V_k \end{array} \begin{pmatrix} L_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & L_k \end{pmatrix}$$

where each $L_i \in \mathrm{Hom}_\Gamma(V_i)$. Furthermore, since each $\mathrm{Hom}_\Gamma(V_i) \simeq \mathbb{R}$ we have

$$L_i = \mu_i I \qquad \mu_i \in \mathbb{R} \qquad i = 1, \ldots, k.$$

In this paper we consider only examples where $S(\Gamma)$ is scalar. A non-scalar problem is studied by Golubitsky and Schaeffer (1984) ch IX. They look at the non-degenerate bifurcation problems in two state variables with no symmetry. Their result for high-order terms is easily recovered using corollary 3.9; indeed the problems are linearly determined. However it is in the S-recognition problem that all the difficulties lie.

In the remainder of this section we look at a straightforward example where Γ does not act irreducibly but where $S(\Gamma)$ is scalar.

7.1. $\Gamma = \mathbb{Z}_2$ acting on $\mathbb{R}^2$, by reflection on one copy of $\mathbb{R}$, trivially on the other (Dangelmayr and Armbruster 1983)

The $\mathbb{Z}_2$ action is generated by

$$(x, y) \mapsto (x, -y).$$

Every $\mathbb{Z}_2$-equivariant germ can be written in the form

$$f(x, y, \lambda) = \begin{pmatrix} f_1(x, y, \lambda) \\ f_2(x, y, \lambda) \end{pmatrix}$$

where

$$f_1(x, y, \lambda) = p(u, v, \lambda) \qquad f_2(x, y, \lambda) = r(u, v, \lambda)y$$

$$u = x \qquad v = y^2.$$

In the invariant coordinate notation

$$f = [p, r].$$

In this notation the unipotent tangent space

$$T(f, U, \mathbb{Z}_2) = \tilde{T}(f, U, \mathbb{Z}_2) + \mathscr{E}_\lambda\{\lambda^2[p_\lambda, r_\lambda]\}$$

where $\tilde{T}(f, U, \mathbb{Z}_2)$ is generated as a $\mathscr{E}_{u,v,\lambda}$ module by

$$z[p, 0] \qquad z[0, r] \qquad z[vp_v, vr_v] \qquad z = u, v \text{ or } \lambda$$

$$[0, p] \qquad [vr, 0] \qquad u^2[up_u, ur_u] \qquad v[up_u, ur_u] \qquad \lambda[up_u, ur_u].$$

Let $\mathscr{M} = \langle u, v, \lambda \rangle$ denote the maximal ideal in $\mathscr{E}_{u,v,\lambda}$. Let I and J consist of sums and products of ideals of the form $\mathscr{M}$, $\langle v \rangle$ and $\langle \lambda \rangle$. Then it is easily seen from the tangent space generators that (I, J) is an intrinsic module if and only if

$$vJ \subset I \subset J.$$

This characterisation of 'obvious' intrinsic modules proves more useful in this particular case than the more general theorem 5.5.

It turns out that the methods of this paper simplify calculations for relatively few of the bifurcation problems. Linear determinacy holds for three out of the five problems of topological codimension 1 or less , but for only three of a further twelve problems of topological codimension 2. There are two types of equivalence that restrict the number of intrinsic subspaces:

$$x \mapsto x + \lambda \qquad \text{and} \qquad [p, 0] \mapsto [0, p].$$

The first of these types also occurs when there is one state variable without symmetry and causes bifurcation problems of low codimension to fail to be linearly determined. This does not happen when there is reflectional symmetry present. For example in our present context we do not have equivalences of the form

$$y \mapsto y + \lambda \qquad \text{or} \qquad [0, q] \mapsto [q, 0].$$

We would expect the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{R}^2$

$$(x, y) \mapsto (-x, y) \qquad (x, y) \mapsto (x, -y)$$

to behave far better, in much the same way that $\mathbb{Z}_2$ behaves better than $\mathbb{1}$ when acting on $\mathbb{R}$.

Example 7.4. $[u^m + \varepsilon_1\lambda + \varepsilon_3 v, \varepsilon_2 u^2 + v]$, $m \geqslant 3$, top. $\operatorname{codim}_{\mathbb{Z}_2} = m - 1$. This is family $(3)_{m2}$ of Dangelmayr and Armbruster (1983). First we solve the $S(\mathbb{Z}_2)$-recognition problem. Note that $S(\mathbb{Z}_2)$ is scalar:

$$\operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{R}^2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

We ususally require that

$$S(0), (dX)_0 \in \mathscr{L}(\mathbb{Z}_2)^\circ \qquad \Lambda'(0) > 0$$

$\mathscr{L}(\mathbb{Z}_2)^\circ$ being the connected component of $\operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{R}^2) n GL(\mathbb{R}^2)$ containing the identity (see chapter XIV, §1 of Golubitsky *et al* (1988)). Then

$$\mathscr{L}(\mathbb{Z}_2)^\circ = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b > 0 \right\}.$$

Dangelmayr and Armbruster (1983) impose the alternative restrictions

$$\det S(0) \neq 0 \qquad (dX)_0 > 0 \qquad \Lambda'(0) > 0.$$

In other words $(S, X, \Lambda) \in S(\mathbb{Z}_2)$ must satisfy

$$S(x, y, \lambda) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \qquad X(x, y, \lambda) = \begin{pmatrix} \nu_1 x & 0 \\ 0 & \nu_2 y \end{pmatrix} \qquad \Lambda(\lambda) = l\lambda$$

where μ_1, $\mu_2 \neq 0$, ν_1, ν_2, $l > 0$. It can be shown that f is $S(\mathbb{Z}_2)$ equivalent to $[u^m + \varepsilon_1\lambda + \varepsilon_3 v, \varepsilon_2 u^2 + v]$ subject to the following conditions:

$$f = [au^m + b\lambda + cv, du^2 + ev] \tag{7.1a}$$

$$\operatorname{sign}(de) = \varepsilon_2 \qquad \operatorname{sign}(ac) = \varepsilon_3 \qquad \operatorname{sign}(ab) = \varepsilon_1 \qquad \text{if } m \text{ is even} \tag{7.1b}$$

$$\operatorname{sign}(de) = \varepsilon_2 \qquad \operatorname{sign}(bc) = \varepsilon_1\varepsilon_3 \qquad \text{if } m \text{ is odd.} \tag{7.1c}$$

As always we now consider the unscaled bifurcation problem

$$f = [au^m + b\lambda + cv, du^2 + ev] \qquad m \geqslant 3, a, b, c, d, e \neq 0.$$

A simple calculation reveals that

$$T(f, U, \mathbb{Z}_2) = [I, J] + \mathbb{R}\{[0, b\lambda + cv]\}$$

where

$$I = \mathscr{M}^{m+1} + \mathscr{M}\langle v, \lambda \rangle \qquad J = \mathscr{M}^3 + \mathscr{M}\langle v, \lambda \rangle.$$

Clearly $vJ \subset I \subset J$ and so $\mathscr{P}(f, U, \mathbb{Z}_2) \supset [I, J]$. Furthermore it is easily checked from the tangent space generators that if $p \in \mathbb{R}\{[0, b\lambda + cv]\}$ then

$$T(p, U, \mathbb{Z}_2) \subset [I, J].$$

By theorem 4.4

$$U \cdot f = f + A[0, b\lambda + cv] + [I, J].$$

Thus $[p, r] \in U \cdot f$ if and only if

$$\begin{array}{llll} p = p_u = \ldots = p_{u^{m-1}} = 0 & & r = r_u = 0 & \\ p_{u^m} = m!a & p_\lambda = b & p_v = c & r_{uu} = 2d \\ r_v = e + Ac & r_\lambda = Ab. & & \end{array}$$

These conditions are equivalent to

$$\begin{array}{llll} p = p_u = \ldots = p_{u^{m-1}} = 0 & & r = r_u = 0 & \\ p_{u^m} = m!a & p_\lambda = b & p_v = c & r_{uu} = 2d \\ p_\lambda r_v - r_\lambda p_v = p_\lambda e. & & & \end{array}$$

Hence by (7.1) $[p, r]$ is $\mathbb{Z}_2$ equivalent to $[u^m + \varepsilon_1\lambda + \varepsilon_3 v,\ \varepsilon_2 u^2 + v]$ if and only if

$$\begin{array}{l} p = p_u = \ldots = p_{u^{m-1}} = 0 \qquad r = r_u = 0 \\ \operatorname{sign}(r_{uu}p_\lambda(p_\lambda r_v - r_\lambda p_v)) = \varepsilon_2 \end{array}$$

and

$$\begin{array}{lll} \operatorname{sign}(p_{u^m}p_v) = \varepsilon_3 & \operatorname{sign}(p_{u^m}p_\lambda) = \varepsilon_1 & \text{if } m \text{ is even} \\ \operatorname{sign}(p_\lambda p_v) = \varepsilon_1\varepsilon_3 & \text{if } m \text{ is odd.} & \end{array}$$

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