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LETTER TO THE EDITOR

On a one-parameter family of q-exponential functions

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Abstract. We examine the properties of a family of q-exponential functions, which depend on an extra parameter α. These functions have a well defined meaning for both the $0 < |q| < 1$ and $|q| > 1$ cases if only $\alpha \in [0, 1]$. It is shown that any two members of this family with different values of the parameter α are related to each other by a Fourier–Gauss transformation.

The one-parameter family of q-exponential functions

$$E_q^{(\alpha)}(z) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2/2}}{(q;q)_n} z^n \tag{1}$$

with $\alpha \in \Re$ has been considered in [1]. The q-shifted factorial $(q;q)_n$ in (1) is defined as $(z;q)_0 = 1$ and $(z;q)_n = \prod_{j=0}^{n-1}(1 - zq^j)$, $n = 1, 2, 3, \ldots$. Consequently, in the limit when $q \to 1$ we have

$$\lim_{q\to 1} E_q^{(\alpha)}((1-q)z) = \mathrm{e}^z. \tag{2}$$

The possibility of introducing such a family of q-exponential functions has been already mentioned in [2]. Exton defines it as

$$E(q, \lambda; x) = \sum_{n=0}^{\infty} \frac{x^n q^{\lambda n(n-1)}}{[n;q]!} \tag{3}$$

where the symbol $[n;q]!$ denotes the product $\prod_{j=1}^{n}[j;q]$, with $[0;q] = 1$ and the bracket notation

$$[j;q] = \frac{1-q^j}{1-q}. \tag{4}$$

Since $(q;q)_n = (1-q)^n [n;q]!$ by definition, it is obvious that the relationship between these two notations is

$$E(q, \lambda; x) = E_q^{(2\lambda)}(q^{-\lambda}(1-q)x). \tag{5}$$

Two particular cases of this family with $\alpha = 0$ and $\alpha = 1$ are well known: they are the q-exponential function

$$e_q(z) = E_q^{(0)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} \tag{6}$$

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and its reciprocal

$$E_q(z) = e_q^{-1}(z) = E_q^{(1)}(-q^{-\frac{1}{2}}z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n}(-z)^n \tag{7}$$

respectively [3]. Another particular example of (1) corresponds to the value $\alpha = \frac{1}{2}$ and is

$$E_q^{(1/2)}(z) = \mathcal{E}_q(-;0,z) = \varepsilon_q(z) \tag{8}$$

where $\mathcal{E}_q(z;a,b)$ is a two-parameter q-exponential function, introduced in [4]. Exton denotes this q-exponential function as [2]

$$E(q,x) = E_q^{(1/2)}(q^{-1/4}(1-q)x) \tag{9}$$

and he emphasizes that it 'was originally considered in connection with a particular q-generalization of the circular functions which exhibit properties of q-orthogonality'.

By analogy with the exponentiating of Lie algebras into Lie groups, one may consider q-exponentials of the generators of a q-algebra and express their matrix elements in representation space in terms of q-special functions. In this way one manages to interpret algebraically the properties of these q-special functions through symmetry techniques [1]. Futhermore, as we show below, members of the family (1) with different values of the parameter α turn out to be Fourier–Gauss transforms of (6) and (7). In other words, they are of central importance for constructing Fourier–Gauss transforms of a number of q-special functions (cf [5, 6]). Therefore we wish to study here some additional properties of the q-exponential functions (1).

We start with the observation that by the ratio test the infinite series in (1) is convergent for $0 < |q| < 1$ and arbitrary complex z only if the parameter α is positive: $0 < \alpha < \infty$. The case $\alpha = 0$ is a little bit more involved, but $E_q^{(0)}(z) = e_q(z)$ and the properties of the q-exponential function (6) are well studied [2, 3, 7].

The series (1) converges also for $1 < |q| < \infty$ and arbitrary complex z, provided that $-\infty < \alpha < 1$. Thus, the q-exponential functions (1) are well defined for both $0 < |q| < 1$ and $1 < |q| < \infty$, if the parameter α belongs to the line segment $[0, 1]$. Observe that the inversion formula

$$(q^{-1};q^{-1})_n = (-1)^n q^{-n(n+1)/2}(q;q)_n \tag{10}$$

leads to the relation

$$E_{q^{-1}}^{(\alpha)}(z) = E_q^{(1-\alpha)}(-q^{1/2}z). \tag{11}$$

When $\alpha = 0$, (11) reproduces the known relation [2, 7]

$$e_{q^{-1}}(z) = E_q(qz) \tag{12}$$

between the q-exponential functions (6) and (7) for $0 < |q| < 1$ and $1 < |q| < \infty$ (or vice versa), respectively.

There are two types of Fourier–Gauss transforms for the q-exponential functions (1), depending on whether the parameter α belongs to either the interval $[0, \frac{1}{2}]$, or $[\frac{1}{2}, 1]$. Let us consider these cases in turn.

(a) When $0 \leqslant \alpha \leqslant \frac{1}{2}$, it is not hard to show that

$$E_q^{(\alpha+\frac{1}{2})}(t\,\mathrm{e}^{-\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xy-y^2/2} E_q^{(\alpha)}(t\,\mathrm{e}^{\mathrm{i}\kappa y})\,\mathrm{d}y \tag{13}$$

where $q = \exp(-2\kappa^2)$. Indeed, to evaluate the right-hand side of (13) one only needs to use the definition (1) with $z = t\,\mathrm{e}^{\mathrm{i}\kappa y}$ and to integrate this sum termwise by the Fourier transform

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xy-y^2/2}\,\mathrm{d}y = \mathrm{e}^{-x^2/2} \tag{14}$$

for the Gauss exponential function $\exp(-x^2/2)$. Important particular cases of (13) are

$$\varepsilon_q(t\,\mathrm{e}^{-\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}e_q(t\,\mathrm{e}^{\mathrm{i}\kappa y})\,\mathrm{d}y \tag{15}$$

and

$$E_q(-q^{1/2}t\,\mathrm{e}^{-\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}\varepsilon_q(t\,\mathrm{e}^{\mathrm{i}\kappa y})\,\mathrm{d}y. \tag{16}$$

They correspond to the values 0 and $\frac{1}{2}$ of the parameter α, respectively.

(b) In like manner, for $\frac{1}{2} \leqslant \alpha \leqslant 1$ we have

$$E_q^{(\alpha-1/2)}(t\,\mathrm{e}^{\mathrm{i}\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}E_q^{(\alpha)}(t\,\mathrm{e}^{\kappa y})\,\mathrm{d}y. \tag{17}$$

In particular, when $\alpha = \frac{1}{2}$ from (17) follows the inverse Fourier transformation with respect to (15)

$$e_q(t\,\mathrm{e}^{\mathrm{i}\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}\varepsilon_q(t\,\mathrm{e}^{\kappa y})\,\mathrm{d}y \tag{15$'$}$$

whereas the value $\alpha = 1$ yields the inverse to (16), i.e.

$$\varepsilon_q(t\,\mathrm{e}^{\mathrm{i}\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}E_q(-q^{1/2}t\,\mathrm{e}^{\kappa y})\,\mathrm{d}y. \tag{16$'$}$$

Actually, the Fourier–Gauss transforms (13) and (17) may be written in the unified form

$$E_q^{(\alpha+\nu^2/2)}(t\,\mathrm{e}^{-\nu\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}E_q^{(\alpha)}(t\,\mathrm{e}^{\mathrm{i}\nu\kappa y})\,\mathrm{d}y \tag{18}$$

provided that $-\alpha \leqslant \nu^2/2 \leqslant 1-\alpha$. This is easy to prove in exactly the same way as (13), or (17). When $0 \leqslant \alpha \leqslant \frac{1}{2}$ and $\nu = 1$ from (18) one obtains (15) and when $\frac{1}{2} \leqslant \alpha \leqslant 1$ and $\nu = -\mathrm{i}$ from (18) follows (17). Also, for $\alpha = 0$ and $\nu = \sqrt{2}$ the Fourier–Gauss transform (18) gives the following relation between the q-exponential functions $e_q(z)$ and $E_q(z)$ (cf (16))

$$E_q(-q^{1/2}t\,\mathrm{e}^{-\sqrt{2}\kappa x})\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}xy-y^2/2}e_q(t\,\mathrm{e}^{\mathrm{i}\sqrt{2}\kappa y})\,\mathrm{d}y \tag{19}$$

which is a particular case of Ramanujan's integral with a complex parameter [8–10].

Finally, we would like to touch upon two intimately interrelated features of the q-exponential functions (1). One of them is an explicit form of reciprocal q-exponential functions for any parameter α from the interval $[0, 1]$. Unfortunately, we know such reciprocals for the boundary values $\alpha = 0$ and $\alpha = 1$ only. The formal solution of this problem is known: for any $\alpha \in [0, 1]$ the function reciprocal to (1) is represented by the infinite series

$$1/E_q^{(\alpha)}(z) = \sum_{n=0}^{\infty} c_n^{(\alpha)}(q)z^n \tag{20}$$

with the coefficients $c_0^{(\alpha)}(q) = 1$ and

$$c_n^{(\alpha)}(q) = (-1)^n \begin{vmatrix} a_1^{(\alpha)}(q) & 1 & 0 & \dots & 0 \\ a_2^{(\alpha)}(q) & a_1^{(\alpha)}(q) & 1 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{n-1}^{(\alpha)}(q) & a_{n-2}^{(\alpha)}(q) & a_{n-3}^{(\alpha)}(q) & \dots & 1 \\ a_n^{(\alpha)}(q) & a_{n-1}^{(\alpha)}(q) & a_{n-2}^{(\alpha)}(q) & \dots & a_1^{(\alpha)}(q) \end{vmatrix} \qquad n = 1, 2 \dots \tag{21}$$

where $a_n^{(\alpha)}(q) = q^{\alpha n^2/2}(q;q)_n^{-1}$ are the corresponding coefficients in the expansion (1) for $E_q^{(\alpha)}(z)$.

Another interesting point is the possibility of representing $E_q^{(\alpha)}(z)$ as an infinite product. Again, we know that in the particular cases of $\alpha = 0$ and $\alpha = 1$ the affirmative answer to this question is given by Euler's formulae [3]

$$e_q(z) = (z;q)_\infty^{-1} \qquad E_q(z) = (z;q)_\infty \tag{22}$$

for the q-exponential functions (6) and (7). Observe that Euler's formulae (22) follow from the functional relations

$$e_q(qz) = (1-z)e_q(z) \qquad E_q(z) = (1-z)E_q(qz) \tag{23}$$

and the side conditions $e_q(0) = E_q(0) = 1$. One may try to combine one of the relations (23) with either (15), or (16), respectively , in order to get the corresponding representation at least for the parameter $\alpha = \frac{1}{2}$. However, in both cases this results in the functional relation (cf [11])

$$\varepsilon_q(qz) = \varepsilon_q(z) - q^{1/4} z \varepsilon_q(q^{1/2}z) \tag{24}$$

with $z = t\,\mathrm{e}^{-\kappa x}$ and $z = t\,\mathrm{e}^{\mathrm{i}\kappa x}$, respectively. Actually, this type of functional relation holds for arbitrary z and $\alpha \in [0,1]$ (not for $\alpha = \frac{1}{2}$ only!) and has the form

$$E_q^{(\alpha)}(qz) = E_q^{(\alpha)}(z) - q^{\alpha/2} z E_q^{(\alpha)}(q^\alpha z). \tag{25}$$

The validity of (25) can be readily verified by using the definition (1). When $\alpha = 0$ and $\alpha = 1$, from (25) follow the functional relations (23) for the q-exponential functions $e_q(z)$ and $E_q(z)$, respectively.

The relation (25) is equivalent to a q-difference equation. Indeed, in terms of the Jackson q-difference operator [12]

$$\Delta f(z) = \frac{f(z) - f(qz)}{z(1-q)} \tag{26}$$

it may be represented as

$$\Delta E_q^{(\alpha)}(z) = \frac{q^{\alpha/2}}{1-q} E_q^{(\alpha)}(q^\alpha z). \tag{25$'$}$$

Note also that applying the functional relations (23) n times in succession leads to

$$e_q(q^n z) = (z;q)_n e_q(z) \qquad E_q(z) = (z;q)_n E_q(q^n z) \tag{27}$$

respectively. These simple formulae turn out to be very useful in proving the orthogonality of the classical q-polynomials with respect to measures, containing q-exponential functions $e_q(z)$ and $E_q(z)$ [13–16]. In an analogous manner, from (25) it follows at once that

$$E_q^{(\alpha)}(q^n z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k q^{[(\alpha+1)k-1]k/2} E_q^{(\alpha)}(q^{\alpha k} z) \tag{28}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

are the q-binomial coefficients. This relation is easily verified by induction, upon using the following property of the q-binomial coefficients [3]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ k \end{bmatrix}_q .$$

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