Journal of Physics A: Mathematical and General

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To cite this article: D J Saunders 1987 *J. Phys. A: Math. Gen.* **20** 339

J. Phys. A: Math. Gen. 20 (1987) 339-349. Printed in the UK

An alternative approach to the Cartan form in Lagrangian field theories

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Received 5 February 1986

Abstract. An operator analogous to the almost tangent structure on a tangent bundle is defined on jet bundles of a fibred manifold. This operator is used to construct a Cartan form. The construction is unique for first-order Lagrangians and is also unique when restricted to higher-order mechanics.

1. Introduction

There has recently been some interest in the use of almost tangent structures in Lagrangian dynamics (for example, see Crampin 1983, Cariñena and Ibort 1985). An almost tangent structure on a $2n$-dimensional manifold is a type $(1, 1)$ tensor field S of constant rank n satisfying $S^2 = 0$. If, in addition, the Nijenhuis tensor of S vanishes, then the almost tangent structure is termed integrable (Clark and Bruckheimer 1960). The reason for the name is that any tangent manifold TE has a canonical almost tangent structure, also known as its vertical endomorphism, given by composing the tangent bundle projection $TTE \to TE$ with the vertical lift $TE \to TTE$. In local coordinates $(q^\alpha, \dot{q}^\alpha)$ on TE, this almost tangent structure is given as

$$S = \frac{\partial}{\partial \dot{q}^\alpha} \otimes \mathrm{d}q^\alpha. \tag{1.1}$$

If $L: TE \to R$ is a Lagrangian, then a closed 2-form ω_L can be defined by $\omega_L = \mathrm{d}(S(\mathrm{d}L))$. When L satisfies certain regularity conditions, ω_L is symplectic and equal to $FL(\omega)$, the image of the canonical symplectic form ω on T^*E under the fibre derivative FL of L (de León and Rodrigues 1985). On a higher-order tangent manifold T^kE there is a similar construction leading to the first vertical endomorphism S, where now S has the property $S^k \neq 0$, $S^{k+1} = 0$.

In the time-dependent theory, the vertical endomorphism on the first tangent manifold TE may be extended in a standard way to $TE \times R$, and when this is done the 1-form $S(\mathrm{d}L) + L\,\mathrm{d}t$ corresponding to a time-dependent Lagrangian is just the Cartan form of L:

$$S(\mathrm{d}L) + L\,\mathrm{d}t = (\partial L/\partial \dot{q}^\alpha)(\mathrm{d}q^\alpha - \dot{q}^\alpha\,\mathrm{d}t) + L\,\mathrm{d}t. \tag{1.2}$$

Cartan forms for different Lagrangians may therefore be constructed in a very simple way from a single geometrical object defined in $TE \times R$.

The purpose of the present work is to indicate a generalisation of this construction to field theories defined on jet bundles of a fibred manifold. The type $(1, 1)$ tensor field S (which may of course be regarded as a vector-valued 1-form) is ultimately replaced in the case of first-order theories by a vector-valued m-form S_Ω, where m is the dimension of the base manifold (= number of independent variables) and Ω is a given volume form on that manifold. In the case of higher-order theories, S_Ω is no longer a tensorial object but becomes a differential operator which in general is not unique. Using such an operator, the Cartan form of a Lagrangian can once again be written in the simple form $S_\Omega(\mathrm{d}L)+L\Omega$.

The structure of this paper is as follows. In § 2 we indicate the notation from the theory of jet bundles which will be used. Section 3 shows how to define the vertical endomorphism of a jet manifold corresponding to a given closed 1-form on the base manifold; this construction is fundamental to the rest of the work. In § 4 we use the vertical endomorphism to construct the Cartan form corresponding to a first-order Lagrangian; here, the parallel with mechanics is quite straightforward. By contrast, in § 5 we follow the work of Kuperschmidt (1980) and replace the manifold $J^k\pi$ by its image in $J^{k-1}\pi_1$, allowing us to use the first-order operator in an iterative way. In the final section we consider the uniqueness of this construction, obtaining the familiar result that the Cartan form is unique if (and only if) either k or m equals one.

2. Notation

All manifolds are assumed to be C^∞, finite-dimensional, paracompact and connected; all maps are C^∞. We use the notation $\Lambda^r M$ to denote the module of r-forms on M and $C^\infty(M)$ to denote the real-valued functions on M. Given $f: N\to M$, we write $\Lambda_0^r(f)$ for the module generated by $f^*\Lambda^r M$ over $C^\infty(N)$. Similarly, we use the notation $\Lambda_1^r(f)$ for the module generated by $\Lambda_0^{r-1}(f)\wedge\Lambda^1 N$.

If $\pi: E\to M$ is a locally trivial fibred manifold and ϕ is a local section of π then $j_p^k\phi$ denotes the k-jet of ϕ at a point p in its domain, and the set of all such k-jets is the kth jet manifold $J^k\pi$. The source, target and l-jet projections are denoted by $\pi_k: J^k\pi\to M$, $\pi_{k,0}: J^k\pi\to E$ and $\pi_{k,l}: J^k\pi\to J^l\pi$ $(k>l)$.

We shall also be interested in $\pi_k: J^k\pi\to M$ as a locally trivial fibred manifold in its own right, and hence in its r-jet manifold $J^r\pi_k$. The canonical injection $\iota_{r,k}: J^{r+k}\pi\to J^r\pi_k$ satisfies $\iota_{r,k}(j_p^{r+k}\phi)=j_p^r(j^k\phi)$, where $j^k\phi$ is the k-jet extension of ϕ. To avoid an abundance of parentheses, we shall also use the notation $\pi_1^0=\pi$, $\pi_1^r=(\pi_1^{r-1})_1$ for repeated 1-jets.

As local coordinates on the manifolds M, E and $J^k\pi$ we take the collections of functions (x^i), (x^i, u^α) and (x^i, u_I^α) where lower-case Latin indices run from 1 to m (the dimension of M), Greek indices run from 1 to n (the fibre dimension of E) and $I\in N^m$ is a multi-index. The number $I!$ is defined to equal $\Pi_{i=1}^m(I(i)!)$, 1_i is the multi-index with 1 in its ith position and zeroes everywhere else, and $I-1_i$ is the multi-index defined by $(I-1_i)(i)=\max\{I(i)-1, 0\}$, $(I-1_i)(j)=I(j)$ for $j\neq i$. In coordinate formulae we adopt the usual summation and range conventions for ordinary indices, whereas summation over a repeated multi-index I is always indicated explicitly. Coordinates on the repeated jet manifold $J^{k-1}\pi_1$ will be denoted $(x^i, u_{;I}^\alpha, u_{i;I}^\alpha)$. When referring specifically to mechanics, however, we shall often revert to the more traditional notation $(t, q_{(j)}^\alpha)$ for coordinates on $J^k\pi$ and $(t, q_{(j)}^\alpha, \dot q_{(j)}^\alpha)$ for coordinates on $J^{k-1}\pi_1$. The base manifold M will be assumed orientable with a given volume form Ω and we only consider coordinate systems in which Ω may be expressed locally as $\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^m$ (or $\mathrm{d}t$).

Given a point $j_p^k\phi \in J^k\pi$ and a tangent vector $\xi \in T_pM$, the holonomic lift of ξ is denoted $\xi^{k-1} \in T_{j_p^{k-1}\phi}(J^{k-1}\pi)$ and satisfies $\xi^{k-1} = (j^{k-1}\phi)_*\xi$. The horizontalisation operator for differential forms is denoted by $h : \Lambda^1(J^{k-1}\pi) \to \Lambda_0^1(\pi_k)$ and the corresponding derivation of type d_* (Frölicher and Nijenhuis 1956) mapping r-forms on $J^{k-1}\pi$ to $(r+1)$-forms on $J^k\pi$ is called the horizontal differential and denoted $\mathrm{d_h}$. The horizontal differential is important in the study of Lagrangian theories, for if $L : J^k\pi \to R$ is a Lagrangian then a Cartan m-form Θ_L corresponding to L must have the properties that $(j^{2k-1}\phi)^*\Theta_L = (j^k\phi)^*(L\Omega)$ and that the $(m+1)$-form

$$\delta L = \pi_{2k,k}^* \,\mathrm{d}(L\Omega) + \mathrm{d_h}\Theta_L \tag{2.1}$$

is an element of $\Lambda_0^{m+1}(\pi_{2k,0})$. δL is called the Euler–Lagrange form associated with L, because in coordinates it incorporates the Euler–Lagrange equations familiar from the calculus of variations, and hence vanishes along the extremals of L. Construction of a suitable Cartan form is therefore one of the main tasks of the theory.

3. The vertical endomorphism defined by a 1-form

In Crampin *et al* (1985) the first vertical endomorphism on a higher-order tangent bundle is constructed as the composition of two maps, a projection and a vertical lift. When studying field theories there is no natural 'vertical direction' in which to perform a lift, so we adopt the device of fixing a closed 1-form on the base manifold to specify a direction. The resulting operator is a type (1, 1) tensor field which depends in general on the derivatives of the coefficients of the 1-form.

Proposition 3.1. Given a point $a \in J^k\pi$, a tangent vector ξ at $\pi_{k,k-1}(a) \in J^{k-1}\pi$ vertical over M, and a closed 1-form ω defined in a neighbourhood of $\pi_k(a) \in M$, then there is an intrinsically defined tangent vector at a which is vertical over E. We denote this new vector by the symbol $\xi \circledS_a \omega$ and call it the *vertical lift of ξ by ω to a.* In local coordinates, if

$$\xi = \sum_{|I|=0}^{k-1} \xi_I^\alpha \frac{\partial}{\partial u_I^\alpha}\bigg|_{\pi_{k,k-1}(a)} \qquad \omega = \omega_i \,\mathrm{d}x^i$$

then

$$\xi \circledS_a \omega = \sum_{|I+J|=0}^{k-1} \frac{(I+J+1_i)!}{I!(J+1_i)!} \xi_I^\alpha \frac{\partial^{|J|}\omega_i}{\partial x^J}\bigg|_{\pi_k(a)} \frac{\partial}{\partial u_{I+J+1_i}^\alpha}\bigg|_a. \tag{3.1}$$

Proof. First, since the 1-form ω is closed, there is a function f defined in a neighbourhood of $\pi_k(a)$ satisfying $\mathrm{d}f = \omega$; the germ of f is unique to within an additive constant which we specify by requiring $f(\pi_k(a)) = 0$.

Next, we consider the one-parameter families ϕ_t of local sections of π, defined on a neighbourhood of $\pi_k(a)$ and such that $j_{\pi_k(a)}^{k-1}\phi_t$ is a curve in $J^{k-1}\pi$ which defines the vector ξ; this is possible as ξ is vertical over M. Among all such families we restrict attention to those satisfying $j_{\pi_k(a)}^k\phi_0 = a$, where the existence of families satisfying this additional condition may be seen easily in local coordinates.

Assuming such a family ϕ_t to have been chosen, we now use the function f to define a new family χ_t of local sections of π by the rule

$$\chi_t(p) = \phi_{tf(p)}(p) \tag{3.2}$$

for p suitably close to $\pi_k(a)$. Certainly $j^k_{\pi_k(a)}\chi_0 = j^k_{\pi_k(a)}\phi_0 = a$, so the curve $j^k_{\pi_k(a)}\chi_t$ passes through the point a and hence defines a tangent vector in $T_a(J^k\pi)$ which we denote $\xi \textcircled{v}_a \omega$. Since $(\partial/\partial t)|_{t=0}\chi^\alpha_t(\pi_k(a)) = 0$ this vector is vertical over E. A calculation in local coordinates using the chain rule shows that

$$\left.\frac{\partial^{|I|+1}\chi^\alpha}{\partial t\,\partial x^I}\right|_{(0,\pi_k(a))} = \sum_{J+K=I} \frac{I!}{J!K!} \left.\frac{\partial^{|J|}f}{\partial x^J}\right|_{\pi_k(a)} \left.\frac{\partial^{|K|+1}\phi^\alpha}{\partial t\,\partial x^K}\right|_{(0,\pi_k(a))} \qquad |I| \leq k \tag{3.3}$$

making it clear that $\xi \textcircled{v}_a \omega$ depends on the vector ξ rather than the particular choice of family ϕ_t. The coordinate expression for $\xi \textcircled{v}_a \omega$ follows immediately.

Corollary 3.2. For each closed 1-form ω on M there is a type $(1, 1)$ tensor field $S^{(k)}_\omega$ defined intrinsically on $J^k\pi$. In local coordinates

$$S^{(k)}_\omega = \sum_{|I+J|=0}^{k-1} \frac{(I+J+1_i)!}{I!(J+1_i)!} \pi^*_k\left(\frac{\partial^{|J|}\omega_i}{\partial x^J}\right) \frac{\partial}{\partial u^\alpha_{I+J+1_i}} \otimes (\mathrm{d}u^\alpha_I - u^\alpha_{I+1_j}\,\mathrm{d}x^j). \tag{3.4}$$

Proof. We define $S^{(k)}_\omega$ by its action on the vector fields on $J^k\pi$. If X is such a vector field then, for each $a \in J^k\pi$, $\pi_{k,k-1*}(X_a)$ is a tangent vector to $J^{k-1}\pi$ at $\pi_{k,k-1}(a)$. This vector need not be vertical over M. However, it has a unique vertical representative defined by subtracting the holonomic lift of its image in M to give $\pi_{k,k-1*}(X_a) - (\pi_{k*}(X_a))^{k-1}$. We then put $S^{(k)}_\omega(X)_a = (\pi_{k,k-1*}(X_a) - (\pi_{k*}(X_a))^{k-1}) \textcircled{v}_a \omega$.

It is conceptually desirable to distinguish the tensor field $S^{(k)}_\omega$ from its action as an operator on vector fields and on 1-forms; to do so we shall adopt the notation $\hat{S}^{(k)}_\omega$, $\check{S}^{(k)}_\omega$ for these operators. It is easy to see that both operators behave well with respect to the jet projections, in the sense that $\pi_{k,l*}(\hat{S}^{(k)}_\omega(X)) = \hat{S}^{(l)}_\omega(\pi_{k,l*}(X))$ for any vector field X on $J^k\pi$ which is $\pi_{k,l}$ related to a vector field on $J^l\pi (l<k)$, and that $\check{S}^{(k)}_\omega(\pi^*_{k,l}(\sigma)) = \pi^*_{k,l}(\check{S}^{(l)}_\omega(\sigma))$ for any 1-form σ on $J^l\pi$. In view of this we omit the superscript and refer to $\hat{S}_\omega$, $\check{S}_\omega$ where no confusion is possible. We also note the fundamental property of the operator $\check{S}_\omega$ that its image is always a contact form.

At this stage it is of interest to compare the tensor field S_ω with the corresponding object in mechanics which motivated the present work. The first point to make is that, in first-order theories, S_ω actually depends in a tensorial manner on ω. To see this, note that the vertical lift of a vector ξ on E to a vector $\xi \textcircled{v}_a \omega$ on $J^1\pi$ depends only upon the cotangent vector $\omega_{\pi_1(a)}$ and not on its extension as a 1-form: in coordinates, if

$$\xi = \xi^\alpha \left.\frac{\partial}{\partial u^\alpha}\right|_{\pi_{1,0}(\alpha)} \qquad \omega = \omega_i\,\mathrm{d}x^i$$

then

$$\xi \textcircled{v} \omega = \xi^\alpha \omega_i(\pi_1(a)) \left.\frac{\partial}{\partial u^\alpha_i}\right|_a. \tag{3.5}$$

The vertical endomorphism $S^{(1)}_\omega$ may therefore be defined for an arbitrary 1-form ω on M which need not satisfy any differential condition. In fact we could choose to consider the 'type (2, 1) tensor field' $S^{(1)}$ defined on $J^1\pi$ by its action on a pair of 1-forms:

$$\check{S}^{(1)}(\omega, \sigma) = \check{S}^{(1)}_\omega(\sigma) \tag{3.6}$$

where $\omega \in \Lambda^1(M)$, $\sigma \in \Lambda^1(J^1\pi)$ and the quotation marks indicate that $S^{(1)}$ is really a cross section of the bundle $\pi_1^*(TM) \otimes T(J^1\pi) \otimes T^*(J^1\pi) \to J^1\pi$. In coordinates

$$S^{(1)} = \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u_i^\alpha} \otimes (\mathrm{d}u^\alpha - u_j^\alpha \,\mathrm{d}x^j). \tag{3.7}$$

One recovers the case of classical mechanics from this construction when $M = R$ by choosing for ω the volume form $\mathrm{d}t$, so that

$$S_{\mathrm{d}t}^{(1)} = \frac{\partial}{\partial \dot{q}^\alpha} \otimes (\mathrm{d}q^\alpha - \dot{q}^\alpha \,\mathrm{d}t). \tag{3.8}$$

For higher-order theories the comparison with mechanics, although superficially quite similar, is actually rather more subtle. We can no longer construct a type (2, 1) tensor field since the action on ω is that of a linear differential operator. However, when $M = R$ we can still use the volume form $\mathrm{d}t$ as a (closed) 1-form to define a canonical type (1, 1) tensor field $S_{\mathrm{d}t}$, which is the first vertical endomorphism described in Crampin *et al* (1985) and elsewhere. In coordinates, we have

$$S_{\mathrm{d}t} = \sum_{r=0}^{k-1} (r+1) \frac{\partial}{\partial q_{(r+1)}^\alpha} \otimes (\mathrm{d}q_{(r)}^\alpha - q_{(r+1)}^\alpha \,\mathrm{d}t) \tag{3.9}$$

where of course the coefficient of the volume form in any allowable coordinate system is constant so that terms containing derivatives no longer appear. This aspect of the construction does not occur in field theories.

As a final remark in this section, we point out that the operators $\check{S}_\omega$ are related to certain operators defined by Tulczyjew (1980) in the context of forms on $J^\infty\pi$ and used to construct the Euler–Lagrange form. These latter operators were called θ_I (where I is a multi-index) and were defined locally in a particular coordinate system. In fact $\theta_{1_i}(\pi_{\infty,k}^*\sigma) = \check{S}_{\mathrm{d}x^i}(\sigma)$, so the present work provides a global construction for these operators.

4. Constructing a Cartan form in first-order theories

As mentioned earlier, the Cartan form associated in classical mechanics to a Lagrangian L may be written as $\check{S}(\mathrm{d}L) + L\,\mathrm{d}t$, where $\check{S}$ is the vertical endomorphism regarded as acting on 1-forms. Our intention is to generalise this construction to higher-order field theories. The two obstacles to be overcome in carrying out this generalisation, namely the need to create a mapping from 1-forms to m-forms and the need to incorporate higher derivatives, are considered in turn. We therefore start with the first-order case.

Theorem 4.1. There is a canonically defined vector-valued m-form S_Ω on the first jet manifold $J^1\pi$ which satisfies $(j^1\phi)^*(S_\Omega(\sigma)) = 0$ for each $\sigma \in \Lambda^1(J^1\pi)$ and every local section ϕ of π (where Ω denotes the volume form on M and also its pull-back to $J^1\pi$ by π_1). If $L: J^1\pi \to R$ is a Lagrangian function then the Cartan form associated to L is given by

$$\Theta_L = S_\Omega(\mathrm{d}L) + L\Omega \in \Lambda_0^m(\pi_{1,0}) \cap \Lambda_1^m(\pi_1).$$

Proof. For a fixed 1-form σ on $J^1\pi$, define the operator $\check{S}\sigma$ mapping 1-forms on M to 1-forms on $J^1\pi$ by the rule $\check{S}\sigma(\omega) = \check{S}_\omega(\sigma)$. $\check{S}\sigma$ represents a 'type (1, 1) tensor field along π_1'; it may be obtained from the type (2, 1) tensor field $S^{(1)}$ described in § 3 by contraction of σ with the second contravariant index.

We now obtain S_Ω by letting $i_{\check{S}\sigma}$ be the derivation of type i_* corresponding to $\check{S}\sigma$, and writing $S_\Omega\sigma = i_{\check{S}\sigma}\Omega$. Specifically, the dual operator to $\check{S}\sigma$ is defined to be the map $\hat{S}\sigma$ taking vector fields on $J^1\pi$ to vector fields along π_1 by the rule that, for each vector field X on $J^1\pi$ and each point $a \in J^1\pi$, the tangent vector $\hat{S}\sigma(X)_a \in T_{\pi_1(a)}M$ is defined by

$$\omega_{\pi_1(a)}(\hat{S}\sigma(X)_a) = (\check{S}\sigma(\omega))_a(X_a) \tag{4.1}$$

for each $\omega \in \Lambda^1(M)$. We then define the vector-valued m-form S_Ω by the rule that, for each 1-form σ on $J^1\pi$, each m-tuple $(X_1, \ldots, X_m)$ of vector fields on $J^1\pi$ and each point $a \in J^1\pi$, we have

$$(S_\Omega(\sigma)(X_1, \ldots, X_m))_a = \sum_{j=1}^{m} \Omega_{\pi_1(a)}(\pi_{1*}X_{1a}, \ldots, (\hat{S}\sigma(X_j))_a, \ldots, \pi_{1*}X_{ma}). \tag{4.2}$$

In local coordinates, if $\sigma = \sigma_i\, dx^i + \sigma_\alpha\, du^\alpha + \sigma^i_\alpha\, du^\alpha_i$ and

$$X = X^i\,\partial/\partial x^i + X^\alpha\,\partial/\partial u^\alpha + X^\alpha_i\,\partial/\partial u^\alpha_i$$

then

$$\hat{S}\sigma(X) = \sigma^i_\alpha(X^\alpha - u^\alpha_j X^j)\,\partial/\partial x^i \tag{4.3}$$

and

$$S_\Omega(\sigma) = \sigma^i_\alpha(du^\alpha - u^\alpha_j\, dx^j) \wedge (\partial/\partial x^i \lrcorner \Omega). \tag{4.4}$$

It is clear that, for every local section ϕ of π_1, we have $(j^1\phi)^*(S_\Omega(\sigma)) = 0$. Writing $\Theta_L = S_\Omega(dL) + L\Omega$, it is then immediate that

$$\Theta_L = \frac{\partial L}{\partial u^\alpha_i}(du^\alpha - u^\alpha_j\, dx^j) \wedge (\partial/\partial x^i \lrcorner \Omega) + L\Omega \tag{4.5}$$

which is the standard representation of the Cartan form in first-order theories. It follows that $\Theta_L \in \Lambda^m_0(\pi_{1,0}) \cap \Lambda^m_1(\pi_1)$ and that $d(L\Omega) + d_h\Theta_L = (\partial L/\partial u^\alpha - (d/dx^i)(\partial L/\partial u^\alpha_i))\, du^\alpha \wedge \Omega$ is an element of $\Lambda^{m+1}_0(\pi_{2,0})$.

The proof of this result demonstrates some of the collapsing which occurs in classical mechanics. In first-order theories the vertical endomorphism $\check{S}_\omega$ (defined by a 1-form ω) and the vector-valued m-form S_Ω represent quite different aspects of the same geometric object. When, however, the base manifold is one dimensional with volume form dt then we find that these aspects coalesce and that the operators $\check{S}_{dt}$, S_{dt} are identical.

5. Constructing a Cartan form in higher-order theories

The construction described in § 4 for first-order theories cannot be used directly in the higher-order case as the map $\check{S}_\omega$ used in the proof of theorem 4.1 becomes a differential operator. To make the construction work one would need a tensor which was 'equivalent' to the differential operator in the sense of integration by parts. Rather than proceeding directly along this route, we adapt the technique of Kuperschmidt (1980). The method uses induction on the order k by taking advantage of the injection $\iota_{k-1,k}: J^k\pi \to J^{k-1}\pi_1$. Since this procedure involves successive differentiations of σ, the resulting operator must be regarded as mapping to m-forms on $J^k\pi$ (indeed, on $J^{k-1}\pi$) with coefficients in $J^{2k-1}\pi$, i.e. the image of S_Ω will be contained in $\Lambda_0^m(\pi_{2k-1,k-1})$. Repeated jets have been used elsewhere in the construction of a Cartan form (Aldaya and de Azcárraga 1980) but the method used there is somewhat different.

First we need two technical lemmas.

Lemma 5.1. There is a canonical map (also denoted S_Ω) from $\Lambda_1^{m+1}(\pi_1)$ to $\Lambda_0^m(\pi_{1,0})\cap\Lambda_1^m(\pi_1)$ which satisfies $S_\Omega(\sigma\wedge\Omega)=S_\Omega(\sigma)$ for $\sigma\in\Lambda^1(J^1\pi)$, and $(j^1\phi)^*(S_\Omega(\theta))=0$ for $\theta\in\Lambda_1^{m+1}(\pi_1)$ where ϕ is any local section of π.

Proof. Suppose $\theta\in\Lambda_1^{m+1}(\pi_1)$. Define $\sigma\in\Lambda^1(J^1\pi)$ to be a representative of θ if $\theta=\sigma\wedge\Omega$. Such a representative always exists for in a coordinate neighbourhood

$$(-1)^{m(m+1)/2}\frac{\partial}{\partial x^1}\lrcorner\ldots\lrcorner\frac{\partial}{\partial x^m}\lrcorner\theta \tag{5.1}$$

is certainly a representative of θ and a partition of unity can be used to construct a global representative σ. If σ_1,σ_2 are both representatives of θ then $(\sigma_1-\sigma_2)\wedge\Omega=0$ so that $\sigma_1-\sigma_2\in\Lambda_0^1(\pi_1)$ and hence $S_\Omega(\sigma_1)=S_\Omega(\sigma_2)$. We therefore define $S_\Omega(\theta)$ to equal $S_\Omega(\sigma)$ where $S_\Omega(\sigma)$ where σ is any representative of θ.

Lemma 5.2. The map $S_\Omega:\Lambda_1^{m+1}(\pi_1)\to\Lambda_0^m(\pi_{1,0})\cap\Lambda_1^m(\pi_1)$ can be lifted to a map (also called S_Ω) from $\Lambda_0^{m+1}(\pi_{s,1})\cap\Lambda_1^{m+1}(\pi_s)$ to $\Lambda_0^m(\pi_{s,0})\cap\Lambda_1^m(\pi_s)$, satisfying $(j^s\phi)^*(S_\Omega(\theta))=0$ where ϕ is any local section of π.

Proof. $\Lambda_0^{m+1}(\pi_{s,1})\cap\Lambda_1^{m+1}(\pi_s)$ is the module generated by $\pi_{s,1}^*(\Lambda_1^{m+1}(\pi_1))$ over $C^\infty(J^s\pi)$, so define $S_\Omega(\pi_{s,1}^*(\theta))$ to equal $\pi_{s,1}^*(S_\Omega(\theta))$ and extend by linearity.

In the following theorem we construct an operator $S_\Omega^{(k)}:\Lambda^1(J^k\pi)\to\Lambda_0^m(\pi_{2k-1,k-1})\cap\Lambda_1^m(\pi_{2k-1})$ where again we indicate the manifold on which S_Ω is defined by a superscript.

Theorem 5.3. Suppose we have a family of tubular neighbourhoods of $J^{k-r}\pi_1^r$ in $J^{k-r-1}\pi_1^{r+1}$ for $0\leq r\leq k-2$. Then corresponding to this family there is a local R-linear operator

$$S_\Omega^{(k)}:\Lambda^1(J^k\pi)\to\Lambda_0^m(\pi_{2k-1,k-1})\cap\Lambda_1^m(\pi_{2k-1})$$

satisfying the conditions

$$\pi_{2k,k}^*(\sigma\wedge\Omega)+\mathrm{d}_h(S_\Omega^{(k)}(\sigma))\in\Lambda_0^{m+1}(\pi_{2k,0})\cap\Lambda_1^{m+1}(\pi_{2k}) \tag{5.2}$$

and, for every local section ϕ of π, $(j^{2k-1}\phi)^*(S_\Omega^{(k)}(\sigma))=0$.

Proof. When $k=1$ this is just theorem 4.1, so suppose $k>1$. The induction hypothesis is that, for every locally trivial fibred manifold $\nu:F\to M$ and family of tubular neighbourhoods of $J^{k-r-1}\nu_1^r$ in $J^{k-r-2}\nu_1^{r+1}$ for $0\leq r\leq k-3$, there is a local R-linear operator $S_\Omega^{(k-1)}:\Lambda^1(J^{k-1}\nu)\to\Lambda_0^m(\nu_{2k-3,k-2})\cap\Lambda_1^m(\nu_{2k-3})$ such that, for $\tilde\sigma\in\Lambda^1(J^{k-1}\nu)$,

$$\nu_{2k-2,k-1}^*(\tilde\sigma\wedge\Omega)+\mathrm{d}_h(S_\Omega^{(k-1)}(\tilde\sigma))\in\Lambda_0^{m+1}(\nu_{2k-2,0})\cap\Lambda_1^{m+1}(\nu_{2k-2}) \tag{5.3}$$

and for every local section ψ of ν, $(j^{2k-3}\psi)^*(S_\Omega^{(k-1)}(\tilde\sigma))=0$.

Choose ν to be $\pi_1:J^1\pi\to M$. Given $\sigma\in\Lambda^1(J^k\pi)$, transfer σ to $\iota_{k-1,1}(J^k\pi)$ along $\iota_{k-1,1}$. Extend it to the tubular neighbourhood using the neighbourhood's projection and then extend it as a smooth 1-form $\tilde{\sigma}$ over the whole of $J^{k-1}\pi_1$. By the induction hypothesis we have

$$S_\Omega^{(k-1)}(\tilde{\sigma}) \in \Lambda_0^m((\pi_1)_{2k-3,k-2}) \cap \Lambda_1^m((\pi_1)_{2k-3}) \subset \Lambda^m(J^{2k-3}\pi_1) \tag{5.4}$$

and writing $E^{(k-1)}\tilde{\sigma}$ for $(\pi_1)^*_{2k-2,k-1}(\tilde{\sigma} \wedge \Omega) + \mathrm{d_h}(S_\Omega^{(k+1)}(\tilde{\sigma}))$ we have

$$E^{(k-1)}\tilde{\sigma} \in \Lambda_0^{m+1}((\pi_1)_{2k-2,0}) \cap \Lambda_1^{m+1}((\pi_1)_{2k-2}) \subset \Lambda^{m+1}(J^{2k-2}\pi_1). \tag{5.5}$$

Now $\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma}) \in \Lambda^{m+1}(J^{2k-1}\pi)$. From the relations $(\pi_1)_{2k-2,0} \circ \iota_{2k-2,1} = \pi_{2k-1,1}$ and $\pi_1 \circ (\pi_1)_{2k-2,0} \circ \iota_{2k-2,1} = \pi_{2k-1}$ we find that

$$\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma}) \in \Lambda_0^{m+1}(\pi_{2k-1,1}) \cap \Lambda_1^{m+1}(\pi_{2k-1}). \tag{5.6}$$

By lemma 5.2 we can therefore apply $S_\Omega^{(1)}$ to obtain

$$S_\Omega^{(1)}(\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma})) \in \Lambda_0^m(\pi_{2k-1,0}) \cap \Lambda_1^m(\pi_{2k-1}). \tag{5.7}$$

We also find, using the relation $(\pi_1)_{2k-3,k-2} \circ \iota_{2k-3,1} = \iota_{k-2,1} \circ \pi_{2k-2,k-1}$, that

$$\iota^*_{2k-3,1}(S_\Omega^{(k-1)}(\tilde{\sigma})) \in \Lambda_0^m(\pi_{2k-2,k-1}) \cap \Lambda_1^m(\pi_{2k-2}). \tag{5.8}$$

Consequently we can define $S_\Omega^{(k)}(\sigma)$ by

$$\begin{aligned} S_\Omega^{(k)}(\sigma) = {} & \pi^*_{2k-1,2k-2}\iota^*_{2k-3,1}(S_\Omega^{(k-1)}(\tilde{\sigma})) \\ & + S_\Omega^{(1)}(\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma})) \in \Lambda_0^m(\pi_{2k-1,k-1}) \cap \Lambda_1^m(\pi_{2k-1}). \end{aligned} \tag{5.9}$$

Using $(\pi_1)_{2k-2,k-1} \circ \iota_{2k-2,1} = \iota_{k-1,1} \circ \pi_{2k-1,k}$ and $\iota^*_{k-1,1}\tilde{\sigma} = \sigma$ we find that

$$\pi^*_{2k,k}(\sigma \wedge \Omega) + \mathrm{d_h}(S_\Omega^{(k)}(\sigma)) = (\pi^*_{2k,2k-1} + \mathrm{d_h}S_\Omega^{(1)})(\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma})) \tag{5.10}$$

which is an element of $\Lambda_0^{m+1}(\pi_{2k,0}) \cap \Lambda_1^{m+1}(\pi_{2k})$ by virtue of the properties of $S_\Omega^{(1)}$. We also note that $S_\Omega^{(k)}(\sigma)$ does not depend on the particular extension of σ to the whole of $J^{k-1}\pi_1$ but only on its value in a neighbourhood of $\iota_{k-1,1}(J^k\pi)$ because the operations used in the construction are all local. Consequently $S_\Omega^{(k)}$ is a well defined, local R-linear operator. Finally, if ϕ is a local section of π,

$$(j^{2k-1}\phi)^*(S_\Omega^{(k)}(\sigma)) = (j^{2k-3}(j^1\phi))^*S_\Omega^{(k-1)}(\tilde{\sigma}) + (j^{2k-1}\phi)^*(S_\Omega^{(1)}(\iota^*_{2k-2,1}(E^{(k-1)}\tilde{\sigma}))) \tag{5.11}$$

where the first term vanishes by the induction hypothesis because $j^1\phi$ is a local section of π_1, and the second term vanishes by virtue of lemma 4.2.

In general we should not expect the operator $S^{(k)}\Omega$ to be unique as it depends on the choices of tubular neighbourhood; this question is considered further in § 6. However, locally we can be specific because each coordinate chart on E defines a local system of tubular neighbourhoods. If (x^i, u^α) is the coordinate chart on $U \subset E$ and if, for each s with $1 \leqslant s \leqslant k$, (x^i, u^α_I) and $(x^i, u^\alpha_{;J}, u^\alpha_{i;J})$ are the corresponding charts on $U^s \subset J^s\pi$ and $U_1^{s-1} \subset J^{s-1}\pi_1$ respectively, where $|I| \leqslant s$, $|J| \leqslant s-1$, then we can define a projection $\tau_s : U_1^{s-1} \to U^s$ by the rule

$$\begin{aligned} & x^i(\tau_s(a)) = x^i(a) \\ & u^\alpha_I(\tau_s(a)) = \tfrac{1}{2}u^\alpha_{;I}(a) + \frac{1}{2}\left(\sum_{i=1}^m \frac{I(i)}{|I|} u^\alpha_{i;I-1_i}(a)\right) \qquad |I| \leqslant s-1 \\ & u^\alpha_I(\tau_s(a)) = \sum_{i=1}^m \frac{I(i)}{|I|} u^\alpha_{i;I-1_i}(a) \qquad |I| = s. \end{aligned} \tag{5.12}$$

We may extend each τ_s to define a tubular neighbourhood of the whole of $J^s\pi$ in $J^{s-1}\pi_1$. If we then construct the corresponding operator $S_\Omega^{(k)}$ and consider a Lagrangian $L: J^k\pi \to R$, we find that in the particular coordinate neighbourhood U^{2k-1} we can write $S_\Omega^{(k)}(\mathrm{d}L) + \pi_{2k-1,k}^*(L\Omega)$ as

$$\sum_{|I|=0}^{k-1} \sum_{|J|=0}^{k-|I|-1} (-1)^{|J|} \frac{(I+J+1_i)!|I|!|J|!}{|I+J+1_i|!I!J!} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J}\left(\frac{\partial L}{\partial u^\alpha_{I+J+1_i}}\right)(\mathrm{d}u^\alpha_I - u^\alpha_{I+1_j}\,\mathrm{d}x^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right) + L\Omega \tag{5.13}$$

which is the local coordinate expression for a Cartan form given by Shadwick (1982). The difference in constant factors arises as a result of a different convention for summing over a multi-index. The corresponding Euler–Lagrange form δL defined by

$$\delta L = \pi_{2k,k}^*\,\mathrm{d}(L\Omega) + \mathrm{d_h}(S_\Omega^{(k)}(\mathrm{d}L) + \pi_{2k-1,k}^*(L\Omega)) \tag{5.14}$$

then has the familiar form

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J}\left(\frac{\partial L}{\partial u^\alpha_J}\right) \mathrm{d}u^\alpha \wedge \Omega. \tag{5.15}$$

6. Uniqueness of the Cartan form

The uniqueness of any global Cartan form in higher-order field theories has been a matter of some discussion recently. For example, Shadwick (1982) provides a construction which attempts to specify a unique Cartan form, although a review of that paper (Chrastina 1984) has pointed out the technique used there is only valid globally for first- and second-order field theories. We therefore consider this question in the context of the present construction.

The first observation to make is that, given a Lagrangian L, the Euler–Lagrange form δL does not depend on any particular choice of Cartan form Θ_L in cases where this latter form is not unique. This implies, of course, that δL always has the coordinate representation given in (5.15) even if the tubular neighbourhoods are not constructed in the way described at the end of § 5. Formally, we have the following result.

Lemma 6.1. Let $L: J^k\pi \to R$ be a Lagrangian and let $\Theta_1, \Theta_2 \in \Lambda_0^m(\pi_{2k-1,k-1}) \cap \Lambda_1^m(\pi_{2k-1})$ have the property that both the $(m+1)$-forms $\delta L_1 = \mathrm{d}(L\Omega) + \mathrm{d_h}\Theta_1$ and $\delta L_2 = \mathrm{d}(L\Omega) + \mathrm{d_h}\Theta_2$ are elements of $\Lambda_0^{m+1}(\pi_{2k,0}) \cap \Lambda_1^{m+1}(\pi_{2k})$. Then $\delta L_1 = \delta L_2$.

Proof. We use local coordinates to show that $\mathrm{d_h}(\Theta_1 - \Theta_2) = 0$. First, from $\Theta_1 - \Theta_2 \in \Lambda_0^m(\pi_{2k-1,k-1}) \cap \Lambda_1^m(\pi_{2k-1})$ we may write

$$\Theta_1 - \Theta_2 = {}^i\sigma \wedge (\partial/\partial x^i \lrcorner \Omega) \tag{6.1}$$

where the 1-forms ${}^i\sigma$ are elements of $\Lambda_0^1(\pi_{2k-1,k-1})$. If the coordinate representation of each ${}^i\sigma$ is ${}^i\sigma_j\,\mathrm{d}x^j + \Sigma_{|I|=0}^{k-1}\,{}^i\sigma^I_\alpha\,\mathrm{d}u^\alpha_I$, then

$$\mathrm{d_h}{}^i\sigma = \frac{\mathrm{d}^i\sigma_j}{\mathrm{d}x^m}\,\mathrm{d}x^m \wedge \mathrm{d}x^j + \sum_{|I|=0}^{k-1}\left(\frac{\mathrm{d}^i\sigma^I_\alpha}{\mathrm{d}x^m}\,\mathrm{d}x^m \wedge \mathrm{d}u^\alpha_I + {}^i\sigma^I_\alpha\,\mathrm{d}x^m \wedge \mathrm{d}u^\alpha_{I+1_m}\right) \tag{6.2}$$

and so

$$\mathrm{d_h}(\Theta_1 - \Theta_2) = -\sum_{|I|=0}^{k-1}\left(\frac{\mathrm{d}^i\sigma^I_\alpha}{\mathrm{d}x^i}\,\mathrm{d}u^\alpha_I + {}^i\sigma^I_\alpha\,\mathrm{d}u^\alpha_{I+1_i}\right) \wedge \Omega. \tag{6.3}$$

Since $d_h(\Theta_1 - \Theta_2) \in \Lambda_0^{m+1}(\pi_{2k,0})$ the only non-zero terms in this expression are those in $du^\alpha \wedge \Omega$ with coefficients $-(d/dx^i)({}^i\sigma_\alpha)$. From the vanishing of the other terms, we find recursively that these coefficients equal $\Sigma_{|I|=k-1}(-1)^k(d/dx^{I+1_i})({}^i\sigma_\alpha^I)$. But for each fixed multi-index J with $|J| = k$, we have $\Sigma_{I+1_i=J}\, {}^i\sigma_\alpha^I$ equal to the coefficient of $du_J^\alpha \wedge \Omega$, which is zero. The coefficient of $du^\alpha \wedge \Omega$ is then a sum of derivatives of the coefficients of the $du_J^\alpha \wedge \Omega$ ($|J| = k$) and so itself is zero.

We can now consider the uniqueness question for the various operators $S_\Omega^{(k)}$ and the corresponding Cartan forms. We start with the first-order case.

Lemma 6.2. The operator S_Ω defined in theorem 4.1 for first-order theories is the unique operator $\Lambda^1(J^1\pi) \to \Lambda_0^m(\pi_{1,0}) \cap \Lambda_1^m(\pi_1)$ satisfying $(j^1\phi)^*(S_\Omega\sigma) = 0$ for every local section ϕ of π and $\pi_{2,1}^*(\sigma \wedge \Omega) + d_h(S_\Omega\sigma) \in \Lambda_0^{m+1}(\pi_{1,0}) \cap \Lambda_1^{m+1}(\pi_1)$.

Proof. Again we use local coordinates. If S is such an operator then for each $\sigma \in \Lambda^1(J^1\pi)$, since $(S_\Omega - S)\sigma \in \Lambda_0^m(\pi_{1,0}) \cap \Lambda_1^m(\pi_1)$, we have locally

$$(S_\Omega - S)\sigma = {}^i\sigma_\alpha\, du^\alpha \wedge (\partial/\partial x^i \lrcorner \Omega) + f\Omega \tag{6.4}$$

for some functions ${}^i\sigma_\alpha, f$ on $J^1\pi$. Since $(j^1\phi)^*((S_\Omega - S)\sigma) = 0$ for each local section ϕ of π we find that $f = -{}^i\sigma_\alpha u_i^\alpha$. Now

$$d_h((S_\Omega - S)\sigma) = -(d\,{}^i\sigma_\alpha/dx^i)\, du^\alpha \wedge \Omega - {}^i\sigma_\alpha\, du_i^\alpha \wedge \Omega \tag{6.5}$$

and from $d_h((S_\Omega - S)\sigma) \in \Lambda_0^{m+1}(\pi_{1,0})$ it follows that each ${}^i\sigma_\alpha = 0$.

Corollary 6.3. The Cartan form in first-order theories is unique.

We can also show that the Cartan form in higher-order mechanics is unique.

Lemma 6.4. Suppose the base manifold M is one dimensional. Then the operator $S_{dt}: \Lambda^1(J^k\pi) \to \Lambda_0^1(\pi_{2k-1,k-1})$ defined in theorem 5.3 is unique.

Proof. Suppose S_1, S_2 are two operators satisfying the conditions given in the theorem. Let $\sigma \in \Lambda^1(J^k\pi)$; then $(j^{2k-1}\phi)^*(S_1(\sigma)) = (j^{2k-1}\phi)^*(S_2(\sigma)) = 0$, so that $S_1(\sigma) - S_2(\sigma)$ is a contact form. By lemma 6.1 we must have $d_h(S_1(\sigma) - S_2(\sigma)) = 0$, so locally $S_1(\sigma) - S_2(\sigma) = d_h f$ for some function f on $J^{2k-2}\pi$. But from the definition of d_h and the horizontalisation operator h, $\pi_{2k,2k-1}^*\, d_h f = \pi_{2k,2k-1}^* h\, df = h^2\, df = h\, d_h f = h(S_1(\sigma) - S_2(\sigma)) = 0$ since the horizontal component of any contact form is zero.

Corollary 6.5. The Cartan form $S_{dt}(dL) + L\, dt$ is unique and has coordinate representation

$$\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^j \frac{d^j}{dt^j}\left(\frac{\partial L}{\partial q_{(i+j+1)}^\alpha}\right)(dq_{(i)}^\alpha - q_{(i+1)}^\alpha\, dt) + L\, dt. \tag{6.6}$$

On the other hand, the construction for higher-order field theories is never unique.

Example 6.6. Suppose $k = 2$ and $m = 2$. Choose a particular coordinate patch U, let S_1 be an operator $S_\Omega^{(2)}$ defined using the projection $\tau_1: U_1^1 \to U^2$ defined by equations (5.12) and let S_2 be an operator defined using the alternative projection τ_2 where

$u^\alpha_{11}(\tau_2(a)) = u^\alpha_{1;1}(a) + (u^\alpha_{1;2}(a) - u^\alpha_{2;1}(a))$, but the other components of τ_1 and τ_2 are equal. Then in this coordinate patch

$$S_1(\mathrm{d}u^\alpha_{11}) = (\mathrm{d}u^\alpha_1 - u^\alpha_{1i}\,\mathrm{d}x^i) \wedge (\partial/\partial x^1 \lrcorner \Omega)$$

but

$$S_2(\mathrm{d}u^\alpha_{11}) = (\mathrm{d}u^\alpha_1 - u^\alpha_{1i}\,\mathrm{d}x^i) \wedge (\partial/\partial x^1 \lrcorner \Omega + \partial/\partial x^2 \lrcorner \Omega) - (\mathrm{d}u^\alpha_2 - u^\alpha_{2i}\,\mathrm{d}x^i) \wedge (\partial/\partial x^1 \lrcorner \Omega) \quad (6.7)$$

so that S_1, S_2 both satisfy the conditions of theorem 5.3, but $S_1 \neq S_2$. A similar example can obviously be constructed in cases where $k \geqslant 2$ and $m \geqslant 2$.

Corollary 6.7. The global Cartan form in higher-order Lagrangian field theories is not unique.

Acknowledgment

I should like to thank Mike Crampin for his continual advice and encouragement.

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