Journal of Physics A: Mathematical and General

On the stationary axially symmetric Einstein-Weyl field equations

To cite this article: C A Kolassis 1983 *J. Phys. A: Math. Gen.* **16** 749

J. Phys. A: Math. Gen. **16** (1983) 749–766. Printed in Great Britain

On the stationary axially symmetric Einstein–Weyl field equations

Charalampos A Kolassis

Institut Henri Poincaré, Laboratoire de Physique Théorique, ERA 533, 11 rue Pierre et Marie Curie, 75231 Paris, Cedex 05, France

Received 26 April 1982

Abstract. The combined gravitational-neutrino field equations in general relativity are solved under the following two assumptions: (i) space–time is stationary axially symmetric and the line element of the metric can be put into a canonical form, (ii) the energy flow vector of the neutrino field is time-like or null for all observers. The resulting metric is uniquely determined and asymptotically non-flat; the Weyl tensor is of Petrov type D, and the Ricci tensor belongs to the class $[2T-2S]_2$ in the Plebanski classification.

1. Introduction

The classical massless neutrino field is represented by a two-spinor ξ^A which in a curved space–time satisfies Weyl's equation

$$\sigma^{\mu}_{A\dot{X}}\xi^{A}_{;\mu}=0 \tag{1.1}$$

where $\sigma^{\mu}_{A\dot{X}}$ are the generalised Pauli matrices and the semicolon denotes covariant differentiation. The symmetrised neutrino energy–momentum tensor is given by

$$T_{\mu\nu}=\tfrac{1}{4}\mathrm{i}(\bar{\xi}_{\dot{X}}\sigma_{\mu}^{A\dot{X}}\xi_{A;\nu}+\bar{\xi}_{\dot{X}}\sigma_{\nu}^{A\dot{X}}\xi_{A;\mu}-\mathrm{cc}) \tag{1.2}$$

and by virtue of the Weyl equation is trace-free. The uniqueness of the expression (1.2) for $T_{\mu\nu}$ is investigated by Anderson (1974). If we suppose that the neutrino field interacts with a classical gravitational field then besides equation (1.1) we must also consider the Einstein field equations, which in this case can be written in the form

$$R_{\mu\nu}=-T_{\mu\nu}. \tag{1.3}$$

Here, we attempt to resolve the coupled equations (1.1) and (1.3) in the case of a stationary axially symmetric space–time. A consequence of the expression (1.2) for the neutrino energy–momentum tensor is that the sign of the neutrino energy density is observer dependent. Because of this, Wainwright (1971) introduced the two energy conditions E_1 and E_2 for the neutrino field. We shall consider only the neutrino fields satisfying the energy condition E_2. This condition describes fields with causal behaviour and is defined as follows: A field is said to satisfy the energy condition E_2 or equivalently to be of class E_2 if its energy flow vector $T_{\mu\nu}u^{\nu}$ is time-like or null for all unit, future pointing, time-like vectors u^{ν} at each event for which $T_{\mu\nu}\neq 0$. For the treatment of equations (1.1) and (1.3) in conjunction with the energy condition E_2 it appears that the most powerful method is the spin coefficient formalism (Newman and Penrose 1962). Griffiths and Newing (1970, 1971) pioneered the use of the null tetrad and spin coefficient formalism for neutrino fields. According to these authors, the adaptation of the spin coefficient formalism to the interaction of the neutrino with the gravitational field is realised by the introduction of a two-spinor χ^A so that together with ξ^A it forms a spinor frame. This spinor frame gives rise to a null tetrad as follows:

$$l^\mu = \sigma^\mu_{A\dot{X}} \xi^A \bar{\xi}^{\dot{X}} \tag{1.4a}$$

$$\kappa^\mu = \sigma^\mu_{A\dot{X}} \chi^A \bar{\chi}^{\dot{X}} \tag{1.4b}$$

$$m^\mu = \sigma^\mu_{A\dot{X}} \xi^A \bar{\chi}^{\dot{X}} \tag{1.4c}$$

$$\bar{m}^\mu = \sigma^\mu_{A\dot{X}} \chi^A \bar{\xi}^{\dot{X}}. \tag{1.4d}$$

The vector l^μ is the neutrino flux vector. The completeness relation for the null tetrad reads

$$l_\mu \kappa_\nu + l_\nu \kappa_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu = g_{\mu\nu}. \tag{1.5}$$

The transformations of the null tetrad which preserve the metric, the neutrino flux vector and the neutrino energy–momentum tensor constitute a two-parameter subgroup of the proper Lorentz group and are called 'null rotations about l^μ'. They are given by

$$l^\mu = l'^\mu \tag{1.6a}$$

$$\kappa^\mu = \kappa'^\mu + \Psi m'^\mu + \bar{\Psi} \bar{m}'^\mu + \Psi \bar{\Psi} l'^\mu \tag{1.6b}$$

$$m^\mu = m'^\mu + \bar{\Psi} l'^\mu \tag{1.6c}$$

where Ψ is any complex function of the coordinates. With respect to the null tetrad the Weyl equation reduces to the following conditions on the spin coefficients

$$\rho = \varepsilon \tag{1.7a}$$

$$\beta = \tau. \tag{1.7b}$$

The spin coefficients also enter in the expansion of $T_{\mu\nu}$ on the null tetrad; assuming that the neutrino field is of class E_2 and taking into account the Weyl equation we can prove (Wainwright 1971) that by means of equations (1.6a)–(1.6c) the null tetrad can be chosen so that

$$T_{\mu\nu} = -\tfrac{1}{4}[2\mathrm{i}(\bar{\gamma} - \gamma) l_\mu l_\nu + 2\omega g_{\mu\nu} - 4\omega (l_\mu \kappa_\nu + l_\nu \kappa_\mu)] \tag{1.8}$$

where

$$\omega = \tfrac{1}{2}\mathrm{i}(\rho - \bar{\rho}). \tag{1.9}$$

At the same time the spin coefficients $\alpha, \tau, \kappa, \sigma, \gamma$ and ρ must satisfy the following conditions

$$\alpha - 2\bar{\tau} = 0 \tag{1.10}$$

$$\kappa = 0 \tag{1.11}$$

$$\sigma = 0 \tag{1.12}$$

$$\mathrm{i}\omega(\gamma - \bar{\gamma}) \geqslant 0. \tag{1.13}$$

Conversely, if the neutrino energy–momentum tensor is given by equation (1.8) and condition (1.13) is fulfilled, then the neutrino field is of class E_2. Equation (1.10) is due to the particular choice of the null tetrad while equations (1.11) and (1.12) are due to the fact that the neutrino field is of class E_2 and have an interesting geometrical meaning. The first, equation (1.11), means that the neutrino flux vector is tangential to a geodesic null congruence and the second that this geodesic congruence is shear-free. Furthermore part of the Goldberg–Sachs theorem can be easily generalised in the case of the gravitational–neutrino interaction; so from equations (1.11) and (1.12) it follows that the neutrino flux vector forms a repeated principal null direction of the Weyl tensor which is algebraically special, i.e., such that

$$\Psi_0 = 0 \tag{1.14}$$

$$\Psi_1 = 0. \tag{1.15}$$

(The quantities Ψ_i, $i = 0, \ldots, 4$ are the tetrad components of the Weyl tensor. For their definition, see Newman and Penrose (1962, equations (4.3*a*)).) Using equation (1.8) and the expansion of $R_{\mu\nu}$ on the null tetrad we can put the Einstein field equations into the equivalent tetrad form

$$\Phi_{00} = 0 \tag{1.16a}$$

$$\Phi_{01} = 0 \tag{1.16b}$$

$$\Phi_{02} = 0 \tag{1.16c}$$

$$\Phi_{11} = \tfrac{1}{4}\omega \tag{1.16d}$$

$$\Phi_{12} = 0 \tag{1.16e}$$

$$\Phi_{22} = \tfrac{1}{4}\mathrm{i}(\gamma - \bar{\gamma}). \tag{1.16f}$$

(The Hermitian quantities Φ_{ij}, $i, j = 0, 1, 2$ are the tetrad components of the Ricci tensor. For their definition, see Newman and Penrose (1962, equations (4.3*b*)).)

2. Stationary axially symmetric space–time interacting with a neutrino field of class E_2

A stationary axially symmetric space–time is characterised by the existence of a two-parameter abelian group of isometries with Killing vector fields ξ^μ and η^μ. The vector field ξ^μ is time-like with open trajectories surrounding the rotation axis. The rotation axis is a time-like two-dimensional submanifold of the space–time on which the vector field η^μ vanishes. The fact that the isometry group is abelian is expressed by the equation

$$\xi^\mu \eta_{\mu;\nu} - \eta^\mu \xi_{\mu;\nu} = 0. \tag{2.1}$$

Using equation (2.1) and the Frobenius theorem we can prove that the trajectories of the two Killing vector fields form a family of two-dimensional surfaces $\mathscr{P}$. Furthermore, equation (2.1) implies the existence of ignorable coordinates x^0 and x^3 such that $\xi^\mu = \delta^\mu_0$ and $\eta^\mu = \delta^\mu_3$. If $\mathscr{P}$ admits a family $\mathscr{P}^*$ of two-dimensional orthogonal two-surfaces then in the system of the coordinates (x^0, x^1, x^2, x^3) the line element of the metric can be written in the canonical form

$$\mathrm{d}s^2 = g_{mn}(x^l)\,\mathrm{d}x^m\,\mathrm{d}x^n + g_{qr}(x^l)\,\mathrm{d}x^q\,\mathrm{d}x^r \tag{2.2}$$

where $l, m, n = 1, 2$ and $q, r = 0, 3$. The existence of the line element (2.2) implies important simplifications for the stationary axially symmetric gravitational field equations. For the case of an empty space–time Papapetrou (1966) has proved that a line element of the form of equation (2.2) always exists. Papapetrou's result was generalised by Carter (1969, 1972) for a non-empty space–time as follows.

Generalised Papapetrou theorem. The line element of a stationary axially symmetric space–time can be put into the canonical form of equation (2.2) in a simply connected subdomain $\mathscr{D}$ of the space–time which intersects the rotation axes, if and only if the following circularity condition is satisfied everywhere in $\mathscr{D}$

$$\xi^{\mu}R_{\mu[\nu}\xi_{\lambda}\eta_{\rho]}=0 \tag{2.3a}$$

$$\eta^{\mu}R_{\mu[\nu}\xi_{\lambda}\eta_{\rho]}=0 \tag{2.3b}$$

where the brackets on the indices denote antisymmetrisation.

In this article we assume that the space–time interacting with the neutrino field contains at least a simply connected subdomain $\mathscr{D}$ into which the line element of the metric can be put in the canonical form of equation (2.2). Therefore, by virtue of the generalised Papapetrou theorem and the Einstein field equations, the neutrino energy–momentum tensor will also satisfy the circularity condition (equations (2.3a, b)). For convenience, this condition is written in the form

$$-4T_{\mu\nu}\xi^{\mu}=A\xi_{\nu}+B\eta_{\nu} \tag{2.4a}$$

$$-4T_{\mu\nu}\eta^{\mu}=\Gamma\xi_{\nu}+\Delta\eta_{\nu} \tag{2.4b}$$

where A, B, Γ, Δ are real functions of the coordinates. Equations (2.4a, b), together with the assumption that the neutrino field satisfies the weak energy condition E_2 constitute all the restrictions we impose on the neutrino field. For the treatment of the equations (2.4a, b) we must expand ξ^{μ} and η^{μ} in terms of the null tetrad

$$\xi^{\mu}=al^{\mu}+b\kappa^{\mu}-cm^{\mu}-\bar{c}\bar{m}^{\mu} \tag{2.5}$$

$$\eta^{\mu}=dl^{\mu}+e\kappa^{\mu}-fm^{\mu}-\bar{f}\bar{m}^{\mu}. \tag{2.6}$$

As the vector field ξ^{μ} is time-like and non-vanishing everywhere in $\mathscr{D}$ we have

$$a\neq 0 \tag{2.7a}$$

$$b\neq 0. \tag{2.7b}$$

Inserting equations (2.5), (2.6) and (1.8) into the circularity condition (2.4a, b) and taking into account equations (2.7a) and (2.7b) we finally conclude that all the possibilities about the form of the neutrino energy–momentum tensor and the expansion of ξ^{μ} and η^{μ} on the null tetrad are given by the following three cases.

Case A

$$T_{\mu\nu}=-\tfrac{1}{2}\mathrm{i}(\bar{\gamma}-\gamma)l_{\mu}l_{\nu}$$

$$\xi^{\mu}=al^{\mu}+b\kappa^{\mu}-cm^{\mu}-\bar{c}\bar{m}^{\mu}$$

$$\eta^{\mu}=dl^{\mu}+e\kappa^{\mu}-fm^{\mu}-\bar{f}\bar{m}^{\mu}$$

$$e=hb \qquad f=hc$$

where h is a real function of the coordinates.

Case B

$$T_{\mu\nu} = -\tfrac{1}{4}\{2\mathrm{i}(\bar{\gamma}-\gamma)l_\mu l_\nu + 2\omega[g_{\mu\nu} - 2(l_\mu \kappa_\nu + l_\nu \kappa_\mu)]\}$$
$$\xi^\mu = al^\mu + b\kappa^\mu$$
$$\eta^\mu = dl^\mu + e\kappa^\mu.$$

Case C

$$T_{\mu\nu} = -\tfrac{1}{2}\omega[g_{\mu\nu} - 2(l_\mu \kappa_\nu + l_\nu \kappa_\mu)]$$
$$\xi^\mu = al^\mu + b\kappa^\mu - cm^\mu - \bar{c}\bar{m}^\mu$$
$$\eta^\mu = dl^\mu + e\kappa^\mu - fm^\mu - \bar{f}\bar{m}^\mu$$
$$d = ha \qquad e = hb \qquad c = gf$$

where h, g are real functions of the coordinates. In the following sections the above three cases will be treated separately. The case of a ghost neutrino field is not considered.

From a theorem we proved in a previous article (Kolassis 1982), it follows† that the Lie derivatives of the null tetrad vectors with respect to the Killing vector fields ξ^μ and η^μ are given by

$$\mathscr{L}_\xi l^\mu = pl^\mu \tag{2.8a}$$
$$\mathscr{L}_\xi \kappa^\mu = -p\kappa^\mu \tag{2.8b}$$
$$\mathscr{L}_\xi m^\mu = -\mathrm{i}sm^\mu \tag{2.8c}$$
$$\mathscr{L}_\eta l^\mu = ul^\mu \tag{2.9a}$$
$$\mathscr{L}_\eta \kappa^\mu = -u\kappa^\mu \tag{2.9b}$$
$$\mathscr{L}_\eta m^\mu = -\mathrm{i}wm^\mu \tag{2.9c}$$

where p, s, u, w are, in general, real functions of the coordinates. If the neutrino field is not a pure radiation field then p and u vanish while w and s reduce to real constants. With the help of equations (2.5), (2.6) and the Killing equations, equations (2.8a)–(2.9c) can be written equivalently in the form

$$b_{,\mu} = (l_{\nu;\mu} - l_{\mu;\nu})\xi^\nu + pl_\mu \tag{2.10a}$$
$$a_{,\mu} = (\kappa_{\nu;\mu} - \kappa_{\mu;\nu})\xi^\nu - p\kappa_\mu \tag{2.10b}$$
$$\bar{c}_{,\mu} = (m_{\nu;\mu} - m_{\mu;\nu})\xi^\nu - \mathrm{i}sm_\mu \tag{2.10c}$$
$$e_{,\mu} = (l_{\nu;\mu} - l_{\mu;\nu})\eta^\nu + ul_\mu \tag{2.11a}$$
$$d_{,\mu} = (\kappa_{\nu;\mu} - \kappa_{\mu;\nu})\eta^\nu - u\kappa_\mu \tag{2.11b}$$
$$\bar{f}_{,\mu} = (m_{\nu;\mu} - m_{\mu;\nu})\eta^\nu - \mathrm{i}wm_\mu. \tag{2.11c}$$

Using the expansions of ξ^μ, n^μ, $l_{\mu;\nu}$, $\kappa_{\mu;\nu}$ and $m_{\mu;\nu}$ on the null tetrad (see the appendix), equations (2.10a)–(2.11c) can be written in a form convenient for the calculations which follow.

$$\mathrm{D}b = (\rho + \bar{\rho})b \tag{2.12a}$$

† If $\omega = 0$, the quantity r which enters in equations (III.3b) and (III.3c) in this reference can be eliminated by a null rotation about l^μ.

$$\Delta b = -(\rho+\bar\rho)a + 2\tau c + 2\bar\tau\bar c + p \tag{2.12b}$$

$$\delta b = 2\tau b + (\rho-\bar\rho)\bar c \tag{2.12c}$$

$$\mathrm{D}a = (\gamma+\bar\gamma)b + (\bar\pi - 3\tau)c + (\pi - 3\bar\tau)\bar c - p \tag{2.13a}$$

$$\Delta a = -(\gamma+\bar\gamma)a + \bar\nu c + \nu\bar c \tag{2.13b}$$

$$\delta a = (\bar\pi - 3\tau)a + \bar\nu b + (\mu-\bar\mu)\bar c \tag{2.13c}$$

$$\mathrm{D}\bar c = (\bar\pi+\tau)b - \bar\rho\bar c \tag{2.14a}$$

$$\Delta\bar c = -(\bar\pi+\tau)a + \bar\lambda c + (\bar\mu+\gamma-\bar\gamma)\bar c \tag{2.14b}$$

$$\delta\bar c = \bar\lambda b - \tau\bar c \tag{2.14c}$$

$$\bar\delta\bar c = (\bar\mu+\gamma-\bar\gamma)b - \bar\rho a + \tau c + \mathrm{i}s \tag{2.14d}$$

$$\mathrm{D}e = (\rho+\bar\rho)e \tag{2.15a}$$

$$\Delta e = -(\rho+\bar\rho)d + 2\tau f + 2\bar\tau\bar f + u \tag{2.15b}$$

$$\delta e = 2\tau e + (\rho-\bar\rho)\bar f \tag{2.15c}$$

$$\mathrm{D}d = (\gamma+\bar\gamma)e + (\bar\pi-3\tau)f + (\pi-3\bar\tau)\bar f - u \tag{2.16a}$$

$$\Delta d = -(\gamma+\bar\gamma)d + \bar\nu f + \nu\bar f \tag{2.16b}$$

$$\delta d = (\bar\pi-3\tau)d + \bar\nu e + (\mu-\bar\mu)\bar f \tag{2.16c}$$

$$\mathrm{D}\bar f = (\bar\pi+\tau)e - \bar\rho\bar f \tag{2.17a}$$

$$\Delta\bar f = -(\bar\pi+\tau)d + \bar\lambda f + (\bar\mu+\gamma-\bar\gamma)\bar f \tag{2.17b}$$

$$\delta\bar f = \bar\lambda e - \tau\bar f \tag{2.17c}$$

$$\bar\delta\bar f = -\bar\rho d + (\bar\mu+\gamma-\bar\gamma)e + \tau f + \mathrm{i}w. \tag{2.17d}$$

Finally, using equations (2.12a)–(2.17d) and the restrictions satisfied by the spin coefficients, the components of the equation (2.1) with respect to the null tetrad can be written in the form

$$ep - bu + (bd-ae)(\rho+\bar\rho) + (c\bar f - \bar c f)(\rho-\bar\rho) + 2(ec-bf)\tau + 2(e\bar c - b\bar f)\bar\tau = 0 \tag{2.18a}$$

$$\begin{aligned}(bd-ae)(\gamma+\bar\gamma) &+ (cd-af)(\pi+\bar\tau-4\tau) + (ec-bf)\bar\nu + (e\bar c - b\bar f)\nu \\ &+ (\bar c d - a\bar f)(\bar\pi+\tau-4\bar\tau) + (c\bar f - \bar c f)(\mu-\bar\mu) + au - dp = 0\end{aligned} \tag{2.18b}$$

$$\begin{aligned}(bd-ae)(\bar\pi+\tau) &- (\bar c e - b\bar f)(\bar\mu+\gamma-\bar\gamma) - (\bar c d - a\bar f)\bar\rho + (ec-bf)\bar\lambda \\ &- (c\bar f - \bar c f)\tau + \mathrm{i}(w\bar c - s\bar f) = 0.\end{aligned} \tag{2.18c}$$

In the sequel, the various restrictions satisfied by the spin coefficients, i.e. equations (1.7a), (1.7b), (1.10), (1.11), (1.12) etc, the Einstein field equations (1.16a)–(1.16f), the restrictions (1.14), (1.15), (2.7a), (2.7b) and equations (2.12a)–(2.17d) will be used extensively without explicit reference. The restriction (1.13) is not used in what follows. At first sight the energy condition E_2 is needed when $\omega = 0$, because in this case the condition (1.10) cannot be achieved by means of the null rotation (1.6a)–(1.6c), and consequently the terms $\mathrm{i}(\alpha - 2\bar\tau)(l_\mu m_\nu + l_\nu m_\mu) + \mathrm{cc}$ enter into the expansion (1.8) of $T_{\mu\nu}$. However, a straightforward calculation shows that in this case (i.e. $\kappa = \sigma = \omega = 0$) the condition (1.10) follows from the circularity conditions (2.4a, b)

and the restriction (2.7*b*). Therefore the results obtained here are valid not only for neutrino fields of class E_2 but for all neutrino fields with geodesic and shear-free rays. The reference notation adopted in our previous articles (Kolassis 1982a, b) for the Ricci and Bianchi identities is used again here; e.g. (R1) and (B1), by which we mean the first Ricci identity and the first Bianchi identity, respectively, in the listing given by Pirani (1965) or by Flaherty (1976). In the next sections, the Einstein–Weyl equations are investigated on $\mathscr{D}$ for the cases A, B and C separately and the following theorem is proved.

Theorem. If, in a connected subdomain of space–time, the following conditions are satisfied:

(i) the Einstein–Weyl equations hold and the neutrino field has geodesic and shear-free rays and

(ii) the metric is stationary axially symmetric and can be put into a canonical form, then the metric is uniquely determined and asymptotically non-flat, the Weyl tensor is of Petrov type D, and the Ricci tensor belongs to the class $[2T\text{–}2S]_2$ in the Plebanski classification.

3. Case A

In this section we investigate case A of the preceding section i.e. the interaction of a pure radiation neutrino field with a stationary axially symmetric space–time. The two commuting Killing vector fields are related by

$$e = hb \tag{3.1}$$

and

$$f = hc \tag{3.2}$$

where h is a real function of the coordinates. As the vector field ξ^μ cannot be collinear with η^μ on $\mathscr{D}$, from equations (3.1) and (3.2) it follows that

$$d - ha \neq 0. \tag{3.3}$$

Because the neutrino field is a pure radiation field we have

$$\rho = \bar{\rho}. \tag{3.4}$$

The integrability conditions of equations (3.1) and (3.2) (i.e., their D, Δ and δ derivatives) give

$$\mathrm{D}h = 0 \tag{3.5a}$$

$$b\Delta h = (\rho + \bar{\rho})(ha - d) + u - hp \tag{3.5b}$$

$$\delta h = 0 \tag{3.5c}$$

$$c(u - hp) + (d - ha)[b(\pi + \bar{\tau}) - c(\rho + \bar{\rho})] = 0 \tag{3.5d}$$

$$\mathrm{i}(w - hs) + (d - ha)\rho = 0. \tag{3.5e}$$

Inserting equations (3.1) and (3.2) into the commutation equations (2.18*a*)–(2.18*c*), we obtain

$$u - hp = (\rho + \bar{\rho})(d - ha) \tag{3.6a}$$

$$ua-pd+(d-ha)[b(\gamma+\bar{\gamma})+c(\pi+\bar{\tau}-4\tau)+\bar{c}(\bar{\pi}+\tau-4\bar{\tau})]=0 \tag{3.6b}$$

$$(d-ha)[b(\bar{\pi}+\tau)-\bar{c}\bar{\rho}]+i\bar{c}(w-hs)=0. \tag{3.6c}$$

With the help of equations (3.3) and (3.4) equations (3.5a)–(3.5e) and (3.6a)–(3.6c) yield

$$\rho=0 \tag{3.7a}$$

$$\pi+\bar{\tau}=0 \tag{3.7b}$$

$$u=hp \tag{3.8a}$$

$$w=hs \tag{3.8b}$$

$$h=\text{constant} \tag{3.8c}$$

$$p=b(\gamma+\bar{\gamma})-4c\tau-4\bar{c}\bar{\tau}. \tag{3.8d}$$

Now, as on the rotation axis the vector field η^{μ} vanishes, we have $e=0$ which by virtue of equation (3.1) in turn implies that h also vanishes on the axis. (It is clear that a physically significant neutrino flux vector cannot be singular on the rotation axis. Therefore the null tetrad will also be non-singular on the axis except perhaps at some isolated points.) Therefore, by virtue of equation (3.8c),

$$h=0 \tag{3.9}$$

everywhere in $\mathscr{D}$. But this last equation has as a consequence that η^{μ} is collinear with l^{μ} everywhere in $\mathscr{D}$ and this is in contradiction with our hypotheses. Finally, we can conclude that the pure radiation neutrino field is incompatible with a stationary axially symmetric configuration of the space–time.

4. Case B

In this case the neutrino energy–momentum tensor has the general form corresponding to the class E_2 fields while the two Killing vector fields satisfy the restrictions

$$c=0 \tag{4.1}$$

and

$$f=0. \tag{4.2}$$

In view of the results of the preceding section we adopt the restriction

$$\rho\neq\bar{\rho}. \tag{4.3}$$

This restriction means that the neutrino field cannot be a pure radiation field and therefore as a direct consequence we have

$$p=u=0 \tag{4.4}$$

while the quantities s and w are reduced to real constants. As the two Killing vector fields cannot be collinear, the substitution of equations (4.1), (4.2) and (4.4) into equation (2.18a) yields

$$\rho+\bar{\rho}=0. \tag{4.5}$$

But this last equation together with (R1) implies that the spin coefficient ρ vanishes, and this contradicts equation (4.3). Thus case B is unrealisable by a neutrino field interacting with a stationary axially symmetric space–time.

5. Case C

In this case, apart from the other restrictions satisfied by the spin coefficients, we also have

$$\gamma = \bar{\gamma} \tag{5.1}$$

and

$$\rho \neq \bar{\rho}. \tag{5.2}$$

The restriction (5.2) excludes from our discussion the case of a ghost neutrino field. As, by virtue of equation (5.1), the neutrino field cannot be a pure radiation field, it follows that

$$p = u = 0 \tag{5.3a}$$

$$s = \text{constant} \tag{5.3b}$$

$$w = \text{constant}. \tag{5.3c}$$

The two Killing vector fields are related by

$$d = ha \tag{5.4a}$$

$$e = hb \tag{5.4b}$$

$$c = gf \tag{5.4c}$$

where h and g are real functions of the coordinates. It is clear that except perhaps for the points on the axis we have on $\mathscr{D}$

$$f \neq 0 \tag{5.5}$$

otherwise the vector field η^μ would be time-like on $\mathscr{D}$. As the two Killing vector fields cannot be collinear on $\mathscr{D}$ we have the additional restriction

$$1 - hg \neq 0. \tag{5.6}$$

For convenience, in addition to the conventions adopted at the end of § 2, we agree that in this section the restrictions (5.1)–(5.6) will also be used without explicit reference. The integrability conditions of equations (5.4a)–(5.4c) respectively yield

$$a\mathrm{D}h = (1-hg)[(\bar{\pi} - 3\tau)f + (\pi - 3\bar{\tau})\bar{f}] \tag{5.7a}$$

$$a\Delta h = (1-hg)(\bar{\nu}f + \nu\bar{f}) \tag{5.7b}$$

$$a\delta h = (1-hg)(\mu - \bar{\mu})\bar{f} \tag{5.7c}$$

$$\mathrm{D}h = 0 \tag{5.8a}$$

$$b\Delta h = 2(1-hg)(\tau f + \bar{\tau}\bar{f}) \tag{5.8b}$$

$$b\delta h = (1-hg)(\rho - \bar{\rho})\bar{f} \tag{5.8c}$$

$$f\mathrm{D}g = (1-hg)(\pi+\bar{\tau})b \tag{5.9a}$$

$$f\Delta g = -(1-hg)(\pi+\bar{\tau})a \tag{5.9b}$$

$$\bar{f}\delta g = (1-hg)\bar{\lambda}b \tag{5.9c}$$

$$\bar{f}\bar{\delta}g = (1-hg)(\bar{\mu}b-\bar{\rho}a)+\mathrm{i}(s-gw). \tag{5.9d}$$

Moreover the commutation equations (2.18*a*)–(2.18*c*) can be written respectively

$$f\tau+\bar{f}\bar{\tau}=0 \tag{5.10a}$$

$$a[f(\pi+\bar{\tau}-4\tau)+\bar{f}(\bar{\pi}+\tau-4\bar{\tau})]+b(f\bar{\nu}+\bar{f}\nu)=0 \tag{5.10b}$$

$$(1-hg)[\bar{f}(a\bar{\rho}-b\bar{\mu})-bf\bar{\lambda}]=\mathrm{i}(s-gw)\bar{f}. \tag{5.10c}$$

From equations (5.7*a*)–(5.10*c*) we obtain the following restrictive equations on the spin coefficients

$$\pi+\bar{\tau}=0 \tag{5.11}$$

$$f\tau+\bar{f}\bar{\tau}=0 \tag{5.12}$$

$$f\bar{\nu}+\bar{f}\nu=0 \tag{5.13}$$

$$f^2\bar{\lambda}+\bar{f}^2\lambda=0 \tag{5.14}$$

$$a\rho=b\mu. \tag{5.15}$$

In order to obtain further restrictions on the spin coefficients we must consider the integrability conditions of equations (5.12)–(5.15). Thus, from the D derivative of equation (5.12) and with the help of equation (5.11) and (R3) we obtain

$$\tau=0. \tag{5.16}$$

From the D derivative of equation (5.13) and with the help of equations (5.11) and (5.16) and (R9), (R15) and (R18) we obtain

$$\nu=0 \tag{5.17}$$

and

$$\Psi_3=0. \tag{5.18}$$

From the D derivative of equation (5.14) and with the help of equations (5.11), (5.16) and (5.17) and (R7) and (R10) we obtain

$$\lambda=0 \tag{5.19}$$

and

$$\Psi_4=0. \tag{5.20}$$

Finally from the D derivative of equation (5.15) and with the help of equations (5.11) and (5.16) and (R1) and (R8) we obtain

$$\Psi_2=2\gamma\rho+2\rho\mu. \tag{5.21}$$

Using the Ricci identities and the various restrictions satisfied by the spin coefficients we can show by straightforward calculations that the Δ and $\bar{\delta}$ derivatives of equation (5.15) reduce to identities while from its δ derivative it follows that the quantity $\delta\mu$

vanishes. Introducing equations (5.16) and (5.21) into (R12) we obtain

$$\Phi_{11} = \rho\bar{\mu} + \gamma(\rho + \bar{\rho}). \tag{5.22}$$

From (R6), (R15) and (R17) and with the help of equations (5.16), (5.17) and (5.22), we obtain $\delta\gamma = \bar{\delta}\gamma = \mathrm{D}\gamma = 0$. On the other hand, substituting equation (5.22) into (B11) and taking into account equations (5.16), (5.17), (5.19) and (5.21) and (R14) and (R17) we obtain $(\rho + \bar{\rho})\Delta\gamma = 0$. From this equation and from the fact that by virtue of (R1) the spin coefficient ρ cannot be purely imaginary it follows that $\Delta\gamma = 0$. Therefore, it is clear that

$$\gamma = \text{real constant}. \tag{5.23}$$

We have not used yet the ten Bianchi identities (B1)–(B10) and the commutation relations of the D, Δ, δ and $\bar{\delta}$ operators acting on scalars, but by straightforward calculations we can see that by virtue of the results we have obtained, all these equations are identically satisfied. Recapitulating, we can say that in case C all the spin coefficients vanish except ε, ρ, μ and γ, and the Weyl tensor is of Petrov type D. Now, if we wish to proceed we have to introduce a coordinate system and resolve the metric equations. The resulting metric will admit at least two commuting Killing vector fields and we must check whether or not these Killing vector fields satisfy equations (5.4*a*)–(5.4*c*).

Following the methods of Collinson and Morris (1973) we introduce a coordinate system (x^1, x^2, x^3, x^4) adapted to the geodesic null congruence defined by l^μ. The coordinate $x^2 \equiv r$ is one affine parameter along each null geodesic so that

$$l^\mu = A\,\partial x^\mu/\partial r \tag{5.24}$$

with

$$\mathrm{D}(\log A) = \rho + \bar{\rho} \tag{5.25}$$

where A is a real function of the coordinates. The remaining three coordinates of the congruence label the geodesics of the null congruence. The null tetrad vectors may be considered in the form

$$l^\mu = A\delta_2^\mu \tag{5.26a}$$

$$\kappa^\mu = U\delta_2^\mu + X^i\delta_i^\mu \tag{5.26b}$$

$$m^\mu = \varphi\delta_2^\mu + Y^i\delta_i^\mu \tag{5.26c}$$

where the index i takes the values $i = 1, 3, 4$. They are invariant in form under the following coordinate transformations

$$r' = r \qquad x'^i = x^i(x^1, x^3, x^4) \tag{5.27a}$$

$$r' = r + f(x^1, x^3, x^4) \qquad x'^i = x^i \tag{5.27b}$$

$$r' = g(x^1, x^3, x^4)r \qquad x'^i = x^i. \tag{5.27c}$$

From equations (5.25) and (R1) the spin coefficient ρ is uniquely determined. It can be written in the form

$$\rho = -A/(r + \mathrm{i}\rho^0) \tag{5.28}$$

where the superscript is used to denote a function independent of r. The coordinate transformations (5.27b) and (5.27c) can be used to put $\rho^0 = 1$. Hence

$$\rho = -A/(r+\mathrm{i}). \tag{5.29}$$

Substituting equation (5.29) into equation (5.25) and resolving the resulting equation we obtain

$$A = A^0/(r^2+1). \tag{5.30}$$

The metric equations are obtained by substitution of the coordinates into the NP commutation relations. They can be written in the form

$$\Delta A - \mathrm{D}U = 2\gamma A + (\rho+\bar{\rho})U \tag{5.31}$$

$$\delta A - \mathrm{D}\varphi = -\rho\varphi \tag{5.32}$$

$$\delta U - \Delta\varphi = \mu\varphi \tag{5.33}$$

$$\bar{\delta}\varphi - \delta\bar{\varphi} = (\bar{\mu}-\mu)A + (\bar{\rho}-\rho)U \tag{5.34}$$

$$\mathrm{D}X^i = -(\rho+\bar{\rho})X^i \tag{5.35}$$

$$\mathrm{D}Y^i = \rho Y^i \tag{5.36}$$

$$\delta X^i - \Delta Y^i = \mu Y^i \tag{5.37}$$

$$\bar{\delta}Y^i - \delta\bar{Y}^i = (\bar{\rho}-\rho)X^i. \tag{5.38}$$

Integrating the radial equations (5.35) and (5.36) we obtain

$$X^i = X^{0i}(r^2+1) \tag{5.39}$$

and

$$Y^i = Y^{0i}/(r+\mathrm{i}). \tag{5.40}$$

By virtue of equation (5.39) the integration of equation (5.31) yields

$$U = r(\tfrac{1}{3}r^2+1)(X^{0i}/A^0)\partial_i A^0 - 2\gamma r + U^0. \tag{5.41}$$

On the other hand, by virtue of equation (5.40) the integration of equation (5.32) yields

$$\varphi = (r-\mathrm{i})(r^2+1)^{-2}[r(\tfrac{1}{3}r^2+1)(Y^{0i}/A^0)\partial_i A^0 + \mathrm{i}\varphi^0]. \tag{5.42}$$

Inserting equation (5.29) into equation (5.22) we obtain

$$\mu = (\tfrac{1}{4}-2\gamma r)(r-\mathrm{i})(r^2+1)^{-1}. \tag{5.43}$$

Substituting equations (5.39) and (5.40) into equations (5.37) and (5.38) we obtain

$$Y^{0i}\partial_i X^{0j} = X^{0i}\partial_i Y^{0j} \tag{5.44a}$$

$$(Y^{0i}/A^0)\partial_i A^0 X^{0j} = 0 \tag{5.44b}$$

$$\varphi^0 X^{0i} = 0 \tag{5.44c}$$

$$(X^{0i}/A^0)\partial_i A^0 Y^{0j} = 0 \tag{5.44d}$$

$$(U^0 - \tfrac{1}{4})Y^{0i} = 0 \tag{5.44e}$$

$$\varphi^0 \bar{Y}^{0i} = 0 \tag{5.45a}$$

$$(Y^{0i}/A^0)\partial_i A^0 \bar{Y}^{0j} = 0 \tag{5.45b}$$

$$2\mathrm{i}A^0 X^{0j} + \bar{Y}^{0i}\partial_i Y^{0j} - Y^{0i}\partial_i \bar{Y}^{0j} = 0. \tag{5.45c}$$

Now we notice that if $\varphi^0 \neq 0$ from equations (5.44*c*) and (5.45*a*), it follows that $X^{0i} = Y^{0i} = 0$. But from these two equations and equation (5.33) the contradiction $\varphi^0 = 0$ follows. Hence

$$\varphi^0 = 0. \tag{5.46}$$

On the other hand if $Y^{0i} = 0$ then by virtue of equations (5.40), (5.42) and (5.46) the vector field m^μ will be identically zero on $\mathscr{D}$. Hence

$$Y^{0i} \neq 0 \tag{5.47}$$

everywhere on $\mathscr{D}$. By virtue of equation (5.47), equations (5.44*e*), (5.44*d*) and (5.45*b*) respectively yield

$$U^0 = \tfrac{1}{4} \tag{5.48}$$

$$X^{0i}\partial_i A^0 = 0 \tag{5.49}$$

$$Y^{0i}\partial_i A^0 = 0. \tag{5.50}$$

Since X^{0i} and Y^{0i} span the subspace $r = \text{constant}$, from equations (5.49) and (5.50) it follows that $A^0 = \text{constant}$. Thus equations (5.42) and (5.41) reduce to

$$\varphi = 0 \tag{5.51}$$

and

$$U = \tfrac{1}{4} - 2\gamma r. \tag{5.52}$$

Now equations (5.33) and (5.34) are identically satisfied. The remaining equations (5.44*a*) and (5.45*c*) will be used to determine X^{0i} and Y^{0i}. Since X^{0i} has a non-zero magnitude, by performing the transformation (5.27*a*) we achieve

$$X^{0i} = \delta^i_1. \tag{5.53}$$

The remaining coordinate freedom is given by

$$r' = r \qquad x'^i = x^1\delta^i_1 + q^i(x^3, x^4). \tag{5.54}$$

By virtue of equation (5.53), equation (5.44) reduces to $\partial_1 Y^{0j} = 0$. Therefore, the coordinate transformation (5.54) can be used to put

$$Y^{03} = P \qquad Y^{04} = \mathrm{i}P \tag{5.55}$$

where P is a complex function of x^3, x^4. The remaining coordinate freedom is given now by

$$r' = r \tag{5.56a}$$

$$x'^1 = x^1 + q^1(x^3, x^4) \tag{5.56b}$$

$$z' = z'(z) \tag{5.56c}$$

where $z = x^3 + \mathrm{i}x^4$. Considering equation (5.45*c*) for $j = 3, 4$ we obtain $P = P(\bar{z})$. Then, with the use of equation (5.56*c*) we can put

$$P = 1. \tag{5.57}$$

The remaining coordinate freedom can be written now

$$r' = r \tag{5.58a}$$

$$x'^1 = x^1 + q^1(x^3, x^4) \tag{5.58b}$$

$$z' = z + \text{constant}. \tag{5.58c}$$

Finally, for $j = 1$ equation (5.45c) reduces to

$$\frac{\partial Y^{01}}{\partial z} - \frac{\partial \bar{Y}^{01}}{\partial \bar{z}} = -\mathrm{i}A^0. \tag{5.59}$$

Now under the coordinate transformation (5.58a)–(5.58c) the quantity Y^{01} transforms as follows

$$Y'^{01} = Y^{01} + 2\frac{\partial q^1}{\partial \bar{z}}. \tag{5.60}$$

However, we notice that equation (5.59) is just the integrability condition for q^1 to be chosen so that $Y'^{01} = -\frac{1}{2}\mathrm{i}A^0 z + \text{constant}$, and the constant can be eliminated by means of equation (5.58c). Therefore we can put

$$Y^{01} = -\tfrac{1}{2}\mathrm{i}A^0 z, \tag{5.61}$$

and the only remaining coordinate freedom is $x'^1 = x^1 + \text{constant}$. All the metric equations are now satisfied. The null tetrad is given by

$$l^\mu = \frac{A^0}{r^2+1}\delta^\mu_2 \tag{5.62a}$$

$$\kappa^\mu = (\tfrac{1}{4} - 2\gamma r)\delta^\mu_2 + (r^2+1)\delta^\mu_1 \tag{5.62b}$$

$$m^\mu = \frac{1}{r+\mathrm{i}}(-\tfrac{1}{2}\mathrm{i}A^0 z\delta^\mu_1 + \delta^\mu_3 + \mathrm{i}\delta^\mu_4). \tag{5.62c}$$

Now equations (2.12a)–(2.17d) and (5.8a)–(5.8c) can be easily integrated to obtain the following uniquely determined solutions

$$c = (r+\mathrm{i})(\bar{C} - \tfrac{1}{2}\mathrm{i}s\bar{z}) \tag{5.63a}$$

$$b = \frac{A^0}{r^2+1}\left(\mathrm{i}(C\bar{z} - \bar{C}z) - \tfrac{1}{2}sz\bar{z} + \frac{B}{A^0}\right) \tag{5.63b}$$

$$a = -(\tfrac{1}{4} - 2\gamma r)\left(\mathrm{i}(C\bar{z} - \bar{C}z) - \tfrac{1}{2}sz\bar{z} + \frac{B}{A^0}\right) \tag{5.63c}$$

$$f = (r+\mathrm{i})(\bar{C}' - \tfrac{1}{2}\mathrm{i}w\bar{z}) \tag{5.64a}$$

$$e = \frac{A^0}{r^2+1}\left(\mathrm{i}(C'\bar{z} - \bar{C}'z) - \tfrac{1}{2}wz\bar{z} + \frac{B'}{A^0}\right) \tag{5.64b}$$

$$d = -(\tfrac{1}{4} - 2\gamma r)\left(\mathrm{i}(C'\bar{z} - \bar{C}'z) - \tfrac{1}{2}wz\bar{z} + \frac{B'}{A^0}\right) \tag{5.64c}$$

$$h = \frac{1}{g}\left(1 + \frac{H}{(r^2+1)b}\right) \tag{5.65}$$

where C, B, C', B' and H are constants of integration. In order that the circularity conditions (2.3a) and (2.3b) are satisfied, equations (5.4a)–(5.4c) and (5.6) must be fulfilled by equations (5.63a)–(5.65). By a straightforward calculation we can see that this is so if

$$C = gC' \tag{5.66a}$$

and

$$B + H = gB'. \tag{5.66b}$$

We have, therefore, a four-parameter group of isometries. The four linearly independent Killing vector fields can be written in the form

$$\xi^{\mu} = \delta_1^{\mu} \tag{5.67}$$

$$\eta^{\mu} = -\tfrac{1}{2}s[\mathrm{i}(z-\bar{z})\delta_3^{\mu} + (z+\bar{z})\delta_4^{\mu}] \tag{5.68}$$

$$K_{(1)}^{\mu} = \tfrac{1}{2}\mathrm{i}A^0(z-\bar{z})\delta_1^{\mu} + 2\delta_3^{\mu} \tag{5.69}$$

$$K_{(2)}^{\mu} = \tfrac{1}{2}A^0(z+\bar{z})\delta_1^{\mu} + 2\delta_4^{\mu}. \tag{5.70}$$

The line element of the metric written in the coordinate system $(x^1, x^2, x^3, x^4) = (u, r, \frac{1}{2}(z+\bar{z}), \frac{1}{2}\mathrm{i}(\bar{z}-z))$ assumes the form

$$\begin{aligned} ds^2 = {} & (2/A^0)F(r)\,du^2 + (2/A^0)\,du\,dr - \tfrac{1}{8}A^0F(r)(z\,d\bar{z} - \bar{z}\,dz)^2 - \tfrac{1}{2}(r^2+1)\,dz\,d\bar{z} \\ & + \mathrm{i}(\tfrac{1}{2}\,dr + F(r)\,du)(z\,d\bar{z} - \bar{z}\,dz) \end{aligned} \tag{5.71}$$

where

$$F(r) = (2\gamma r - \tfrac{1}{4})(r^2+1)^{-1}. \tag{5.72}$$

The investigation of the asymptotic behaviour of the Riemann curvature tensor for large values of the radial coordinate r shows that this metric is asymptotically non-flat. Let us now introduce angular coordinates θ, φ by

$$z = 2\,e^{\mathrm{i}\varphi}\tan\tfrac{1}{2}\theta. \tag{5.73}$$

The line element (5.71) then assumes the form

$$\begin{aligned} ds^2 = {} & (2/A^0)F(r)\,du^2 + (2/A^0)\,du\,dr - \tan^2\tfrac{1}{2}\theta[2(r^2+1) - \tfrac{1}{8}A^0F(r)\tan^2\tfrac{1}{2}\theta]\,d\varphi^2 \\ & - \tfrac{1}{2}(r^2+1)\cos^{-4}\tfrac{1}{2}\theta\,d\theta^2 + 8\tan^2\tfrac{1}{2}\theta(\tfrac{1}{2}\,dr + F(r)\,du)\,d\varphi \end{aligned} \tag{5.74}$$

and the two commuting Killing vector fields become $\xi^{\mu} = \delta_1^{\mu}$ and $\eta^{\mu} = \delta_4^{\mu}$. Finally, the line element (5.74) can be put into a canonical form by introducing a time coordinate t defined by

$$dt = du + \frac{1}{2F(r)}\,dr. \tag{5.75}$$

Thus, we obtain

$$\begin{aligned} ds^2 = {} & (2/A^0)F(r)\,dt^2 - \tan^2\tfrac{1}{2}\theta[2(r^2+1) - \tfrac{1}{8}A^0F(r)\tan^2\tfrac{1}{2}\theta]\,d\varphi^2 \\ & + 8F(r)\tan^2\tfrac{1}{2}\theta\,d\varphi\,dt - (1/2A^0)(1/F(r))\,dr^2 \\ & - \tfrac{1}{2}(r^2+1)\cos^{-4}\tfrac{1}{2}\theta\,d\theta^2. \end{aligned} \tag{5.76}$$

6. Discussion

From the investigations of the previous sections it is clear that there exist no solutions of the Einstein–Weyl equations under the following assumptions.

(i) The neutrino field has geodesic and shear-free rays.

(ii) Space–time is stationary axially symmetric and the line element of the metric can be put into a canonical form.

(iii) Space–time is asymptotically flat.

A result similar to this, but less general, has been obtained by Trim and Wainwright (1974). They proved that under some appropriate restrictions imposed on the asymptotic behaviour of the Weyl and Ricci tensors and for a neutrino field with geodesic and shear-free rays there exists no solution of the Einstein–Weyl equations analogous to the charged Demianski–Newman solutions of the Einstein–Maxwell equations.

Recently, two Kerr-like solutions of the Einstein–Weyl equations were presented by Lun (1980). The corresponding neutrino fields are time-independent with geodesic and shear-free rays and violate the energy conditions E_2 and E_1 as the restriction (1.13) is asymptotically not satisfied. Both metrics are not in contradiction with our results as they cannot be put into a canonical form.

Finally, we must notice that the Einstein–Weyl equations are solved explicitly with respect to the affine parameter r and with the only restriction that the neutrino field be of class E_2 by Trim and Wainwright (1971, see equations (4.1) and (4.2) in this reference). The dependence of the metric with respect to the other three coordinates is determined by a system of 'reduced equations' (see equations (4.6)–(4.8) in this reference). The authors pointed out that these equations admit a particularly simple partial solution which is actually of the form of equation (5.71). However, his character of the unique stationary axisymmetric and canonical solution of the Einstein–Weyl equations with geodesic and shear-free rays was not revealed.

Appendix

For convenience, we list the expansions of $l_{\mu;\nu}$, $\kappa_{\mu;\nu}$ and $m_{\mu;\nu}$ on the null tetrad.

$$\begin{aligned} l_{\mu;\nu} = {} & (\varepsilon+\bar\varepsilon)l_\mu\kappa_\nu - \kappa\bar m_\mu\kappa_\nu - \bar\kappa m_\mu\kappa_\nu + (\gamma+\bar\gamma)l_\mu l_\nu - \tau\bar m_\mu l_\nu - \bar\tau m_\mu l_\nu \\ & -(\bar\alpha+\beta)l_\mu\bar m_\nu - (\alpha+\bar\beta)l_\mu m_\nu + \rho\bar m_\mu m_\nu + \bar\rho m_\mu\bar m_\nu + \sigma\bar m_\mu\bar m_\nu + \bar\sigma m_\mu m_\nu \\ \kappa_{\mu;\nu} = {} & -(\varepsilon+\bar\varepsilon)\kappa_\mu\kappa_\nu + \pi m_\mu\kappa_\nu + \bar\pi\bar m_\mu\kappa_\nu - (\gamma+\bar\gamma)\kappa_\mu l_\nu + \nu m_\mu l_\nu + \bar\nu\bar m_\mu l_\nu \\ & +(\bar\alpha+\beta)\kappa_\mu\bar m_\nu + (\alpha+\bar\beta)\kappa_\mu m_\nu - \lambda m_\mu m_\nu - \bar\lambda\bar m_\mu\bar m_\nu - \mu m_\mu\bar m_\nu - \bar\mu\bar m_\mu m_\nu \\ m_{\mu;\nu} = {} & -\kappa\kappa_\mu\kappa_\nu + \bar\pi l_\mu\kappa_\nu - (\bar\varepsilon-\varepsilon)m_\mu\kappa_\nu - \tau\kappa_\mu l_\nu + \bar\nu l_\mu l_\nu - (\bar\gamma-\gamma)m_\mu l_\nu \\ & +\sigma\kappa_\mu\bar m_\nu - \bar\lambda l_\mu\bar m_\nu + (\bar\alpha-\beta)m_\mu\bar m_\nu + \rho\kappa_\mu m_\nu - \bar\mu l_\mu m_\nu + (\bar\beta-\alpha)m_\mu m_\nu. \end{aligned}$$

References

Anderson I M 1974 *Gen. Rel. Grav.* **6** 621–42

Carter B 1969 *J. Math. Phys.* **10** 70–81

—— 1972 in *Les astres occlus, Les Houches 1972* ed C De Witt and B S De Witt (New York: Gordon and Breach) part II § 7

Collinson C D and Morris P B 1973 *Nuovo Cimento* B **16** 273–84
Flaherty E 1976 *Hermitian and Kalherian Geometry in Relativity* in *Lecture Notes in Physics* ed J Ehlers *et al* (New York: Springer)
Griffiths J B and Newing R A 1970 *J. Phys. A: Gen. Phys.* **3** 269–73
—— 1971 *J. Phys. A: Gen. Phys.* **4** 208–13
Kolassis C A 1982a *J. Math. Phys.* **23** 1630–8
—— 1982b *J. Math. Phys.* **23** 2147–50
Lun A W C 1980 *J. Math. Phys.* **21** 2257–60
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566–78
Papapetrou A 1966 *Ann. Inst. Henri Poincaré* **4** 83–105
Pirani F A E 1965 in *Lectures on General Relativity 1964 Brandeis Summer Institute* vol 1 ed S Deser and K Ford (Englewood Cliffs NJ: Prentice-Hall)
Trim D W and Wainwright J 1971 *J. Math. Phys.* **12** 2494–8
—— 1974 *J. Math. Phys.* **15** 535–46
Wainwright J 1971 *J. Math. Phys.* **12** 828–35. Here the scalar ϕ associated with the neutrino field (see equations (2.2) and (2.4) in this reference) is chosen so that $\phi = 1$.