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Stability in the linearized problem of quantitative elastography

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Abstract
The goal of quantitative elastography is to identify biomechanical parameters from interior displacement data, which are provided by other modalities, such as ultrasound or magnetic resonance imaging. In this paper, we analyze the stability of several linearized problems in quantitative elastography. Our method is based on the theory of redundant systems of linear partial differential equations. We analyze the ellipticity properties of the corresponding PDE systems augmented with the interior displacement data; we explicitly characterize the kernel of the forward operators and show injectivity for particular linearizations. Stability criteria can then be deduced. While joint reconstruction of all parameters suffers from non-ellipticity even for more measurements, our results show stability of the separate reconstruction of shear modulus, pressure and density; they indicate that singular strain fields should be avoided, and show how additional measurements can help in ensuring stability of particular linearized problems.

Keywords: elastography, linearization, stability

1. Introduction

Elastography is a medical imaging technology; its current applications range from detection of cancer in the breast and in the prostate, liver cirrhosis and characterization of atherosclerotic plaque in hardened coronary vessels [4, 16, 20, 45, 57, 58, 60].
Elastography is based on the fact that tissue has high contrast in biomechanical quantities and the health state of organs is reflected in their elastic properties [2, 23]. The most important parameter for diagnosis is the shear modulus $\mu$, which is the dominant factor in the propagation of shear waves in tissue; shear wave speed in tissue can change up to a factor of 4 with disease [50].

Elastography is performed by coupling with various established imaging techniques, such as ultrasound [28, 44], MRI [30, 40] or OCT [42, 54]. What is common to these elastography techniques is that they provide interior data of the displacement $u|_\Omega$ of the tissue on the imaging domain $\Omega$. According to the specific excitation used, $u$ can be space or space-time-dependent (both cases are considered in this work).

In some applications, knowledge of the displacement $u$ already gives qualitative diagnostic information (see, e.g., [61] for a dermatological application). More accurate information is provided by quantitative information of the underlying biomechanical parameters. For this, one formulates the elasticity problem as a model; various models based on linear elasticity, viscoelasticity or hyperelasticity have been considered for elastography [20].

To recover the material parameters, an inverse problem based on an elasticity model has to be solved. Given the displacement $u$, the mathematical problems in quantitative elastography are the recovery of parameters such as the Lamé parameter $\lambda$, the shear modulus $\mu$, the density $\rho$, or recovery of the shear wave speed $\sqrt{\frac{\mu}{\rho}}$ [13, 20]. These problems are nonlinear inverse problems.

Much effort in elastography research has been concerned with developing adequate numerical inversion schemes for all kind of experimental varieties of elastography (see, e.g., [6, 32–34, 41] and the reviews in [20, 52]). Among the proposed algorithms, optimization procedures, which need the linearized inverse problem, form an important class.

Mathematical results about uniqueness and stability of the inverse problems in elastography have been gathered recently, mostly for the simplest models in linear elasticity. In [21, 36], it was proved, that—given one piece of dynamic interior information $u$ subject to restrictions such as $u \neq 0$—one can uniquely recover material parameters. Other uniqueness results have been reported in [14] for two and more measurements or with apriori knowledge of parameters in [15], and in [49] for a hyperelastic model. The recent paper [10] showed unique reconstruction of $\lambda$ and $\mu$ given two sets of exact measurements subject to some non-trivial conditions on the displacements. The stability of the nonlinear problem has been studied in [10] using ODE-based and variational tools.

Elastography can be seen as part of hybrid (or coupled-physics) imaging methods (see [7] and the discussion in [31]). The coupling phenomenon can be simple advection for ultrasound and OCT elastography, or a more complicated coupling in magnetic resonance elastography [30].

The body of literature in hybrid or coupled-physics imaging centers on novel imaging methods involving more than one physical modality; emphasis is laid upon the quantitative imaging problems for mechanical, optical or electrical parameters that have high diagnostic contrast. For reviews on the typical problems in hybrid imaging, see [5, 7, 9, 24, 25, 59].

In many hybrid inverse problems, high resolution in the reconstructions was observed. To explain the high resolution in a unified manner, a general strategy is to linearize the corresponding nonlinear quantitative problems. Then the stability properties of the linearized problem have been treated with tools of linear PDE theory [8, 12, 26, 27]. In [26], problems in conductivity imaging and quantitative photoacoustic imaging were treated. The linearized forward operators were studied with pseudodifferential theory and shown to be Fredholm, i.e., with stable inversion up to a finite-dimensional kernel. In [8], a general framework was
proposed for treating linearizations of quantitative problems with interior data; in this method, the parameters are considered as variables and the information is recast into a single redundant PDE system, to which theory such as [53] is applied. This method was applied to the power density problem in [8] and to the problem of acousto-optic imaging in [12]. In these analyses, information from multiple experiments has been assumed, as the ellipticity condition was easier to obtain in this case.

In this article, we treat the linearized problem in quantitative elastography using the general coupled-physics approach in [8], using [53]. This is the first time that this technique is applied to a problem in elasticity imaging. We treat stability for several kinds of operators with one or more measurements. Depending on the operator, we explicitly characterize the finite-dimensional kernel, and show injectivity in several versions of the linearized inverse problems.

In future work, we investigate the impact of these results on the nonlinear problem and on iterative reconstruction algorithms.

The structure of this paper is as follows: in section 2, we review the elasticity model which we are using, and in section 3, we review necessary background of linear PDE theory. In section 4, we apply this theory to the elasticity equation. This section contains the main results of the paper. We investigate how several kinds of linearizations perform analytically, discuss the ellipticity conditions and derive the stability results as well as the characterization of the kernel. The appendix contains topological lemmas.

2. Modelling quantitative elastography

2.1. Experiments and interior information in elastography

The general principle in elastography [20] is to

- perturb the tissue using a suitable mechanical source,
- determine the internal tissue displacement using an ultrasound, magnetic resonance or optical displacement estimation method,
- infer the mechanical properties from the interior information, using a mechanical model.

Note that in elastography, there are two forms of excitation: an elastic deformation from the mechanical source, and the excitation from the ground modality. Also, the reconstruction procedure involves two steps: the recovery of the mechanical displacement \( u(x) \mid_{\Omega} \) resp. \( u(x, t) \mid_{\Omega} \) from measurements on the boundary, and the recovery of the mechanical parameters and properties from \( u \mid_{\Omega} \). In this last step of quantitative reconstruction (which is treated in this article), the mechanical displacement \( u \mid_{\Omega} \) is also referred to as interior information.

2.2. A linear elasticity model for inhomogeneous linear isotropic media

There exist different variants of elastography using quasi-static, transient or time-harmonic mechanical excitations, but they can be described by common PDE models [20, 45], which can be deduced from the equation of motion [29]

\[
\nabla \cdot \sigma - \rho u_n = F. \tag{1}
\]

Here, \( F(x, t) \) is the excitation force density in \( \text{N m}^{-3} \). (We use the convention that letters printed in bold denote vectors in \( \mathbb{R}^3 \).) The mechanical displacement \( u(x, t) \) of the material point which is at position \( x \) at time \( t \) is measured in \( [u] = \text{m} \). \( \rho(x) \) is the density in \( \text{Kg m}^{-3} \).
\[ \sigma = (\sigma_{ij}(x, t))_{ij} \] is the mechanical (or Cauchy) stress tensor with unit N m\(^{-2}\). Here, the divergence of a second-rank tensor \( A \) is computed column-wise:

\[ \nabla \cdot A = \nabla \cdot (a_1, \ldots, a_n) := (\nabla \cdot a_1, \ldots, \nabla \cdot a_n). \]

The constitutive equation in linearized elasticity (also termed Hooke’s law) is

\[ \sigma = C : \varepsilon \]

with the stress tensor \( \sigma = \sigma_{ij}(x, t) \) and the dimensionless strain tensor \( \varepsilon = \varepsilon_{ij}(x, t) \). Here, : denotes the tensor multiplication. The material properties are incorporated into \( C = C_{ijkl}(x) \), which is the rank-four stiffness tensor (with unit N m\(^{-2}\)).

In linear elasticity, one works with small deformations, and uses the following representation of the strain \( \varepsilon \):

\[ \varepsilon = \frac{1}{2} (\nabla u + \nabla u^T). \]

Assuming that the tissue is isotropic, one can reduce \( C \) to knowledge of two scalar quantities \( \lambda \) and \( \mu \):

\[ \sigma = 2\mu \varepsilon + \lambda \text{tr} \varepsilon \text{Id} \]

instead of (2). Here, \( \lambda(x) \) is called the first Lamé parameter, and \( \mu(x) \) is called the shear modulus or the second Lamé parameter. The physical units of \( \lambda \) and \( \mu \) are the same as of \( \sigma \) and \( C \), i.e. N m\(^{-2}\).

For tissue, it is actually sometimes assumed that the incompressibility condition \( \nabla \cdot u = 0 \) holds, or one sets \( \lambda \nabla \cdot u = 0 \) [15, 30, 31].

An alternative in the compressible case is to change the quantities and define the pressure \( p(x, t) := \lambda(x) \nabla \cdot u(x, t) \) [14, 15, 37, 48]. Then, instead of (4), one uses

\[ \sigma = 2\mu \varepsilon + p \text{Id}. \]

Note that \( p(x, t) \) is an elastic quantity, but not a material parameter.

Combining (1) with (3) and (4) results in the equation

\[ \nabla (\lambda \nabla \cdot u) + 2 \nabla \cdot (\mu \varepsilon(u)) - \rho u_{tt} = F \quad \text{on } \Omega. \]

Alternatively, combining (1) with (3) and (5), one has

\[ \nabla p + 2 \nabla \cdot (\mu \varepsilon(u)) - \rho u_{tt} = F \quad \text{on } \Omega. \]

Then this model is augmented with boundary conditions and appropriate sources. There are different choices for initial and boundary conditions. One option is

\[ u|_{t=0} = 0 \]
\[ u|_{\partial \Omega \times \{t=0\}} = g \]
\[ \partial_t u|_{\partial \Omega \times \{t=0\}} = h. \]

Existence, uniqueness and regularity properties for either (6)+(8) or (7)+(8) follow from the theory in [35] (for an earlier result for elastodynamic problems, see [18]). The initial and boundary values \( g \) and \( h \) are prescribed only for the complete mathematical analysis. In practical physical experiments, the varying excitations enter in the source term \( F \).

The standard problem of quantitative elastography is to determine the material parameters \( \lambda(x) \), \( \mu(x) \) and \( \rho(x) \) in the model (6)+(8), given the interior information \( u(x, t)|_{\Omega} \).

The modified inverse problem, which we address in this article, is to recover \( p(x, t) \), as well as \( \mu(x) \) and \( \rho(x) \), in the model (7)+(8), given \( u(x, t)|_{\Omega} \).
3. A result from linear PDE theory

We first treat the background from the general theory of linear PDE systems in $\mathbb{R}^n$. This will later be applied to $n=3$, $n=4$. The essential information for the quick reader is: We write overdetermined redundant systems of form (9) like (18). If they satisfy the conditions in definition 1 and 2, a stability estimate as in theorem 1 holds.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ (smoothness requirements on $\Omega$ are specified later). We consider the redundant system of linear partial differential equations

$$L(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) \mathbf{w} = \mathbf{S} \quad \text{on } \Omega$$

$$B(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) \mathbf{w} = \varphi \quad \text{on } \partial \Omega$$

for $m$ unknown functions $\mathbf{w}(\mathbf{x}) = (w_1(\mathbf{x}), \ldots, w_m(\mathbf{x}))$, comprising in total $M$ equations. Here, $L(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}})$ is a matrix differential operator of dimension $M \times m$,

$$L(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) = \begin{pmatrix} L_{11}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) & \cdots & L_{1m}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) \\ \vdots & \ddots & \vdots \\ L_{M1}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) & \cdots & L_{Mm}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}}) \end{pmatrix}$$

(10)

For each $1 \leq i \leq M$, $1 \leq j \leq m$ and for each point $\mathbf{x}$, $L_{ij}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}})$ is a polynomial in $\frac{\partial}{\partial \mathbf{x}} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Redundancy of the system means that there are possibly more equations than unknowns: $M \geq m$.

Similarly, $B(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}})$ has entries $B_{kj}(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}})$ for $1 \leq k \leq Q$, $1 \leq j \leq m$, consisting of $Q$ equations at the boundary. The operations are again polynomial in the second variable. $-\mathbf{S}(\mathbf{x})$ is a vector of length $M$, and $\varphi(\mathbf{x})$ is a vector of length $Q$.

We now define the notions of ellipticity and the principal part of $L$ and $B$, respectively, in the sense of Douglis and Nirenberg [19].

**Definition 1.** Let integers $s_i, t_j \in \mathbb{Z}$ be given for each row $1 \leq i \leq M$ and column $1 \leq j \leq m$ with the following property: For $s_i + t_j \geq 0$, the order of $L_{ij}$ does not exceed $s_i + t_j$. For $s_i + t_j < 0$, one has $L_{ij} = 0$. Furthermore, the numbers are normalized so that for all $i$ one has $s_i \leq 0$. Such numbers $s_i, t_j$ are called Douglis–Nirenberg numbers.

The **principal part** of $L$ for this choice of numbers $s_i, t_j$ is defined as the matrix operator $L_0$ whose entries $L_{0,ij}$ are composed of those terms in $L_{ij}$ which are exactly of order $s_i + t_j$.

The principal part $B_0$ of $B$ is composed of the entries $B_{0,ij}$, which are composed of those terms in $B_{kj}$ which are exactly of order $\sigma_k + t_j$. The numbers $\sigma_k$, $1 \leq k \leq Q$ are computed as

$$\sigma_k := \max_{1 \leq j \leq m} \left( b_{kj} - t_j \right)$$

(11)

where $b_{kj}$ denotes the order of $B_{kj}$.

Real directions $\xi \neq 0$ with rank $L_0(\mathbf{x}, 1\xi) < m$ are called characteristic directions of $L$ at $\mathbf{x}$. (The complex unit is denoted by the symbol $1 = i$.) The operator $L(\mathbf{x}, \frac{\partial}{\partial \mathbf{x}})$ is said to be **overdetermined elliptic** in $\Omega$ if for all $\mathbf{x} \in \Omega$ and for all real vectors $\xi \neq 0$ one has that

$$\text{rank } L_0(\mathbf{x}, 1\xi) = m$$

(12)

for the $M \times m$ matrix $L_0(\mathbf{x}, 1\xi)$. 


Douglis–Nirenberg numbers allow that terms of order less than \( \max \{\text{ord } L_{ij}\} \) appear in the symbol \( \mathcal{L}_0 \). The maximum number of entries in the symbol is attained if one has that \( s_i + t_j = \text{ord } L_{ij} \).

Next we define the condition of \( \mathcal{B} \) covering \( \mathcal{L} \), or the Lopatinskii boundary condition [53].

**Definition 2.** Fix \( y \in \partial \Omega \), and let \( \nu \) be the inward unit normal vector at \( y \). Let \( \zeta \) be any non-zero tangential vector to \( \Omega \) at \( y \). Consider the half-line \( \{ y + z \nu, z > 0 \} \) and the following system of ordinary differential equations on it:

\[
\begin{align*}
\mathcal{L}_0 \left( y, 1\zeta + \nu \frac{d}{dz} \right)w(z) &= 0 & z > 0 \\
\mathcal{B}_0 \left( y, 1\zeta + \nu \frac{d}{dz} \right)\tilde{w}(z) &= 0 & z = 0.
\end{align*}
\]  

Consider the vector space of all solutions \( \tilde{w} \) of (13) which satisfy \( \tilde{w}(z) \to 0 \) for \( z \to \infty \). If this vector space consists just of the trivial solution \( \tilde{w}(z) \equiv 0 \), then the Lopatinskii condition is said to be fulfilled for the pair \( (\mathcal{L}, \mathcal{B}) \) at \( y \), or \( \mathcal{B} \) covers the operator \( \mathcal{L} \) at \( y \).

A typical example of an overdetermined elliptic systems with Lopatinskii boundary conditions is

\[
\begin{align*}
\nabla \times w &= f \\
\nabla \cdot w &= g
\end{align*}
\]  

on a domain \( \Omega \) with normal component \( w \cdot v \rvert_{\partial \Omega} \) given on the boundary [53].

In the context of hybrid imaging, examples of overdetermined elliptic systems with Lopatinskii boundary conditions have been considered in [8, 12].

For investigating the stability for linearized quantitative elastography, we are going to use the a-priori estimate in [53] for the solutions of system (9). This theory does not need smooth coefficients, but coefficients in the Sobolev spaces \( W^\alpha_p(\Omega) \) (for the usual definition, also for noninteger values of \( \alpha \), see [1]). In the setting of [53] with Douglis–Nirenberg numbers \( t_j, s_i, \alpha_k \), one has that the operator \( \mathcal{A} \) with

\[
\mathcal{A} w = \begin{pmatrix} \mathcal{L} w \\ \mathcal{B} w \end{pmatrix}
\]  

acts on the space

\[
D(p, l) := W_p^{l+s_l}(\Omega) \times \ldots \times W_p^{l+s_l}(\Omega),
\]  

where \( l \geq 0, p > 1 \). Under suitable restrictions on the coefficients \( L_{ij} \) and \( B_{ij} \) (specified below in the conditions of the theorem), the operator \( \mathcal{A} \) is bounded with range in

\[
R(p, l) := W_p^{l-s_l}(\Omega) \times \ldots \times W_p^{l-s_l}(\Omega) \times W_p^{l-s_l}(\partial \Omega) \times \ldots \times W_p^{l-s_l}(\partial \Omega).
\]  

Using the operator \( \mathcal{A} \) in (15), the equations (9) read

\[
\mathcal{A} w = \begin{pmatrix} S \\ \varphi \end{pmatrix}.
\]  

In formulating the restrictions on the coefficients of \( \mathcal{L} \) and \( \mathcal{B} \), we simplify the version of [53 thm. 1.1] for the following result.
Theorem 1. Let integers \( l \geq 0, p > 1 \) be given. Let \((S, qr)\), the data from (9), be in \( R(p, 1) \) as defined in (17). Let Douglas–Nirenberg numbers \( s_i \) and \( t_j \) be given for \( L \) in (9), and let \( \sigma_k \) be as in definition 1. Let \( \Omega \) be a bounded domain with boundary in \( C^{1+\text{max} i/j} \). Assume furthermore that \( p(l - s_i) > n \) and \( p(l - \sigma_k) > n \) for all \( i \) and \( k \). Let the coefficients of \( L_{ij} \) be in \( W^{l-s_{ij}}(\Omega) \), and let the coefficients of \( B_{kj} \) be in \( W^{l-\sigma_k}(\Omega) \). Then the following statements are equivalent:

1. \( L \) in (9) is overdetermined elliptic (see (12)) and the Lopatinskii covering condition (13) is fulfilled for \((L, B)\) on \( \partial \Omega \).
2. There exists a left regularizer \( R \) for the operator \( A = L \times B \) in (15), that is, we have
   \[
   RA = I - D
   \]
   with \( D \) compact from \( R(p, 1) \) in (17) to \( D(p, 1) \) in (16).
3. The following a-priori estimate holds
   \[
   \sum_{j=1}^{m} \|w_j\|_{W_p^{l-\tau_j}(\Omega)} \leq C_1 \left( \sum_{i=1}^{M} \|S_i\|_{W_p^{l-\tau_i}(\Omega)} + \sum_{k=1}^{O} \|\phi_k\|_{W_p^{l-\tau_k - \tau_i}(\Omega)} \right) + C_2 \sum_{i,t>0} \|w_j\|_{L^\tau(\Omega)}, \tag{20}
   \]
   where \( w_j \) is the \( j \)th component of the solution \( w \) of (15).

The assertion of the theorem gives a criterion for the existence of a left regularizer for the overdetermined redundant systems. For the case of boundary value problems for square systems with \( M = m \), this equivalence is a classical result (see [3]). For square systems, one has the stronger statement that ellipticity and Lopatinskii condition are equivalent to the Fredholm property of a differential operator (which also needs the existence of a right regularizer).—The criterion for redundant systems with \( M \geq m \) was established in [53], and investigates the stability estimate, and even gives a representation formula for the solution, provided it exists. The existence of a right regularizer \( Q \) with \( AQ = I - D \) (which would yield local existence) cannot be assured in general for overdetermined systems.

We will exploit this criterion for the linearized version of quantitative elastography.

4. Stability analysis

4.1. Setting and notation

For treating the hybrid imaging problem described in 2.1, we take the adapted forward model with (7)+(8). We recast the equations of the forward problem for the displacement, and the interior information, respectively, into a single system of partial differential equations describing one experiment:

\[
\begin{align*}
Vp + 2V \cdot (\mu \varepsilon(u)) - \mu \varepsilon(u) &= F \\
u(x, t) &= H \\
\frac{\partial u}{\partial t} |_{\partial \Omega} &= 0 \\
\frac{\partial u}{\partial t} |_{\partial \Omega_{\text{r=0}}} &= g \\
\frac{\partial u}{\partial t} |_{\partial \Omega_{\text{r=0}}} &= h.
\end{align*}
\tag{21}
\]

Here, we are given \( F \) (the excitation force), \( g \) and \( h \), as well as the interior information \( H \). We formally keep \( u \) and \( H \) distinct in the second equation, as \( u \) is treated as variable of the system, and \( H \) represents the interior information as a separate object.—In this we keep the
analogy with the formalism in [8]. We aim at a quantitative estimate such as (20) with the given interior information in the inhomogeneity.

We consider the system (21), for the variables \( u, p, \mu, \rho \); therefore, we consider it as nonlinear, involving multiplication of the unknowns. In order to make it tractable for the analysis, we linearize this system to provide equations of the form (9). Several of these linearizations will be considered in the subsequent theory.

We first consider the forward operator

\[
\mathcal{V} : (p(x, t), \mu(x), \rho(x)) \mapsto u(x, t),
\]

which maps the triple \((p, \mu, \rho)\) to the displacement field \( u \) satisfying (21). Then we consider the linearization of \( \mathcal{V} \) at a reference state \((p, \mu, \rho)\),

\[
\mathcal{V}(p, \mu, \rho) : \begin{pmatrix} \delta p \\ \delta \mu \\ \delta \rho \end{pmatrix} \mapsto \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} =: \delta u.
\]

By formal differentiation of (21), we find that, at the reference state \( u = \mathcal{V}(p, \mu, \rho) \), the increment \( \delta u \) satisfies the equations

\[
\begin{align*}
\nabla \delta p + 2 \nabla \cdot (\delta \mu \varepsilon(u)) + 2 \nabla \cdot (\mu \varepsilon(\delta u)) - \delta \rho \ u_n - \rho(\delta u)_n &= 0 \\
\delta u &= \delta H \\
\delta u \big|_{\partial \Omega(t=0)} &= 0 \\
\partial_t \delta u \big|_{\partial \Omega(t=0)} &= 0.
\end{align*}
\]

Note that \( F \) does not depend on the reference state, therefore no inhomogeneity appears in the first equation in (23). The quantity \( \delta H \) is the derivative of the interior information.

Observe that (23) is a system of differential equations for the functions \((\delta p, \delta \mu, \delta \rho, \delta u_1, \delta u_2, \delta u_3)=(\delta p, \delta \mu, \delta \rho, \delta u)\). The system is linear in these unknowns.

We now introduce the operator

\[
\mathcal{L}_{ppp}(\delta p, \delta \mu, \delta \rho, \delta u) := \begin{pmatrix} \nabla \delta p + 2 \nabla \cdot (\delta \mu \varepsilon(u)) + 2 \nabla \cdot (\mu \varepsilon(\delta u)) - \delta \rho \ u_n - \rho(\delta u)_n \end{pmatrix} \delta u.
\]

This operator is the linearization of the redundant system (21) with respect to \( p, \mu \) and \( \rho \). Note that \( \mathcal{L}_{ppp} \) is a matrix differential operator like \( \mathcal{L} \) in (9).

In our analysis, we will treat the operator \( \mathcal{L}_{ppp} \) as well as several specializations, corresponding to the directional derivatives with respect to only one parameter; these operators are denoted \( \mathcal{L}_p, \mathcal{L}_\mu, \mathcal{L}_\rho \). We also treat the combination \( \mathcal{L}_{pp} \). These operator specifications are summarized in table 1.

In the model (21) which we started from, we incorporated the pressure \( \rho \) in (4). An alternative is to use the original model (5) only, which involved the first Lamé parameter \( \lambda \). This corresponds to re-substituting \( p = \lambda \nabla \cdot u \) in (21). In exact analogy to constructing \( \mathcal{L}_{ppp} \), one can form the forward operator \( \mathcal{V}_\lambda \). One then considers its linearization

\[
\mathcal{V}_\lambda(\lambda, \mu, \rho) : (\delta \lambda, \delta \mu, \delta \rho) \mapsto \delta u,
\]

and introduces the operator \( \mathcal{L}_{\delta p \delta \mu \delta \rho} \). We particularly will consider
We want to apply the methodology of section 4.3 to the various linearizations and compare their properties. We keep the formalism consistent with (9) resp. (18) to study redundant systems of the form:

\[
\frac{\partial \delta u}{\partial \partial \partial \partial} = \lambda \lambda \mu \rho \delta \mu \epsilon \delta \rho \delta
\]

In table 1, we already defined several operators \( \mathcal{L} \). What rests is that we have to explain the operators \( \mathcal{L} \) and the inhomogeneity \( \delta \).

This is very easily done: looking at (23), the boundary data are for each case

\[
B(\delta u) := \left( \begin{array}{c}
\delta u|_{\partial \Omega} \\
\delta u|_{\partial \Omega} \\
\delta u|_{\partial \Omega}
\end{array} \right).
\]

For the inhomogeneity \( \delta \), we have

\[
\delta = (0, 0, 0, \delta H_1, \delta H_2, \delta H_3)^T.
\]

All of these are linear differential systems which are redundant systems of form (9) with \( M \geq m \). These are summarized in table 2.

In the elastography literature, linearizations as in table 1 have been treated in iterative algorithms to solve the nonlinear problem. See, for example the discussion on the Newton-type algorithms in [20 section 4.1.2], or [43, 56]. In these works, the respective linearizations have been computed in their discretized forms, and for only one experiment. Because most applications aim at reconstructing the shear modulus, often only the linearization with respect to \( \mu \) is employed.

In the literature on hybrid imaging, often multiple measurements are considered. To incorporate the additional information, we can use different excitations \( \mathbf{F} \) and possibly different functions \( \mathbf{g}, \mathbf{h} \) in (21) and obtain different versions of \( u(x, t) \) and interior information \( \mathbf{H}(x, t) \). For each experiment, we also have a different variable \( p \) in the system. While these quantities change with each excitation, the material parameters \( \lambda, \mu \) and \( \rho \) remain the same.
For example, we write $\mu_i$ for the operator corresponding to the reference state $u_i$ and

\[
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\mu_5 \\
\mu_6 \\
\mu_7 \\
\mu_8 \\
\end{pmatrix}
= \mu = \mu_0,
\]

for the system corresponding to 2 experiments. In the inhomogeneity, we have the quantities $\delta_i = H(0,0,0,0)$ for $i = 1, 2$.

Note that the superscript in brackets denotes the total number of experiments, not the single equations for the experiments. Similarly, we also write $\mu_0(\delta_t, \delta u_1, \delta u_2) = \mu_0(\delta_t, \delta u_1, \delta u_2)$.

Comparison of $\mu$ with $\mu^{(2)}$ in table 2 shows the effect of adding one more experiment in the system: there are 6 new equations and 3 new variables: together 12 equations and 7 unknowns.

The shown procedure of adding experiments can be applied to any of the operators $A$ previously introduced. For each experiment we add, the inequality $M \geq m$ in the formalism of section 3 is fulfilled and the system is redundant. We summarize these operators also in table 2. The reason why we study precisely these operators with several measurements will be apparent from the results in the next section 4.2.

Note that in this section, we have given the general form of the linearization operators for the dynamic case on a cylindrical domain $\Omega \times T$. For (quasi-)static elastography, we will consider these operators in the stationary case with

\[
u_{\tau} = (\delta u)_{\tau} = 0,
\]

using only the spatial domain $\Omega$. This is specially indicated in each case.

### 4.2. Ellipticity

We want to apply the methodology of section 3, and use the criterion in theorem 1. Therefore, we have to determine the ellipticity condition in definition 1 for the operators $L_p, L_{\mu}, L_{\rho}$ in table 1. We first determine the principal symbol and possible characteristic directions for the

<table>
<thead>
<tr>
<th>Operator name</th>
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<th># Eqn. in $\Omega$</th>
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<tbody>
<tr>
<td>$A_p$</td>
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<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$A_{p\mu}$</td>
<td>$L_p(\delta \mu, \delta \mu) \times B(\delta \mu)$</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$A_{p\rho}$</td>
<td>$L_p(\delta \rho, \delta \mu) \times B(\delta \rho)$</td>
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<tr>
<td>$A_{\mu}^{(2)}$</td>
<td>$L_{\mu}^{(2)}(\delta \mu, \delta u_1, \delta u_2) \times B(\delta u_1) \times B(\delta u_2)$</td>
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<td>7</td>
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<tr>
<td>$A_{\rho}^{(r)}$</td>
<td>$L_{\rho}^{(r)} \times B \times \ldots \times B$</td>
<td>$r$ times</td>
<td>$4r + 1$</td>
</tr>
</tbody>
</table>

Table 2. Summary of the operators used to study the redundant systems (9) resp. (18); specification of the number of equations in the interior and the number of unknowns; due to space, the arguments of the operator $A_{p\mu}^{(r)}$ have not been given explicitly.
operator $L_{p\rho \mu}$, which is treated in proposition 1. From this analysis, we then draw some corollaries concerning the ellipticity of $L_{p\mu}$, $L_{\rho\mu}$, as well as ellipticity of $L_{p\rho}$.

For the analysis of $L_{p\rho \mu}$, we choose Douglis–Nirenberg numbers $(t_{ij})_{i,j=1}^6 = (1, 1, 0, 2, 2, 2)$ corresponding to the variables $(\delta p, \delta \mu, \delta \rho, \delta u_1, \delta u_2, \delta u_3)$ and numbers $(s_{ij})_{i,j=1}^6 = (0, 0, 0, -2, -2, -2)$ corresponding to the six equations. If there are fewer variables in the system (as in $L_{p\rho}$, $L_{\rho\mu}$, $L_{p\rho}$), then only the corresponding subset of Douglis–Nirenberg numbers is used (e.g., for the analysis of $L_{p\rho}$, we have the numbers $(1, 2, 2, 2)$ for the variables $\delta \rho, \delta u_2, \delta u_3$ in $L_{p\rho}$).—We will see below in proposition 1 that this choice of the numbers $s_{ij}$ equals exactly the order of $L_{ij}$; therefore, it is ensured that we treat the Douglis–Nirenberg symbol $L_0$ with the maximum number of entries (see also definition 1).

**Proposition 1.** Let $L$ be the operator $L_{p\rho \mu}$ in table 1.

(a) The principal symbol of $L_{p\rho \mu}$ (in the dynamic case) is

$$
L_0(x, t, \xi) = \begin{pmatrix}
1 \xi_1 & 2 \xi_2 & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_1^2 & -\mu |\xi|^2 + \rho \xi_1^2 & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_2^2 & \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_3^2 & \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \cdot \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

where $\xi_i := (\xi_1, \xi_2, \xi_3)$ for $\xi \in \mathbb{R}^4$.

(b) For every point $(x, t)$, the operator $L_{p\rho \mu}$ is not overdetermined elliptic; also, the operator $L_{p\rho \mu}^{(r)}$ corresponding to $r$ experiments is not overdetermined elliptic.

(c) In the stationary case (29), we have $L_{p\rho \mu} = L_{pp}$, and the principal symbol is

$$
L_0(x, 1 \xi) = \begin{pmatrix}
1 \xi_1 & 2 \xi_2 & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_1^2 & -\mu |\xi|^2 & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_2^2 & \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \cdot \\
-\mu \xi_3^2 & \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \cdot \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

for $\xi \in \mathbb{R}^3$.

(d) For every point $x$, consider $L = L_{p\rho \mu} = L_{pp}$ in the stationary case (29). Then $L$ is not overdetermined elliptic at $x$.

(e) For every point $x$, consider the operator $L = L_{p\rho \mu}^{(r)}$ corresponding to $r$ measurements. Then $L$ is also not overdetermined elliptic at $x$. 
Proof. (a) To compute the principal symbol (30), we refer to the definition of $L_{pup}$ in table 1.

We write the first three equations in $L_{pup}(\delta \rho, \delta \mu, \delta \rho, \delta u) = S$ as
\[
\partial_i \delta \rho + 2 \nabla \cdot (\delta \mu \epsilon(u)) + 2 \nabla \cdot (\mu \epsilon(\delta u)) - \delta \rho(u_i)_H - \rho(\delta u_i)_H = 0,
\]
for $i = 1, 2, 3$ (32)

where we denote the columns of the (symmetric) strain as $\epsilon(u) = \frac{1}{2} (\nabla u + \nabla u^T) = (\epsilon(u)_1, \epsilon(u)_2, \epsilon(u)_3).

The meaning of $\epsilon(\delta u)$, is analogous.

The last three equations in $L_{pup}(\delta \rho, \delta \mu, \delta \rho, \delta u) = S$ are written as
\[
\delta u_i = \delta H_i \quad i = 1, 2, 3.
\]

We now want to determine the entries $L_{pup}$ of the principal symbol $L_0((x, t), \xi)$. Note that each of the columns in $L_0$ exactly corresponds to one of the variables $(\delta \rho, \delta \mu, \delta \rho, \delta u_1, \delta u_2, \delta u_3)$. Starting with $j = 1$, we go through the variable list until $j = 6$.

For each variable, we determine the term where the unknown appears in the equations (32) resp. in (33). Then we choose that component of the term which has order $\mu^j$ and $x_j$. We substitute $L_j$ for $\frac{d}{\partial x_j}$ in that component gives the corresponding entries $(L_{0,0})_j^0 = 1$ in the $j$th column of $L_0$.

For the first column corresponding to $\partial \rho$, the term $\partial_i \delta \rho$ in (32) is translated to $(1 \xi_1, 1 \xi_2, 1 \xi_3)$ in $(L_{0,0})_1^0$. The second column corresponding to $\partial \mu$, and the summand of highest order in the term $2 \nabla \cdot (\delta \mu \epsilon(u))$ translates to $2 \xi_\mu \cdot \epsilon(u)$ in $(L_{0,0})_2^0$ with $\xi_\mu = (\xi_1, \xi_2, \xi_3)$. In the third column corresponding to $\partial \rho$, no differentiation occurs, so we just have $-u_j$ as entries in $(L_{0,0})_3^0$.

The last three columns correspond to the variables $(\delta u_1, \delta u_2, \delta u_3)$. The relevant terms in (32) are

\[
2 \nabla \cdot (\mu \epsilon(\delta u)) - \rho(\delta u_i)_H = \nabla \cdot \left( \mu \begin{pmatrix} \partial_1 \delta u_1 \\ \partial_2 \delta u_2 \\ \partial_3 \delta u_3 \end{pmatrix} \right) + \nabla \cdot \left( \mu \begin{pmatrix} \partial_1 \delta u_1 \\ \partial_2 \delta u_2 \\ \partial_3 \delta u_3 \end{pmatrix} \right) - \rho(\delta u_i)_H.
\]

We substitute $1 \xi$ for differentiation in (34) and take the terms of highest order to find the entries of the columns $(L_0)_j^i, j = 4, 5, 6$; these are the terms $-\mu \xi_i^2 \xi_j - \mu \xi_i^2 \xi_j^2$ in $(L_0)_i^j, i = 1, 2, 3$, and the term $-\mu \xi_i^2 \xi_j$ in the entries $(L_0)_j^i, j = 4, 5, 6, i \neq j$. The last three equations (33) contain no derivatives and give rise to the identity matrix in the entries $(L_0)_{ij}, 4 \leq i, j \leq 6$ of the principal symbol.

(b) For $L_{pup}$, we have the characteristic direction $\xi = (0, 0, 0, \xi_4), \xi_4 \neq 0$. The same is true of $L_{pup}^{0,0}$.

(c) The calculations for the symbol (31) in the stationary case are exactly the same as for the dynamic case. The only change in this case is that there is no temporal derivative. Therefore there is no variable $\delta \rho$, and one column less than in (30), and $\xi_4$ can be set to zero everywhere.

(d) We first observe that in (31), three of the columns are clearly linearly independent.

The first two columns, though, can be linearly dependent.

Consider the symmetric strain $\epsilon = \epsilon^T$. As the entries are real, there always exist an eigenvector $v$ such that

12
\[ \varepsilon \ast \mathbf{v} = (\varepsilon_1 \cdot \mathbf{v}, \varepsilon_2 \cdot \mathbf{v}, \varepsilon_3 \cdot \mathbf{v}) = \kappa \mathbf{v}, \]  
\[(35)\]

where \(\ast\) denotes matrix multiplication. Choosing \((\xi_1, \xi_2, \xi_3) = \mathbf{v}\) gives linear dependence of \(L_0(1\xi)\) in the first two columns.

In conclusion, at each point \((\mathbf{x}, t)\) resp. \(\mathbf{x}\), there are choices of \(\xi\) such that the symbols in (31) do not have full rank. Therefore \(L_{\mu\rho}\) is not overdetermined elliptic.

(e) Let \(L = L_{\mu\rho}^{(r)}\) be the operator corresponding to \(r\) measurements \((\mathbf{u^1}, ..., \mathbf{u^r})\). Organising the unknowns in the order \(\delta_{\mu} \delta_{\rho_1} \delta_{\rho_2} \delta_{\rho_3} ...\) we get the following \(6r \times 4r + 1\) matrix as symbol:

\[
L_0(\mathbf{x}, \xi) = \begin{pmatrix}
2\varepsilon(\mathbf{u_1})^n \xi & \xi & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \mathbf{Id}_{3\times 3} & 0 & 0 & \ldots & 0 & 0 \\
2\varepsilon(\mathbf{u_2})^n \xi & 0 & 0 & \xi & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{Id}_{3\times 3} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2\varepsilon(\mathbf{u_r})^n \xi & 0 & 0 & 0 & 0 & \ldots & \xi & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Id}_{3\times 3}
\end{pmatrix}
\]  
\[(36)\]

Note that each experiment contributes 6 rows to \(L_0\)—ellipticity of \(L\) is equivalent to the \(4r + 1\) columns being linearly independent.

First, the blocks with \(\mathbf{Id}_{3\times 3}\) in the matrix immediately yield that \(3r\) columns of the matrix are a linearly independent set. Also, these columns are independent of the remaining \(r + 1\) columns corresponding to \((\delta_{\mu}, \delta_{\rho_1}, ..., \delta_{\rho_r})\).

To investigate the ellipticity of \(L_{\mu\rho}^{(r)}\), we therefore build sub-matrices \(D = D(\mathbf{x}, 1\xi)\) of size \(r + 1 \times r + 1\) in the following way: We take the indexes of those remaining \(r + 1\) columns and make any choice \(C\) of \(r + 1\) row indexes. Then we analyze whether the rank of such submatrices \(D\) can be maximal.

**Case 1:** in the index choice \(C\), there are 3 consecutive entries \((6(j_0 - 1) + k)_{j=1,2,3}\), corresponding to one single experiment \(j_0\). Taking \(\xi\) an eigenvector of \(\varepsilon(\mathbf{u_{j_0}})\) yields linear dependence of the columns \(1\) and \(j_0 + 1\) in \(D(\mathbf{x}, 1\xi)\).

**Case 2:** there exists an experiment \(j_0\) not represented in \(C\). In this case, the column \(j_0 + 1\) of \(D\) is degenerate.

**Case 3:** of each experiment, there is one row in our choice \(C\), and from one experiment, there are two rows in \(C\). Without loss of generality, assume that it is the last experiment of which two rows have been chosen. Then we have that \(C = (k_1, 6 + k_2, ..., 6(r - 1) + k_r, 6(r - 1) + l), (k_1, ..., k_r, l) \in \{1, 2, 3\}, l \neq k_r\)

In this case, the resulting submatrix \(D_l\) is

\[
D_l(\mathbf{x}, 1\xi) = \begin{pmatrix}
2\varepsilon(\mathbf{u_1})_{k_1} \cdot \xi & \xi & 1\xi_{k_1} & 0 & \ldots & 0 \\
2\varepsilon(\mathbf{u_2})_{k_2} \cdot \xi & 0 & 1\xi_{k_2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2\varepsilon(\mathbf{u_r})_{k_r} \cdot \xi & 0 & 0 & \ldots & 1\xi_{k_r} \\
2\varepsilon(\mathbf{u_l})_{l} \cdot \xi & 0 & 0 & \ldots & 1\xi_{l}
\end{pmatrix}
\]  
\[(37)\]

We now take an eigenvector \(\xi \neq 0\) of \(\varepsilon(\mathbf{u_i})\) and claim that the matrix \(D_l(\mathbf{x}, 1\xi)\) does not have full rank.
Inductively, we begin with $r = 1$. With $\kappa$ being the corresponding eigenvalue, we have that
\[
\begin{pmatrix}
21 \xi_1 \\
21 \xi_2 \\
21 \xi_3
\end{pmatrix}
\begin{pmatrix}
1 \xi_1 \\
1 \xi_2 \\
1 \xi_3
\end{pmatrix} = \begin{pmatrix}
1 \xi_1 \\
1 \xi_2 \\
1 \xi_3
\end{pmatrix}
\begin{pmatrix}
\kappa \\
\kappa \\
\kappa
\end{pmatrix}
\text{is rank-deficient.}
\]

Now let $\det D_{r-1}(x, 1\xi) = 0$ for $r - 1$. Choosing an experiment $j_0 \neq r$, and evaluating the Laplace expansion of the determinant of (37) along the column $j_0 + 1$, the only nontrivial cofactor resulting equals $i\xi_1 \cdot \det D_{r-1}$ (where $D_{r-1}$ is the matrix corresponding to experiments $(u_1, \ldots, u_{j_0-1}, u_{j_0+1}, \ldots, u_r)$). But the induction assumption tells us that this expression vanishes. Therefore, with our choice of $\xi$, the matrix $D_r(x, 1\xi)$ has a rank-deficiency for any $r \geq 1$.

The cases 1-3 exhaust all possibilities of forming the relevant sub-matrices $D$ of $L_0$. In the proof, we have seen that neither of them has full rank. Therefore, $L = L^{(r)}_\mu$ is not over-determined elliptic.

We now restrict the focus on linearizations in only one direction, which were introduced in section 4.1. Then the corresponding principal symbol contains fewer columns and results on ellipticity can be obtained.

**Corollary 1.** The operator $L_p$ in table 1, considered in the stationary case, is elliptic everywhere.

**Proof.** Let $L = L_p$. The principal symbol consists of the first and the three last columns of the matrix in (31):
\[
L_0(x, 1\xi) = \begin{pmatrix}
1 \xi_1 \\
1 \xi_2 \\
1 \xi_3
\end{pmatrix} \ast Id_{3 \times 3}.
\]

For $\xi \neq 0$, this symbol has maximal rank 4 everywhere, therefore $L_p$ is elliptic.

**Corollary 2.** The operator $L_\mu$ in table 1, considered in the stationary case, is elliptic exactly at points which satisfy
\[
\det(\epsilon(u(x, t))) \neq 0.
\]

The operator $L^{(r)}_\mu$, corresponding to $r$ measurements $(\delta u)_k, 1 \leq k \leq r$, is elliptic exactly at points where at least one of $\det(\epsilon(u_k(x, t))) \neq 0$.

**Proof.** Let $L = L_\mu$ (the case of one measurement). Then the principal symbol consists of the second and the three last columns of the matrix in (31):
\[
L_0(x, 1\xi) = \begin{pmatrix}
1 \xi_1 \cdot \epsilon(u_1) \\
1 \xi_2 \cdot \epsilon(u_2) \\
1 \xi_3 \cdot \epsilon(u_3)
\end{pmatrix} \ast Id_{3 \times 3}.
\]

This symbol has rank 4 provided that the first column is non-degenerate, which is equivalent to the condition (39) of non-singular strain.

Now let $L = L^{(r)}_\mu$ (the case of multiple measurements). This can be treated by induction on $r$. Let the principal symbol $L^{(r-1)}_\mu$ corresponding to $r - 1$ measurements have dimension $a \times b$. Adding one measurement means adding three more columns (and six new lines).
in the matrix, such that it has dimension \((a + 6) \times (b + 3)\). These three new columns are independent because of the identity component in \((L_{0,i})\), \(a + 4 \leq i \leq a + 6\), \(b + 1 \leq j \leq b + 3\).

In the first column of \(L^{(1)}_{0}\), there are three new entries \((21 \xi_i \cdot e_i \xi_i)_{i=1}^{3r, r-2}\). If we have that for \(1 \leq k \leq r\), at least one of \(\det(r \xi_i(x, t)) \neq 0\) is non-zero, then the principal symbol \(L^{(1)}_{0}\) has full rank: in the case \(k < r\) because of the induction assumption, and in the case \(k = r\) because of non-degeneracy in the first column due to the new measurement.

**Corollary 3.** The operator \(L_{\rho}\) in table 1 is elliptic exactly at points with \(u_{\rho} \neq 0\). For the case of several measurements, \(L^{(r)}_{\rho}\) is elliptic exactly at points \((x, t)\) where at least one of \((u_{\rho})_k(x, t) \neq 0\), \(1 \leq k \leq r\).

**Proof.** Let \(L = L_{\rho}\). The principal symbol consists of the last four columns of the matrix in (30):

\[
L_{0}((x, t), 1\xi) = \begin{pmatrix}
-(u_{1})_t \\
-(u_{2})_t \\
-(u_{3})_t \\
\text{Id}_{3x3}
\end{pmatrix}.
\]  

This symbol has rank 4 iff \(u_{\rho}\) is nonzero. The statement for multiple measurements is proved by induction, analogous to the case of \(L^{(1)}_{\rho}\). □

**Remark 1.** As starting point of our analysis, we have chosen the modified model (7) with the substitution \(p = \lambda \nabla \cdot u\). Of course, we could also analyze the system corresponding to the original model (6). Using linearization in direction of \(\delta \lambda\), we have the operator \(L_{\lambda}\) in (25).

Now let \(L = L_{\lambda}\) in the stationary case. Doing calculations as in proposition 1, and restricting the focus on the first and last three columns as in corollary 1, one finds the resulting principal symbol as

\[
L_{0}(x, 1\xi) = \begin{pmatrix}
1\xi_1 \nabla \cdot u \\
1\xi_2 \nabla \cdot u \\
1\xi_3 \nabla \cdot u \\
\text{Id}_{3x3}
\end{pmatrix}.
\]  

This symbol may seem to be degenerate in many cases: In fact, the literature often assumes the incompressibility condition \(\nabla \cdot u = 0\) on the whole of \(\Omega\) [6, 14, 30, 48]. The symbol then degenerates. (In this case, of course, the measurement data are not dependent on \(\lambda\), so this parameter cannot be reconstructed anyway.)

But in the compressible case, where \(\nabla \cdot u \neq 0\) on the whole of \(\Omega\), there still might be single points \(x\) at which \(\nabla \cdot u(x) = 0\). The ellipticity analysis then entails that at such points \(x\), ellipticity is lost and every direction is a characteristic.—Concerning the particular data one has, it might then be better to reconstruct the pressure \(p = \lambda \nabla \cdot u\), with the operator \(L_{\rho}\) being always elliptic.

**Remark 2.** Similarly for \(\rho\): If \(u_{\rho} = 0\) on the whole of \(\Omega\), the parameter \(\rho\) does not appear in the model, so it cannot be reconstructed from the measurements. If, on the other hand,
\( u_\rho(x) = 0 \) only for particular points \( x \), the analysis says that ellipticity is lost at these points \( x \), and every direction is a characteristic for \( \mathcal{A}_\rho \), there.

Remark 3. Corollary 2 gives the criterion (39) for the reference state (which will be used later in theorem 2). This criterion means that at each point, at least one of the measured elastic displacement fields has non-singular strain. The requirement of such qualitative conditions for the solutions is typical for the coupled-physics literature. In fact, the condition (39) for \( r = 2 \) is a generalization of the invertibility condition for the nonlinear reconstruction problem in elastography, which was found in the research of [10], namely

\[
\det(\tau \epsilon(u_1) - \tau \epsilon(u_2)) \neq 0. \tag{43}
\]

Here, we have for \( k = 1, 2 \) that \( \tau_k := \text{tr}(\epsilon(u_k)) = V \cdot u_k \). It can be verified by simple calculation that violation of (39) for both \( u = u_1 \) and \( u = u_2 \) leads to violation of (43).

It is unknown whether (43) can be ensured with two vector fields for every distribution of material parameters. For several parameter classes, existence of boundary conditions ensuring (43) can be justified, see the discussion and examples in [10 section 3.3]. As (48) is a consequence of (43), the special argumentation for (43) can also be invoked for arguing for the premise of corollary 2 (and later theorem 2) in our case.

Apart from the characterization in [10], points of singular strain, violating (39) have been found in experiments, namely at the intersection of nodal lines or surfaces in early experiments of elastography using eigenmodes (see [46, 47, 55]). Empirically, it was observed that these patterns could be avoided by choosing multi-frequency excitation functions \( F \) [47].

4.3. Lopatinskii condition

We want to use the stability criterion in theorem 1 for the systems \( \mathcal{A}_p, \mathcal{A}_\mu \) and \( \mathcal{A}_\rho \) in table 2. For this purpose, we are checking the covering condition in definition 2 for the various differential operators \( \mathcal{L} \) introduced in section 4.1, together with the relevant boundary data where necessary.

Proposition 2. The systems \( \mathcal{L}_p \) and \( \mathcal{L}_\mu \) in table 1 satisfy the Lopatinskii condition with arbitrary boundary data.

**Proof.** For checking the Lopatinskii (or covering) condition in definition 2, we have to consider the vector space of functions satisfying the system (13) and show that it is trivial.

Let \( \mathcal{L} = \mathcal{L}_\mu \) with principal symbol (38), and let \( y \) be a point on the boundary. Then the system of equations \( \mathcal{L}_0(y, 1\xi + \nu \frac{d}{dz})\tilde{w}(z) = 0 \) in (13) comprises 6 equations for 4 unknowns \( (\tilde{w}_1(z), \tilde{w}_2(z), \tilde{w}_3(z), \tilde{w}_d(z)) \).

In the entries \( (L_0), 4 \leq i \leq 6, 2 \leq j \leq 4 \) of (38), there is a 3-by-3 identity matrix. The three last equations of (13) therefore mean that

\[
\tilde{w}_2(z) = \tilde{w}_3(z) = \tilde{w}_4(z) = 0. \tag{44}
\]

The first three equations in (13) then reduce to

\[
\left(1\xi_k + \nu_k \frac{d}{dz}\right)\tilde{w}_1(z) = 0, \quad k = 1, 2, 3. \tag{45}
\]

Any functions which are in the vector space considered in definition 2 have to satisfy (45).
The only possible solutions of (45) consist of functions of form $e^{i\lambda z}$, where the parameter $\lambda = \frac{\nu}{\nu_i}$ is a real number; neither of these functions tends to 0 for $z \to \infty$. In this argument, we did not use any boundary constraint. Therefore the vector space to be considered in definition 2 is trivial for every $y \in \partial \Omega$, and the Lopatinskii covering condition is always satisfied.

Now let $L = \mathcal{L}_\mu$ with principal symbol (40). Then, for $y$ on the boundary, consider the system $\mathcal{L}_0(y, 1\xi + \nu \frac{d}{dz}) \tilde{w}(z) = 0$ in (13). Similarly to (44), the last three components vanish, and the system reduces to

$$
\begin{align*}
g_j \cdot \left( 1\xi_j + \nu \frac{d}{dz} \right) \tilde{w}_1(z) &= 0, \\
& \quad j = 1, 2, 3.
\end{align*}
$$

(46)

Here, we use fixed vectors $g_j = \nu(\nu(y))_j$, $j = 1, 2, 3$ and $\xi_j = (\xi_1, \xi_2, \xi_3)$.

Suppose that the relation $\nu \neq 0$ holds for all $\nu$. Then we have solutions of (46) of form $e^{i\frac{\nu}{\nu_i}z}$. The numbers $\frac{g_j}{\nu_j}$ are real, so neither of these functions tends to 0.

Suppose, on the other hand, that we have $\nu_j = 0$ for one $j_0$. Then, using $\nu \cdot \xi_j = 0$, we have $g_{j_0} \cdot \xi_j \neq 0$. Inserting this information in (46), we directly get $\tilde{w}_1(z) = 0$.

Therefore the Lopatinskii condition for $\mathcal{L}_\mu$ is satisfied with arbitrary boundary data. □

**Proposition 3.** The system $\mathcal{L}_\mu$ in table 1 satisfies the Lopatinskii condition with arbitrary boundary data if and only if $u(y, t_\mu) \neq 0$ for all $(y, t) \in \partial(\Omega \times T)$.

**Proof.** Consider the system $\mathcal{L}_0(y, 1\xi + \nu \frac{d}{dz}) \tilde{w}(z) = 0$ in (13) for the operator $\mathcal{L}_\mu$. As in the proof of proposition 2, we get

$$
\tilde{w}_2(z) = \tilde{w}_1(z) = \tilde{w}_4(z) = 0.
$$

The equations for $\tilde{w}_1(z)$ therefore reduce to

$$
(\nu_i)_i \tilde{w}_1(z) = 0 \quad \text{for } i = 1, 2, 3
$$

(47)

If we suppose $u(y, t_\mu) \neq 0$, then (47) implies $\tilde{w}_1(z) = 0$, therefore the Lopatinskii condition is satisfied.

Conversely, suppose that $u(y, t_\mu) = 0$ for one $(y, t)$. Then one can choose $\tilde{w}(z) = (w_1(z), 0, 0, 0)$ satisfying (47) for any function $w_1$ that satisfies $w_1(z) \to 0$. Therefore the Lopatinskii condition would be violated. □

**4.4. Stability estimates and kernel characterization**

In this section, we derive the main results of the paper. In sections 4.2 and 4.3, we collected results about the systems $\mathcal{A}_\rho$, $\mathcal{A}_\mu$, and $\mathcal{A}_\nu$, in table 2. We now treat these operators on a case-by-case basis. For each operator, we give a stability estimate. In general, this estimate holds up to a finite dimensional kernel for operators with one experiment. We characterize this kernel, and show also injectivity of the operators for a larger number of measurements in the linearized problem in quantitative elastography.

We choose $p = 2$ and a number $l > 0$ with $p l > n$, and a bounded and connected domain $\Omega$ with $C^{l+2}$-boundary.

The theory of [53], which we apply, requires that the solution variables lie in Sobolev spaces. In the following statements, we therefore suppose that we have a reference state $(p, \mu, \rho)$ with $\mu \in C^{2+l+3}(\Omega)$, $\rho \in C^{2+l+2}(\Omega)$ and for the reference displacement...
\(\mathbf{u} = \mathcal{V}(p, \mu, \rho)\) in (22) we require \(\mathbf{u} \in H^{l+2}(\Omega), \ \mathbf{u}_n \in H^l(\Omega)\). The existence of such a displacement field \(\mathbf{u}\) can be ensured by the regularity theory [35 thm.8.1]. Moreover, we consider the solutions of the linearized equations (23) as follows: \(\delta \rho \in H^{l+1}(\Omega), \ \delta \mu \in H^{l+1}(\Omega), \ \delta \phi \in H^l(\Omega), \ \delta u_i \in H^{l+2}(\Omega)\) for \(i = 1, 2, 3\).

Note that, as exposed in section 4.1, the variables of the relevant systems are \(\delta \mu, \delta \rho, \delta \phi, \delta u, \delta u_2, \delta u_3\). For subsequent analysis, we use the Douglis–Nirenberg numbers \((t_j)^{\mu}_m = (1, 1, 0, 2, 2, 2)\), corresponding to the variables \(\delta \mu, \delta \rho, \delta \phi, \delta u, \delta u_2, \delta u_3\), as well as \((s_j)^{\mu}_m = (0, 0, 0, -2, -2, -2)\).—Note that for the particular operators \(\mathcal{A}_\mu, \mathcal{A}_\rho, \mathcal{A}_\phi, \mathcal{A}_x\); only four of those variables are used in the corresponding system.

In the following results, we apply theorem 1 to obtain a left regularizer the particular operators \(\mathcal{A}\), as in (19). Another name for \(\mathcal{A}\) possessing a left regularizer are that \(\mathcal{A}\) is a left semi-Fredholm operator [17 ch. XI], [22].

**Theorem 2.** The operator \(\mathcal{A}_\mu^{(r)}\) for \(r \geq 1\) measurements, described in (28) and considered in the stationary case, has a left regularizer on \(\Omega\) precisely when, for all \(\Omega \in \Omega\),

\[
\det (\mathbf{e} (\mathbf{u}_k(x))) \neq 0 \quad \text{for at least one measurement } k \leq r. \quad (48)
\]

Then the stability estimate

\[
\| \delta \mu \|_{H^{l+1}(\Omega)/K} \leq C \sum_{k=1}^{r} \| \delta \mathbf{u}_k \|_{H^{l+2}(\Omega)} \quad (49)
\]

holds with a finite-dimensional kernel \(K_f\).

**Proof.** We first treat the case of \(\mathcal{A}_\mu\).

Recall that we are treating the equation

\[
\mathcal{A}_\mu (\delta \mu, \delta \mathbf{u}) = (\mathbf{S}, \phi),
\]

and that the operator

\[
\mathcal{A}_\mu = \mathcal{L}_\mu \times \mathcal{B} : D(p, l) \rightarrow R(p, l)
\]

as described in table 2, section 4.1, has six equations in the interior and four variables \((\delta \mu, \delta u_1, \delta u_2, \delta u_3) = (\delta \mu, \delta \mathbf{u})\). The inhomogeneity is \(\mathbf{S} = (0, 0, 0, \delta K_1, \delta K_2, \delta K_3)^T\) and \(\phi = 0\).

The choice of Douglis–Nirenberg numbers corresponding to these variables is \((t_j)^{\mu}_m = (1, 2, 2, 2)\) and \((s_j)^{\mu}_m = (0, 0, 0, -2, -2, -2)\), \((\sigma_j)^{\mu}_m = (0, 0, 0, -2, -2, -1)\), and the principal symbol of \(\mathcal{L}_\mu\) is (40). Therefore, we have (according to (16) and (17)) the domain and range, respectively, as

\[
D(p, l) = H^{l+1}(\Omega) \times \left( H^{l+2}(\Omega) \right)^3
\]

\[
R(p, l) = H^l(\Omega) \times \left( H^{l+2}(\Omega) \right)^3 \times H^{l+\frac{2}{2}}(\Omega) \times H^{l+\frac{2}{2}}(\Omega) \times H^{l+\frac{2}{2}}(\Omega).
\]

By corollary 2, the assumption (48) on the determinant of the strain \(\mathbf{e}(\mathbf{u})\) of the reference state, is equivalent to \(\mathcal{L}_\mu\) being elliptic. According to proposition 2, the Lopatinskii condition is satisfied for \(\mathcal{A}_\mu = \mathcal{L}_\mu \times \mathcal{B}\). Therefore, condition 1 of theorem 1 holds precisely when (48) holds, and this is true also for the two equivalent conditions 2–3 in theorem 1.

Theorem 1 therefore implies the existence of a bounded operator \(R_\mu : R(p, l) \rightarrow D(p, l)\) with
and compact $D_p: R(p, l) \rightarrow D(p, l)$.

The spectral theory for compact operators asserts that the kernel of $I - D_p$ is finite-dimensional [17 ch.VII]. By (50), the kernel $K = \ker(A_\mu)$ is a subspace of $\ker(I - D_p)$. Consequently, $K$ is finite-dimensional also, and hence closed. Therefore we can consider the quotient $D(p, l)/K$ as a Hilbert space.

Existence of the left regularizer in (50) implies that $\mathcal{A}_\mu$ is closed [17 ch.XI]. Therefore $\mathcal{A}_\mu$ is a Hilbert space. We apply the open mapping theorem [17 ch.III] to find that the inverse $\mathcal{A}_\mu^{-1}: \text{ran}(A_\mu) \rightarrow D(p, l)/K$ is continuous. By the third condition in theorem 1, we therefore have the existence of a real number $C$ such that for all $(\delta\mu, \delta u) \in D(p, l)$, the estimate

$$
\| (\delta\mu, \delta u) \|_{D(p,l)/K} = \| \mathcal{A}_\mu^{-1} \mathcal{A}_\mu (\delta\mu, \delta u) \|_{D(p,l)/K} \leq C \| \mathcal{A}_\mu (\delta\mu, \delta u) \|_{R(p,l)} = C \| (S, \varphi) \|_{R(p,l)} .
$$

holds. By (23), we have $\delta H = \delta u$ in $S$. We set $K = K_1 \times K_2$ with $K_1 \subset H^{l+1}(\Omega)$, $K_2 \subset (H^{l+2}(\Omega))^3$. Then from (51), we obtain the estimate (49) for $r = 1$.

The case of $\mathcal{A}_\mu^{(r)}$ follows straight-forwardly by induction.

The availability of reference states satisfying (48) for $r = 2$ has already been discussed in remark 3.

Note that the estimate (49) shows that there is the loss of only one derivative in the linearized problem of reconstructing $\delta\mu$ from $\delta u$, thus this inverse problem is mildly ill-posed.

We now give an explicit characterization of the kernel of the operator $\mathcal{A}_\mu = \mathcal{A}_\mu^{(1)}$, which will be exploited in corollary 4 to show injectivity of $\mathcal{A}_\mu^{(2)}$, the operator corresponding to two measurements.

**Theorem 3.** Consider $\mathcal{A}_\mu$, and suppose that the condition (48) with $r = 1$ holds. Then the estimate (49) holds with a one-dimensional kernel $K_1$. The subspace $K_1$ is generated by the element

$$
\exp \int_p^x a(y) dy
$$

with fixed $p \in \Omega$. Here, the vector field $a(x)$ is uniquely determined by

$$
a \cdot \varepsilon (u)_i = -\nabla \cdot \varepsilon (u)_i , \quad i = 1, 2, 3 ,
$$

where $u$ is a reference state for which (48) holds.

**Proof.** In the proof of the statement, we derive a representation for $\delta\mu$ on a connected set and then infer that the representation is valid on $\Omega$ by a topological argument.

Suppose, to begin with, that $(\delta\mu, \delta u) \in D(p, l)$ is in the kernel of $\mathcal{A}_\mu = \mathcal{L}_\mu \times B$.

As we have $p l > n$, the Sobolev imbedding theorems (see [1 thm.5.4.C]) imply that $\delta\mu \in H^{l+1}(\Omega)$ is continuously differentiable on $\Omega$. In particular, we have that the set

$$
A = \{ x \in \Omega : \delta\mu (x) \neq 0 \}
$$

is open in $\Omega$.

If $\delta\mu \equiv 0$, then the assertion is trivially satisfied. Otherwise, there exists a point $p \in A$. In this case, consider the connected component $V$ of $p$ in the topology of $A \subset \Omega$, that is
\[ V = \bigcup \{ U; \, p \in U \subset A \text{ with } U \text{ connected in } A \}. \]

Lemma 1 implies that \( V \subset A \subset \mathbb{R}^n \) is open; therefore, \( V \) is also path-connected.

Suppose now \( x \in V \). We analyze the 6 equations
\[
\mathcal{L}_p(\delta \mu, \delta u) = \left( 2V \cdot (\delta \mu \cdot \varepsilon(u)) + 2V \cdot (\mu \cdot \varepsilon(\delta u)) \right) \delta u = 0 \quad \text{on } V.
\]
From the last three equations, we immediately get \( \delta u|_V = 0 \). From the first three equations, we then get that
\[ V \cdot (\delta \mu \cdot \varepsilon(u)) = 0 \quad \text{on } V \]
for the element \( \delta \mu \), and hence
\[ V \delta \mu \cdot \varepsilon(u)_i = -\delta \mu \cdot V \cdot \varepsilon(u)_i \quad \text{on } V, \text{ for } i = 1, 2, 3. \quad (55) \]
Evaluating (55) at the point \( x \in V \subset A \) and dividing by \( \delta \mu(x) \) (which, by (54), is non-zero) shows that
\[
\frac{V \delta \mu(x)}{\delta \mu(x)} = V \log \delta \mu(x) = a(x), \quad (56)
\]
with \( a \) determined by (53).

Actually, the conditions (48) and (53) can be used to define \( a(x) \) uniquely for \( x \in \Omega \) and that \( a \) is continuous. To see this, observe that from (53), we have that \( \varepsilon(u) \ast a = -[V \cdot \varepsilon(u)_1, V \cdot \varepsilon(u)_2, V \cdot \varepsilon(u)_3] \). With (48), we then have
\[
a = -\left(\varepsilon(u)^{-1}\right)[V \cdot \varepsilon(u)_1, V \cdot \varepsilon(u)_2, V \cdot \varepsilon(u)_3]. \quad (57)
\]
Now \( u \in (H^{s+2}(\Omega))^3 \) implies that the entries of \( \varepsilon(u) \) lie in \( H^{s+1}(\Omega) \) and that \( V \cdot \varepsilon(u)_i \in H^s(\Omega) \) for \( i = 1, 2, 3. \) Applying Cramer’s rule to (57) yields that \( a \in H^s(\Omega) \).
Using the inequality \( pl > n \) and the Sobolev embedding theorem [1 thm.5.4.C], we then conclude that \( a \) is continuous on \( \Omega \).

Now consider the vector field \( a \) and calculate the path integral from \( p \) to \( x \) to find
\[
\int_p^x a(y)dy \overset{(56)}{=} \int_p^x V \log \delta \mu(y)dy = \log \delta \mu(x) - \log \delta \mu(p) = \log \left( \frac{\delta \mu(x)}{\delta \mu(p)} \right).
\]
From this identity, we have the representation
\[
\delta \mu(x) = \delta \mu(p) \exp \int_p^x a(y)dy, \quad x \in V \quad (58)
\]
for the values \( \delta \mu \) on the set \( V \subset A \subset \Omega \). Note that the function on the right hand side of (58) is continuous and defined on the whole domain \( \Omega \).

We now claim that actually, we have
\[ V = \Omega, \quad (59) \]
such that the representation formula (58) holds for \( x \in \Omega \).
Assume, on the contrary, that \( V \subsetneq \Omega \). We then also have that
\[ A \subsetneq \Omega \]
(otherwise \( \Omega = A = V \cup V_i \), with \( V_i = A \setminus V \) open and nontrivial, \( V \cap V_i = \emptyset \), so \( \Omega \) would not be connected).

Therefore, the assumptions of lemma 2 are satisfied. Consequently, there exists a point \( q \in \partial V \setminus A \), where \( \partial V \) is the boundary of \( V \) in \( \Omega \). As \( q \not\in A \), we have, by (54), that
\[ \delta \mu(q) = 0. \quad (60) \]

As \( q \in \partial V \), there exists a sequence \( v_n \in V \) with

\[ v_n \to q \quad \text{in} \ \Omega. \quad (61) \]

By the representation (58), together with (61), we have

\[ \delta \mu(v_n) = \delta \mu(p) \exp \int_p^q a(y)dy \to \delta \mu(p) \exp \int_p^q a(y)dy \neq 0 \quad (62) \]

On the other hand, the continuity of \( \delta \mu \), equation (60) and (61) imply that

\[ \delta \mu(v_n) \to 0. \quad (63) \]

As (62) and (63) contradict each other, we infer that the assumption \( V \subset \subset \Omega \) is wrong. Therefore, as asserted in (59), \( V = \Omega \) holds.

Therefore, for any \( (\delta \mu, \delta u) \in \ker \mathcal{A}_\mu = K_1 \times \{0\} \) with \( \delta \mu \neq 0 \), the representation of \( \delta \mu \) in (58) is valid for \( x \in \Omega \). This shows that (52) is a generating element for \( K_1 \). Therefore \( \dim(K_1) = 1 \).

We note that the equation (55) and the analytical solution (58) has been found in [15] in a different context for the analysis of the nonlinear problem with constant coefficients \( \lambda \) and \( \rho \), using one measurement. — We now investigate the effect of adding a second measurement in the inverse problem:

**Corollary 4.** Let \( u_1 \neq u_2 \) be two quasi-static elastic deformations satisfying (7) and (8) with different force terms \( F_1, F_2 \). Let the condition (48) hold.

As described in (28), let \( \mathcal{A}^{(2)}_\mu(\delta \mu, \delta u_1, \delta u_2) \) be the corresponding linearized operator. Then we have that

\[ \ker(\mathcal{A}^{(2)}_\mu) = \{ (0, 0, 0) \}. \]

**Proof.** Let \( (\delta \mu, \delta u_1, \delta u_2) \in \ker(\mathcal{A}^{(2)}_\mu) \).

From (28), we immediately get that \( \delta u_1 = \delta u_2 = 0 \) on \( \Omega \). The other equations in (28) yield

\[ \nabla \cdot (\delta \mu \varepsilon(u_1)) = 0 \]
\[ \nabla \cdot (\delta \mu \varepsilon(u_2)) = 0. \]

such that together with the boundary data we have

\[ \nabla \cdot (\delta \mu \varepsilon(u_1 - u_2)) = 0 \]
\[ (u_1 - u_2)|_{\partial \Omega} = 0. \quad (64) \]

Suppose that \( \delta \mu \neq 0 \). Then there exists a point \( p \in \Omega \) with \( \delta \mu(p) \neq 0 \). As in the proof of theorem 3, where we derived the representation formula (58) for \( x \in \Omega \), there exists a continuous vector field \( a \) such that
\[ \delta \mu(x) = \delta \mu(p) \exp \int_{p}^{x} a(y) \, dy, \quad x \in \Omega. \] (65)

This implies that \( \delta \mu(x) > 0 \) for all \( x \in \Omega \). Therefore, the condition \( \text{ess inf}_{p} \delta \mu = \text{ess inf}_{p} \delta \mu_{0} > 0 \) in [35 (2.2)] is satisfied.

The uniqueness result [35 thm.5.2] then implies that, from (64), we have that \( u_{1} = u_{2} \). But this is contradiction to our assumption. \( \square \)

Therefore, we have \( \ker(A^{(2)}_{\rho}) = \{ (0, 0, 0) \} \).

In the subsequent part of the section we give the stability criteria for the operators \( A_{\rho} \), \( A_{\rho} \) and \( A_{\lambda} \).

**Theorem 4.** The operator \( A^{(r)}_{\rho} \) for \( r \) measurements has a left regularizer on any smooth subdomain \( W \subset \Omega \times [0, T] \) precisely when, for all \( (x, t) \in W \),

\[ (u_{k})_{t}(x, t) \neq 0 \quad \text{for at least one measurement} \quad 1 \leq k \leq r. \] (66)

One has the stability estimate

\[ \| \delta p \|_{H^{\gamma+1}(W)} \leq C \sum_{k=1}^{r} \| \delta u_{k} \|_{H^{\gamma+1}(W)}. \] (67)

**Proof.** We first treat \( A_{\rho} \). The case of \( A^{(r)}_{\rho} \) follows by induction.

The stability criterion in theorem 1 is established for domains with \( C^{l_{j}} \max j \) boundary. Upon careful checking of the proof [53, section 6], the only place where this assumption enters is the existence of a partition of unity. Now our domain is \( \Omega \times [0, T] \). The construction of a partition of unity easily generalizes to cylindrical domains \( \Omega \times [0, T] \), where \( \Omega \) has \( C^{l_{j}+\max j} \) boundary. Therefore, we can apply theorem 1 to the problems with cylindrical domains.

The ellipticity condition has been assured in corollary 3, and the Lopatinskii condition is satisfied according to proposition 3. The assumptions in these results give the requirement \( u_{j}(x, t) \neq 0 \). With that, the equivalent conditions of theorem 1 are fulfilled and we apply the result as in the proof of theorem 2.

There appears no kernel in (67) for the following reason: The Douglis–Nirenberg numbers for the operator \( A_{\rho} \) are \( (l_{j})_{j=1}^{4} = (0, 2, 2, 2) \) and \( (s_{j})_{j=1}^{6} = (0, 0, -2, -2, -2) \). On the right hand side of estimate (20), only the variables with \( t_{j} > 0 \) appear, which are in this case \( \delta u_{k} \) for \( k = 1, 2, 3 \).

**Theorem 5.** The operator \( A_{\rho} \) in table 2, considered in the stationary case, has a left regularizer on \( \Omega \), and we have the estimate

\[ \| \delta p \|_{H^{\gamma+1}(\Omega)/K_{3}} \leq C \| \delta u \|_{H^{\gamma+1}(\Omega)}. \] (68)

Here, the kernel \( K_{3} \) consists of the (one-dimensional) space of constant functions on \( \Omega \).

**Proof.** The proof of the stability estimate with a finite-dimensional kernel is the same as in theorem 2, using corollary 1 and proposition 2 in this case.

Suppose that \( (\delta \mu, \delta u) \) is in the kernel \( K = \ker(A_{\rho}) = L_{\rho} \times B \). Consideration of the system

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\[ \delta u = \frac{V \partial p + 2V \cdot (\mu \varepsilon(\delta u))}{\delta u} = 0 \text{ on } \Omega \]

shows that \( \delta u = 0 \), and consequently \( V \partial p = 0 \). Therefore, we have \( K = K_3 \times \{0\} \), with \( K_3 \) the constant functions on \( \Omega \). □

Note that, with the same method, but using remark 1, one obtains a conditional stability result for the operator \( A_\lambda \):

**Theorem 6.** The operator \( A_\lambda^{(r)} \) corresponding to \( r \) measurements, considered in the stationary case, has a left regularizer on \( \Omega \) precisely when, for all \( \Omega \in x \neq k \in r \),

\[ V \cdot u_k(x) \neq 0 \text{ for at least one measurement } 1 \leq k \leq r, \quad (69) \]

Then the stability estimate

\[ \| \delta \lambda \|_{\mathcal{H}^{s+1}(\Omega)/K_k} \leq C \sum_{k=1}^r \| \delta u_k \|_{\mathcal{H}^{s+1}(\Omega)} \quad (70) \]

holds for a finite-dimensional kernel \( K_\lambda \).

5. Conclusion

We have applied a general method of linear PDE to linearized problems in quantitative elastography in \( \mathbb{R}^3 \), with interior data given. We analyzed ellipticity conditions of the PDE problem augmented with the interior data. We deduced simple criteria for the stability of the linearization (see table 3). This analysis revealed non-ellipticity for the joint reconstruction of all parameters, but stable reconstruction of the shear modulus \( \mu \) and the hydrostatic pressure \( p = \lambda V \cdot u \). For the reconstruction of \( \mu \) and \( \rho \), the kernel in the linearization was shown to be trivial for choice of two measurements. The results give a mathematical explanation which biomechanical parameters can be stably reconstructed from interior measurement data \( u \).

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**Table 3.** Stabilites estimates for various linearized problems in elastography; conditions for the reference state in theorems 2–6 to hold for every point for at least one displacement field \( u_k \), \( 1 \leq k \leq r \) in an imaging experiment in elastography.

<table>
<thead>
<tr>
<th>Operator</th>
<th>( A^{(s)}_\mu )</th>
<th>( A^{(s)}_\rho )</th>
<th>( A^{(s)}_\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipticity condition</td>
<td>( \det(\varepsilon(u_k)) \neq 0 ) ( (u_k)_h \neq 0 )</td>
<td>( - V \cdot u_k \neq 0 )</td>
<td></td>
</tr>
</tbody>
</table>
Appendix A

We give here the proof of two topological lemmas which we use in the determination of the kernel in theorem 3.

**Lemma 1.** Let $A \subset \Omega$ be open, and let $p \in A$. Let $V$ be the connected component of $p$ in the topology of $A \subset \Omega$. Then $V$ is open in $\Omega$.

**Proof.** Let $x \in V \subset A$ be an arbitrary point in $V$. As $x \in A$ and $A$ is open, there exists an $\epsilon > 0$ such that $U_1 := \{z \in \Omega : |x - z| < \epsilon\} \subset A$.

Observe that the set $U_1$ is connected and $x \in V \cap U_1$. From [39 thm.23.3], it then follows that $V \cup U_1 \subset A$ is a connected set.

Among all subsets of $A$ which are connected and contain $p$, the component $V$ is maximal. Therefore $p \in V \cup U_1 = V$, or equivalently $U_1 \subset V$. This shows that $V$ is open in $\Omega$. $\square$

**Lemma 2.** Let $A \not\subset \Omega \subset \mathbb{R}^n$ be open and bounded, and let $p \in A$. Let $V$ be the connected component of $p$ in the topology of $A \subset \Omega$. Let $\partial V$ be the boundary of $V$ in the topology of $\Omega$. Then there exists a point $q \in \partial V \setminus A$.

**Proof.** We use lemma 1 and prove the statement in two steps: first, we find a point $q \in \partial V \setminus A$; second, we show that $q \notin A$.

**Claim 1:** there exists a point $q \in \partial V \setminus A$.

We have that $\Omega \setminus \not\subset V$. Therefore, there exists an element $y \in \Omega \setminus V$. Consider the mapping

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad v \mapsto |y - v|.$$ 

Observe that $v \subset \Omega$ is closed and bounded, hence a compact set; observe also that $f$ is continuous. Therefore, a minimum exists, that is:

$$\exists \ q \in \mathbb{R}^n : |y - q| = \min_{v \in \mathbb{R}^n} \{|y - v|\}. \quad (A.1)$$

We now show that, actually, the point $q \in V = \partial V \setminus \not\subset \Omega \setminus V$ is not contained in $V$. Once this is shown, claim 1 is proven.

Assume, on the contrary, that $q \in V$. According to lemma 1, we then would have an $\epsilon$, such that $U_2 := \{z : |q - z| < \epsilon\} \subset V$.

Without loss of generality, we can assume $\epsilon < 2$. Now, using the element $y$ defined above, set

$$w := q + \frac{\epsilon}{2}(y - q), \quad w \in U_2.$$ 

Then calculate

$$|y - w| = |y - q - \frac{\epsilon}{2}(y - q)| = |y - q|\left(1 - \frac{\epsilon}{2}\right)$$

$$\leq |y - q|\left(1 - \frac{\epsilon}{2}\right) < |y - q|.$$ 

This would contradict (A.1). Therefore, $q \notin V$. 

\[24\]
Claim 2: The point $q$ in (A.1) does not belong to $A$.

We prove this claim indirectly. Assume that $q \in A$.

Recall that, according to Claim 1, $q \in \partial V$, where $\partial V$ is the boundary of $V$ in $\Omega$. Hence there exists a sequence $v_n \in V$ with $v_n \to q$ in the topology of $\Omega$.

We assert that

$$v_n \to q$$

in the topology of $A$. (A.2)

To see this, choose an open set $U_1 \subset A$ with $q \in U_1$. Because $A$ is open in $\Omega$, $U_1$ is open in $\Omega$ as well. Now the elements $v_n$ converge to $q$ in $\Omega$; therefore, there exists an $N$, such that for all $n \geq N$: $v_n \in U_1$; hence we have (A.2).

The set $V$, which is the connected component of the point $p$, is closed in the topology of $A$ [39 thm. 23.4]. But a closed set contains all its limit points. Therefore, with (A.2), we would have that the limit of the sequence $v_n \in V$ lies in $V$, so $q \in V$. But this is a contradiction to claim 1.—Therefore, contrary to our assumption, we have $q \not\in A$.

References

[22] Krein S G 1982 Linear Equations in a Banach Space (Boston: Birkhäuser)
[26] Kuchment P and Steinhauer D 2012 Stabilizing inverse problems by internal data Inverse Problems 28 084007
[36] McLaughlin J, Zhang N and Manduca A 2010 Calculating tissue shear modulus and pressure by 2D log-elastographic methods Inverse Problems 26 085007
[37] Montalto C and Stefanov P 2013 Stability of coupled-physics inverse problems with one internal measurement Inverse Problems 29 125004
[59] Widlak T and Scherzer O 2012 Hybrid tomography for conductivity imaging Inverse Problems 28 084008
[61] Zhang Y, Brodell R T, Mostow E N, Vinyard C J and Marie H 2009 In vivo skin elastography with high-definition optical videos Skin Res. Technol. 15 271–82