A primal–dual hybrid gradient method for nonlinear operators with applications to MRI

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A primal–dual hybrid gradient method for nonlinear operators with applications to MRI

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Abstract
We study the solution of minimax problems min\(x\), max\(y\), \(G(x) + \langle K(x), y \rangle - F^*(y)\) in finite-dimensional Hilbert spaces. The functionals \(G\) and \(F^*\) we assume to be convex, but the operator \(K\) we allow to be nonlinear. We formulate a natural extension of the modified primal–dual hybrid gradient method, originally for linear \(K\), due to Chambolle and Pock. We prove the local convergence of the method, provided various technical conditions are satisfied. These include in particular the Aubin property of the inverse of a monotone operator at the solution. Of particular interest to us is the case arising from Tikhonov type regularization of inverse problems with nonlinear forward operators. Mainly we are interested in total variation and second-order total generalized variation priors. For such problems, we show that our general local convergence result holds when the noise level of the data \(f\) is low, and the regularization parameter \(\alpha\) is correspondingly small. We verify the numerical performance of the method by applying it to problems from magnetic resonance imaging (MRI) in chemical engineering and medicine. The specific applications are in diffusion tensor imaging and MR velocity imaging. These numerical studies show very promising performance.

Keywords: primal–dual, nonlinear, non-convex, convergence, MRI

(Some figures may appear in colour only in the online journal)
1. Introduction

Let us be given convex, proper, lower-semicontinuous functionals $G : X \to \mathbb{R}$ and $F^* : Y \to \mathbb{R}$ on finite-dimensional Hilbert spaces $X$ and $Y$. We then wish to solve the minimax problem

$$\min_x \max_y G(x) + \langle K(x), y \rangle - F^*(y), \quad (1.1)$$

where we allow the operator $K \in C^2(X; Y)$ to be nonlinear. If $K$ were linear, this problem could be solved, among others, by the primal–dual method due to Chambolle and Pock [1]. In section 2 of this paper, we derive two extensions of the method for nonlinear $K$.

The aforementioned Chambolle–Pock algorithm is an inertial primal–dual backward–backward splitting method, classified in [2] as the modified primal–dual hybrid gradient method (PDHGM). It can also seen as a preconditioned ADMM (alternating directions method of multipliers). In the linear case, for step sizes $\tau, \sigma > 0$, each iteration of the algorithm consists of the updates

$$x_i^{i+1} := (I + \tau \partial G)^{-1}(x_i - \tau K^* y_i),$$
$$x_{i+1}^i := x_i^{i+1} + \omega(x_i^{i+1} - x_i),$$
$$y_i^{i+1} := (I + \sigma \partial F^*)^{-1}(y_i + \sigma K x_i^{i+1}).$$
The first and last update are the backward (proximal) steps for the primal \((x)\) and dual \((y)\) variables, respectively, keeping the other fixed. However, the dual step is not taken with the primal variable fixed at \(x^{i+1}\) but at the point \(x_{\omega}^{i+1}\). This includes some "inertia" or over-relaxation from the previous iterate \(x^i\), as specified by the parameter \(\omega\). Doing so, the algorithm can to some extent avoid the problem common to first-order methods that the steps become increasingly shorter. If \(G\) or \(F^*\) is uniformly convex, by smartly choosing for each iteration the step length parameters \(\tau, \sigma\), and the inertia \(\omega\), the method can be shown to have convergence rate \(O(1/N^2)\). This is similar to Nesterov’s optimal gradient method [3]. In the general case the rate is \(O(1/N)\). In practise the method has rather good properties on imaging problems, producing solutions of acceptable visual quality relatively quickly.

Our first, simpler extension of the algorithm for nonlinear \(K\) consists of the analogous updates

\[
x^{i+1} := (I + \tau G)^{-1}(x^i - \tau [K(x^i)]^*)^y),
\]
\[
x_{\omega}^{i+1} := x^{i+1} + \omega(x^{i+1} - x^i),
\]
\[
y^{i+1} := (I + \sigma F^*)^{-1}(y^i + \sigma K(x_{\omega}^{i+1})).
\]

The second variant linearizes \(K(x_{\omega}^{i+1})\). Through a technical analysis in section 3, we prove local convergence of both variants of the method to critical points \((\hat{x}, \hat{y})\) of the system \((1.1)\). To do this, in addition to trivial conditions familiar from the linear case, we need two non-trivial estimates. For one, defining the set-valued map

\[
H: (x, y) := \left( \frac{\partial G(x) + \nabla K(\hat{x})^y}{\partial F^*(y) - \nabla K(\hat{x})} \right),
\]

we require that the inverse \(H^{-1}\) is pseudo-Lipschitz [4], a condition also known as the Aubin property [5]. The second, more severe, estimate is that the dual variable \(\hat{y}\) has to be small in the range of the nonlinear part of \(K\). In section 4, we study the satisfaction of these estimates for \(G\), \(F^*\) and \(K\) of forms most relevant to image processing applications that we study numerically in section 5.

Problems of the form \((1.1)\) with nonlinear \(K\) arise, for instance, from various inverse problems in magnetic resonance imaging (MRI). As a motivating example, we introduce the following problem from velocity-encoded MRI. Other applications include the modelling of the Stejskal–Tanner equation in diffusion tensor imaging (DTI). We will discuss this application in more detail in section 5. In velocity-encoded MRI, we seek to reconstruct a complex image \(v = r \exp(i\varphi) \in L^1(\Omega; \mathbb{C})\) from sub-sampled \(k\)-space (Fourier transform) data \(f\). In this application we are chiefly interested in the phase \(\varphi\), and eventually the difference of phases of two suitably acquired images, as the velocity of an imaged fluid can be encoded into the phase difference [6]. Let us denote by \(S\) the sub-sampling operator, and by \(\mathcal{F}\) the Fourier transform.

We observed in [7] that instead of, let’s say, defining total variation (TV) for complex-valued functions \(v\) similarly to vector-valued functions, and solving

\[
\min_v \frac{1}{2} \| f - S\mathcal{F}v \|^2 + \alpha \text{TV}(v),
\]

it may be better to regularize \(r\) and \(\varphi\) differently. This leads us to the problem

\[
\min_{r, \varphi} \frac{1}{2} \| f - T(r, \varphi) \|^2 + \alpha_r R_r(r) + \alpha_\varphi R_\varphi(\varphi).
\]

Here \(R_r\) and \(R_\varphi\) are suitable regularization functionals for the amplitude map \(r\) and phase map \(\varphi\), respectively, of a complex image \(v = r \exp(i\varphi)\). Correspondingly, we define the operator \(T\) by

\[
T(r, \varphi) := S\mathcal{F}(x \mapsto r(x) \exp(i\varphi(x))).
\]
Observe that we may rewrite \( \frac{1}{2} \| f - T(r, \varphi) \|^2 = \max_{\lambda} \{ \langle T(r, \varphi), \lambda \rangle - \langle f, \lambda \rangle - \frac{1}{2} \| \lambda \|^2 \} \).

Generally also the regularization terms can be written in terms of an indicator function and a bilinear part in the form
\[
\alpha_r R_r(r) = \max_{\psi} \langle K_r, \psi \rangle - \delta_{C_r}(\psi).
\]
In case of TV regularization, \( R_r(r) = TV(r) \), we have \( C_r = \{ \psi : \sup_x \| \psi(x) \| \leq \alpha_r \} \) and \( K_r = \nabla \). With these transformations, the problem (1.3) can be written in the form (1.1) with \( G \equiv 0 \),
\[
K(r, \varphi) = (T(r, \varphi), K_r, K_{\varphi, \varphi}),
\]
and
\[
F^*(\lambda, \psi_r, \psi_\varphi) = \langle f, \lambda \rangle + \frac{1}{2} \| \lambda \|^2 + \delta_{C_r}(\psi_r) + \delta_{C_{\varphi}}(\psi_\varphi).
\]
Observe that \( F^* \) is strongly convex in the range of the nonlinear part of \( K \), corresponding to \( T \). Under exactly this kind of structural assumptions, along with strict complementarity and non-degeneracy assumptions from the solution, we can show in section 4 that \( H_{\hat{x}}^{-1} \) possesses the Aubin property required for the general convergence theorem, theorem 3.2, to hold. Moreover, in this case the condition on \( \bar{y} \) being small in the nonlinear range of \( K \) corresponds to \( \| f - T(\hat{r}, \hat{\varphi}) \| \) being small. This can be achieved under low noise and a small regularization parameter.

Computationally (1.3) is significantly more demanding than (1.2), as it is no longer a convex problem due to the nonlinearity of \( T \). One option for locally solving problems of the form (1.3) is the Gauss–Newton scheme. In this, one linearizes \( T \) at the current iterate, solves the resulting convex problem, and repeats until a stopping criterion is satisfied. Computationally such schemes combining inner and outer iterations are expensive, unless one can solve the inner iterations to a very low accuracy. Moreover, the Gauss–Newton scheme is not guaranteed to converge even locally—a fact that we did occasionally observe when performing the numerical experiments for section 5. The scheme is however very useful when combined with iterative regularization, and behaves in that case well for almost linear \( K \) [8, 9]. It can even be combined with Bregman iterations for contrast enhancement [10]. Unfortunately, our operators of interest are not almost linear. Another possibility for the numerical solution of (1.3) would be an infeasible semismooth Newton method, along the lines of [11], extended to nonlinear operators. However, second-order methods quickly become prohibitively expensive as the image size increases, unless one can employ domain decomposition techniques—something that to our knowledge has not yet been done for semismooth Newton methods relevant to TV type regularization. Based on this, we find it desirable to start developing more efficient and provably convergent methods for non-convex problems on large data sets. We now study one possibility.

2. The basics

We describe the proposed method, algorithm 2.1, in section 2.1 below. To begin its analysis, we study in section 2.2 the application of the Chambolle–Pock method to linearizations of our original problem (1.1). We then derive in section 2.3 basic descent estimates that motivate a general convergence result, theorem 2.1, stated in section 2.4. This result will form the basis of the proof of convergence of algorithm 2.1. Our task in the following section 3 will be to derive the estimates required by theorem 2.1. We follow the theorem with a collection of remarks in section 2.5.
2.1. The proposed method

Let $X$ and $Y$ be finite-dimensional Hilbert spaces. Suppose we are given two convex, proper, lower-semicontinuous functionals $G : X \to \mathbb{R}$ and $F^* : X \to \mathbb{R}$, and a possibly nonlinear operator $K \in C^2(X; Y)$. We are interested in solving the problem

$$
\min_x \max_y G(x) + \langle K(x), y \rangle - F^*(y).
$$

(P)

The first-order optimality conditions for $(\hat{x}, \hat{y})$ to solve (P) may be formally derived as

$$
\begin{align*}
- [\nabla K(\hat{x})]^* \hat{y} &\in \partial G(\hat{x}), \quad (2.1a) \\
K(\hat{x}) &\in \partial F^*(\hat{y}). \quad (2.1b)
\end{align*}
$$

Under a constraint qualification, which is satisfied for example when $G$ is $C^1$ and either $[\nabla K(x)]^*$ has empty nullspace or $\text{dom } F = X$, these conditions can be seen to be necessary; cf [5, 10.8]. For linear $K$ in particular, the conditions are necessary and reduce to

$$
\begin{align*}
- K^* \hat{y} &\in \partial G(\hat{x}), \quad (2.2a) \\
K \hat{x} &\in \partial F^*(\hat{y}). \quad (2.2b)
\end{align*}
$$

The modified PDHGM due to [1], solves this problem by iterating for $\sigma, \tau > 0$ satisfying $\sigma \tau \|K\|^2 < 1$ the system

$$
\begin{align*}
x^{i+1} &:= (I + \tau \partial G)^{-1}(x^i - \tau K^* y^i), \quad (2.3a) \\
x^{i+1}_w &:= x^{i+1} + \omega (x^{i+1} - x^i), \quad (2.3b) \\
y^{i+1} &:= (I + \sigma \partial F^*)^{-1}(y^i + \sigma K x^{i+1}_w). \quad (2.3c)
\end{align*}
$$

As such, the method is closely related to a large class of methods including in particular the Uzawa method and the ADMM. For an overview, we recommend [2].

In case the reader is wondering, the order of the primal $(x)$ and dual $(y)$ updates in (2.3) is reversed from the original presentation in [1]. The reason is that reordered the updates can, as discovered in [12], be easily written in a proximal point form. We will exploit this. Indeed, (2.3) already contains two proximal point sub-problems, specifically the computation of the resolvents $(I + \tau \partial G)^{-1}$ and $(I + \sigma \partial F^*)^{-1}$. We recall that they may be written as

$$
(I + \tau \partial G)^{-1}(x) = \arg \min_{x'} \left\{ \frac{\|x' - x\|^2}{2\tau} + G(x') \right\}.
$$

For the good performance of (2.3), it is crucial that these sub-problems can be solved efficiently. Usually in applications, they turn out to be simple projections or linear operations. Resolvents reducing to small pointwise quadratic semidefinite problems have also been studied [13, 14].

Observe the correspondence between the (merely necessary) optimality conditions (2.1) for the problem (P) with nonlinear $K$ and the optimality conditions (2.2) for the linear case. It suggests that we could obtain a numerical method for solving (2.1) by replacing the applications $K^* y'$ and $K x^{i+1}_w$ in (2.3) by $[\nabla K(x')]^* y'$ and $K(x^{i+1})$. We would thus linearize the dual application, but keep the primal application nonlinear. We do exactly that and propose the following method.

**Algorithm 2.1** (Exact NL–PDHGM). Choose $\omega \geq 0$ and $\sigma, \tau > 0$. Repeat the following steps until a convergence criterion is satisfied

$$
x^{i+1} := (I + \tau \partial G)^{-1}(x^i - \tau [\nabla K(x^i)]^* y^i), \quad (2.4a)
$$
\[ x_{i+1}^\omega := x_{i+1}^\omega + \omega (x_{i+1}^\omega - x_i), \quad (2.4b) \]
\[ y^{i+1} := (I + \sigma \partial F^*)^{-1}(y^i + \sigma K(x^i_{\omega})). \quad (2.4c) \]

In practise we require \( \omega = 1 \). Exact conditions on the step length parameters \( \tau \) and \( \sigma \) will be derived in section 3 along the course of the proof of local convergence; in the numerical experiments of section 5, we make \( \tau \) and \( \sigma \) depend on the iteration, choosing \( \tau^i \) and \( \sigma^i \) to satisfy
\[ \sigma^i \tau^i \left( \sup_{k=1,...,i} \| \nabla K(x_k^i) \|^2 \right) < 1 \]
with the ratio \( \sigma^i / \tau^i \) unaltered. We discuss the justification for this kind of strategies in remark 3.6 after the convergence proof.

We will base the convergence proof on the following fully linearized method, where we replace the application \( K(x^i_{\omega}) \) also by a linearization.

**Algorithm 2.2** (Linearized NL–PDHGM). Choose \( \omega \geq 0 \) and \( \tau, \sigma > 0 \). Repeat the following steps until a convergence criterion is satisfied,
\[ x^{i+1} := (I + \tau \partial G)^{-1}(x^i - \tau [\nabla K(x^i)]^* y^i), \quad (2.5a) \]
\[ x^i_{\omega} := x^{i+1} + \omega (x^{i+1} - x^i), \quad (2.5b) \]
\[ y^{i+1} := (I + \sigma \partial F^*)^{-1}(y^i + \sigma [K(x^i) + \nabla K(x^i)](x^{i+1}_{\omega} - x^i)). \quad (2.5c) \]

In numerical practise, as we will see in section 5, the convergence rate of both variants of the algorithm is the same. Algorithm 2.1 is however faster in terms of computational time, as it needs less operations per iteration in the evaluation of \( K(x^i_{\omega}) \) versus \( K(x^i) + \nabla K(x^i)(x^{i+1}_{\omega} - x^i) \).

### 2.2. Linearized problem and proximal point formulation

To start the convergence analysis of algorithm 2.1, we study the application of the standard Chambolle–Pock method (2.3) to linearizations of problem (P) at a base point \( \bar{x} \in X \). Specifically, we define
\[ K_{\bar{x}} := \nabla K(\bar{x}), \quad \text{and} \quad c_{\bar{x}} := K(\bar{x}) - K_{\bar{x}} \bar{x}. \]

Then we consider
\[ \min_{x} \max_{y} G(x) + \langle c_{\bar{x}} + K_{\bar{x}}x, y \rangle - F^*(y). \quad (2.6) \]

This problem is of the form required by the method (2.3), if we write \( F^*_x(y) = F^*(y) - \langle c_{\bar{x}}, y \rangle \). Indeed, we may write the updates (2.3) for this problem as
\[ x^{i+1} := (I + \tau \partial G)^{-1}(x^i - \tau K_{\bar{x}}^* y^i), \quad (2.7a) \]
\[ x^i_{\omega} := x^{i+1} + \omega (x^{i+1} - x^i), \quad (2.7b) \]
\[ y^{i+1} := (I + \sigma \partial F^*)^{-1}(y^i + \sigma (c_{\bar{x}} + K_{\bar{x}} x^{i+1}_{\omega})). \quad (2.7c) \]

Observe how (2.5c) corresponds to (2.7c) with \( \bar{x} = x^i \).

From now on we use the general notation
\[ u = (x, y), \]
We now fix

2.3. Basic descent estimate

and define

\[ H_z(u) := \left( \frac{\partial G(x) + K_i^v y}{\partial F^*(y) - K_i x - c_i} \right) \]

as well as

\[ M_z := \begin{pmatrix} I/\tau & -K_i^v \\ -\omega K_i & I/\sigma \end{pmatrix}. \]

With these operators, \( 0 \in H_z(\tilde{u}) \) characterizes solutions \( \tilde{u} \) to (2.6), and \( u^{i+1} \) computed by (2.8) is according to [12] characterized as the unique solution to the proximal point problem

\[ 0 \in H_z(u^{i+1}) + M_z(u^{i+1} - u^i). \tag{2.8} \]

In fact, returning to the original problem (P), the optimality conditions (2.1) may be written

\[ 0 \in H_z(\tilde{u}), \]

and (2.8) with \( \tilde{x} = x^i \) characterizes the update (2.6) of linearized NL–PDHGM (algorithm 2.2). For the update (2.5) of exact NL–PDHGM (algorithm 2.1), we derive the characterization

\[ 0 \in H_v(u^{i+1}) + D_v(u^{i+1}) + M_v(u^{i+1} - u^i) \tag{2.9} \]

with

\[ D_v(x, y) := \begin{pmatrix} 0 \\ K_i x_0 + c_i - K(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ K(\tilde{x}) + \nabla K(\tilde{x})(x_0 - \tilde{x}) - K(x_0) \end{pmatrix} \tag{2.10} \]

and \( x_0 := x + \alpha(x - \tilde{x}) \). We therefore study next basic estimates that can be obtained from (2.8) with an additional general discrepancy term \( v^i \). These form the basis of our convergence proof.

2.3. Basic descent estimate

We now fix \( \omega = 1 \) in order to force \( M_z \) symmetric, and study solutions \( u^{i+1} \) to the general system

\[ 0 \in H_z(u^{i+1}) + v^i + M_z(u^{i+1} - u^i). \tag{2.11} \]

In lemma 2.1 below, we show that \( u^{i+1} \) is better than \( u^i \) in terms of distance to the ‘perturbed local solution’ \( \tilde{u} \) solving \( 0 \in H_z(\tilde{u}) + v^i \). Here we use the word perturbation to refer to \( v^i \), and local to refer to the linearization point \( \tilde{x} \). Observe that \( \tilde{u} \) depends on both \( u^i \) and \( u^{i+1} \) in case of algorithm 2.1, resp. (2.9). In section 3 we will lessen these dependences, and convert the statement to be in terms of local (unperturbed) optimal solutions \( \tilde{u} \), satisfying \( 0 \in H_z(\tilde{u}) \).

For the statement of the lemma, we use the notation

\[ \langle a, b \rangle_M := \langle a, M b \rangle, \quad \| a \|_M := \sqrt{\langle a, a \rangle_M}, \]

and denote by \( P_0 \) the linear projection operator into a subspace \( V \) of \( Y \). We also say that \( F^* \) is strongly convex on the subspace \( V \) with constant \( \gamma > 0 \) if

\[ F^*(y') - F^*(y) \geq \langle z, y' - y \rangle + \frac{\gamma}{2} \| P_0 (y' - y) \|^2 \]

for all \( y, y' \in Y \) and \( z \in \partial F^*(y) \).

This is equivalent to saying that the operator \( \partial F^* \) is strongly monotone on \( V \) in the sense that

\[ \langle \partial F^*(y') - \partial F^*(y), y' - y \rangle \geq \frac{\gamma}{2} \| P_0 (y' - y) \|^2 \]

for all \( y, y' \in Y \).
2.4. Idea of convergence proof

Changes on each iteration, as do the local perturbed solution.

Lemma 2.1. Fix $\omega = 1$. Let $u' \in X \times Y$ and $\tilde{x} \in X$. Suppose $u'^{i+1} \in X \times Y$ solves (2.11) for some $v' \in X \times Y$, and that $\tilde{u}' \in X \times Y$ is a solution to

$$0 \in H_\varepsilon(\tilde{u}') + v'.$$

(2.12)

Then

$$\|u' - \tilde{u}'\|_{M_i}^2 \geq \|u'^{i+1} - u'\|_{M_i}^2 + \|u'^{i+1} - \tilde{u}'\|_{M_i}^2. \quad (\tilde{D}^2\text{-loc})$$

If $F'$ is additionally strongly convex on a subspace $V$ of $Y$ with constant $\gamma > 0$, then we have

$$\|u' - \tilde{u}'\|_{M_i}^2 \geq \|u'^{i+1} - u'\|_{M_i}^2 + \|u'^{i+1} - \tilde{u}'\|_{M_i}^2 + \gamma \|\nu\|^2_{P_Y(y'^{i+1} - \tilde{y}')}. \quad (\tilde{D}^2\text{-loc-}\gamma)$$

Proof. Since the operator $H_\varepsilon$ is monotone, we have

$$\langle (H_\varepsilon(u'^{i+1}) + v') - (H_\varepsilon(\tilde{u}') + v'), u'^{i+1} - \tilde{u}' \rangle = \langle H_\varepsilon(u'^{i+1}) - H_\varepsilon(\tilde{u}'), u'^{i+1} - \tilde{u}' \rangle \geq 0. \quad (2.13)$$

It thus follows from (2.11), (2.12), and the symmetricity of $M_i$ that

$$0 \geq \langle u'^{i+1} - \tilde{u}', u'^{i+1} - u' \rangle_{M_i} = \langle u'^{i+1} - u', u'^{i+1} - \tilde{u}' \rangle_{M_i}. \quad (2.14)$$

This yields $(\tilde{D}^2\text{-loc})$. The strong convexity estimate $(\tilde{D}^2\text{-loc-}\gamma)$ is proved analogously, using the fact that instead of (2.13), we have the stronger estimate

$$\langle H_\varepsilon(u'^{i+1}) - H_\varepsilon(\tilde{u}'), u'^{i+1} - \tilde{u}' \rangle \geq \frac{\gamma}{2} \|\nu\|^2_{P_Y(y'^{i+1} - \tilde{y}')}. \quad \square$$

Following [15], see also [16], if $v' = 0$, then it is not difficult to show from $(\tilde{D}^2\text{-loc})$ the convergence of the iterates $\{u'^i\}_{i=0}^\infty$ generated by (2.11) to a solution of the linearized problem (2.6). The estimate $(\tilde{D}^2\text{-loc})$ also forms the basis of our proof of local convergence of algorithm 2.1 and algorithm 2.2. However, we have to improve upon it to take into account that $\tilde{x} = x'$ changes on each iteration in (2.5), and that the dual update in (2.4c) is not linearized. The consequence of these changes is that also the weight operator $M_{i\tau}$ of the local norm $\|\cdot\|_{M_{i\tau}}$ changes on each iteration, as do the local perturbed solution $\tilde{u}'$ and the local (unperturbed) solution $\tilde{u}$. Taking these differences into account, it turns out that the correct estimate that we have to derive is $(\tilde{D})$ in the next theorem. There we have improved $(\tilde{D}^2\text{-loc})$ by these changes and additionally, due to proof-technical reasons, by the removal of the squares on the norms.

2.4. Idea of convergence proof

Theorem 2.1. Suppose that the operator $K \in C^1(X; Y)$ and the constants $\sigma, \tau > 0$ satisfy for some $\Theta > \theta > 0$ the bounds

$$\theta I \leq M_i \leq \Theta^2 I, \quad (i = 1, 2, 3, \ldots). \quad (C-M)$$

Let the sequence $\{u'^i\}_{i=0}^\infty$ solve (2.11) for $\tilde{x} = x'$ and some $\{v'^i\}_{i=0}^\infty \subset X \times Y$ satisfying

$$\lim_{i \to \infty} v'^i = 0. \quad (C-v')$$

Suppose, moreover, that for some constant $\zeta > 0$ and points $[\tilde{u}']_{i=0}^\infty$ we have the estimate

$$\|u' - \tilde{u}'\|_{M_{i\tau}} \geq \zeta \|u'^{i+1} - u'\|_{M_{i\tau}} + \|u'^{i+1} - \tilde{u}'\|_{M_{i\tau+1}}. \quad (\tilde{D})$$

Then the iterates $u' \to \tilde{u}$ for some $\tilde{u} = (\tilde{x}, \tilde{y})$ that solves (2.1).
Proof. It follows from (D) that
\[ \sum_{i=1}^{\infty} \| u_i^{t+1} - u_i^t \|_{M_{ij}} < \infty. \]

Consequently, an application of (C-M) shows that
\[ \sum_{i=1}^{\infty} \| u_i^{t+1} - u_i^t \| \leq \Theta \sum_{i=1}^{\infty} \| u_i^{t+1} - u_i^t \|_{M_{ij}} < \infty. \] (2.15)

This says that \( \{ u_i^t \}_{i=1}^{\infty} \) is a Cauchy sequence, and hence converges to some \( \hat{u} \).

Let
\[ z_i^t := u_i^t + M_{ij}(u_i^{t+1} - u_i^t). \]

Since \( v_i^t \to 0 \) by (C-vi), and
\[ \| M_{ij}(u_i^{t+1} - u_i^t) \| \leq \Theta \| u_i^{t+1} - u_i^t \| \to 0, \]
it follows that \( z_i \to 0 \). By (2.11), we moreover have \(-z_i \in H_{ij}(u_i^{t+1})\). Using \( K \in C^1(X; Y) \) and the outer semicontinuity of the subgradient mappings \( \partial G \) and \( \partial F^* \), we see that
\[ \limsup_{i \to \infty} H_{ij}(u_i^{t+1}) \subset H_{ij}(\hat{u}). \]

Here the lim sup is in the sense of an outer limit [5], consisting of the limits of all converging subsequences of elements \( v_i^t \in H_{ij}(u_i^{t+1}) \). As by (2.11) we have \(-z_i \in H_{ij}(u_i^{t+1})\), it follows in particular that \( 0 \in H_{ij}(\hat{u}) \). This says precisely that (2.1) holds.

2.5. A few remarks

Remark 2.1 (Reference points). Observe that we did not need to assume the reference points \( \{ \hat{y}_i \}_{i=1}^{\infty} \) in the above proof to solve anything. We did not even need to assume boundedness, which follows from (D) and (C-M).

Remark 2.2 (Gauss–Newton). Let \( v_i^t = -M_{ij}(u_i^{t+1} - u_i^t) \) and \( \bar{x} = x_i^t \). In fact, we can even replace \( M_{ij} \) by the identity \( I \). Then \( u_i^{t+1} \) solves the local linearized problem (2.6), that is
\[ 0 \in H_{ij}(u_i^{t+1}), \]
and lemma 2.1 together with theorem 2.1 show the convergence of the Gauss–Newton method to a critical point of (P), provided \( v_i^t \to 0 \). That is, either the iterates diverge, or the Gauss–Newton method converges to a solution.

Remark 2.3 (Varying over-relaxation parameter). We have assumed that \( \omega = 1 \). It is however possible to accommodate a varying parameter \( \omega_i \to 1 \) through \( v_i^t \). In particular, one can easily show the convergence (albeit merely sublinear) of the accelerated algorithm of [1] this way. In this variant, dependent on the strong convexity of \( F \) or \( G \), one updates the parameters at each step as \( \omega_i = 1/\sqrt{1 + 2\gamma t_i}, \ t_i^{t+1} = \omega_i t_i, \) and, \( \sigma_i^{t+1} = \sigma_i^t/\tau_i \) for \( \gamma \) the factor of strong convexity of either \( F^* \) or \( G \).

Remark 2.4 (Interpolated PDHGM for nonlinear operators). Our analysis, forthcoming and to this point, applies to a yet further variant of algorithm 2.1. Here we replace \( K(x_i^{t+1}) \) in (2.4c) by
\[ (1 + \omega)K(x_i^{t+1}) - \omega K(x_i^t) = K(x_i^t) + (1 + \omega)(K(x_i^{t+1}) - K(x_i^t)) \approx K(x_i^{t+1}). \]

For this method
\[ v_i^t = \left( 0, \nabla K(x_i^t)(x_i^{t+1} - x_i^t) - (1 + \omega)(K(x_i^{t+1}) - K(x_i^t)) \right) = (1 + \omega)D_{ij}(x_i^{t+1}). \]
Remark 2.5 (An alternative update). It is also interesting to consider using \([\nabla K(x^i)]^*\) instead of \([\nabla K(x)]^*\) in (2.4a). From the point of view of the convergence proof, this however introduces major difficulties, as \((x^{i+1}, y^{i+1})\) no longer depends on just \((x^i, y^i)\), but also on \(x^{i-1}\) through \(x^i\). This kind of dependence also makes analysis using the original ordering of the PDHGM updates in [1] difficult, but is avoided by the reordering due to [12] that we employ in (2.4) and (2.5).

3. Detailed analysis of the nonlinear method

We now proceed to verifying the assumptions of theorem 2.1 for algorithms 2.1 and 2.2, provided our initial iterate is close enough to a solution which satisfies certain technical conditions, to be derived along the course of the proof. We will begin with the formal statement of our running assumptions in section 3.1, after which we prove some auxiliary results in section 3.2. Our first task in verifying the assumptions of theorem 2.1 is to show that the discrepancy term \(v^i = D_i(\hat{u}^{i+1}) \rightarrow 0\). This we do in section 3.3. Then in section 3.4 we begin deriving the estimate (D) by analysing the switch to the new local norm at the next iterate. In section 3.5 we introduce and study Lipschitz type estimates on \(H^{-1}_{\hat{x}}\). We then use these in section 3.6 and section 3.7, respectively, to remove the squares from the estimate (\(D^2_{loc}-\gamma\)) and to bridge from one local solution to the next one. The Lipschitz type estimates themselves we will derive in section 4 to follow. We state and prove our main convergence theorem, combining all the above-mentioned estimates, in section 3.8. This we follow by a collection of remarks in section 3.9.

3.1. General assumptions

We take \(G : X \rightarrow \mathbb{R}\) and \(F^* : Y \rightarrow \mathbb{R}\) to be convex, proper, lower-semicontinuous functionals on finite-dimensional Hilbert spaces \(X\) and \(Y\), satisfying \(\text{int dom } G, \text{int dom } F^* \neq \emptyset\). We fix \(\omega = 1\) and study the sequence of iterates generated by solving

\[ 0 \in S_i(u^{i+1}) := H_i(u^{i+1}) + D_i(u^{i+1}) + M_i(x^i - u^i), \]

where we expect the discrepancy functional

\[ D_i : X \times Y \rightarrow \{0\} \times Y, \quad (i = 1, 2, 3, \ldots), \]

to satisfy for any fixed \(i, C, \epsilon > 0\), the existence of \(\rho > 0\) such that

\[ \|D_i(u)\| \leq \epsilon \|x^i - x\|, \quad (\|x^i - x\| \leq \rho, \|u^i\| \leq C). \quad (A-D_i) \]

For brevity, we denote

\[ v^i := D_i(u^{i+1}). \]

We then aim to take \(\tilde{u}^i\) as a solution of the perturbed linearized problem

\[ 0 \in H_i(\tilde{u}^i) + v^i, \]

and \(\tilde{u}\) as a solution of the linearized problem

\[ 0 \in H_i(\tilde{u}). \]

Note that these solutions do not necessarily exist. We will later prove that they exist near a solution \(\tilde{u}\) for which \(H^{-1}_{\tilde{x}}\) satisfies the Aubin property. As theorem 2.1 makes no particular requirement on \(\tilde{u}\), we will begin with \(\tilde{u}\) and \(\tilde{u}\) arbitrary elements of \(X \times Y\), and state the above inclusions explicitly when we require or have them.
Regarding the operator $K : X \rightarrow Y$ and the step length parameters $\sigma, \tau > 0$, we require that

\[ K \in C^2(X; Y) \quad \text{and} \quad \sigma \tau \left( \sup_{\|x\| \leq R} \|\nabla K(x)\| \right) < 1. \quad \text{(A-K)} \]

Here we fix $R > 0$ such that there exists a solution $\hat{u}$ to

\[ 0 \in H_\xi(\hat{u}) \quad \text{with} \quad \|\hat{u}\| \leq R/2. \quad \text{(3.1a)} \]

We then denote by $L_2$ the Lipschitz factor of $x \mapsto \nabla K(x)$ on the closed ball $B(0, R) \subset X$, namely

\[ L_2 := \sup_{\|x\| \leq R} \|\nabla^2 K(x)\| \quad \text{(3.1b)} \]

in operator norm. By (A-K), the supremum is bounded. Finally, we define the ‘linear’ and ‘nonlinear’ subspaces

\[ Y_L := \{ z \in Y \mid \text{the map } x \mapsto \langle z, K(x) \rangle \text{ is linear} \}, \quad \text{and} \quad Y_{NL} := Y_L^\perp, \]

and denote by $P_{NL}$ the orthogonal projection into $Y_{NL}$.

### 3.2. Auxiliary results

The assumption (A-K) guarantees in particular that the weight operators $M_{x^i}$ are uniformly bounded, as we state in the next lemma. For $\Theta$ and $\theta$ as in the lemma, we also define the uniform condition number

\[ \kappa := \Theta/\theta. \quad \text{(3.1c)} \]

Observe that $\kappa \to 1$ as $\tau, \sigma \to 0$.

**Lemma 3.1.** Suppose (A-K) holds. Then there exist $\Theta_1 \geq \theta > 0$ such that

\[ \Theta_1 I \leq M_x \leq \Theta^2 I, \quad (\|x\| \leq R). \quad \text{(3.1d)} \]

**Proof.** This follows immediately from the fact that $\nabla K$ is bounded on $B(0, R)$.

In case of algorithm 2.2, showing (A-D$^0$) is trivial because $v^i = 0$. For algorithm 2.1, we show this in the next lemma.

**Lemma 3.2.** Suppose (A-K) holds, and let $D^i = D_{x^i}$, i.e., suppose $\{\hat{u}^i\}_{i=1}^\infty$ is generated by algorithm 2.1. Then (A-D$^0$) holds. The same is the case with $D^i = 0$, i.e., algorithm 2.2.

**Proof.** We recall the definition of $D_x$ from (2.10), and notice that

\[ D_x(x, y) = (0, Q_x(x + \omega(x - \bar{x}))) \]

for

\[ Q_x(x) := K(\bar{x}) + \nabla K(\bar{x})(x - \bar{x}) - K(x). \]

To show (A-D$^0$), we observe that thanks to $K$ being twice continuously differentiable, for any $C > 0$ there exist $L > 0$ and $\rho_1 > 0$ such that

\[ \|Q_x(x)\| \leq L\|x - \bar{x}\|^2, \quad (\|x - \bar{x}\| \leq \rho_1, \|\bar{x}\| \leq C). \]

In particular, minding that

\[ [x^{i+1} + \omega(x^{i+1} - x^i)] - x^i = (1 + \omega)(x^{i+1} - x^i), \]
setting \( \rho_2 := \rho_1 / (1 + \omega) \), we find
\[
\| D_{\omega}(u^{i+1}) \| \leq L(1 + \omega) \| x^{i+1} - x^i \|, \quad (\| x^{i+1} - x^i \| \leq \rho_2, \| x^i \| \leq C).
\]
Choosing \( \rho \in (0, \rho_2) \) small enough, (A-D') follows.

We will occasionally use the auxiliary mapping \( E \) defined in the following lemma. The motivation behind it is that if \( v \in H_{\omega} (u) \) then \( v + E (u; \bar{u}, \bar{u}') \in H_{\omega} (u) \).

**Lemma 3.3.** Suppose (A-K) holds. Let \( \bar{u}, \bar{u}' \) satisfy \( \| \bar{u} \|, \| \bar{u}' \| \leq R \). Define
\[
E (u) := E (u; \bar{u}, \bar{u}'): = \begin{pmatrix} (K_{\bar{x}} - K_{\bar{x}})^* y \\ (K_{\bar{x}} - K_{\bar{x}}) (x - \bar{x}) \end{pmatrix}.
\]
Then \( E \) is Lipschitz with Lipschitz factor \( \ell_E \leq L_2 \| \bar{u} - \bar{u}' \| \), and
\[
\| E (u) \| \leq L_2 \| \bar{x} - \bar{x}' \| (\| P_{NL} y \| + \| x - \bar{x} \| + \| \bar{x} - \bar{x}' \|).
\]

**Proof.** The fact that \( E \) is Lipschitz with the claimed factor is easy to see; indeed
\[
E (u) - E (u') = \begin{pmatrix} (K_{\bar{x}} - K_{\bar{x}})^* y \\ (K_{\bar{x}} - K_{\bar{x}}) (x - \bar{x}) \end{pmatrix}.
\]
Since (A-K) holds and \( \| \bar{x} \|, \| \bar{x}' \| \leq R \), as we have assumed, we have
\[
\| K_{\bar{x}} - K_{\bar{x}} \| \leq L_2 \| \bar{x} - \bar{x}' \|.
\]
It follows that
\[
\| E (u) - E (u') \| \leq L_2 \| \bar{x} - \bar{x}' \| \| u - u' \|.
\]
That is, the claimed Lipschitz estimate holds.

Regarding (3.3), we may rewrite
\[
E (u) = \begin{pmatrix} (K_{\bar{x}} - K_{\bar{x}})^* y \\ (K_{\bar{x}} - K_{\bar{x}}) (x - \bar{x}) - Q_{\bar{x}}(\bar{x}) \end{pmatrix}.
\]
As in the proof of lemma 3.2, the quantity
\[
Q_{\bar{x}}(\bar{x}') = (K_{\bar{x}} - K_{\bar{x}}) \bar{x}' + c_{\bar{x}} - c_{\bar{x}} = K(\bar{x}) + \nabla K(\bar{x}) (\bar{x}' - \bar{x}) - K(\bar{x})
\]
satisfies for some \( L_2' > 0 \) that
\[
\| Q_{\bar{x}}(\bar{x}') \| \leq L_2' \| \bar{x} - \bar{x}' \|^2.
\]
In fact, since \( \| \bar{x} \|, \| \bar{x}' \| \leq R \), we may choose \( L_2' = L_2 \). We then observe that
\[
(K_{\bar{x}} - K_{\bar{x}})^* y = (K_{\bar{x}} - K_{\bar{x}})^* P_{NL} y.
\]
Thus
\[
\| E (u) \| \leq \| K_{\bar{x}} - K_{\bar{x}} \| (\| P_{NL} y \| + \| x - \bar{x} \|) + L_2 \| \bar{x} - \bar{x}' \|^2.
\]
Using (3.4), we get (3.3).
3.3. Convergence of the discrepancy term

The convergence $\nu^i \to 0$ required by theorem 2.1 follows from the following simple result that we will also use later on.

**Lemma 3.4.** Let $R$ and $\kappa$ be as in (3.1). Suppose (A-D') and (A-K) hold, and let $u^1 \in X \times Y$ and $u^{i+1} \in S_{v}^{-1}(0)$, $(i = 1, \ldots, k-1)$. If (D) holds for $i = 1, \ldots, k-1$ for some $\tilde{u}^1, \ldots, \tilde{u}^k$ and $\zeta > 0$ with

$$
\|u^1 - \tilde{u}\| \leq R/4 \quad \text{and} \quad \|u^1 - \tilde{u}^1\| \leq R\zeta/(4\kappa),
$$

then

$$
\|u^j\| \leq R, \quad (3.6a)
$$

$$
\|u^j - u^1\| \leq (\kappa/\zeta)\|u^1 - \tilde{u}\|, \quad \text{and} \quad (3.6b)
$$

$$
\|\tilde{u}^j - u^1\| \leq \kappa\|u^1 - \tilde{u}\|. \quad (3.6c)
$$

**Proof.** Choose $\epsilon > 0$, and let $\rho > 0$ be prescribed by (A-D') for $C = R$ and $\epsilon$. Since (D) holds, we have

$$
\|u^i - \tilde{u}\|_{M_{j}} \geq \zeta\|u^{i+1} - u^i\|_{M_{j}} + \|u^{i+1} - \tilde{u}^{i+1}\|_{M_{j}}, \quad (i = 1, \ldots, k-1).
$$

Taking $j \in \{1, \ldots, k\}$, this gives

$$
\|u^i - \tilde{u}\|_{M_{j}} \geq \zeta \sum_{i=1}^{j-1} \|u^{i+1} - u^i\|_{M_{j}} + \|u^i - \tilde{u}^i\|_{M_{j}}. \quad (3.7)
$$

By (3.5), we have $\|u^1\| \leq 3R/4$. Making the induction assumption

$$
\sup_{i=1, \ldots, j} \|u^i\| \leq R, \quad (3.8)
$$

we get using lemma 3.1 and (3.7) that

$$
\Theta\|u^1 - \tilde{u}^k\| \geq \|u^1 - \tilde{u}^i\|_{M_{i}} \geq \theta \zeta \sum_{i=1}^{j-1} \|u^{i+1} - u^i\| \geq \theta \zeta \|u^i - u^1\|. \quad (3.9)
$$

This gives

$$
\|u^i\| \leq \|u^1 - u^1\| + \|u^1\| \leq (\kappa/\zeta)\|u^1 - \tilde{u}\| + \|u^1\|.
$$

Using (3.5), we thus have $\|u^i\| \leq R$, so that the induction assumption (3.8) is satisfied for $j$. Taking $j = k$, (3.8) shows (3.6a) and (3.9) shows (3.6b). Furthermore, using (3.7) and lemma 3.1, we get

$$
\kappa\|u^1 - \tilde{u}^k\| \geq \|u^k - \tilde{u}\|. \quad (3.6c)
$$

This shows (3.6c). \qed

**Lemma 3.5.** Under the assumptions of lemma 3.4, $\nu^i \to 0$. In particular, the conditions (C-M) and (C-\nu') of theorem 2.1 hold.

**Proof.** Firstly, applying (3.6a) and lemma 3.1, we see that (C-M) holds. Secondly, by (3.7) we find that $\|u^{i+1} - u^i\| \to 0$. Thus eventually $\|u^{i+1} - u^i\| < \rho$. Using (A-D'), we now see that $\nu^i \equiv D'(u^{i+1}) \to 0$. This proves (C-\nu'). \qed
3.4. Switching local norms: estimates from strong convexity

We finally begin deriving the descent inequality \((\tilde{D})\) that we so far have assumed. As a first step, we set \(\bar{x} = x^t\) in \((\tilde{D}\text{-loc}-\gamma)\), and use the extra slack that strong convexity gives to replace \(M_j\) by \(M_{j+1}\) in the term \(||u^{t+1} - \bar{u}||^2_{M_j}\).

**Lemma 3.6.** Suppose \((A-K)\) and \((\tilde{D}\text{-loc}-\gamma)\) hold for some \(\bar{u}\). Let \(R, L_2, \kappa\) be as in (3.1), and choose \(\zeta_1 \in (0, 1)\). If

\[
||u^t - \bar{u}|| \leq R/4 \quad \text{and} \quad ||u^t - \bar{u}|| \leq \min[\sqrt{\gamma}/(1 - \zeta_1 \theta/\sqrt{\gamma})], \quad R/(4\kappa), \quad (3.10)
\]

then

\[
||u^t - \bar{u}||^2_{M_j} \geq \zeta_1 ||u^{t+1} - u^t||^2_{M_j} + ||u^{t+1} - \bar{u}||^2_{M_{j+1}}.
\]

\((\tilde{D}\text{-M})\)

**Proof.** Using (3.10) and \(|\tilde{u}| \leq R/2\), we have

\[
||u^t|| \leq ||u^t - \bar{u}|| + ||\tilde{u}|| \leq 3R/4.
\]

The estimate \((\tilde{D}\text{-loc}-\gamma)\) implies

\[
||u^{t+1} - u^t|| \leq \kappa ||\tilde{u}||^2 - u^t||^2.
\]

Using (3.10) and (3.11), we thus get

\[
||u^{t+1}|| \leq ||u^{t+1} - u^t|| + ||u^t|| \leq \kappa ||\tilde{u}||^2 - u^t||^2 + ||u^t|| \leq R.
\]

As both \(||u^t||, ||u^{t+1}|| \leq R\), by \((A-K)\) we have again

\[
||K_{\gamma}^t - \tilde{K}_t|| \leq L_2 ||x^{t+1} - x^t||.
\]

This allows us to deduce

\[
||u^{t+1} - \tilde{u}||^2_{M_j} = ||u^{t+1} - \tilde{u}||^2_{M_{j+1}} = -2(y^{t+1} - \gamma \tilde{y}, (K_{\gamma}^{t+1} - \tilde{K}_t)(x^{t+1} - \tilde{x}))
\]

\[
\leq -2(\gamma^{t+1} - \gamma \tilde{y}, P_{NL}(K_{\gamma}^{t+1} - \tilde{K}_t)(x^{t+1} - \tilde{x}))
\]

\[
\geq -2(P_{NL}(y^{t+1} - \gamma \tilde{y}) ||K_{\gamma}^{t+1} - \tilde{K}_t|| ||x^{t+1} - \tilde{x}||
\]

\[
\geq -2L_2 ||P_{NL}(y^{t+1} - \gamma \tilde{y})|| ||x^{t+1} - \tilde{x}|| ||x^{t+1} - \tilde{x}||. \quad (12)
\]

Using Young’s inequality we therefore have

\[
||u^{t+1} - \tilde{u}||^2_{M_j} \geq ||u^{t+1} - u^t||^2_{M_j} + \gamma ||P_{NL}(y^{t+1} - \gamma \tilde{y})||^2 \geq \frac{2L_2^2}{\gamma} ||u^{t+1} - u^t||^2 ||u^{t+1} - \tilde{u}||^2.
\]

By \((\tilde{D}\text{-loc}-\gamma)\) it follows

\[
||u^t - \bar{u}||^2_{M_j} \geq ||u^{t+1} - u^t||^2_{M_j} + ||u^{t+1} - \tilde{u}||^2_{M_{j+1}} - \frac{2L_2^2}{\gamma} ||u^{t+1} - u^t||^2 ||u^{t+1} - \tilde{u}||^2. \quad (13)
\]

An application of lemma 3.1 and \((\tilde{D}\text{-loc}-\gamma)\) shows that

\[
||u^{t+1} - \tilde{u}||^2 \leq \theta^{-1} ||u^{t+1} - \tilde{u}||^2_{M_j} \leq \kappa^2 ||u^t - \tilde{u}||^2,
\]

and

\[
||u^{t+1} - u^t||^2 \leq \theta^{-1} ||u^{t+1} - u^t||^2_{M_j}.
\]

Using (3.10), therefore

\[
\frac{2L_2^2}{\gamma} ||u^{t+1} - u^t||^2 ||u^{t+1} - \tilde{u}||^2 \leq \frac{2L_2^2\kappa^2}{\gamma \theta^2} ||u^{t+1} - u^t||^2 ||u^{t+1} - \tilde{u}||^2 \leq (1 - \zeta_1) ||u^{t+1} - u^t||^2_{M_j}.
\]

Applying this to (3.13) yields \((\tilde{D}\text{-M})\). \(\square\)
3.5. Aubin property of the inverse

In [16], the Lipschitz continuity of the equivalent of the map $H_{\tilde{F}}^{-1}$ was used to prove strong convergence properties of basic proximal point methods for maximal monotone operators. In order to remove the squares from $(\tilde{D}^2-M)$, and to bridge with $\tilde{u}$ between the local solutions $\tilde{u}$ and $\tilde{u}^{-1}$, we follow the same rough ideas. We however replace the basic form of Lipschitz continuity by a weaker version that is localized in the graph of $H_{\tilde{F}}$. Namely, with $0 \in H_{\tilde{F}}(\tilde{u})$, we assume that the map $H_{\tilde{F}}^{-1}$ has the Aubin property at $0$ for $\tilde{u}$ [4, 5]. This is also called metric regularity.

Generally, the inverse $S^{-1}$ of a set-valued map $S : X \rightrightarrows Y$ having the Aubin property at $\tilde{w}$ for $\tilde{u}$ means that Graph $S$ is locally closed, and there exist $\rho, \delta, \ell > 0$ such that

$$\inf_{v : w \in S(v)} \|u - v\| \leq \ell \|w - S(u)\|, \quad (\|u - \tilde{u}\| \leq \delta, \|w - \tilde{w}\| \leq \rho).$$

(3.14)

We denote the infimum over valid constants $\ell$ by $\ell_{S^{-1}}(\tilde{w}|\tilde{u})$, or $\ell_{S^{-1}}$ for short, when there is no ambiguity about the point $(\tilde{w}, \tilde{u})$. For bijective single-valued $S$ the Aubin property reduces to Lipschitz-continuity of the inverse $S^{-1}$. For single-valued Lipschitz $S$, we also use the notation $\ell_{S}$ for the (local) Lipschitz factor.

We can translate the Aubin property of $H_{\tilde{F}}^{-1}$ to $H_{\tilde{F}}^{-1}$, based on the following general lemma. This is needed in order to perform estimation at an iterate $u'$, only assuming the Aubin property of $H_{\tilde{F}}$ at a known solution $\tilde{u}$.

**Lemma 3.7.** Suppose $S^{-1}$ has the Aubin property at $0$ for $\tilde{u}$. Let $T(u) = S(u) + \Delta(u)$ for a single-valued Lipschitz map $\Delta : X \rightarrow Y$ with $\ell_{S^{-1}} \ell_{\Delta} < 1$. Then $T^{-1}$ has the Aubin property at $\Delta(\tilde{u})$ for $\tilde{u}$, and

$$\ell_{T^{-1}} \leq \frac{\ell_{S^{-1}}}{1 - \ell_{S^{-1}} \ell_{\Delta}}.$$  

(3.15)

The proof is a minor modification of the proof of the following result.

**Lemma 3.8 ([17]).** Suppose $S^{-1}$ has locally closed values and the Aubin property at $0$ for $\tilde{u}$. Let $\Delta : X \rightarrow Y$ be a single-valued Lipschitz map, supposing that $\Delta$ is strictly stationary at $\tilde{u}$. This means that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\Delta(u) - \Delta(u')\| \leq \epsilon \|u - u'\|, \quad (\|\tilde{u} - u\|, \|\tilde{w} - u'\| < \delta).$$

Let $T(u) = S(u) + \Delta(u)$. Then $T^{-1}$ has the Aubin property at $\Delta(\tilde{u})$ for $\tilde{u}$, and $\ell_{T^{-1}} = \ell_{S^{-1}}$.

**Proof.** This result is the main theorem of [17] combined with the remark following the theorem. □

**Proof of Lemma 3.7** We show how to modify the proof of lemma 3.8 in [17] into a proof of lemma 3.7. There are two differences in assumptions between the lemmas. The first is the apparently different closedness condition on $S$. But, obviously, $S^{-1}$ has locally closed values if Graph$S$ is locally closed, which our definition of the Aubin property includes. Therefore that part of the assumptions of lemma 3.8 is satisfied.

The second difference is the strict stationarity condition on $\Delta$. In lemma 3.7, this is replaced by the weaker condition $\ell_{S^{-1}} \ell_{\Delta} < 1$. We however observe that, indeed, the strict stationarity condition is only used in the proof of lemma 3.8 to show that $\Delta$ is Lipschitz with a given constant $\epsilon > 0$ in a neighbourhood of $\tilde{u}$ [17, (3)], and then

$$\ell_{T^{-1}} < \frac{\ell_{S^{-1}}}{1 - \epsilon \ell_{S^{-1}}}.$$
With our assumptions, we may only take \( \epsilon > \ell \Delta \). This gives us the weaker result (3.15) instead of \( \ell \Delta = \ell \Delta \).

We conclude that the proof of lemma 3.8 in [17] is easily modified into a proof of lemma 3.7.

**Lemma 3.9.** Suppose \( H^{-1} \) has the Aubin property at 0 for \( \hat{u} \), and that (A-K) holds. Given \( \ell^* > \ell H^{-1} \), there exists \( \delta \in (0, R/2) \) and \( \rho > 0 \) such that if

\[
\| \theta' - \hat{\theta} \| \leq \delta,
\]

then

\[
\inf_{v : w \in H_{\ell^*}(v)} \| u - v \| \leq \ell^* \| w - H_{\ell^*}(u) \|, \quad (\| u - \hat{\theta} \| \leq \delta, \| w \| \leq \rho).
\]

(3.16)

**Remark 3.1.** The property (3.16) is formally the Aubin property of \( H_{\ell^*}^{-1} \) at 0 for \( \hat{u} \), but cannot strictly be called that, because generally \( 0 \notin H_{\ell^*}(\hat{u}) \).

**Proof.** We use lemma 3.7 on \( S = H_{\ell^*} \), and \( \Delta = E(\cdot, \hat{u}, \theta') \), where \( E \) is defined in (3.2). We pick \( \ell > 0 \) such that

\[
\ell^* \geq \frac{\ell H_{\ell^*}}{1 - \ell H_{\ell^*}} \ell > 0.
\]

Clearly the same condition then holds with \( \ell \) replaced by any \( \ell \in (0, \ell^*) \). We exploit this. By lemma 3.3, \( \Delta \) is Lipschitz with factor \( \ell_\Delta \leq L_\Delta \delta \), so that \( \ell_\Delta < \ell \) provided \( \delta < \ell / L_\Delta \). Thus by lemma 3.7, we have

\[
\inf_{v : w \in H_{\ell^*}(v)} \| u - v \| \leq \ell^* \| w - H_{\ell^*}(u) \|, \quad (\| u - \hat{\theta} \| \leq \delta', \| w - \Delta(\hat{\theta}) \| \leq \rho')
\]

for some \( \rho', \delta' > 0 \). Referring to lemma 3.3, we have

\[
\| \Delta(\hat{\theta}) \| \leq L_\Delta \| x - x' \| (\| P_{\Omega} \hat{y} \| + 2 \| x - x' \|).
\]

For \( \delta > 0 \) small enough, we can thus force \( \| \Delta(\hat{\theta}) \| \leq \rho' / 2 \). Thus \( \| w \| \leq \rho \) guarantees \( \| w - \Delta(\hat{\theta}) \| \leq \rho' \) for \( \rho < \rho' / 2 \). This proves (3.16).

The next lemma bounds step lengths near a solution.

**Lemma 3.10.** Suppose (A-D') and (A-K) hold, and that \( H_{\ell^*}^{-1} \) has the Aubin property at 0 for \( \hat{u} \). Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that the following holds. If

\[
\| \theta' - \hat{\theta} \| \leq \delta,
\]

then there exist \( \hat{\theta} \in H_{\ell^*}^{-1}(0) \) and \( \theta' \in H_{\ell^*}^{-1}(-\nu') \). If also

\[
\| \theta' - \hat{\theta} \| \leq \delta,
\]

then with the specific choice

\[
\hat{\theta} = \arg \min\{\| \theta' - v \| \mid v \in H_{\ell^*}^{-1}(-\nu')\},
\]

we have the estimates

\[
\| \theta' \| \leq 3R / 4,
\]

\[
\| \theta' - \theta^{i+1} \| \leq \epsilon,
\]

\[
\| \hat{\theta} - \theta^{i} \| \leq \epsilon,
\]

\[
\| \theta' \| \leq 3R / 4,
\]

\[
\| \theta' - \theta^{i+1} \| \leq \epsilon,
\]

\[
\| \hat{\theta} - \theta^{i} \| \leq \epsilon,
\]

\[
(3.19)
\]

\[
(3.20a)
\]

\[
(3.20b)
\]

\[
(3.20c)
\]
\[ \| u' - \hat{u}' \| \leq \epsilon, \quad (3.20d) \]
\[ \| \hat{u} - \hat{u}' \| \leq \epsilon, \quad \text{and} \quad (3.20e) \]
\[ \| v' \| \leq \epsilon. \quad (3.20f) \]

**Proof.** Observe that in each of the estimates (3.20), we may take \( \epsilon > 0 \) smaller than prescribed. Along the course of the proof, we will accordingly assume \( \epsilon \) as small as required.

We begin by proving (3.20a), (3.20b) and (3.20f), using (3.18a) without (3.18b). Indeed, observe that since \( \| \hat{u} \| \leq R/2 \), (3.20a) trivially holds by choosing \( \delta \in (0, R/4) \). To bound \( \| u' - u'^{+} \| \), we return to the algorithmic approach for computing \( u'^{+} = (x'^{+}, y'^{+}) \in S^{-1}_{\epsilon}(0) \). Indeed, since \( D'(u'^{+}) = (0, v') \in X \times Y \) for some \( v' = v'(x'^{+}) \), we have

\[ x'^{+} := (I + \tau \partial G)^{-1} (x' - \tau [\nabla K(x')] y'), \quad (3.21a) \]
\[ x'^{+}_u := x'^{+} + \omega (x'^{+} - x'), \quad (3.21b) \]
\[ y'^{+} := (I + \sigma \partial F^{*})^{-1}(y' + \sigma [K(x') + K_v(x'^{+}_u - x') + v']). \quad (3.21c) \]

From (3.21a) we see that \( x'^{+} \) solves for \( x \) the problem

\[ \min_x \left\{ \| x - x' + \tau K_v y' \|^2 + \tau G(x) \right\}. \quad (3.22) \]

Therefore

\[ \| x'^{+} - x' + \tau K_v y' \|^2 + \tau G(x'^{+}) \leq \| \hat{x} - x' + \tau K_v y' \|^2 + \tau G(\hat{x}), \]

which leads to

\[ \| x'^{+} - x' \|^2 \leq \| \hat{x} - x' \|^2 + \tau \{ G(\hat{x}) - G(x'^{+}) + \langle (\hat{x} - x') + (x' - x'^{+}), K_v y' \rangle \}. \]

We have \( -K_v^2 \hat{y} \in \partial G(\hat{x}) \). Therefore

\[ G(\hat{x}) - G(x'^{+}) \leq \langle (\hat{x} - x') + (x' - x'^{+}), -K_v^2 \hat{y} \rangle, \]

so that

\[ \| x'^{+} - x' \|^2 = \| \hat{x} - x' \|^2 + \tau \langle (\hat{x} - x'), K_v y' - K_v^2 \hat{y} \rangle + \tau \langle x' - x'^{+}, K_v y' - K_v^2 \hat{y} \rangle. \]

An application of Young’s inequality gives

\[ \frac{1}{2} \| x'^{+} - x' \|^2 + \frac{1}{2} \| \hat{x} - x' \|^2 \leq \frac{\tau}{4} \| K_v y' - K_v^2 \hat{y} \|^2. \quad (3.23) \]

The right-hand side of (3.23) can be made arbitrarily close to zero by application of (A-K) and (3.18a). That is, for any \( \epsilon' > 0 \), there exists \( \delta' > 0 \) such that if (3.18) holds for some \( \delta \in (0, \delta') \), then

\[ \| x'^{+} - x' \| \leq \epsilon'. \quad (3.24) \]

We now have to bound \( \| y'^{+} - y' \| \) through (3.21c). Similarly to (3.22), \( y'^{+} \) solves for \( y \) the problem

\[ \min \left\{ \| y - y' + \sigma [K(x') + K_v (x'^{+}_u - x') + v'] \|^2 + \sigma F(x) \right\}. \quad (3.25) \]
Proceeding as above, using the fact that $K(\tilde{x}) \in \partial F^*(\tilde{y})$ we get

$$\frac{1}{2} \| y' - y' \|^2 \leq \frac{1}{2} \| \tilde{y} - \tilde{y}' \|^2 + \sigma^2 \| K(x') + K(\tilde{x}_{\omega}^{-1} - x') + v' - K(\tilde{x}) \|^2.$$ 

We approximate

$$\| K(x') + K'(\tilde{x}_{\omega}^{-1} - x') + v' - K(\tilde{x}) \| \leq \| K(x') - K(\tilde{x}) \| + \omega \| K'(\tilde{x}_{\omega}^{-1} - x') \| + \| v' \|.$$ 

(3.26)

By taking $\delta, \epsilon > 0$ small enough, which we may do, we can make the term $\| K(x') - K(\tilde{x}) \|$ arbitrarily small by application of (A-K) and (3.18a). Likewise, we can make the term $\| K'(\tilde{x}_{\omega}^{-1} - x') \|$ approach zero by application of (A-K), (3.18a) and (3.24). Finally, the term $\| v' \| = \| v' \|$ we can make small by additionally using (A-D'). Indeed, choosing $\epsilon'' > 0$, and employing (A-D'), we have

$$\| v' \| \leq \epsilon'' \| \tilde{x} - \tilde{x} \|,$$

provided $\| \tilde{x} - \tilde{x} \|$ is small enough. This can be guaranteed by (3.24) above. Thus, taking $\epsilon > 0$ small enough, we can make $\| v' \|$ arbitrarily small. This proves (3.20f), and shows that (3.26) can be made arbitrarily small. In summary, there exists $\delta'' \in (0, \delta')$ such that if (3.18a) holds for $\delta \in (0, \delta'')$, then

$$\| y' - y \|^2 + \| \tilde{x} - \tilde{x} \|^2 \leq \epsilon^2.$$ 

This proves (3.20b).

To prove the remaining estimates in (3.20), we require (3.18b) as well as the existence of $\tilde{u}$ and $\tilde{w}$. In fact, since $\text{int dom } G \neq \emptyset$ and $\text{int dom } F^* \neq \emptyset$, we may find a point $u' \in (\text{int dom } G) \times (\text{int dom } F^*)$ arbitrarily close to $\tilde{u}$. Then the set $H_{\ell}(u')$ is bounded. If $\delta > 0$ is small enough, we thus find from (3.16) that

$$\inf_{v \in \text{dom } H_{\ell}(v)} \| u' - v \| < \infty, \quad (\| w \| \leq \rho).$$

Minding that we have shown (3.20f), choosing $\epsilon > 0$ small enough and setting $w = 0$ resp. $w = -v'$ shows the existence of $\tilde{w} \in H_{\ell}^{-1}(0)$ resp. $v \in H_{\ell}^{-1}(-v')$. In fact, we have the existence of a minimizer $\tilde{u}$ to (3.19). This follows simply from $v \mapsto \| \tilde{w} - v \|$ being level bounded, $\partial G$ and $\partial F^*$ outer semicontinuous set-valued mappings, and $K_{\ell}$ a continuous linear operator.

We may now move on to the proof of (3.20e). By lemma 3.9, for any $\ell > \ell_{\ell^{-1}}$, there exist $\bar{\delta}, \bar{\delta} > 0$ such that

$$\inf_{v \in \text{dom } H_{\ell}(v)} \| u - v \| \leq \ell^* \| w - H_{\ell}(u) \|, \quad (\| u - \tilde{u} \| \leq \bar{\delta}, \ | \| w \| \leq \bar{w}).$$

Minding the choice of $\tilde{w}$ in (3.19), with $w = -v'$, $v = \tilde{w}$ and $u = \tilde{w}$, we thus have

$$\| \tilde{w} - \tilde{w} \| \leq \ell \| v' \|,$$

provided that $\| \tilde{w} - \tilde{w} \| \leq \bar{\delta}$ and $\| v' \| \leq \bar{w}$. The former follows from (3.18b) and taking $\delta$ small enough. The latter follows from (3.24), (3.27) and choosing $\epsilon' > 0$ small enough. In fact, taking $\epsilon' \leq \min\{\epsilon/(\ell \epsilon''), \bar{\rho}/\epsilon''\}$, we also show that $\| \tilde{w} - \tilde{w} \| \leq \epsilon$. This completes the proof of (3.20c).

Finally, to show (3.20d) and (3.20e) we simply bound

$$\| u' - \tilde{w} \| \leq \| u - u' \| + \| \tilde{u} - \tilde{w} \|, \quad \text{and}$$

$$\| \tilde{u} - \tilde{w} \| \leq \| u - \tilde{u} \| + \| \tilde{w} - \tilde{w} \||.$$

Then we use (3.18) and (3.20d)-(3.20c), assuming that these estimates hold to the higher accuracy $\epsilon/2$ instead of $\epsilon$. By the arguments above, this can be done by making $\delta > 0$ small enough.
3.6. Removing squares

With the Aubin property assumed, we are now able to remove the squares from $(D^2\text{-}M)$. Later in section 4 we will prove the Aubin property for important classes of $G$, $F^*$ and $K$.

Lemma 3.11. Suppose (A-K) holds, and that $H^{-1}_x$ has the Aubin property at 0 for $\hat{u}$. Pick $\ell^* > \ell_{H^{-1}_x}$. Then there exists $\delta > 0$ such that if (3.18) holds for $\delta$, and $(D^2\text{-}M)$ holds for $\tilde{\nu} \in H^{-1}_x(\nu')$ and some $\xi > 0$, then

$$
\|\tilde{\nu} - u\|_{M,v} \geq \xi \|u^{i+1} - u\|_{M,v} + \|\tilde{\nu} - u^{i+1}\|_{M,v},
$$

(\tilde{D}\text{-}M)

for $\xi := 1 - 1/\sqrt{1 + \xi_1/(\ell^*\Theta^2)^2}$. \hfill \Box

Proof. Choosing $\delta > 0$ small enough and applying lemma 3.9, we have for some $\rho', \delta' > 0$ that

$$
\inf_{\nu \in H^{-1}_x(\nu') \setminus H^{-1}_x(\nu)} \|u - \nu\| \leq \ell^* (w - H^x(u)), \quad (\|u - \tilde{u}\| \leq \delta', \|w\| \leq \rho').
$$

(3.28)

Lemma 3.10 provides a further upper bound on $\delta$ such that if (3.18) holds with such a $\delta$, then $\tilde{\nu} \in H^{-1}_x(\nu')$ exists,

$$
\|\nu\| \leq 3R/4, \quad \|u\| \leq 3R/4,
$$

(3.29a)

$$
\|\nu - u\| \leq \rho'/(2\Theta),
$$

(3.29b)

$$
\|\nu - u\| \leq \rho'/2, \quad \text{and}
$$

(3.29c)

$$
\|\tilde{\nu} - \tilde{u}\| \leq \delta'.
$$

(3.29d)

Moreover, $u^{i+1} \in H^{-1}_x(\nu')$ for $w = M_v(u^{i} - u^{i+1}) - \nu'$ is unique, as can be seen from the strictly convex problems (3.22), (3.25) for calculating $u^{i+1}$. Thus $H^{-1}_x(\nu')$ is single-valued. By (3.29a), (A-K) and lemma 3.1, we have $M_v \leq \Theta^2 I$. Therefore, using (3.29b) and (3.29c), we have

$$
\|u\| \leq \|M_v(u^{i} - u^{i+1})\| + \|\nu\| \leq \rho'.
$$

Minding (3.29d), we may thus apply (3.28) to get

$$
\|\tilde{\nu} - u^{i+1}\|_{M,v} \leq \Theta \|\tilde{\nu} - u^{i+1}\|
$$

$$
\leq \ell^* (\Theta \|M_v(u^{i} - u^{i+1}) - \nu\| + \|\nu\|)
$$

$$
\leq \ell^* \Theta^2 \|u^{i} - u^{i+1}\|_{M,v}.
$$

(3.30)

Squaring this and applying it to $(D^2\text{-}M)$, we get for $\lambda := \sqrt{1 + \xi_1/(\ell^*\Theta^2)^2}$ that

$$
\lambda \|\tilde{\nu} - u^{i+1}\|_{M,v} \leq \|\tilde{\nu} - u^{i}\|_{M,v},
$$

(3.31)

By application of $(D^2\text{-}loc)$, minding that $\lambda > 1$, we also have

$$
\|\tilde{\nu} - u^{i}\|_{M,v} = \lambda \|\tilde{\nu} - u^{i}\|_{M,v} - (\lambda - 1) \|\tilde{\nu} - u^{i}\|_{M,v}
$$

$$
\leq \lambda \|\tilde{\nu} - u^{i}\|_{M,v} - (\lambda - 1) \|u^{i+1} - u^{i}\|_{M,v}.
$$

(3.32)

Together, (3.31) and (3.32) give $(D\text{-}M)$ for $\xi := 1 - 1/\lambda$. \hfill \Box
3.7. Bridging local solutions

In order to finalize the proof of (D), we will now use the perturbed local linearized solution \( \tilde{\alpha} \) to bridge between the local linearized solutions \( \tilde{\alpha} \) and \( \tilde{\alpha}^{i+1} \). For this we again need the Aubin property of \( H_{\tilde{\alpha}}^{-1} \).

**Lemma 3.12.** Let \( \kappa \) and \( L_2 \) be as in (3.3). Assume that (A-D\( ^0 \)) and (A-K) hold, and that \( H_{\tilde{\alpha}}^{-1} \) has the Aubin property at 0 for \( \tilde{\alpha} \). Suppose, moreover, that (D-M) holds for any choice of \( \tilde{\alpha} \in H_{\tilde{\alpha}}^{-1}(-\nu') \), and that

\[
\ell H_{\tilde{\alpha}}^{-1} \kappa L_2 p_{NL} \tilde{\alpha} < \zeta_2. 
\]

(3.33)

Under these conditions there exist \( \delta > 0 \) and \( \zeta_3 \in (0, 1] \) such that if \( \tilde{\alpha}^{i+1} \in S_{\nu}^{-1}(0) \) and \( \tilde{\alpha} \in H_{\tilde{\alpha}}^{-1}(0) \) are given and (3.18) holds with this \( \delta \), then there exists \( \tilde{\alpha}^{i+1} \in H_{\tilde{\alpha}}^{-1}(0) \) satisfying

\[
\|\tilde{\alpha} - \tilde{\alpha}^{i+1}\|_{M_{\tilde{\alpha}}} \geq \zeta_3 \|\tilde{\alpha}^{i+1} - u\|_{M_{\tilde{\alpha}}} + \|\tilde{\alpha}^{i+1} - u^{i+1}\|_{M_{\tilde{\alpha}^{i+1}}}. 
\]

(3.34)

That is, the descent inequality (D) holds.

**Proof.** Suppose there exists \( \eta \in (0, \zeta_2) \), independent of \( i \), and some \( \tilde{\alpha}^i \in H_{\tilde{\alpha}}^{-1}(-\nu') \) and \( \tilde{\alpha}^{i+1} \in H_{\tilde{\alpha}}^{-1}(0) \), such that the double sensitivity estimate

\[
\|\tilde{\alpha}^i - \tilde{\alpha}^{i+1}\|_{M_{\tilde{\alpha}^{i+1}}} + \|\tilde{\alpha} - \tilde{\alpha}^{i+1}\|_{M_{\tilde{\alpha}}} \leq \eta \|u^i - u^{i+1}\|_{M_{\tilde{\alpha}}}. 
\]

(3.35)

holds. Then we may proceed as follows. We recall that by (D-M), we have

\[
\|\tilde{\alpha}^i - u^i\|_{M_{\tilde{\alpha}}} \geq \zeta_3 \|u^i - u\|_{M_{\tilde{\alpha}}} + \|\tilde{\alpha}^i - u^{i+1}\|_{M_{\tilde{\alpha}^{i+1}}}. 
\]

Two applications of the triangle inequality give

\[
\|\tilde{\alpha}^i - u^i\|_{M_{\tilde{\alpha}}} \geq (\zeta_3 \|u^i - u\|_{M_{\tilde{\alpha}}} + \|\tilde{\alpha}^i - u^{i+1}\|_{M_{\tilde{\alpha}^{i+1}}}) + \|\tilde{\alpha}^i - u^{i+1}\|_{M_{\tilde{\alpha}^{i+1}}}. 
\]

Employing the sensitivity estimate (3.35), we now get (3.34) with \( \zeta_3 := (\zeta_2 - \eta) > 0 \).

We still have to show (3.35). Using lemma 3.10, with \( \delta > 0 \) small enough, we may take

\[
\tilde{\alpha} \in \arg \min \{\|\tilde{\alpha} - v\|_H | v \in H_{\tilde{\alpha}}^{-1}(-\nu')\}. 
\]

We then have \( E - \nu' \in H_{\tilde{\alpha}^{i+1}}(\tilde{\alpha}) \) for

\[
E := E(\tilde{\alpha}^i, x^{i+1}; x') := \frac{(K_{\alpha}^{i+1} - K_{\alpha})^{\gamma}}{(K_{\alpha} - K_{\alpha}^{i+1})^{\gamma} + c_{\alpha} - c_{\alpha}^{i+1}}. 
\]

We pick \( \epsilon^* > \ell H_{\tilde{\alpha}}^{-1} \). Again with \( \delta > 0 \) small enough, an application of lemma 3.9 yields for some \( \rho' \), \( \delta^* > 0 \) that

\[
\inf_{v \in H_{\tilde{\alpha}}(v)} \|u - v\| \leq \epsilon^* \|w - H_{\tilde{\alpha}}(u)\|, \quad (\|u - \tilde{\alpha}\| \leq \delta', \|w\| \leq \rho'). 
\]

(3.36)

With \( w = -\nu' \), \( v = \tilde{\alpha} \), and \( u = \tilde{\alpha} \), minding that \( \tilde{\alpha} \in H_{\tilde{\alpha}}^{-1}(0) \), we thus have

\[
\|\tilde{\alpha}^i - \tilde{\alpha}^{i+1}\|_{M_{\tilde{\alpha}}} \leq \epsilon^* \Theta \|
u'\|, 
\]

(3.37)

provided that \( \|\tilde{\alpha}^i - \tilde{\alpha}^{i+1}\| \leq \delta^* \) and \( \|\nu'\| \leq \rho' \). These conditions are guaranteed by lemma 3.10 and choosing \( \epsilon, \delta > 0 \) small enough.

With \( \epsilon, \delta > 0 \) small, applying (3.18) and (3.20b), we can also make \( \|\tilde{\alpha} - u^{i+1}\| \) small enough that another application lemma 3.9 guarantees the existence of

\[
\tilde{\alpha}^{i+1} \in \arg \min \{\|\tilde{\alpha}^{i+1} - v\|_H | v \in H_{\tilde{\alpha}^{i+1}}^{-1}(0)\}, 
\]

and shows

\[
\inf_{v \in H_{\tilde{\alpha}^{i+1}}(v)} \|u - v\| \leq \epsilon^* \|w - H_{\tilde{\alpha}^{i+1}}(u)\|, \quad (\|u - \tilde{\alpha}\| \leq \delta', \|w\| \leq \rho'). 
\]

20
We may assume that \( \rho', \delta' > 0 \) are the same as in (3.36). With \( w = 0 \), \( v = \hat{u}^{i+1} \) and \( u = \bar{u} \), we obtain
\[
\|\hat{u} - \hat{u}^{i+1}\|_{M_{\rho,1}} \leq \ell^* \Theta (\|E\| + \|v'\|),
\] (3.38)
provided \( \|\hat{u} - \bar{u}\| \leq \delta' \). This condition was already verified for (3.37).

To start approximating \( \|E\| \) and \( \|v'\| \), we use lemma 3.3 with \( \bar{x} = x^{i+1} \) and \( \bar{x}' = x' \), to obtain
\[
\|E\| \leq L_2 \|x' - x^{i+1}\| \left( \|P_{\text{NL}}(\bar{y})\| + \|\bar{y}' - x'|\| + \|x' - x^{i+1}\| \right).
\] (3.39)
We approximate
\[
\|P_{\text{NL}}(\bar{y})\| \leq \|P_{\text{NL}}(\bar{y})\| + \|\bar{y} - \bar{y}'\|
\]
and
\[
\|\bar{y}' - x'|\| \leq \|x' - \bar{x}\| + \|\bar{x} - \bar{x}'\|
\]
Inserting these estimates back into (3.39), it follows for some constant \( C > 0 \) that
\[
\|E\| \leq L_2 \|x' - x^{i+1}\| A,
\] (3.40)
where
\[
A := \|P_{\text{NL}}(\bar{y})\| + C \|\bar{u} - \bar{u}\| + \|u' - \bar{u}\| + \|x' - x^{i+1}\|
\]
Using (A-D), we can for any \( \epsilon > 0 \) find \( \delta'' > 0 \) such that
\[
2 \|v'\| \leq \epsilon L_2 \|x' - x^{i+1}\|, \quad (\|u' - u^{i+1}\| < \delta'').
\]
The condition \( \|u' - u^{i+1}\| < \delta'' \) can be guaranteed through lemma 3.10 and choosing \( \delta > 0 \) small enough. Thus, using (3.37), (3.38) and (3.40), we have
\[
\|\hat{u} - \hat{u}^{i+1}\|_{M_{\rho,1}} + \|\bar{u}' - \hat{u}'\|_{M_{\rho}} \leq 2 \ell^* \Theta \|v'\| + \ell^* \Theta L_2 \|x' - x^{i+1}\| A
\]
\[
\leq \ell^* \Theta L_2 \|x' - x^{i+1}\| (A + \epsilon)
\]
\[
\leq \ell^* \Theta L_2 (A + \epsilon) \|u' - u^{i+1}\|_{M_{\rho}}.
\]
In order to prove (3.35), we thus need to force \( \eta := \ell^* \Theta L_2 (A + \epsilon) < \zeta_2 \). As \( \epsilon > 0 \) and \( \ell^* > \ell_{H^1} \) were arbitrary, it suffices to show \( \ell_{H^1} \Theta L_2 A < \zeta_2 \). Minding (3.33), we have
\[
0 < \zeta' := \zeta_2 - \ell_{H^1} \Theta L_2 \|P_{\text{NL}}(\bar{y})\|
\]
Thus it remains to force
\[
C \|\bar{u} - \hat{u}\| + \kappa \|u' - \bar{u}\| + \|x' - x^{i+1}\| < \zeta' / (\ell_{H^1} \Theta L_2).
\]
By lemma 3.10, this holds for \( \delta > 0 \) small enough. Thus (3.35) holds, and we may conclude the proof.

\[ \square \]

3.8. Combining the estimates

**Lemma 3.13.** Suppose (A-K) holds, and that given any choice of \( \hat{u}^i \in H^{-1}_{\text{reg}} (0) \), (D) holds for some \( \hat{u}^{i+1} \in H^{-1}_{\text{reg}} (0) \), \( i = 1, \ldots, k - 1 \). Suppose, moreover, that \( 0 \in H_{\text{reg}}(\hat{u}) \) and that \( H_{\text{reg}}^{-1} \) has the Aubin property at \( 0 \) for \( \hat{u} \). In this case, given \( \epsilon > 0 \), there exists \( \delta_1 > 0 \), independent of \( k \), such that if
\[
\|u' - \bar{u}\| \leq \delta_1,
\] (3.41)
then there exists some \( \hat{u}^i \in H^{-1}_{\text{reg}} (0) \) satisfying the bounds
\[
\|\hat{u} - \bar{u}\| \leq \epsilon,
\] (3.42a)
\[
\|\hat{u} - \hat{u}'\| \leq \epsilon,
\] (3.42b)
\[
\|\hat{u} - \bar{u}\| \leq \epsilon.
\] (3.42c)
Proof. First of all, we require that $\delta_1 \in (0, R/4)$, so that $\|u^1\| \leq 3R/4$. We then show that
\[
\|\hat{u}^1 - u^1\| \leq c\delta_1 \tag{3.43}
\]
for a choice of $\hat{u}^1 \in H_{i,1}^{-1}(0)$ and some constant $c > 0$. Indeed, letting $w := E(\hat{\nu}; u^1, \hat{\nu}) \in H_{i,1}(\hat{\nu})$ and choosing $\delta_1 \in (0, \delta)$ for $\delta$ small enough, by lemma 3.9 we have
\[
\inf_{\hat{\nu} \in H_{i,1}^{-1}(0)} \|\hat{\nu} - \hat{u}^1\| \leq \ell \|w\|.
\]
Referring to lemma 3.3, there exists $\delta' > 0$ such that if $\delta_1 \in (0, \delta')$, and (3.41) holds, then $\ell \|w\| < L_2(R + 1)\|u^1 - \hat{\nu}\|$. Consequently we see that (3.43) by estimating
\[
\|\hat{u}^1 - u^1\| \leq \|u^1 - \hat{\nu}\| + \|\hat{\nu} - \hat{u}^1\| \leq (1 + L_2(R + 1))\delta_1.
\]
Choosing $\delta_1 \leq \delta' := \min\{\delta, \delta', R/4, R\zeta/(4\kappa \zeta)\}$, lemma 3.4 now shows that
\[
\|u^k - u^1\| \leq (\kappa/\zeta)\|\hat{u}^1 - u^1\|, \quad \text{and}
\]
\[
\|\hat{u}^k - \hat{u}^1\| \leq \kappa\|u^1 - \hat{\nu}\|. \tag{3.44}
\]
Thus, using (3.41), (3.43), and (3.44), we get
\[
\|\hat{u}^k - \hat{\nu}\| \leq \|\hat{u}^1 - u^1\| + \|u^1 - \hat{\nu}\| \leq (\kappa \zeta + 1)\delta_1.
\]
Likewise (3.41), (3.43), and (3.45) give
\[
\|\hat{u}^k - u^1\| \leq \kappa\|\hat{u}^1 - u^1\| \leq c\kappa \delta_1.
\]
Finally, these two estimates give
\[
\|\hat{u}^k - \hat{\nu}\| \leq \|\hat{u}^1 - \hat{u}^k\| + \|\hat{u}^k - \hat{\nu}\| \leq C\delta_1
\]
for $C := 1 + \kappa \zeta(1 + 1/\zeta)$. Choosing $\delta_1 \leq \min\{\delta', \epsilon/C\}$, we get (3.42). \qed

The following two theorems form our main convergence result.

**Theorem 3.1.** Let $R, \Theta, \kappa$ and $L_2$ be as in (3.3). Suppose (A-D') and (A-K) hold, $\omega = 1$, and that $F^*$ is strongly convex on the subspace $Y_{\text{NL}}$. Let $\hat{\nu}$ solve $0 \in H_{2}(\hat{\nu})$, and $H_{i,1}^{-1}$ has the Aubin property at 0 for $\hat{\nu}$ with
\[
\ell H_{i,1}^{-1} L_2 \|B_{\text{NL}}\hat{\nu}\| < 1 - 1/\sqrt{1 + 1/(2\hat{H}_{i,1}^2, \Theta^2)}. \tag{3.46}
\]
Under these conditions, there exist $\delta_1 > 0$ and $\zeta \in (0, 1)$, such that if the initial iterate $u^1 \in X \times Y$ satisfies
\[
\|u^1 - \hat{\nu}\| \leq \delta_1, \tag{3.47}
\]
and we let $u^{i+1} \in S^{-1}_i(0), (i = 1, 2, 3, \ldots)$, then ($\hat{D}$) holds, i.e.,
\[
\|u^i - \hat{\nu}\|_{M_i} \geq \zeta \|u^{i+1} - u^i\|_{M_{i+1}} + \|u^{i+1} - \hat{\nu}\|_{M_{i+1}}, \quad (i = 1, 2, 3, \ldots).
\]

Proof. We prove the claim inductively using lemma 3.13. Indeed, we let $\delta > 0$ be small enough for (3.18) to be satisfied for application in lemmas 3.11 and 3.12, and for (3.10) to be satisfied for lemma 3.6. Then we first we apply lemma 2.1 to get the estimate ($\hat{D}^{\hat{2}}$-loc-$\gamma$). Then we pick $\zeta_1 = 1/2$ in lemma 3.6, so that
\[
\zeta_2 = 1 - 1/\sqrt{1 + 1/(2\hat{H}_{i,1}^2, \Theta^2)}.
\]
Choosing $\zeta > \zeta H_{i,1}$ small enough, (3.46) then guarantees (3.33). Inductively assuming that ($\hat{D}$) holds for $i = 1, \ldots, k - 1$, we use lemma 3.13 to show that
\[
\|u^i - \hat{\nu}\| \leq \delta \quad \text{and} \quad \|\hat{u}^k - \hat{\nu}\| \leq \delta,
\]
provided $\delta_1 > 0$ is small enough. We then use lemma 3.6 to show that the squared local norm descent inequality ($\hat{D}^{\hat{2}}$-$M$) holds. Next we apply lemma 3.11 to show that unsquared local descent inequality ($\hat{D}^{\hat{2}}$-$M$) holds. Finally, we employ lemma 3.12 to derive ($\hat{D}$) for $i = k$. As $k$ was arbitrary, we may conclude the proof. \qed

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Theorem 3.2. Suppose the conditions of theorem 3.1 hold for \( \tilde{u} = (\tilde{x}, \tilde{y}) \) and \( u^1 = (x^1, y^1) \). In that case there exists \( \delta_1 > 0 \) such that provided the initialization \( u^1 \) satisfies \( \|u^1 - \tilde{u}\| \leq \delta_1 \), then the iterates \( (x', y') \) produced by algorithm 2.1 or algorithm 2.2 converge to a solution \( u^* = (x^*, y^*) \) of (2.1), i.e., a critical point of the problem \( (P) \).

Proof. We pick \( \delta_1 > 0 \) small enough that (3.47) is satisfied, and that lemma 3.13 guarantees the assumption (3.5) of lemma 3.5. Then theorem 3.1 and lemma 3.5 verify the assumptions of the general convergence result theorem 2.1, from which the claim follows.

3.9. Some remarks

Remark 3.2 (Convergence to another solution). In principle the solution \( u^* \) discovered in theorem 3.2 may be distinct from \( \tilde{u} \), also solving (2.1).

Remark 3.3 (Small nonlinear dual). The condition (3.46) forces \( \|P_{NL}(\tilde{x})\| \) to be small. As we will further discuss in section 4.3, in the applications that we are primarily interested in, involving solving \( \min \|f - T(x)\|^2/2 + \alpha R(x) \), this corresponds to \( \|f - T(\tilde{x})\| \) being small. This says that the noise level of the data \( f \) and regularization parameter \( \alpha \) have to be low.

Remark 3.4 (Inexact solutions). It is possible to accommodate for inexact solutions \( u^{i+1} \) in the proof, if we relax the requirement \( D'(u') = 0 \) to \( D'(u') \rightarrow 0 \), \( \|D'(u')\| \leq \epsilon \), and take \( \tilde{u} \) to solve \( 0 \in H_f(\tilde{u}) + D'(u') \). Since this involves significant additional technical detail that complicates the already very technical proof, we have opted not to include this generalization.

Remark 3.5 (Switch of local metric). The shift to the new local metric \( M_{\sigma^{i+1}} \), done in lemma 3.6 using the strong convexity of \( F^* \) on \( Y_{NL} \), can also be done similarly to the removal of squares in lemma 3.11. This suggests that the strong convexity might not be necessary. In practice we however need the strong convexity for the Lipschitz continuity, so there is little benefit from that. Moreover, the required strong convexity exists in case of regularization problems of the form discussed in remark 3.3. As we are primarily interested in applying the method to such problems, assuming strong convexity is natural.

Remark 3.6 (Varying step lengths). Strictly speaking, we do not need the whole force of (A-K) for the bound (C-M). We can allow for \( \sigma, \tau \) vary on each iteration, and even preconditioning operators in place of simple step length parameters, cf [18]. Indeed, with \( \sigma^i, \tau^i \) dependent on \( u^i \), we see that \( \theta^2 I \leq M_e \), if

\[
\frac{1}{2} \theta^2 \leq \frac{1}{\sigma^i \tau^i} - \|K_e\|^2.
\]

If now \( \epsilon, \theta > 0 \) are such that \( \epsilon \|K_e\|^2 > \theta^2/2 > 0 \), we see that it suffices to have

\[
\sigma^i \tau^i (1 + \epsilon) \|K_e\|^2 < 1.
\]

(3.48)

Likewise \( M_e \leq \Theta^2 I \) if \( \sigma^i, \tau^i \) are bounded from below, and \( \|K_e\| \infty \) is bounded. Since \( \|u^i\| \leq R \), the latter follows from the assumption \( K \in C^2(X; Y) \). Likewise, \( \|K_e\| \infty \) is bounded away from zero if

\[
d := \inf_{\|v\| \leq R} \|\nabla K(x)\| > 0.
\]

(3.49)

This is the case with the operators in section 5. If (3.49) is satisfied, to obtain (C-M), it thus suffices to choose \( \sigma^i \) and \( \tau^i \) such that (3.48) holds for fixed \( \epsilon > 0 \), and to choose \( \theta \) such that \( \epsilon d > \theta^2/2 > 0 \).
This alone is however not enough to show convergence of the method with varying step lengths. There is one further difficulty in the switch of local norms from $\| \cdot \|_{M_j}$ to $\| \cdot \|_{M_{j+1}}$ in lemma 3.6. Specifically, the first equality in (3.12) does not hold. Defining

$$M'_{\nu+1} := \left( \frac{1/\nu}{1/\nu} - \frac{K_{\nu}}{I/\sigma} \right),$$

and otherwise using $\sigma = \sigma^i$ and $\tau = \tau^i$ in the definition of $M_{\nu}$ for each $i$, we can however calculate

$$\| u^{i+1} - \tilde{u} \|^2_{M_{\nu+1}} - \| u^{i+1} - \tilde{u} \|^2_{M_{\nu+1}} \geq \left( \frac{1}{\tau^i} - \frac{1}{\tau^{i+1}} \right) \| x^{i+1} - \tilde{x} \|^2 + \left( \frac{1}{\sigma^i} - \frac{1}{\sigma^{i+1}} \right) \| y^{i+1} - \tilde{y} \|^2.$$

Then estimating $\| u^{i+1} - \tilde{u} \|^2_{M_{\nu}} - \| u^{i+1} - \tilde{u} \|^2_{M_{\nu+1}}$ similarly to (3.12), we can prove following lemma 3.6 that

$$\| u^{i+1} - \tilde{u} \|^2_{M_{\nu}} \geq \frac{\zeta_1}{2} \| u^{i+1} - \tilde{u} \|^2_{M_{\nu+1}} + \| u^{i+1} - \tilde{u} \|^2_{M_{\nu+1}}$$

provided

$$\left| \frac{1}{\tau^i} - \frac{1}{\tau^{i+1}} \right|, \left| \frac{1}{\sigma^i} - \frac{1}{\sigma^{i+1}} \right| \leq (1 - \zeta_1)\theta / 2.$$  (3.50)

Since $x \mapsto \| K_x \|$ is Lipschitz close to $\tilde{x}$, if we start close enough to $\tilde{u}$ that $\| x^i - x^{i+1} \|$ stays small, (3.50) can be made to hold for good choices of $\tau^i, \sigma^i$. Indeed, let us choose $\epsilon, \tau_0, \sigma_0 > 0$ such that $\tau_0 \tau_0 < 1/(1 + \epsilon)$, and then

$$\tau^i := \tau_0 / L^i, \quad \sigma^i := \sigma_0 / L^i, \quad \text{for} \quad L^i := \sup_{j=1, \ldots, i} \| K_{\nu} \|. \quad (3.51)$$

Then (3.48) holds, $\{ \tau^i \}_{i=1}^\infty$ and $\{ \sigma^i \}_{i=1}^\infty$ are decreasing, and (3.50) amounts to

$$| L^i - L^{i+1} | \leq \frac{(1 - \zeta_1) \theta}{2 \max(\sigma_0, \tau_0)}.$$

By the above considerations, this holds if we initialize $u^i$ close enough to $\tilde{u}$.

4. Lipschitz estimates

We now need to show the Aubin property of the inverse $H_{\nu}^{-1}$ of the set-valued map $H_{\nu}$, and to bound $\| P_{H_{\nu}} \|$. We will calculate $\ell_{H_{\nu}^{-1}}(0)\tilde{u}$ through the Mordukhovich criterion, which brings us to the topic of graphical differentiation of set-valued maps. We will introduce the necessary tools in section 4.1, following [5]; another treatment also covering the infinite-dimensional case can be found in [19]. Afterwards we derive bounds on $\ell_{H_{\nu}^{-1}}(0)\tilde{u}$ for some general class of maps in section 4.2. These will then be used in the following sections 4.3 and 4.4 to study important special cases. These include in particular TV and total generalized variation (TGV2) [20] regularization.

4.1. Differentials of set-valued maps

Let $S : X \rightrightarrows Y$ be a set-valued map between finite-dimensional Hilbert spaces $X, Y$. The graph of $S$ is

$$\text{Graph } S := \{ (x, y) \mid y \in H(x) \}.$$

The outer limit at $x$ is defined as

$$\limsup_{x \to y} S(x) := \{ y \in Y \mid \text{there exist } x' \to x \text{ and } y' \in S(x') \text{ with } y' \to y \}.$$
The inner limit is defined as
\[
\liminf_{x \to x} S(x) := \{ y \in Y \mid \text{for every } x' \to x \text{ there exist } y' \in S(x') \text{ with } y' \to y \}.
\]

Pick \( x \in X \) and \( y \in S(x) \). The graphical derivative of \( S \) at \( x \) for \( y \), denoted \( DS(x|y) : X \rightrightarrows Y \), is defined by
\[
DS(x|y)(w) = \limsup_{\tau \downarrow 0, u' \to w} \frac{S(x + \tau u') - y}{\tau}.
\]

Geometrically, Graph \( DS(x|y) \) is the tangent cone to Graph \( S \) at \( (x, y) \). If \( S \) is single-valued and differentiable, then \( DS(x|y) = \nabla S(x) \) for \( y = S(x) \). Observe that \( DS(x|y) \) satisfies
\[
z \in DS(x|y)(w) \iff w \in D(S^{-1})(y|z).
\]

There are also various other definitions of differentials for set-valued maps. In particular, the regular derivative of \( S \) at \( x \) for \( y \), denoted \( \tilde{DS}(x|y) : X \rightrightarrows Y \), is defined by
\[
\tilde{DS}(x|y) = \liminf_{(x', y') \to (x, y)} \text{Graph}(DS(x'|y')).
\]

The importance of the regular derivative to us lies in following. The map \( S \) is said to be graphically regular at \( (x, y) \) if \( \tilde{DS}(x|y) = DS(x|y) \). We stress that this correspondence does not hold generally. If it does, we may express the coderivative \( D^*S(x|y) \) as
\[
D^*S(x|y) = [DS(x|y)]^{\perp^+},
\]
where
\[
H^{\perp^+}(w) := \{ z \mid \langle z, q \rangle \leq \langle w, v \rangle \text{ when } v \in H(q) \}.
\]

Geometrically, Graph \( D^*S(x|y) \) is a normal cone to Graph \( S \) at \( (x, y) \) rotated such that it becomes adjoint to \( DS(x|y) \) in the sense (4.1). Without graphical regularity, the coderivative has to be defined through other means [5]; we will however always assume graphical regularity.

In our forthcoming analysis, we will occasionally employ the tangent and normal cones to a convex set \( A \) at \( y \). These are denoted \( T_A(y) \) and \( N_A(y) \), respectively.

Finally, with the above concepts defined, we may state a version of the Mordukhovich criterion [5, theorem 9.40] sufficient for our purposes.

**Theorem 4.1.** Let \( S : X \rightrightarrows Y \). Suppose Graph \( S \) is locally closed at \( (x, y) \) and
\[
D^*S(x|y)(0) = [0].
\]

Then
\[
\ell_S(x|y) = [D^*S(x|y)]^{\perp^+},
\]
where the outer norm
\[
[H]^+ := \sup_{\|w\| \leq 1} \sup_{z \in H(w)} \|z\|.
\]

We want to translate this result to be stated in terms of \( DS(x|y) \) for \( S^{-1} \), since the graphical derivative is easier to obtain than the coderivative.

**Proposition 4.1.** Suppose \( S \) is graphically regular and Graph \( S \) locally closed at \( (x, y) \). Then
\[
\ell_{S^{-1}}(y|x) \leq \sup \{ \|z\| \mid \langle z, q \rangle \leq \|v\| \text{ when } q \in DS(x|y)(v) \}.
\]

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Proof. From (4.1), we have that \( z \in D^*S^{-1}(y|x)(w) \) if \( \langle z, q \rangle \leq \langle w, v \rangle \) when \( (q, v) \) satisfy \( v \in DS^{-1}(y|x)(q) \).

By the symmetricity of graphical differentials for \( S \) and \( S^{-1} \), this is the same as \( \langle z, q \rangle \leq \langle w, v \rangle \) when \( (q, v) \) satisfy \( q \in DS(x|y)(v) \).

Thus, if (4.2) is satisfied, theorem 4.1 gives

\[
\ell_{S^{-1}}(y|x) = \sup_{\|z\| \leq 1} \sup \{ \|z\| \mid z \in D^*S^{-1}(y|x)(w) \}.
\]

\[
= \sup_{\|z\| \leq 1} \sup \{ \|z\| \mid \langle z, q \rangle \leq \langle w, v \rangle \text{ when } q \in DS(x|y)(v) \}
\]

\[
\leq \sup \{ \|z\| \mid \langle z, q \rangle \leq \|v\| \text{ when } q \in DS(x|y)(v) \}.
\]

If (4.2) is not satisfied, then by (4.1) the supremum in (4.4) is infinite. Thus (4.4) holds whether (4.2) holds or not.

\[\square\]

4.2. Bounds on Lipschitz factors

We now want to approximate the local Lipschitz factor \( \ell_{H^{-1}}(0)\hat{u}(v) \) of \( H^{-1} \) at 0 for \( \hat{u} \). We apply proposition 4.1, assuming that \( H \) is graphically regular and Graph \( H \) is locally closed at \( (\hat{u}, 0) \). Then

\[
\ell_{H^{-1}}(0)\hat{u}(v) \leq \sup \{ \|z\| \mid \langle z, q \rangle \leq \|v\| \text{ when } q \in DH_{\hat{u}}(0)(v) \}.
\]

Writing \( u = (x, y) \) and \( v = (\xi, \nu) \), we have

\[
DH_{\hat{u}}(0)(v) = \left( DG(\hat{u}) - K_i^\top\hat{y}(\xi) + K_i^\top v \right)
\]

\[
(4.5)
\]

Suppose there exist self-adjoint linear maps \( G : X \rightarrow X \) and \( F : Y \rightarrow Y \) and (possibly trivial) subspaces \( V_G \subset X \) and \( V_F \subset Y \) such that

\[
DG(\hat{u}) - K_i^\top\hat{y}(\xi) = \begin{cases} G\xi + V_G^\perp, & \xi \in V_G \\ \emptyset, & \xi \notin V_G \end{cases}, \quad \text{and}
\]

\[
DF(\hat{y})K_i^\top + c_G(v) = \begin{cases} Fv + V_F^\perp, & v \in V_F \\ \emptyset, & v \notin V_F \end{cases}
\]

Then

\[
DH_{\hat{u}}(0)(v) = \begin{cases} Av + V^\perp, & v \in V \\ \emptyset, & v \notin V \end{cases}, \quad (4.6)
\]

for

\[
A = \begin{pmatrix} G & K_i^\top \\ \emptyset & F \end{pmatrix}, \quad \text{and} \quad V = V_G \times V_F.
\]

With \( P_V \) the orthogonal projection operator into \( V \), this allows us to approximate

\[
\ell_{H^{-1}}(0)\hat{u}(v) \leq \sup \{ \|z\| \mid \langle z, Av + p \rangle \leq \|v\| \text{ when } v \in V, \ p \in V^\perp \}
\]

\[
= \sup \{ \|z\| \mid \|P_VA^*z\| \leq 1 \text{ when } z \in V \}
\]

\[
= \sup \{ \|P_Vz\| \mid \|P_VA^*P_Vz\| \leq 1 \}
\]

\[
\leq \inf \{ c^{-1} \mid c\|P_Vz\| \leq \|P_VA^*P_Vz\| \text{ for all } z \}.
\]

(4.7)

In the following lemma, we show that \( c > 0 \). To do so, we have to assume various forms of boundedness from the involved operators. We introduce the operator \( \Xi \) as a way to make the estimate hold for a range of regularization parameters; the details of the procedure will follow the lemma in section 4.3.
Lemma 4.1. Let $\Xi : X \times Y \to X \times Y$ be a self-adjoint positive definite linear operator, $\Xi(x, y) = (\Xi_G(x), \Xi_F(y))$. Suppose that $\Xi_G$ commutes with $\overline{G}$ and $P_{V_G}$, that $\Xi_F$ commutes with $\overline{F}$ and $P_{V_F}$, and that one of the following conditions hold.

1. $\overline{G} \geq \gamma \Xi_G$ and $\overline{F} \geq \gamma \Xi_F$ for some $\gamma > 0$.
2. $\overline{G} = 0$, $V_G = X$, $\Xi_G = I$ and $M \Xi_F \geq \overline{F}$ for some $M$, $\gamma > 0$, as well as $\|P_{V_G} K_G \Xi \| \geq a \| \Xi \|, \quad (\xi \in X)$.

Then there exists a constant $c = c(M, \gamma, a)$, such that $\|P_{V_G} A^+ P_{V_G} \| \geq c \| \Xi P_{V_G} \| \quad (z \in X \times Y)$.

Proof. Let us write $A_Y := P_{V_G} A_{V_G} P_{V_G}$ for $A_{V_G} := P_{V_G} G P_{V_G}$, $F_{V_G} := P_{V_G} F P_{V_G}$, and $K_{V_G} := P_{V_G} K_G P_{V_G}$. Then $z = (\zeta, \eta)$ satisfies

$$\|A_{V_G} \Xi \| = \|A_{V_G} \Xi - K_{V_G} \Xi\| + \|F_{V_G} \Xi + K_{V_G} \Xi\|^2$$

$$= \|A_{V_G} \Xi\|^2 + \|K_{V_G} \Xi\|^2 - 2\langle A_{V_G} \Xi, K_{V_G} \Xi \rangle + \|K_{V_G} \Xi\|^2 + \|F_{V_G} \Xi\|^2 + 2\langle K_{V_G} \Xi, F_{V_G} \Xi \rangle.$$

We consider point (i) first. We write $\overline{G} = G_G + \gamma \Xi_G$ and $\overline{F} = F_G + \gamma \Xi_F$ for $\Xi_{V_G} := \Xi_G P_{V_G}$ and $\Xi_{V_F} := \Xi_G P_{V_F}$. Observe that the self-adjointness and positivity of $\Xi_{V_G}$ and the commutativity with $\overline{G}$ yield

$$\langle \Xi_{V_G} \zeta, G_G \zeta \rangle = \langle \Xi_{V_G}^{1/2} \zeta, G_G \Xi_{V_G}^{1/2} \zeta \rangle \geq 0.$$ Therefore

$$\|G_G \zeta\|^2 - 2\langle G_G \zeta, K_G \eta \rangle + \|K_G \eta\|^2 = \|G_G \zeta\|^2 + 2\gamma \langle \Xi_{V_G} \zeta, G_G \zeta \rangle + \gamma^2 \|\Xi_{V_G} \zeta\|^2$$

$$- 2\langle G_G \zeta, K_G \eta \rangle - 2\gamma \langle \zeta, K_G \eta \rangle + \|K_G \eta\|^2$$

$$\geq \gamma^2 \|\Xi_{V_G} \zeta\|^2 - 2\gamma \langle \zeta, K_G \eta \rangle.$$ Analogously

$$\|K_G \zeta\|^2 + 2\langle K_G \zeta, F_G \eta \rangle + \|F_G \eta\|^2 \geq \gamma^2 \|\Xi_{V_F} \eta\|^2 + 2\gamma \langle K_G \zeta, \eta \rangle,$$

so that

$$\|A_{V_G} \Xi\|^2 \geq \left( \gamma^2 \|\Xi_{V_G} \zeta\|^2 - 2\gamma \langle \zeta, K_G \eta \rangle \right) + \left( \gamma^2 \|\Xi_{V_F} \eta\|^2 + 2\gamma \langle K_G \zeta, \eta \rangle \right) = \gamma^2 \|P_{V_G} \Xi\|^2.$$

This shows that (4.9) holds with $c = \gamma$ in case (i).

Consider then case (ii). Now $P_{V_G} = I$, so $K_G = P_{V_G} K_G$ and $\Xi_{V_G} = I$. Let us pick arbitrary $\gamma \in (0, \gamma/2)$. Then $\overline{F} = F_G + \gamma \Xi_{V_F}$. Expanding, we have

$$\|A_{V_G} \Xi\|^2 = \|K_G \Xi\|^2 + \|K_G \Xi\|^2 + \|F_G \Xi\|^2 + 2\langle K_G \zeta, F_G \eta \rangle$$

$$= \|K_G \Xi\|^2 + \|K_G \Xi\|^2 + \|F_G \Xi\|^2$$

$$+ 2\gamma \langle \Xi_{V_F} \eta, F_G \eta \rangle + \gamma^2 \|\Xi_{V_F} \eta\|^2 + 2\gamma \langle K_G \zeta, F_G \eta \rangle + 2\gamma \langle K_G \zeta, \eta \rangle.$$ Using the Young’s inequalities

$$2\gamma \langle K_G \zeta, \eta \rangle \leq \gamma^2 \|\zeta\|^2 + \|K_G \eta\|^2,$$

and

$$2\gamma \langle K_G \zeta, F_G \eta \rangle \leq \mu \|K_G \zeta\|^2 + (1/\mu) \|F_G \eta\|^2, \quad (\mu > 0),$$

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give
\[ \|A_v z\|^2 \geq (1 - \mu) \|K_v \xi\|^2 + (1 - 1/\mu) \|F_v \eta\|^2 + 2\gamma (\|\Sigma_{V,F} \eta, F_v \eta\| + \gamma^2 \|\Sigma_{V,F} \eta\|^2 - \gamma^2 \|\xi\|^2). \]

(4.10)

Observe, that the self-adjointness and positivity of \( \Sigma_{V,F} \) and the commutativity with \( \overline{F} \) yield
\[ M \Sigma_{V,F} \geq F_v \geq (\gamma/2) \Sigma_{V,F}, \]
as well as
\[ (\Sigma_{V,F} \eta, F_v \eta) = (\Sigma_{V,F}^{1/2} \eta, F_v \Sigma_{V,F}^{1/2} \eta) \geq \frac{\gamma}{2} \|\Sigma_{V,F} \eta\|^2. \]
Therefore, using these estimates and (4.8) in (4.10), we obtain for \( \mu < 1 \) the estimate
\[ \|A_v z\|^2 \geq ((1 - \mu) a - \gamma^2) \|\xi\|^2 + ((1 - 1/\mu) M^2 + \gamma \gamma + \gamma^2) \|\Sigma_{V,F} \eta\|^2. \]
We require
\[ (1 - \mu) a - \gamma^2 > 0 \quad \text{and} \quad (1 - 1/\mu) M^2 + \gamma \gamma + \gamma^2 > 0. \]
Solving for \( \gamma \), we get the conditions
\[ \gamma < g_1(\mu) := \sqrt{(1 - \mu) a} \]
and
\[ \gamma > g_2(\mu) := \frac{\sqrt{\gamma} + \gamma + 4(1/\mu - 1) M^2}{2}. \]
We have \( g_1(1) = g_2(1) = 0 \). Further \( g_1(\mu)/g_2(\mu) \to \infty \) as \( \mu \to 1 \) (L’Hôpital). Thus there exists \( \mu < 1 \) and \( \gamma \) satisfying \( g_2(\mu) < \gamma < g_1(\mu) \), and dependent only on \( M \), \( \gamma \), and \( a \). The existence of \( c > 0 \) satisfying
\[ \|A_v z\|^2 \geq c\|\xi\|^2 + c\|\Sigma_{V,F} \eta\|^2 \]
follows. Since \( \Sigma_{V,F} = I \), the proof is finished. \( \square \)

4.3. Regularization functionals with \( L^1 \)-type norms

Writing \( Y = Y_{NL} \times Y_L \) and \( y = (\lambda, \varphi) \), we now restrict our attention to
\[ F^*(\gamma) = (F_{NL}^*(\lambda), F_{L,a}^*(\varphi)) \quad \text{and} \quad K(x) = (T(x), K_x), \]
where the operator \( K_L : X \to Y_L \) is linear, \( \varphi = (\varphi_1, \ldots, \varphi_N) \in \prod_{i=1}^N \mathbb{R}^m \), and
\[ F_{L,a}^*(\varphi) = \sum_{i=1}^N \left( \delta_{B(0,\alpha)}(\varphi_i) + \frac{\gamma}{2a} \|\varphi_i\|^2 \right) \]
(4.12)
for some \( \gamma \geq 0 \) and \( a > 0 \). Here \( B(0, \alpha) \subset \mathbb{R}^m \) are closed Euclidean unit balls of radius \( \alpha \). We assume that \( T \in C^2(X; Y_{NL}) \), and the functionals \( G \in C^2(X) \) and \( F_{NL} \in C^2(Y_{NL}) \) to be strongly convex, however also allowing \( G = 0 \).

**Example 4.1** (Total variation, TV). Let \( \Omega_d = \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2 - 1\} \) be a discrete domain, \( f \in \mathbb{R}^m \), and \( T : (\Omega_d \to \mathbb{R}) \to \mathbb{R}^m \) a possibly nonlinear forward operator. We are interested in the TV regularized reconstruction problem
\[ \min_v \frac{1}{2} \|f - T(v)\|^2 + \alpha \text{TV}(v). \]
(4.13)

Denoting by
\[ \nabla_d : (\Omega_d \to \mathbb{R}) \to (\Omega_d \to \mathbb{R}^2) \]
a discrete gradient operator, we define the discrete TV by
\[ \text{TV}(v) := \| \nabla u \|_{L^1(\Omega_d)}. \]

We may then write
\[ \alpha \text{TV}(v) = \max_{\psi} (K_L u, \varphi) - F^*_{L,\alpha}(\varphi), \]
where \( \varphi : \Omega_d \to \mathbb{R}^2, K_L = \nabla_d, \) and
\[ F^*_{L,\alpha}(\varphi) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \delta_{B(0,\alpha)}(\varphi(i, j)). \]

A non-zero \( \gamma > 0 \) in (4.12) corresponds to Huber regularization of the \( L^1 \)-norm, that is to the functional
\[ \text{TV}_\gamma(v) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |\nabla_d u(i, j)|_\gamma, \]
where
\[ |g|_\gamma = \begin{cases} \|g\| - \frac{\gamma}{2}, & \|g\| \geq \gamma, \\ 1 + \frac{\|g\|^2}{2\gamma}, & \|g\| < \gamma. \end{cases} \]

**Example 4.2** (Second-order total generalized variation, TGV\(^2\)). Let us now replace TV by TGV\(^2\) in (4.13). This regularization functional was introduced in [20] as a higher-order extension of TV that tends to avoid the stair-casing effect while still preserving edges. Parametrized by \( \alpha(h) = (\beta, \alpha) \), it can according to [21], see also [22], be written as the differentiation cascade
\[ \text{TGV}_{\alpha}^2(v) = \min_{u} \alpha \| Dv - w \| + \beta \| Ew \| \]
for \( E \) the symmetrized gradient. The parameter \( \alpha \) is the conventional regularization parameter, whereas the ratio \( \beta/\alpha \) controls the smoothness of the solution. With a large ratio, one obtains results similar to TV, but as the ratio becomes smaller, TGV\(^2\) better reconstructs smooth features of the image.

In the discrete setting, on the two-dimensional domain \( \Omega_d = [1, \ldots, n_1] \times [1, \ldots, n_2], \) we may write
\[ \text{TGV}_{\alpha}^2(v) = \min_{u} \max_{\varphi, \psi} (K_L (v, w), (\varphi, \psi)) - F^*_{L,\alpha}(\varphi, \psi), \]
where \( w, \varphi : \Omega_d \to \mathbb{R}^2, \psi : \Omega_d \to \mathbb{R}^{2 \times 2} \) and
\[ F^*_{L,\alpha}(\varphi) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \delta_{B(0,\alpha)}(\varphi(i, j)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \delta_{B(0,\alpha)}(\psi(i, j)). \quad (4.14) \]

The operator \( K_L \) is defined by
\[ K_L(v, w) = (\nabla_d u - w, (\beta/\alpha) E_d w), \quad E_d w = (\nabla_j w + (\nabla_j w)^T)/2, \]
for \( \nabla_j : (\Omega_d \to \mathbb{R}^2) \to (\Omega_d \to \mathbb{R}^{2 \times 2}) \) another discrete gradient operator. Instead of the ratio \( \beta/\alpha \) in front of \( E_d \), we could remove this and set the radii in (4.14) for \( \psi_{i,j} \) to \( \beta \). The present approach however makes the forthcoming analysis simpler.

We now wish to apply lemma 4.1 to obtain bounds on the Lipschitz factor of \( H_{\xi_0}^{-1} \) at a solution \( \tilde{u} = (\tilde{x}, \lambda, \tilde{\varphi}) \) to \( 0 \in H_{\xi}^2(\tilde{u}) \). From (4.5), we see that we have to calculate \( D(\partial F^*_{\alpha}(\varphi))(\tilde{\varphi}|K_L \tilde{x}) \). We begin by calculating \( D(\partial \delta_{B(0,\alpha)}) \) and studying conditions for the graphical regularity of \( \partial \delta_{B(0,\alpha)} = N_{\delta_{B(0,\alpha)}} \).
Lemma 4.2. Let \( f(y) = \delta_{B(0, \alpha)}(y), \ (y \in \mathbb{R}^m) \), and pick \( v \in \partial f(y) \). Then

\[
D(\partial f)(y|v)(w) = \begin{cases} \| v \| w/\alpha + \mathbb{R}y, & \| y \| = \alpha, \| v \| > 0, \ (y, w) = 0, \\ [0, \infty) y, & \| y \| = \alpha, \| v \| > 0, \ (y, w) \leq 0, \\ 0, & \| y \| < \alpha \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4.15}
\]

Moreover \( \partial f \) is graphically regular and \( \text{Graph} \ \partial f \) is locally closed at \( (y, v) \) for \( v \in \partial f(y) \) whenever either \( \| v \| > 0 \) or \( \| y \| < \alpha \).

Remark 4.1. The condition in the final statement can be seen as a form of strict complementarity, commonly found in the context of primal–dual interior point methods [23].

Proof. We first show graphical regularity and local closedness assuming (4.15). Indeed, local closedness of \( \partial f \) is a direct consequence of \( f \) being convex and lower semicontinuous [24].

With regard to graphical regularity, let \( (y', v') \rightarrow (y, v) \). Then for large enough \( i \), \( \| v' \| > 0 \). This forces \( \| y' \| = \alpha \), because \( v' \notin \mathbb{R}^m \). Consequently we get from (4.15) the expression

\[
D(\partial f)(y'|v')(w') = \| v' \| w'/\alpha + \mathbb{R}y'
\]

for any large enough index \( i \) and any \( w' \) with \( \langle y', w' \rangle = 0 \). Choosing \( w \) with \( (y, w) = 0 \), and \( z \in \| v \| w/\alpha + \mathbb{R}y \), we can find \( w' \rightarrow w \) and \( z' \rightarrow z \) with \( \langle y', w' \rangle = 0 \) and \( z' \in \| v \| w'/\alpha + \mathbb{R}y' \). It is now immediate that

\[
\lim_{i \rightarrow \infty} \text{Graph} \ D(\partial f)(y'|v') \supset \text{Graph} \ D(\partial f)(y|v).
\]

The inclusion in the other direction is obvious from the definitions. This proves that \( \overline{D}(\partial f)(y|v) = D(\partial f)(y|v) \), i.e., graphical regularity in the case \( \| v \| > 0 \). The case \( \| y \| < \alpha \) is trivial, because \( \partial f(y') = \emptyset \) for any \( y' \) close enough to \( y \).

Let us now prove (4.15). We do this by calculating the second-order subgradient \( d^2 f(y|v) \).

In the present situation, writing

\[
C := B(0, \alpha) = \{ x \in \mathbb{R}^m \mid g(y) \in D \}, \quad D := (-\infty, \alpha^2/2], \quad g(y) = \| y \|^2/2,
\]

the latter is given by [5, 13.17] as

\[
d^2 f(y|v)(w) = \delta_{K(v,y)}(w) + \max_{x \in \mathbb{R}^m} \langle w, x \nabla^2 g(y)w \rangle, \tag{4.16}
\]

provided the following constraint qualification is satisfied:

\[
x \in N_D(g(y)), \quad x \nabla g(y) = 0 \Rightarrow x = 0. \tag{4.17}
\]

Here

\[
K(y, v) := \{ w \in \mathbb{R}^m \mid \nabla g(y)w \in T_D(g(y)), \ (w, v) = 0 \}
\]

is the normal cone to \( N_C(y) \) at \( v \), and

\[
N_C(y) := \{ x \in \mathbb{R}^m \mid v - s \nabla g(y) = 0 \}.
\]

The constraint qualification (4.17) is trivially satisfied: if \( 0 \neq x \in N_D(g(y)) \), then necessarily \( \| y \| = \alpha \), so that \( x \nabla g(y) = xy \neq 0 \). We may thus proceed to expanding (4.16).

We find for \( v \in N_C(y) \) that

\[
K(y, v) = \begin{cases} \{ w \in \mathbb{R}^m \mid \langle y, w \rangle = 0 \}, & \| y \| = \alpha, \| v \| > 0 \\ \{ w \in \mathbb{R}^m \mid \langle y, w \rangle \leq 0 \}, & \| y \| = \alpha, \| v \| = 0 \\ \mathbb{R}^m, & \| y \| < \alpha. \end{cases}
\]
and
\[ X(y, v) = \|v\|/\alpha. \]

Thus for \( v \in N_C(y) \) we have \( \max_{w \in X(y,v)} (w, x\nabla^2 g(y)w) = \|v\|\|w\|^2/\alpha \). It follows from (4.16) that
\[
d^2 f(y|v)(w) = \begin{cases} 
\|v\|\|w\|^2/\alpha, & \|y\| = \alpha, \|v\| > 0, \langle y, w \rangle = 0, \\
0, & \|y\| = \alpha, \|v\| = 0, \langle y, w \rangle \leq 0, \\
0, & \|y\| < \alpha, \\
\emptyset, & \text{otherwise.}
\end{cases}
\] (4.18)

We still have to calculate \( D(\partial f)(y|v) \). Since \( f \) is proper, convex, and lower semicontinuous, it is also prox-regular and subdifferentiably continuous \([5, 13.30]\) in the senses defined in \([5]\), that we introduce here by name just for the sake of binding various results from that book rigorously together. By \([5, 13.17]\), \( f \) is also twice epi-differentiable. It therefore follows from \([5, 13.40]\) that
\[
D(\partial f)(y|v) = \partial h \quad \text{for} \quad h := \frac{1}{2}d^2 f(y|v).
\]

Applying this to (4.18), we easily calculate (4.15).

**Lemma 4.3.** If \( \varphi = (\varphi_1, \ldots, \varphi_N) \in Y_\lambda \) and \( v = (v_1, \ldots, v_N) \in \partial F_{\lambda,\alpha}(\varphi) \) satisfy
\[
either \|\varphi_j\| < \alpha \lor \|v_j - g\varphi_j/\alpha\| > 0, \quad (j = 1, \ldots, N),
\] (4.19)
then \( \partial F_{\lambda,\alpha}^* \) is graphically regular and \( \text{Graph} \partial F_{\lambda,\alpha}^* \) locally closed at \( (\varphi, v) \), and
\[
D(\partial F_{\lambda,\alpha}^*)(\varphi|v)(w) = \begin{cases} 
A_{\lambda,\alpha}^*(\varphi, v)w + V_{\lambda,\alpha}(\varphi)^\perp, & w \in V_{\lambda,\alpha}(\varphi), \\
\emptyset, & w \notin V_{\lambda,\alpha}(\varphi).
\end{cases}
\]

Here \( A_{\lambda,\alpha}^*(\varphi, v) := (A_{\lambda,\alpha}^*(\varphi_1, v_1), \ldots, A_{\lambda,\alpha}^*(\varphi_N, v_N)) \) and \( V_{\lambda,\alpha}(\varphi) := V_{\lambda,\alpha}^v(\varphi_1) \times \cdots \times V_{\lambda,\alpha}^v(\varphi_N) \) with
\[
A_{\lambda,\alpha}^*(\varphi_j, v_j)w_j := \begin{cases} 
\frac{\|v_j - g\varphi_j/\alpha\|^2}{\varphi_j}w_j, & \|\varphi_j\| = \alpha, \\
\emptyset, & \|\varphi_j\| < \alpha,
\end{cases}
\]
\[ \text{and} \quad V_{\lambda,\alpha}^v(\varphi_j) := \begin{cases} 
(\mathbb{R}\varphi_j)^\perp, & \|\varphi_j\| = \alpha, \\
\mathbb{R}_{\emptyset}, & \|\varphi_j\| < \alpha.
\end{cases} \]

**Proof.** Let \( g_j(\varphi_j) := g/2\alpha\|\varphi_j\|^2 \). Then \( F_{\lambda,\alpha}^*(\varphi) = \sum_{j=1}^N f_j(\varphi_j) + g_j(\varphi_j) \), for \( f_j = f \) as in lemma 4.2.

It follows that
\[
D(\partial F_{\lambda,\alpha}^*)(\varphi|v)(w) = \prod_{j=1}^N D(\partial (f_j + g_j))(\varphi_j|v_j)(w_j),
\]
and that \( \partial F_{\lambda,\alpha}^* \) is graphically regular and the graph locally closed if \( \partial (f_j + g_j) \) satisfies the same for each \( i = 1, \ldots, N \). By sum rules for graphical differentiation \([5, 10.43]\), we have
\[
D(\partial (f_j + g_j))(\varphi_j|v_j)(w_j) = D(\partial f_j)(\varphi_j|v_j - \nabla g_j(\varphi_j))(w_j) + \nabla^2 g(\varphi_j).
\]

Moreover, since \( g \) is twice continuously differentiable, \( \partial (f_j + g_j) \) is graphically regular if \( \partial f_j \) is. By lemma 4.2, this is the case if \( \|\varphi_j\| < \alpha, \) or \( \|v_j - \nabla g_j(\varphi_j)\| > 0 \). Minding that \( \nabla g_j(\varphi_j) = \gamma\varphi_j/\alpha \), referring to lemma 4.2 once again for the expression of \( D(\partial f_j)(\varphi_j|v_j - \gamma\varphi_j/\alpha)(w_j) \), the claim of the present lemma follows.

We now have the necessary results to bound \( \ell_{H^{-1}}(0|\tilde{u}) \).
Proposition 4.2. Suppose $F_{NL}^*$ is twice continuously differentiable and strongly convex, and $G = 0$ or $G$ is twice continuously differentiable and strongly convex. Define $F^*$ and $K$ by (4.11) for some $\alpha > 0$. Suppose $0 \in H_2(\hat{u})$. If $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_N)$ and $\hat{x}$ satisfy for some $a, b > 0$ the conditions

$$γ(1 - \|\hat{\varphi}_j\|/α) + \|[K_0 \hat{x}]_j\| - γ\hat{\varphi}_j/α > b, \quad (j = 1, \ldots, N), \quad (4.20)$$

and

$$\|\nabla K_{NL}(\hat{x})\|_2^2 + \|P_{NL}(\hat{\varphi}, K_0, \hat{x})\|_2^2 \geq α^2\|\hat{x}\|_2^2 \quad (4.21)$$

then $H^{-1}_2$ has the Aubin property at $0$ for $\hat{u}$. In particular, given $\bar{a} > α$, $(j = 1, \ldots, N)$, there exists a constant $c = c(a, b + γ, R, \bar{a}) > 0$ such that

$$\ell_{H^{-1}_2}(0;\hat{u}) \leq c^{-1}. \quad (4.22)$$

Proof. We calculate $D(\partial F_{NL}^*)((\hat{\varphi}, K_0, \hat{x}))$ using lemma 4.3, whose condition (4.19) at $u = K_0 \hat{x}$ is guaranteed by (4.20). Then we use (4.5), (4.6) to obtain

$$DH_2(\hat{u})(v) = \begin{cases} Av + V^\perp, & v \in V, \\ 0, & v \notin V, \end{cases} \quad A = \begin{pmatrix} \overline{G} & K_0^2 \\ K_0 & F \end{pmatrix},$$

where

$$\overline{G} = \nabla^2 G(\hat{x}) \quad V = X \times Y_{NL} \times V_{NL}(\hat{\varphi}).$$

$$\mathcal{F} = \begin{pmatrix} \nabla^2 F_{NL}^*(\hat{x}) & 0 \\ 0 & A_{NL}(\hat{\varphi}, K_0, \hat{x}) \end{pmatrix} K_\mathcal{T} = \begin{pmatrix} \nabla T(\hat{x}) & 0 \\ 0 & K_0 \end{pmatrix}.$$ 

Here $A_{NL}$ and $V_{NL}$ are given by lemma 4.2. The lemma together with (4.20) also show that $\partial F_1$ is graphically regular and Graph $\partial F_1$ is locally closed at $(\hat{\varphi}, K_0, \hat{x})$. From our standing assumptions, both $F^*$ and $G$ are convex and lower semicontinuous. Therefore Graph $\partial F^*$ and $\partial G$ are locally closed. Moreover $\partial F_{NL}^*$ and $\partial G$ are graphically regular, $F_{NL}^*$ and $G_{NL}$ being twice continuously differentiable. It follows that $H^{-1}_2$ is graphically regular and Graph $H^{-1}_2$ is locally closed at $(0, \hat{x})$.

Let us define

$$\Xi_a(x, λ, ϕ) = (x, Ξ_{a,F}(λ, ϕ)) \quad \text{with} \quad Ξ_{a,F}(λ, ϕ) = λ, ϕ/α, \ldots, ϕ/N/α.$$

The operator $Ξ_{a,F}$ commutes with $\mathcal{F}$ and $P_{V}$. Moreover, using (4.20) and the strong convexity of $F_{NL}^*$, we see that there exist $M = M(R) > 0$ and $\tilde{γ} = \tilde{γ}(\bar{a}, b + γ)$ such that

$$M\Xi_{a,F} ≥ F ≥ \tilde{γ}\Xi_{a,F}.$$ 

The dependence of $M$ on $R$ comes through sup$_{|λ| < R} \|\nabla^2 F_{NL}^*(λ)\|$. Since commutativity with $G = 0$ is automatic, and (4.21) guarantees (4.8), we may therefore apply lemma 4.1 to derive the bound

$$\|P_{V}A^*P_{V}z\| ≥ c\|\Xi P_{V}z\|, \quad (z \in X \times Y).$$

Here the constant $c = c(M, \tilde{γ}, a) = c(a, b + γ, R, \bar{a})$. Recalling (4.7), this proves (4.22). □

Remark 4.2 (Huber regularization). Condition (4.20) is difficult to satisfy without Huber regularization. Indeed, with $γ = 0$, the condition becomes $\|[K_0 \hat{x}]_j\| > 0$ for each $j = 1, \ldots, N$. In case of TV regularization in example 4.1, this says that we cannot have $[\nabla d \hat{x}]_j = 0$; there
can be no flat areas in the image $x$. This kind of requirement is, however, almost to be expected: the dual variable $\hat{\varphi}_j$, solving
\[
\max_{\varphi_j \in B(0, \alpha)} \langle \varphi_j, [K_L \hat{x}_j] \rangle
\]
is not uniquely defined. Any small perturbation of $x$ can send it anywhere on the boundary $\partial B(0, \alpha)$.

This oscillation is avoided by Huber regularization, i.e., $\gamma > 0$. In this case optimal $\hat{\varphi}_j$ for $\hat{x}$ solves
\[
\max_{\varphi_j \in B(0, \alpha)} \langle \varphi_j, [K_L \hat{x}_j] \rangle + \gamma \|\varphi_j\|^2/(2\alpha).
\]
If $\|K_L \hat{x}_j\| < \gamma$, necessarily $\|\hat{\varphi}_j\| < \alpha$. Clearly (4.20) follows. If, on the other hand, $\|K_L \hat{x}_j\| > \gamma$, necessarily $\|K_L \hat{x}_j - \gamma \hat{\varphi}_j/\alpha\| > 0$. Thus (4.20) holds again.

Even with $\gamma > 0$, we do however have a problem when $\|K_L \hat{x}_j\| = \gamma$: the solution is not necessarily strictly complementary in the sense (4.19). A way to avoid this theoretical problem would be to replace the 2-norm cost in (4.12) by a barrier function of $B(0, \alpha)$. This would, however, cause the resolvent $(I + \sigma F^*)^{-1}(y)$ to become very expensive to calculate. We therefore do not advise this in practise.

4.4. Squared $L^2$ cost functional with $L^1$-type regularization

We now make further assumptions on our problem, and limit ourselves to reformulations of
\[
\min_x \frac{1}{2} \|f - T(x)\|^2 + \alpha R(x).
\]
(4.23)

Here we assume that $T \in C^2(X; \mathbb{R}^m)$, and that the regularization term $\alpha R(x)$ be for some linear operator $K_L : X \to Y_L$ and $F_{\text{NL}}^*$ as in (4.12), be written in the form
\[
\alpha R(x) = \max_{\varphi} \langle \varphi, K_L x \rangle - F_{\text{NL}}^* (\varphi).
\]

This covers in particular examples 4.1 and 4.2.

Setting
\[
F_{\text{NL}}^* (\lambda) := \frac{1}{2} \|\lambda\|^2 + \langle f, \lambda \rangle
\]
we may write
\[
\frac{1}{2} \|f - T(x)\|^2 = \max_{\lambda} \langle \lambda, T(x) \rangle - F_{\text{NL}}^* (\lambda).
\]
Observe that $F_{\text{NL}}^*$ is strongly convex. Thus with $y = (\lambda, \varphi)$, we may reformulate (4.23) in the saddle point form
\[
\min_x \max_y G(x) + \langle K(x), y \rangle - F^*(y)
\]
for
\[
G = 0, \quad K(x) = (T(x), K_L x), \quad F^*(y) = (F_{\text{NL}}^* (\lambda), F_{\text{NL}}^* (\varphi)).
\]

The optimality conditions (2.1) presently expand to $\hat{\alpha}_a = (\hat{\lambda}_a, \hat{\varphi}_a)$ satisfying
\[
[\nabla T(\tilde{x}_a)]^T \hat{\lambda}_a + K_L^T \hat{\varphi}_a = 0, \quad (4.25a)
\]
\[
T(\tilde{x}_a) - f = \hat{\lambda}_a, \quad \text{and} \quad (4.25b)
\]
\[
[K_L \hat{x}_a]_j - \gamma \hat{\varphi}_a/j/\alpha \in N_{B(0, \alpha)}(\hat{\varphi}_a/j), \quad (j = 1, \ldots, N). \quad (4.25c)
\]
Since proposition 4.2 only shows the Aubin property of $H_{\hat{\mathcal{L}}^{-1}}$ without any guarantees of smallness of the Lipschitz factor, we have to make $\|P_{NL}\hat{\mathcal{L}}\| \leq \alpha$, a solution to (4.25) necessarily satisfies
\[
\|P_{NL}\hat{\mathcal{L}}\hat{\mathcal{X}}\| = \|f - T(\hat{\mathcal{X}})\|.
\]
We will discuss how to make this small after the next proposition, rewriting theorem 3.2 for the present setting.

**Proposition 4.3.** Suppose $\hat{\mathcal{U}} = (\hat{\mathcal{X}}, \hat{\lambda}, \hat{\varphi})$ satisfies the optimality conditions (4.25), and for some $a, b > 0$ the strict complementarity condition
\[
\gamma(1 - \|\hat{\varphi}_j\|/\alpha) + \|[K_1\hat{\mathcal{X}}]_j - \gamma \hat{\varphi}_j/\alpha\| > b, \quad (j = 1, \ldots, N), \quad (4.26)
\]
as well as the non-degeneracy condition
\[
\|\nabla T(\hat{\mathcal{X}})\zeta\|^2 + \|P_{KL}(\hat{\varphi})K_L\zeta\|^2 \geq a^2 \|\zeta\|^2. \quad (4.27)
\]
Suppose, moreover, that $K$ and the step lengths $\tau, \sigma > 0$ satisfy (A-K). Pick $\tilde{a} > \alpha$, and let $c = c(a, b + \gamma, R, \tilde{\alpha})$ be given by proposition 4.2. Then there exists $\delta_1 > 0$ such that algorithm 2.1 applied to the saddle point form (4.24) of (4.23) converges provided
\[
\|u^1 - \tilde{u}\| \leq \delta_1,
\]
and
\[
\|f - T(\hat{\mathcal{X}})\| < \frac{1 - 1/\sqrt{1 + c^2/(2c\Theta_t)}}{c^{-1}L_2}. \quad (4.28)
\]

**Proof.** We verify the assumptions of theorem 3.2. By lemma 3.2, it only remains to prove (3.46). As (4.26) and (4.27) hold, proposition 4.2 shows that $H_{\hat{\mathcal{L}}^{-1}}$ has the Aubin property at 0 for $\hat{\mathcal{U}}$ with $\|H_{\hat{\mathcal{L}}^{-1}}\| \leq c^{-1}$ for some constant $c > 0$. Condition (4.28) now implies (3.46). The claim of the present proposition thus follows from theorem 3.2. $\square$

**Remark 4.3 (Non-degeneracy condition).** The non-degeneracy condition (4.27) may also be stated
\[
\nabla T(\hat{\mathcal{X}})\zeta = 0 \quad \text{and} \quad P_{KL}(\hat{\varphi})K_L\zeta = 0 \Rightarrow \zeta = 0.
\]
Verifying this is easy if $\nabla T(\hat{\mathcal{X}})$ has full range, but otherwise it can be quite unwieldy thanks to the projection $P_{KL}(\hat{\varphi})$. Unfortunately, we have found no way to avoid this condition or (4.28).

We conclude our theoretical study with a simple exemplary result on the satisfaction of (4.28). The problem with simply letting $\alpha \searrow 0$ in order to get $\|f - T(\hat{\mathcal{X}})\| \rightarrow 0$ is that the constant $c$ might blow up, depending on $\hat{\mathcal{U}}$ through $a$ and $b$. We therefore need to study the uniform satisfaction of these conditions. In order to keep the present paper at a reasonable length, we limit ourselves to a very simple result that assumes the existence of a convergent sequence $\{\hat{\mathcal{U}}_n\}$ of solutions to (4.25) as $\alpha \searrow 0$. The entire topic of the existence of such a sequence merits an independent study involving set-valued implicit function theorems (e.g. [25]) and source conditions on the data (cf, e.g. [9]). Besides the existence of the minimising sequence, we assume the existence of $x^*$ such that $f = T(x^*)$. It is not difficult to formulate and prove equivalent results for the noisy case, where we only have the bound $\inf_{\delta} \|f - T(x)\| \leq \sigma$ for small $\sigma$. 34
Proposition 4.4. Suppose \( f = T(x^*) \) for some \( x^* \in X \). Let \( \tilde{u}_a = (\tilde{x}_a, \tilde{\lambda}_a, \tilde{\varphi}_a) \) solve (4.25) for \( \alpha > 0 \), and suppose that
\[
(\tilde{x}_a, \tilde{\varphi}_a/\alpha) \to (\tilde{x}, \tilde{\varphi}), \quad (\alpha \searrow 0).
\] (4.29)
If \( \tilde{x} \) and \( \tilde{\varphi} \) satisfy for some \( \tilde{a}, \tilde{b} > 0 \) the strict complementarity condition
\[
\gamma(1 - \|\tilde{\varphi}\|) + \|K\tilde{a}\tilde{\varphi}_j - \gamma\tilde{\varphi}_j\| > \tilde{b}, \quad (j = 1, \ldots, N),
\]
and the non-degeneracy condition
\[
\|\nabla T(\tilde{x})\| + \|P_{\tilde{\varphi}}\| K\tilde{\varphi}\| \geq \tilde{a}\|\xi\|^2,
\]
then there exists \( \alpha^* > 0 \) such that (4.26) and (4.27) hold for \( \alpha = 0, \alpha^* \). In consequence, there exist
\[
\tilde{u}^* = (x^*, \lambda^*, \varphi^*)\] satisfies
\[
\lambda^* = T(x^*) - f \quad \text{and} \quad \|x^*, \varphi^*/\alpha\| \leq \delta_1.
\]
Remark 4.4. If each \( \tilde{x}_a, (\alpha > 0) \), solves the minimization problem (4.23), instead of just the first-order optimality conditions (4.25), so that \( \|f - T(\tilde{x}_a)\| \to 0 \), then the limit \( \tilde{x} \) solves
\[
\min_{X} R(x) \quad \text{such that} \quad T(x) = f.
\]
Proof. Let us begin by defining
\[
\Psi_a(x, \varphi) = (x, \varphi_1/\alpha, \ldots, \varphi_N/\alpha).
\]
Minding the renormalization of \( \varphi \) through \( \Psi_a \) in (4.29), the conditions (4.30) and (4.31) arise in the limit from (4.26) and (4.27). Indeed, if there were sequences \( \alpha' \searrow 0, (\tilde{x}'^{\alpha'}) \to (\tilde{x}, \tilde{\varphi}) \), such that (4.26) or (4.27) would not eventually hold for \( \alpha = \tilde{a}/2 \) and \( b = \tilde{b}/2 \), we would find a contradiction to (4.30) or (4.31), respectively. We may thus pick \( \alpha^* > 0 \) such that (4.26) and (4.27) hold for \( \alpha = \tilde{a}/2 \) and \( b = \tilde{b}/2 \) and \( \alpha \in (0, \alpha^*) \). Denoting by \( H_a \) the operator \( H_{\alpha} \) for a specific choice of \( \alpha \), proposition 4.2 now shows that
\[
\ell_{H_a}(0, \tilde{u}_a) \leq c^{-1}, \quad (\alpha \in (0, \alpha^*)�, \)
for some \( c = c(a, b + \gamma, R, \alpha^*) \) independent of \( a \) and \( \tilde{u}_a \). Observing that \( \kappa \) and \( \Theta \) are independent of \( \alpha \) (which is encoded into \( H_{\alpha} \) in our formulation), it follows that (4.28) holds if
\[
\|f - T(\tilde{x}_a)\| < \epsilon := \frac{1 - 1/\sqrt{1 + c^2/(2\gamma^2)}}{c^{-1}L_2}. \]
Because \( f = T(x^*) \), this can be achieved whenever \( \alpha \in (0, \alpha^*) \) for some \( \alpha^* \in (0, \alpha^*) \). Proposition 4.3 now provides \( \delta > 0 \) such that algorithm 2.1 converges if
\[
\|\tilde{u}_a - u^1\| \leq \delta.
\]
Setting \( \lambda^1 = T(x^1) - f \) and using \( T \in C^2(X; \mathbb{R}^m) \), we have
\[
\|\tilde{\lambda}_a - \lambda^1\| = \|T(\tilde{x}_a) - T(x^1)\| \leq \ell\|\tilde{x}_a - x^1\|
\]
for \( \ell \) the Lipschitz factor of \( T \) on a suitable compact set around \( \tilde{x} \). Therefore, whenever \( \alpha \in (0, \alpha^*) \), we may estimate with \( C = (1 + \ell) \max\{1, \alpha^*\} \) that
\[
\|\tilde{u}_a - u^1\| \leq (1 + \ell)\|\tilde{\lambda}_a - \lambda^1\| \leq (1 + \ell)\|\Psi_a^{-1}\|\|\Psi_a((\tilde{x}_a, \tilde{\varphi}_a) - (x^1, \varphi^1))\| \leq C\|\Psi_a((\tilde{x}_a, \tilde{\varphi}_a) - (\tilde{x}, \tilde{\varphi}))\| + \|\Psi_a((x^1, \varphi^1) - (\tilde{x}, \tilde{\varphi}))\|.
\]
By the convergence \( \Psi_a((\tilde{x}_a, \tilde{\varphi}_a) \to (\tilde{x}, \tilde{\varphi}) \), choosing \( \tilde{a} \in (0, \alpha^*) \) small enough, we can force
\[
C\|\Psi_a((\tilde{x}_a, \tilde{\varphi}_a) - (\tilde{x}, \tilde{\varphi}))\| \leq \delta/2, \quad (\alpha \in (0, \tilde{a})�.
\]
Choosing \( \delta_1 = \delta/(2C) \), we may thus conclude the proof. \( \square \)
5. Applications and computational experience

With convergence theoretically studied, and TV type regularization problems reformulated into the minimax form, we now move on to studying the numerical performance of NL–PDHGM. We do this by applying the method to two problems from MRI: velocity imaging in section 5.1, and DTI in section 5.2.

5.1. Phase reconstruction for velocity-encoded MRI

As our first application, we consider the phase reconstruction problem (1.3) for magnetic resonance velocity imaging. We are given a complex sub-sampled k-space data \( f = SFu^* + v \in \mathbb{R}^n \) corrupted by noise \( v \). Here \( S \) is the sparse sampling operator, \( F \) the discrete two-dimensional Fourier transform, and \( u^* \in L^1(\Omega_d; \mathbb{C}) \) the noise-free complex image in the discrete spatial domain \( \Omega_d = \{0, \ldots, n-1\}^2 \). We seek to find a complex image \( u = r \exp(i\phi) \) approximating \( u^* \). Motivated by the discoveries in [7], we wish to regularize \( r \) and \( \phi \) separately. Introducing nonlinearities into the reconstruction problem, we therefore define the forward operator

\[
T(r, \phi) := SF[x \mapsto r(x) \exp(i\phi(x))].
\]

For the phase \( \phi \), we choose to use second-order total generalized variation regularization, and for the magnitude \( r \), TV regularization. Choosing regularization parameters \( \alpha_r, \alpha_\phi, \beta_\phi > 0 \) appropriate for the data at hand, we then seek to solve

\[
\min_{r,\phi \in L^1(\Omega_d)} \frac{1}{2} \|f - T(r, \phi)\|^2 + \alpha_r TV(r) + TGV^2_{(\beta_\phi, \alpha_\phi)}(\phi). \tag{5.1}
\]

For our experiments, due to trouble obtaining real data, we use a simple synthetic phantom on \( \Omega = [-1, 1]^2 \), depicted in figure 1(a). It simulates the speed along the y-axis of a fluid.
rotating in a ring. Specifically
\[ r(x, y) = \chi_{0.3 < \sqrt{x^2 + y^2} < 0.9}(x, y), \quad \text{and} \quad \varphi(x, y) = x/\sqrt{x^2 + y^2}. \]

The image is discretized on an \( n \times n \) grid with \( n = 256 \). To the discrete Fourier-transformed \( k \)-space image, we add pointwise Gaussian noise of standard deviation \( \sigma = 0.2 \). For the sub-sampling operator \( S \), we choose a centrally distributed Gaussian sampling pattern with variance \( 0.15 \times 128 \) and 15\% coverage of the \( n \times n \) image in \( k \)-space. For the regularization parameters, which we do not claim to have chosen optimally in the present algorithmic paper, we choose \( \alpha_0 = 0.15 \times (2/n) \), \( \beta_0 = 0.20 \times (2/n)^2 \), and \( \alpha_r = 2/n \). (The factors \( 2/n \) are related to spatial step size \( 1/n \) of the discrete differential on the \([-1, 1]^2\) domain. When the step size is omitted, i.e., implicitly taken as 1, the factors disappear.)

We perform computations with algorithm 2.1 (exact NL–PDHGM), algorithm 2.2 (linearized NL–PDHGM), and the Gauss–Newton method. As we recall, the latter is based on linearizing the nonlinear operator \( T \) at the current iterate, solving the resulting convex problem, and repeating until hopeful convergence. We solve each of the inner convex problems by PDHGM, (2.4). We initialize each method with the backprojection \( (S^*)^T f \) of the noisy sub-sampled data \( f \). We use two different choices for the PDHGM step length parameters \( \sigma \) and \( \tau \). The first one has equal primal and dual parameters, both \( \sigma, \tau = 0.95/L \). Recalling remark 3.6, we update \( L = \sup_{k=1,...,d} \| \nabla K(x^k) \| \) dynamically for NL–PDHGM; for linear PDHGM, used within Gauss–Newton, this simply reduces to \( L = \| K \| \). This choice of \( \tau \) and \( \sigma \) is somewhat naïve, and it has been observed that sometimes choosing \( \tau \) and \( \sigma \) in different proportions can improve the performance of PDHGM for linear operators [18]. Therefore, as our second step length parameter choice we use \( \tau = 0.5/L \) and \( \sigma = 1.9/L \).

We limit the number of PDHGM iterations (within each Gauss–Newton iteration) to 100,000, and limit the number of Gauss–Newton iterations to 100. As the primary stopping criterion for the NL–PDHGM methods, we use \( \| x^t - x^{t+1} \| < \rho \) for \( \rho = 1 \times 10^{-4} \). The same criterion is used to stop the outer Gauss–Newton iterations. As the stopping criterion of PDHGM within each Gauss–Newton iteration, we use the decrease of the pseudo-duality gap to less than \( \rho_2 = 1 \times 10^{-3} = \rho/10 \); higher accuracy than this would penalise the computational times of the Gauss–Newton method too much in comparison to NL–PDHGM. Much lower accuracy would almost reduce it to NL–PDHGM, if convergence would be observed at all; we will get back to this in our next application.

The pseudo-duality gap is discussed in detail in [26]. We use it to work around the fact that in reformulations of the linearized problem into the form (P), \( G = 0 \). This causes the duality gap to be in practise infinite. This could be avoided if we could make \( G \) to be non-zero, for example by taking \( G(x) = \delta_{B_{0, M}}(x) \) for \( M \) a bound on \( x \). Often the primal variable \( x \) can indeed be shown to be bounded. The problem is that the exact bound is not known. The idea of the pseudo-duality gap, therefore, is to update the bound \( M \) dynamically. Assuming it large enough, it does not affect the PDHGM itself. It only affects the duality gap, and is updated whenever the duality gap is violated. For the artificially dynamically bounded problem the duality gap is finite, and called the pseudo-duality gap.

We perform the computations with OpenMP parallelization on six cores of an Intel Xeon E5-2630 CPU at 2.3 GHz, with 64GB (that is, enough!) random access memory available. The results are displayed in table 1 for the first choice of \( \sigma \) and \( \tau \), and in table 2 for the second choice. In the tables we report the number of PDHGM and Gauss–Newton iterations taken along with the computational time in seconds, as well as the PSNR of both the magnitude and the phase for the reconstructions. For the calculation of the PSNR of \( \varphi \), we have only included the ring \( 0.3 < \sqrt{x^2 + y^2} < 0.9 \), as the phase is meaningless outside this, the magnitude being zero. The results of the second parameter choice are also visualized in figure 1. It shows
the original noise-free synthetic data, the backprojection of the noisy sub-sampled data, and the obtained reconstructions, which have little difference between the methods, validating the results. Also, because there is no difference linearized and exact NL-PDHGM, we only display the results for the latter.

The main observation from table 1, with equal $\tau$ and $\sigma$, is that exact NL–PDHGM, algorithm 2.1, is significantly faster than Gauss–Newton, taking only around 2 min in comparison to almost an hour for Gauss–Newton. Gauss–Newton nevertheless appears to converge for the present problem, having taken 13 iterations to each the stopping criterion. Another observation is that although linearized NL–PDHGM, algorithm 2.2, takes exactly as many iterations as exact NL–PDHGM and Gauss–Newton is $\|x^* - x^{i+1}\| < \rho$. The stopping criterion for linear PDHGM, used for inner Gauss–Newton iterations, is pseudo-duality gap less than $\rho_2$. 

A full and fair comparison of the nonlinear model (5.1) against the linear model (1.2) is out of the scope of the present paper. Especially, considering that we employed in [7] Bregman iterations [27] for contrast enhancement, without the introduction of a contrast enhancement technique for the nonlinear model as in [10], it is not fair to directly compare against the results therein. Nevertheless, in order to justify the nonlinear model, we have a simple comparison in

<table>
<thead>
<tr>
<th>Method</th>
<th>PDHGM iters.</th>
<th>GN iters.</th>
<th>Time</th>
<th>PSNR(r)</th>
<th>PSNR((\phi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backprojection</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>19.2</td>
<td>41.0</td>
</tr>
<tr>
<td>Exact NL–PDHGM</td>
<td>15 800</td>
<td>–</td>
<td>129.1s</td>
<td>25.5</td>
<td>52.7</td>
</tr>
<tr>
<td>Linearized NL–PDHGM</td>
<td>15 800</td>
<td>–</td>
<td>171.2s</td>
<td>25.5</td>
<td>52.7</td>
</tr>
<tr>
<td>Gauss–Newton</td>
<td>137 900</td>
<td>12</td>
<td>1353.7s</td>
<td>25.5</td>
<td>53.9</td>
</tr>
</tbody>
</table>

Table 2. Phase/magnitude reconstruction using exact NL–PDHGM, linearized NL–PDHGM, and the Gauss–Newton method: unequal primal and dual step length parameters. For Gauss–Newton, PDHGM is used in the inner iterations, and the number of PDHGM iterations reported is the total over all Gauss–Newton iterations. The PSNR of $\phi$ excludes the area outside the ring, where $r = 0$. The stopping criterion for linear PDHGM, used for inner Gauss–Newton iterations, is pseudo-duality gap less than $\rho_2$. 

<table>
<thead>
<tr>
<th>Method</th>
<th>PDHGM iters.</th>
<th>GN iters.</th>
<th>Time</th>
<th>PSNR(r)</th>
<th>PSNR((\phi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backprojection</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>19.2</td>
<td>41.0</td>
</tr>
<tr>
<td>Exact NL–PDHGM</td>
<td>8200</td>
<td>–</td>
<td>57.8s</td>
<td>25.5</td>
<td>51.2</td>
</tr>
<tr>
<td>Linearized NL–PDHGM</td>
<td>8200</td>
<td>–</td>
<td>87.6s</td>
<td>25.5</td>
<td>51.2</td>
</tr>
<tr>
<td>Gauss–Newton</td>
<td>112145</td>
<td>12</td>
<td>1119.1s</td>
<td>25.5</td>
<td>53.7</td>
</tr>
</tbody>
</table>

Table 1. Phase/magnitude reconstruction using exact NL–PDHGM, linearized NL–PDHGM, and the Gauss–Newton method: equal primal and dual step length parameters. For Gauss–Newton, PDHGM is used in the inner iterations, and the number of PDHGM iterations reported is the total over all Gauss–Newton iterations. The PSNR of $\varphi$ excludes the area outside the ring, where $r = 0$. The stopping criterion for NL–PDHGM and Gauss–Newton is $\|x^* - x^{i+1}\| < \rho$. The stopping criterion for linear PDHGM, used for inner Gauss–Newton iterations, is pseudo-duality gap less than $\rho_2$. 

$\rho = 1E^{-4}$, $\rho_2 = 1E^{-3}$, $\tau = 0.95/L$, $\sigma = 0.95/L$ for $L = \sup \|\nabla K(x')\|$.
Figure 2. Phase/magnitude reconstruction using linear and nonlinear model. Also pictured is the original noise-free test data; the backprojection of the noisy data may be found in figure 1(b). The upper image in each column is the magnitude $r$, and the lower image the phase $\phi$. Only results for TV regularization are shown for the linear model.

Table 3. Results for the linear model (1.2) in comparison to the nonlinear model (5.1). Optimal $\hat{\alpha}$ has been chosen by Morozov’s discrepancy principle, and the PSNR for both the amplitude $r$ and phase $\phi$. For each criterion the PSNRs and optimal $\hat{\alpha}$ are reported. In case of the nonlinear model, TV regularization is used for the amplitude, and TGV$^2$ regularization for the phase, with parameters $\alpha_r = 1$, $\alpha_\phi = 0.20$, and $\beta_\phi = 1.1\alpha_\phi$.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Regularizer</th>
<th>$\hat{\alpha} \times n/2$</th>
<th>PSNR($r$)</th>
<th>PSNR($\phi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrepancy principle</td>
<td>TV</td>
<td>0.25</td>
<td>26.7</td>
<td>52.0</td>
</tr>
<tr>
<td>PSNR($r$)</td>
<td>TV</td>
<td>0.20</td>
<td>26.7</td>
<td>51.1</td>
</tr>
<tr>
<td>PSNR($\phi$)</td>
<td>TV</td>
<td>1.25</td>
<td>22.1</td>
<td>57.8</td>
</tr>
<tr>
<td>Discrepancy principle</td>
<td>TGV$^2$</td>
<td>0.25</td>
<td>22.7</td>
<td>53.0</td>
</tr>
<tr>
<td>PSNR($r$)</td>
<td>TGV$^2$</td>
<td>0.15</td>
<td>24.8</td>
<td>50.8</td>
</tr>
<tr>
<td>PSNR($\phi$)</td>
<td>TGV$^2$</td>
<td>2.60</td>
<td>17.8</td>
<td>57.3</td>
</tr>
<tr>
<td>Manual (nonlinear model)</td>
<td>Mixed</td>
<td>25.2</td>
<td>54.0</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 and figure 2. We computed solutions to the linear model for varying $\alpha$ in the interval $[0.01, 3.0] \times (2/n)$ (spacing 0.005 below $\alpha = 1.0 \times (2/n)$, and 0.1 above), $\beta = 1.1\alpha \times (2/n)$, and chose the optimal $\hat{\alpha}$ by three different criteria. These Morozov’s discrepancy principle [28], i.e., the largest $\alpha$ such that $\| f - S^*F u_\alpha \|$ is below the noise level, as well as the optimal PSNR for both the amplitude $r$ and phase $\phi$. For the nonlinear model, we chose the parameters manually. We find this reasonable, because for multi-dimensional parameters we do not at this time have to our avail something practical like the discrepancy principle, and a scan of the parameter space is not doable in practical applications. Moreover, in order to show that the nonlinear model improves over the linear model, it is only necessary to pick parameters for the linear model optimally. With this in mind, we specifically picked $\alpha_r = 1 \times (2/n)$, $\alpha_\phi = 0.20 \times (2/n)$, and $\beta_\phi = 1.1\alpha_\phi \times (2/n)$ for the nonlinear model. We also increased the stopping threshold of NL–PDHGM to $\rho = 1E^{-5}$ for better quality solutions. For the linear model, we set the pseudo-duality gap threshold to $\rho_2 = 1E^{-4}$. This appears to be enough for the comparison, keeping in mind that the stopping criteria are not comparable. What we can draw...
from table 3 is that for most criteria, the linear model fails to balance between PSNR(ϕ) and PSNR(r) as well as the nonlinear model. With the discrepancy principle as parameter choice criterion, and TV as the regularizer, the linear model however beats the nonlinear model in terms of both PSNRs. Inspecting figure 2(c), we however observe extensive stair-casing in both the amplitude and phase. This is avoided by the excellent reconstruction by the nonlinear model in figure 2(b). With the higher α in figure 2(d), the phase reconstruction is good even by the linear model, but the amplitude has become smoothed out. This is also avoided by nonlinear model.

5.2. Diffusion tensor imaging

As a continuation of our earlier work on TGV2 denoising of diffusion tensor MRI (DTI) [13, 26, 29] using linear models, we now consider an improved model. Here the purpose of the nonlinear operator T is to model the so-called Stejskal–Tanner equation

\[ s_j(x) = s_0(x) \exp(\langle b_j, v(x)b_j \rangle), \quad (j = 1, \ldots, N). \]  \hspace{1cm} (5.2)

Here \( v : \Omega \to \text{Sym}^2(\mathbb{R}^3), \Omega \subset \mathbb{R}^3 \) is a mapping to symmetric second-order tensors (representable as symmetric \( 3 \times 3 \) matrices). Each \( v(x) \) models the covariance of a Gaussian probability distribution at \( x \) for the diffusion of water molecules. Each of the diffusion-weighted MRI measurements \( s_j, (j = 1, \ldots, N) \), is obtained with a different non-zero diffusion sensitising gradient \( b_j \), while \( s_0 \) is obtained with a zero gradient. After correct the original \( k \)-space data for coil sensitivities, each \( s_j \) is assumed real. As a consequence, \( s_j \) has in effect Rician noise distribution [30].

Our goal is to denoise \( v \). We therefore consider

\[ \min_{v} \frac{1}{N} \sum_{j=1}^{N} \| s_j - T_j(v) \|^2 + \text{TGV}^2_{\beta,\alpha}(v), \]  \hspace{1cm} (5.3)

where

\[ T_j(v)(x) := s_0(x) \exp(\langle b_j, v(x)b_j \rangle), \quad (j = 1, \ldots, N). \]

Due to the Rician noise of \( s_j \), the Gaussian noise model implied by the \( L^2 \)-norm is not entirely correct. However, the \( L^2 \) model is not necessary too inaccurate, as for suitable parameters the Rician distribution is not too far from a Gaussian distribution. For correct modelling, there would be approaches. One would be to modify the fidelity term to model Rician noise, as was done in [31, 32] for single MR images. The second option would be to include the coil sensitivities in an \( L^2 \) model, either by knowing them, or by estimating them simultaneously, as was done in [33] for parallel MRI. This kind of models with direct tensor reconstruction will be the subject of a future study. For the present work, we are content with the simple \( L^2 \) model, which already presents computational challenges through the nonlinearities of the Stejskal–Tanner equation.

As our test data set, we have an in vivo measurement of the human brain. The data set is three-dimensional with 25 slices of size \( 128 \times 128 \) for 21 different diffusion sensitising gradients, including the zero gradient. Moreover, nine measurement were measured to construct by averaging the ground truth depicted in figure 3(a). Only the first of the measurements is used for the backprojection in figure 3(b) and the reconstructions with (5.3). It has about 8.5 million data points. The diffusion tensor image \( v \) is correspondingly \( 128 \times 128 \times 25 \) with six components for each symmetric tensor element \( v(x) \in \mathbb{R}^{3 \times 3} \). In addition we have the additional variable \( w \), and dual variables. This gives altogether 42 values per voxel in the reconstruction space, or 17 million values. Considering the size of a double
Figure 3. Visualization of one slice of the DTI reconstruction using exact NL–PDHGM and the Gauss–Newton method. Also pictured is the ground truth and the backprojection reconstruction. The data displayed is colour-coded principal eigenvector. The area outside the brain has been masked out. The rectangle in (a) indicates the region displayed in figures 4 and 5.

As in section 5.1, we evaluate all three, algorithm 2.1 (exact NL–PDHGM), algorithm 2.2 (linearized NL–PDHGM), and the Gauss–Newton method. The parametrization and method setup is the same as in the previous section, except for the step length parameter choice \( \tau = \sigma = 0.95/L \), we use the lower accuracy \( \rho = 1E^{-3} \). For the choice \( \tau = 0.5/L \), \( \sigma = 1.9/L \) we use \( \rho = 1E^{-4} \) as before. Also, in both cases, in addition to \( \rho_2 = \rho/10 \), we perform Gauss–Newton computations with the higher accuracy \( \rho_2 = \rho \) for the inner PDHGM iterations. The regularization parameters are also naturally different. We choose \( \alpha = 0.0006/\sqrt{128 \times 128 \times 25} \), and \( \beta = 0.00066/(128 \times 128 \times 25) \).

The results of the computations are reported in tables 4 and 5, and figure 4. They confirm our observations in the velocity imaging example, but we do have some convergence issues. In case of Gauss–Newton, we do not observe convergence with accuracy \( \rho_2 = \rho/10 \) for the inner PDHGM iterations, and have to use \( \rho_2 = \rho \). We find this quite interesting, since NL–PDHGM gives better convergence results, and is essentially Gauss–Newton with just a single step in the inner iteration. Although not reported in the tables, we also observed that if using the more accurate stopping threshold \( \rho = 1E^{-4} \), instead of \( 1E^{-3} \) for the computations in
Table 4. DTI reconstruction using exact NL–PDHGM, linearized NL–PDHGM, and the Gauss–Newton method: equal primal and dual step length parameters. For Gauss–Newton, PDHGM is used in the inner iterations, and the number of PDHGM iterations reported is the total over all Gauss–Newton iterations. The stopping criterion for NL–PDHGM and Gauss–Newton is $\|x^t - x^{t+1}\| < \rho$. The stopping criterion for linear PDHGM within Gauss–Newton iterations is pseudo-duality gap less than $\rho_2$. The number of PDHGM iterations (per Gauss–Newton iteration) is limited to 100 000, and the number of Gauss–Newton iterations to 100.

$\rho = 10^{-3}, \tau = 0.95/L, \sigma = 0.95/L$ for $L = \sup_i \|\nabla K(x^t)\|$

<table>
<thead>
<tr>
<th>Method</th>
<th>PDHGM iters.</th>
<th>GN iters.</th>
<th>Time</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backprojection</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>17.6</td>
</tr>
<tr>
<td>Exact NL–PDHGM</td>
<td>4600</td>
<td>–</td>
<td>1351.8s</td>
<td>20.1</td>
</tr>
<tr>
<td>Linearized NL–PDHGM</td>
<td>4600</td>
<td>–</td>
<td>2068.9s</td>
<td>20.1</td>
</tr>
<tr>
<td>Gauss–Newton; $\rho_2 = \rho$</td>
<td>36 600</td>
<td>6</td>
<td>8222.4s</td>
<td>20.6</td>
</tr>
<tr>
<td>Gauss–Newton; $\rho_2 = \rho/10$</td>
<td>5338</td>
<td>100</td>
<td>1384.8s</td>
<td>21.9</td>
</tr>
</tbody>
</table>

Table 5. DTI reconstruction using exact NL–PDHGM, linearized NL–PDHGM, and the Gauss–Newton method: primal step length smaller than dual. For Gauss–Newton, PDHGM is used in the inner iterations, and the number of PDHGM iterations reported is the total over all Gauss–Newton iterations. The stopping criterion for NL–PDHGM and Gauss–Newton is $\|x^t - x^{t+1}\| < \rho$. The stopping criterion for linear PDHGM within Gauss–Newton iterations is pseudo-duality gap less than $\rho_2$. The number of PDHGM iterations (per Gauss–Newton iteration) is limited to 100 000, and the number of Gauss–Newton iterations to 100.

$\rho = 10^{-4}, \tau = 0.5/L, \sigma = 1.9/L$ for $L = \sup_i \|\nabla K(x^t)\|$

<table>
<thead>
<tr>
<th>Method</th>
<th>PDHGM iters.</th>
<th>GN iters.</th>
<th>Time</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backprojection</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>17.6</td>
</tr>
<tr>
<td>Exact NL–PDHGM</td>
<td>8700</td>
<td>–</td>
<td>2381.9s</td>
<td>20.3</td>
</tr>
<tr>
<td>Linearized NL–PDHGM</td>
<td>8700</td>
<td>–</td>
<td>3302.5s</td>
<td>20.3</td>
</tr>
<tr>
<td>Gauss–Newton; $\rho_2 = \rho$</td>
<td>77 264</td>
<td>8</td>
<td>18 679.1s</td>
<td>20.4</td>
</tr>
<tr>
<td>Gauss–Newton; $\rho_2 = \rho/10$</td>
<td>33 601</td>
<td>100</td>
<td>8442.2s</td>
<td>21.2</td>
</tr>
</tbody>
</table>

Table 4, with equal $\sigma$ and $\tau$, NL–PDHGM did not appear to convergence until reaching the maximum iteration count of 100 000. This may be due to starting too far from the solution, or due to the fact that convergence of even the linear PDHGM can become very slow in the limit. Nevertheless, with the stopping threshold $\rho = 10^{-3}$, we quickly obtained satisfactory solutions, as can be observed by comparing the PSNRs between tables 4 and 5, as well as figure 4, which visualizes the results from the latter. Minding the large scale of the problem, we also had reasonably quick convergence to a greater accuracy with the unequal parameter choice. This gives us confidence for further application and study of NL–PDHGM in future work.

Finally, studying table 4, the reader may observe that the PSNR of the unconverged solution produced by the Gauss–Newton method is better than the other solutions; this is simply because we did not choose the regularization parameters optimally, only being interested in studying convergence of the methods for the present paper. Therefore we also do not compare the nonlinear model completely rigorously against our earlier efforts with linear models. This will be the topic of future research. However, in order to justify the nonlinear model, we have a simple comparison to report in table 6, figures 4 and 5. In order to facilitate comparison, we have zoomed the plots into the rectangle indicated in figure 3(a). We compare the model nonlinear
model (5.3) against the linear model of [26] with fidelity term $\| f - u \|^2$ in tensor space, where we first solve the noisy tensor field $f$ by linear regression from (5.2). Observe that we may linearize the equation by taking the logarithm on both sides. We also include in our comparison the model of [13] involving this linearization in the fidelity term. We calculated solutions to all of these models for $\alpha \in [0.0002, 0.001 10]/\sqrt{128 \times 128 \times 25}$, spacing 0.000 05, and picked again the optimal $\hat{\alpha}$ by both the PSNR and the discrepancy principle. Reading table 6, we see that the nonlinear performs clearly the best when $\hat{\alpha}$ is chosen by the discrepancy principle. When $\hat{\alpha}$ is chosen by optimal PSNR, the tensor space linear model is better. However, studying figure 4(c), the result is significantly smoothed, reminding us that the PSNR is a poor quality criterion for imaging problems. In case of selection of $\hat{\alpha}$ by the discrepancy principle in figure 5, the situation is not so dire, but still the linear models exhibit a level of smoothing. The nonlinear model (5.3) performs visually significantly better.
Acknowledgments

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Software implementation

Our C language implementation of the method for TV and TGV2 regularized problems may be found under http://iki.fi/tuomov/software/.

References


[29] Valkonen T and Liebmann M 2013 GPU-accelerated regularisation of large diffusion tensor volumes Computing 95 771–84 (special issue)


