The definite non-uniqueness results for deterministic EEG and MEG data

To cite this article: George Dassios and A S Fokas 2013 Inverse Problems 29 065012

View the article online for updates and enhancements.

Related content

- Electro-magneto-encephalography for the three-shell model: numerical implementation via splines for distributed current in spherical geometry
  A S Fokas, O Hauk and V Michel

- Electro-magneto-encephalography for the three-shell model: minimal L2-norm in spherical geometry
  A S Fokas and Y Kurylev

- On the non-uniqueness of the inverse problem associated with electroencephalography
  G Dassios and D Hadjiloizi

Recent citations

- On the inverse MEG problem with a 1-D current distribution
  George Dassios and Konstantia Satrazemi

- A hierarchical Krylov–Bayes iterative inverse solver for MEG with physiological preconditioning
  D Calvetti et al

- Inversion of electroencephalography data for a 2-D current distribution
  George Dassios and Konstantia Satrazemi
The definite non-uniqueness results for deterministic EEG and MEG data

George Dassios and A S Fokas
1 Department of Chemical Engineering, University of Patras and FORTH/ICE-HT, Greece
2 Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK

E-mail: gdassios@chemeng.upatras.gr

Received 4 January 2013, in final form 4 April 2013
Published 9 May 2013
Online at stacks.iop.org/IP/29/065012

Abstract

The solvability of the inverse problems of electroencephalography and magnetoencephalography has been studied extensively in the literature using a variety of models, including spherical and non-spherical geometries, homogeneous and inhomogeneous head models, and neuronal excitations involving the discrete and continuous distribution of dipoles. Among the important methods used are the methods based on spectral decompositions, physical arguments and integral representation techniques. Regarding the uniqueness of these inverse problems, a general result, independent of the geometry and the homogeneity of the conducting medium, has been obtained recently by the second author. This paper summarizes this result which appears to be mathematically definitive, in the sense that the geometry is arbitrary, the brain is surrounded by shells of varying conductivities, the neuronal current is arbitrary, the data are complete and the proofs are analytical. Furthermore, this paper includes a summary of the main steps of the proofs leading to the above result. In addition, it demonstrates the consistency of this general uniqueness result with all earlier results known in the literature.

1. Introduction

The mathematical formulations of the brain imaging techniques of electroencephalography (EEG) and magnetoencephalography (MEG) are based on the quasi-static theory of electromagnetism [26, 31, 40, 41]. Inverse problems usually suffer from the lack of uniqueness and this is indeed the case for the inverse problems of EEG and MEG. Actually, the fact that a current within a conductor cannot be completely identified from measurements of the magnetic field it generates in the exterior of the conductor was known to Helmholtz since 1853 [30]. However, a rigorous mathematical proof of this, appeared only in 2004 [23]. Over the last few decades, the authors and their collaborators have investigated extensively the mathematical theory of the direct and the inverse problems of EEG and MEG in different geometries.
In these works, a variety of partial results were obtained regarding the dependence of the exterior electric potential and that of the magnetic field on parts of the neuronal current, which was assumed to be either a collection of dipoles or a continuous distribution of dipoles. The definitive result in this direction was presented by the second author in [19]. It states that in any geometry, the scalar function that defines the irrotational part of the current contributes to both the electric potential and to the magnetic field and that the magnetic field also depends on the radial component of the vector function which defines the solenoidal part of the current. Hence, among the three scalar functions that define the three components of the current vector, one contributes to both brain imaging modalities of EEG and MEG, and one additional component contributes only to MEG. The third component of the current vector is ‘invisible’ by both the EEG and MEG.

The aim of this paper is to (a) elucidate the importance of the results of [19], (b) present the basic steps of the proofs of the above results concerning the uniqueness of EEG and MEG and (c) demonstrate that the above results are consistent with all earlier relevant results.

For a general introduction to EEG and MEG, we refer to [26, 32] and [38].

2. The definitive uniqueness result for EEG

We consider a bounded domain $\Omega_c$ representing the brain tissue which has conductivity $\sigma_c$. A shell $\Omega_f$ filled with cerebrospinal fluid with conductivity $\sigma_f$ surrounds the domain $\Omega_c$. A second shell $\Omega_b$ representing the skull with conductivity $\sigma_b$ surrounds the fluid shell $\Omega_f$. Finally, the skull is surrounded by the shell $\Omega_s$ with conductivity $\sigma_s$ representing the scalp. Let $\partial\Omega_c$, $\partial\Omega_f$, $\partial\Omega_b$ and $\partial\Omega_s$ be the outer boundaries of the corresponding domains which are assumed to be smooth. The exterior to the head domain is denoted by $\Omega_e$ and it is not conductive. The magnetic permeability of all domains is equal to the permeability $\mu_0$ of the empty space.

A neuronal current $J^p$, expressed as a discrete or as a continuous distribution of dipoles with specified moments, is supported within the cerebrum $\Omega_c$. Furthermore, since $\Omega_c$ as well as the three shells $\Omega_f$, $\Omega_b$ and $\Omega_s$ are conductive, it follows that an induction current is generated, which is equal to the conductivity times the electric field that exists in each of the four conductive domains. Physically, this system is governed by the quasi-static theory of Maxwell’s equations [31, 41], and as a consequence of the neuronal activity $J^p$, both the electric and magnetic fields are generated in the interior and the exterior of the head. Both the electric and magnetic fields are irrotational in the exterior domain and therefore in $\Omega_e$ they can be expressed in terms of an electric and a magnetic potential, respectively. The measurement of the electric potential on the boundary $\partial\Omega_e$ is associated with the modality of EEG, and the measurement of the magnetic flux density a few centimeters outside the head is associated with the modality of MEG.

According to the quasi-static theory, the electric potential $u_c$ in the cerebrum satisfies the Poisson equation [12]

$$\sigma_c \Delta u_c(r) = \nabla \cdot J^p(r), \quad r \in \Omega_c,$$

wheras the other potentials $u_f$, $u_b$, $u_s$ and $u_e$ in the domains specified by the corresponding subregions are harmonic functions. On the interface $\partial\Omega_c$, the transmission conditions

$$u_c(r) = u_f(r), \quad r \in \partial\Omega_c,$$

$$\sigma_c \frac{\partial u_c(r)}{\partial n} = \sigma_f \frac{\partial u_f(r)}{\partial n}, \quad r \in \partial\Omega_c,$$

2
ensure the continuity of the electric potential and that of the normal component of the induction current. Similarly, on the interface \( \partial \Omega_f \), the following conditions are valid:

\[
\begin{align*}
u_f(r) &= \nu_0(r), \quad r \in \partial \Omega_f, \\
\sigma_f \frac{\partial \nu_f(r)}{\partial n} &= \sigma_0 \frac{\partial \nu_0(r)}{\partial n}, \quad r \in \partial \Omega_f.
\end{align*}
\]

(4)\hspace{1cm}(5)

On the interface \( \partial \Omega_b \), the following conditions are valid:

\[
\begin{align*}
u_0(r) &= \nu_s(r), \quad r \in \partial \Omega_b, \\
\sigma_b \frac{\partial \nu_0(r)}{\partial n} &= \sigma_s \frac{\partial \nu_s(r)}{\partial n}, \quad r \in \partial \Omega_b.
\end{align*}
\]

(6)\hspace{1cm}(7)

The conductivity outside the head vanishes; thus on the exterior boundary \( \partial \Omega_c \), we have the conditions

\[
\begin{align*}
u_s(r) &= \nu_s(r), \quad r \in \partial \Omega_c, \\
\frac{\partial \nu_s(r)}{\partial n} &= 0, \quad r \in \partial \Omega_c,
\end{align*}
\]

(8)\hspace{1cm}(9)

which express the obvious fact that no current can exit the head.

Finally, at infinity we require the asymptotic condition

\[
u_s(r) = O\left(\frac{1}{r^2}\right), \quad r \to \infty.
\]

(10)

The set of equations (2)–(7) and (9) defines a well-posed problem which determines the potentials \( \nu_c, \nu_f, \nu_0, \) and \( \nu_s \). Then, using the boundary values of \( \nu \) on \( \partial \Omega_c \), under condition (8), as well as condition (10), we obtain a well-posed exterior Dirichlet problem for the determination of the potential \( \nu_c \). Actually, for the EEG problem, it is only the values of \( \nu_c \) on \( \partial \Omega_c \) that is needed. The Neumann problem that governs the interior electric potential is non-unique up to an additive constant; if we choose this constant to be equal to zero, then we arrive at condition (10). Hence, the square of the inverse distance appearing in (10) is due to the particular choice of the reference potential. In each shell, the induction current is given by the expression \( \sigma \nabla \nu \) for the corresponding shell.

Because of linearity, it is sufficient to find the solution of the above problem for a given dipole \( \{r_0, Q\} \), located at the point \( r_0 \in \Omega_c \) with moment \( Q \). Indeed, the solution for the dipolar current plays the role of Green’s function. In the case of a single dipole, the only change in the formulation of the above EEG problem is that equation (1) is now replaced by the equation

\[
s_c \Delta \nu_c(r) = Q \cdot \nabla_c \delta(r - r_0), \quad r \in \Omega_c, \quad r_0 \in \Omega_c,
\]

(11)

where \( \delta \) denotes the usual Dirac measure.

Obviously, each non-trivial solution of the EEG problem is generated by the source activity on the right-hand side of equation (11), which can also be written in the form

\[
Q \cdot \nabla_c \delta(r - r_0) = -Q \cdot \nabla_c \nu_c(r - r_0).
\]

(12)

Hence, it is sufficient to analyze the action of the directional derivative \( Q \cdot \nabla_c \). In physical terminology, since this directional derivative acting on the field of a monopole generates the field of a dipole [42], it follows that we can solve the general problem by considering a monopole source and then by acting with the directional derivative associated with the pair \( \{r_0, Q\} \) to obtain the corresponding solution for a dipole source. For this purpose, we define the potentials \( \nu_j, j = c, f, b, s, e \), corresponding to a unit monopole source at the point \( r_0 \) by the equations

\[
u_c(r; r_0) = \frac{1}{4\pi \sigma_c} \frac{Q \cdot \nabla_c}{\|r - r_0\|} \left[ \frac{1}{\|r - r_0\|} + \nu_c(r; r_0) \right],
\]

(13)
A continuously distributed current $J'$ inside the cerebrum can be decomposed, in the sense of Helmholtz, as the sum of an irrotational and a solenoidal field

$$ J'(r_0) = \nabla \Psi(r_0) + \nabla \times A(r_0), \quad \nabla \cdot A(r_0) = 0, \quad r_0 \in \Omega_c, $$

where $\Psi$ and $A$ are the scalar and vector functions, respectively.

The main result for EEG obtained in [19] is that the electric potential on the boundary $\partial \Omega_c$ can be expressed in the form

$$ u_\nu(r) = -\frac{1}{4\pi} \int_{\Omega_c} (\Delta \Psi(r')) u_\nu(r'; r') \, dV(r'), \quad r \in \partial \Omega_c. $$

For spherical and ellipsoidal geometries, the function $v_\nu$ is given by equations (25) and (2.40) of [9] and [12], respectively. This result demonstrates that for any geometry of the brain-head system, the EEG recordings depend only on the non-harmonic part of the scalar function $\Psi$. Hence, two out of the three components of the current are not ‘visible’ by the EEG modality, and the third component is ‘visible’ only via its non-harmonic part. This is the best possible non-uniqueness result that can be obtained for EEG. The basic idea of the proof of representation (16) is to transfer the directional differentiation (12) from the monopolar potential $v_\nu$ to the current representation (15) (this is achieved using appropriate integral theorems). This procedure eliminates the solenoidal part of the current as well as the harmonic part of the scalar representation function.

### 3. The definitive uniqueness result for MEG

In this section, we summarize the main steps of the proof in [19]. The exterior magnetic field which is recorded by the SQUID outside the head is given by the integral formula

$$ B(r) = -\mu_0 \int_{\Omega_c} J'(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dV(r') $$

$$ + \frac{\mu_0}{4\pi} \sum_{j=c, f, b, s} \sigma_j \int_{\Omega_c} \nabla u_j(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dV(r'), \quad r \in \Omega_c, $$

(17)

where $J'$ represents the neuronal current and $\sigma_j \nabla u_j$ denotes the induction currents in the subregions $\Omega_j$, $j = c, f, b, s$. Using the divergence theorem, formula (17) can be transformed to Gesselowitz-type formulas [24] involving only surface integrals. In the case of a single dipole $\{r_0, Q\}$, this formula takes the following form:

$$ B(r; r_0) = \frac{\mu_0}{4\pi} Q \times \nabla_n \left( \frac{1}{|r - r_0|} \right) - \frac{\mu_0}{(4\pi)^2} Q \cdot \nabla_n H(r; r_0), \quad r \in \Omega_c, r_0 \in \Omega_c, $$

(18)

where the function $H$ is independent of the moment $Q$ and is given in terms of the monopolar potentials $v_j$, $j = c, f, b, s$, by the expression

$$ H(r; r_0) = \int_{\partial \Omega_c} \left[ \frac{1}{|r' - r_0|} + (1 - \sigma_j) v_j(r'; r_0) \right] \hat{n}(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dS(r') $$

$$ + (\sigma_f - \sigma_b) \int_{\partial \Omega_c} v_f(r'; r_0) \hat{n}(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dS(r') $$

$$ + (\sigma_b - \sigma_s) \int_{\partial \Omega_c} v_b(r'; r_0) \hat{n}(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dS(r') $$

$$ + \sigma_s \int_{\partial \Omega_c} v_s(r'; r_0) \hat{n}(r') \times \nabla_r \left( \frac{1}{|r - r'|} \right) dS(r'), \quad r \in \Omega_c, r_0 \in \Omega_c. $$

(19)
The neuronal current has been represented in the literature in four different ways.

Integrating representation (18) over the support of the neuronal current $J^p$, we obtain

$$
B(r) = \frac{\mu_0}{4\pi} \int_{\Omega_c} J^p(r') \times \nabla_r \left( \frac{1}{|r-r'|} \right) \, dV(r') - \frac{\mu_0}{(4\pi)^2} \int_{\Omega_c} J^p(r') \cdot \nabla_r H(r; r') \, dV(r')
$$

(20)

for every $r \in \Omega_c$. Using the vector identity

$$
\nabla_r \times \left( \frac{1}{|r-r'|} J^p(r') \right) = -J^p(r') \times \nabla_r \left( \frac{1}{|r-r'|} \right) + \frac{1}{|r-r'|} \nabla_r \times J^p(r')
$$

(21)

in the first integral, the dyadic identity

$$
\nabla_r \cdot [J^p(r') \otimes H(r; r')] = [\nabla_r \cdot J^p(r')]H(r; r') + J^p(r') \cdot [\nabla_r \otimes H(r; r')]
$$

(22)

in the second integral, as well as the Helmholtz representation (15) and standard integral theorems, equation (20) yields the exterior representation for the magnetic field

$$
B(r) = -\frac{\mu_0}{4\pi} \int_{\Omega_c} (\Delta A(r')) \left( \frac{1}{|r-r'|} \right) \, dV(r') + \frac{\mu_0}{(4\pi)^2} \int_{\Omega_c} (\nabla \Psi(r'))H(r; r') \, dV(r').
$$

(23)

Finally, it is shown in [19] that the radial component of the magnetic field has the representation

$$
\mathbf{r} \cdot B(r) = -\frac{\mu_0}{4\pi} \int_{\Omega_c} [\Delta(r' \cdot A(r'))] \left( \frac{1}{|r-r'|} \right) \, dV(r') + \frac{\mu_0}{(4\pi)^2} \int_{\Omega_c} (\nabla \Psi(r'))\mathbf{r} \cdot H(r; r') \, dV(r').
$$

(24)

which holds at every point outside the head.

The representation (23) involves the non-harmonic parts of the scalar and of the vector functions appearing in the Helmholtz decomposition. However, it is well known [1, 41] that since the exterior magnetic field is irrotational, it can be expressed as the gradient of a potential function, which in turn can be calculated from the radial component of the magnetic field. Thus, the right-hand side of (24) depends only on the non-harmonic parts of the scalar function $\Psi$ and on the non-harmonic part of the radial component of the vector function $A$. Consequently, the MEG recordings depend on the same part of $\Psi$ as the EEG recordings, and they also depend on the non-harmonic part of the radial component of $A$. Hence, we cannot identify completely the active neuronal current, since the tangential component of $A$ is ‘invisible’ for both EEG and MEG modalities.

4. Consistency with earlier results

The neuronal current has been represented in the literature in four different ways.

First, there is the Helmholtz representation [11, 34–36]

$$
J^p(r) = \nabla \Psi(r) + \nabla \times A(r), \quad \nabla \cdot A(r) = 0,
$$

(25)

where $\Psi$ and the two independent components of $A$ are the three scalar functions specifying $J^p$.

Second, there is the Hansen representation [28], also known as the Hodge representation [13, 19, 20, 23],

$$
J^p(r) = J_p(r)\hat{r} + \hat{r} \times \nabla F(r) - \hat{r} \times (\hat{r} \times \nabla G(r)),
$$

(26)

where the current is represented by the three scalar functions $J_p$, $F$ and $G$. This representation is very convenient for the spherical geometry.
Third, there is the representation in terms the complete orthogonal set of the vector surface spherical harmonics $P_n^m$, $B_n^m$ and $C_n^m$ [4, 10, 34–36]:

$$ J^0(r) = p_0^0(r)\hat{r} + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ p_n^m(r) P_n^m(\hat{r}) + b_n^m(r) B_n^m(\hat{r}) + c_n^m(r) C_n^m(\hat{r}) \right], $$

(27)

where the current is represented by the three sequences $\{p_n^m\}$, $\{b_n^m\}$ and $\{c_n^m\}$. In this case, the space $F$ of all functions that can be expanded in terms of vector surface spherical harmonics can be written as the direct sum

$$ F = \mathcal{P} \oplus \mathcal{B} \oplus \mathcal{C}, $$

(28)

where $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{C}$ are the subspaces generated by the sets $\{P_n^m\}$, $\{B_n^m\}$ and $\{C_n^m\}$, respectively. Then, the three components of the current are the three projections of $J^0$ in the subspaces $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{C}$, namely

$$ J^0(r, \theta, \phi) = J_\mathcal{P}(r, \theta, \phi) + J_\mathcal{B}(r, \theta, \phi) + J_\mathcal{C}(r, \theta, \phi). $$

(29)

Finally, the fourth representation uses the complete doubly orthogonal set of vector surface ellipsoidal harmonics $R_n^m$, $D_n^m$ and $T_n^m$ [18]:

$$ J^0(r) = A_0^0(\rho)\hat{\rho} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ A_n^m(\rho)R_n^m(\mu, \nu) + B_n^m(\rho)D_n^m(\mu, \nu) + C_n^m(\rho)T_n^m(\mu, \nu) \right], $$

(30)

where the current is represented by the three sequences $\{A_n^m\}$, $\{B_n^m\}$ and $\{C_n^m\}$, and $(\rho, \mu, \nu)$ are the standard ellipsoidal coordinates [7]. As in the spherical case, there exists the direct sum decomposition

$$ F = \mathcal{R} \oplus \mathcal{D} \oplus \mathcal{T}, $$

(31)

where $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{T}$ are the three subspaces generated by $\{R_n^m\}$, $\{D_n^m\}$ and $\{T_n^m\}$, respectively. Then, in analogy with (29), we now have

$$ J^0(\rho, \mu, \nu) = J_\mathcal{R}(\rho, \mu, \nu) + J_\mathcal{D}(\rho, \mu, \nu) + J_\mathcal{T}(\rho, \mu, \nu), $$

(32)

where $J_\mathcal{R}$, $J_\mathcal{D}$ and $J_\mathcal{T}$ are the three components of the current.

In what follows, we compare the general results stated in sections 2 and 3 with the corresponding results in the existing mathematical literature.

The first rigorous result on the non-uniqueness of the inverse MEG problem, was obtained by Fokas, Gelfand and Kurylev [20, 23]. These authors, using the representation (26), proved that for the case of a sphere, the measured magnetic field in the exterior of the head depends only on the function $F$. This result is consistent with the use of representation (24), since for the sphere

$$ r \cdot \left[ \hat{n}(r') \times \nabla_F \frac{1}{|r - r'|} \right] = r \cdot \left[ \hat{r}' \times \frac{r - r'}{|r - r'|^3} \right] = 0, $$

(33)

which implies that

$$ r \cdot H(r) = 0 $$

(34)

and the only function that is left in the representation formula (24) is the radial component of $A$.

It is shown in [11], using the representation (25), that if the conductor has a star shape and if the electric potential on the boundary is known, then in order to express the exterior magnetic field one only needs the radial part of $A$. Taking into consideration that the knowledge of the electric potential on the boundary provides information equivalent to the knowledge of the scalar function $\Psi$, this result is also consistent with the representation (24).
It is shown in [10], using the representation (25), that within the spherical model of the head, the information obtained from EEG has no overlapping part with the information obtained from MEG. In view of the results obtained in [19], this is consistent with representation (16) which depends only on $\Psi$, and with representation (24) which (because of (34)), depends only on $\hat{r} \cdot A$.

A related result obtained for conductors of any shape is obtained in [4], where it was shown that the electric potential depends on the component of the current that lives in the subspace spanned by the eigenvectors $P_m^m$ and $B_m^m$, whereas the current that generates the exterior magnetic field is expressible in terms of all three eigenvectors $P_m^m$, $B_m^m$ and $C_m^m$. This is also consistent with the results in [19], since it states that MEG involves the part of the current that generates EEG plus an additional component. However, this result is much weaker than the result in [19], since it states that EEG and MEG are generated by two and three components, respectively, instead of one and two components, respectively, as demonstrated in [19]. In order to resolve this discrepancy, we consider the following two decompositions of the current:

$$J^p(r) = \nabla \Psi(r) + \nabla \times A(r) = J_P(r) + J_B(r) + J_C(r),$$

(35)

where $J_P$, $J_B$ and $J_C$ are the components of the current that live in the subspaces spanned by the eigenvectors $P_m^m$, $B_m^m$ and $C_m^m$, respectively. Ignoring some multiplicative constants, these three vectors are given by

$$P_m^m(\hat{r}) = \hat{r} Y_m^m(\hat{r}),$$

(36)

$$B_m^m(\hat{r}) = r \nabla Y_m^m(\hat{r}),$$

(37)

$$C_m^m(\hat{r}) = r \times \nabla Y_m^m(\hat{r}),$$

(38)

where the $Y$'s denote the standard scalar spherical harmonics. On the other hand, the function $\Psi$ can be expressed in the form

$$\Psi(r, \hat{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_n^m(r) Y_n^m(\hat{r}).$$

(39)

Using the identities

$$\nabla (\psi_n^m(r) Y_n^m(\hat{r})) = \frac{d\psi_n^m(r)}{dr} P_n^m(\hat{r}) + \frac{\psi_n^m(r)}{r} r \nabla Y_n^m(\hat{r})$$

$$= \frac{d\psi_n^m(r)}{dr} P_n^m(\hat{r}) + \frac{\psi_n^m(r)}{r} B_n^m(\hat{r}),$$

(40)

it follows that the irrotational part of the current, which is represented by the scalar function $\Psi$, lives in the subspace $\mathcal{P} \oplus \mathcal{B}$. This is the component of the current that appears in both the EEG and MEG modalities. Furthermore, because of (38), the additional component of the current represented by $\hat{r} \cdot A$ belongs to the subspace $\mathcal{C}$. Hence, these results are also consistent with the definitive representations (16) and (23). Completely analogous formulae are also valid for the ellipsoidal representation (32). However, we do not provide the relevant details here, since this would require several identities from the theory of scalar and vector ellipsoidal harmonics [7, 18].

Finally, we consider the results of [13] of the spherical model for the EEG problem, which provide the analogue of the results in [23]. Using the Hansen representation (26) it is shown in [13] that the exterior electric potential depends on the functions $J_r$ and $G$, but not on $F$. In this sense, it is demonstrated once more that for the spherical model there is no overlapping...
information from the EEG and the MEG recordings. In order to demonstrate the consistency of the results in [13] and [23], we first rewrite the representation (26) in the following form:

\[ J^p(r) = \left[ J_r(r) - \frac{\partial G(r)}{\partial r} \right] \hat{r} + \nabla G(r) + \hat{r} \times \nabla F(r). \]  

(41)

The first term on the right-hand side of (41) belongs to the subspace \( \mathcal{P} \), the second term belongs to the subspace \( \mathcal{B} \) and the third term belongs to the subspace \( \mathcal{C} \). As we have explained earlier, the first two terms can be generated from the gradient operator of a scalar function and therefore can be interpreted as the contribution of the function \( \Psi_1 \). The fact that the function \( F \) does not appear in the relevant representation implies that the function \( \Psi \) does not enter the EEG fields, which is consistent with [19].

From the above discussion, it follows that every known partial mathematical result on the uniqueness for the inverse EEG and MEG problems is consistent with the definitive results obtained in [19].

5. Conclusions

The basic inversion problems associated with EEG and MEG consist of estimating the neuronal current \( J^p \) related to brain activity, from measurements of the induced electromagnetic fields on the scalp and outside the human head, respectively. In the ideal case, one assumes that the electric potential \( u_e \) is known on the scalp and that the radial component \( \hat{r} \cdot \mathbf{B} \) of the induced magnetic field is known outside the head, where \( \hat{r} \) denotes the unit vector in the direction of observation \( r \). The fundamental question here is, is it possible to identify the neuronal current from knowledge of the above fields even in the complete absence of noise?

EEG and MEG recordings have high temporal resolution and also are directly correlated with brain excitation; thus EEG and MEG play a crucial role in a variety of clinical situations in neurology, as well as in efforts to elucidate various important open questions in neuroscience. The Achilles heel of EEG/MEG is the luck of uniqueness of the inverse problem, namely the fact that even if one ignores the important practical considerations of incomplete and noisy measurements, as well as the existence of external and internal artifacts and signal interference, and one assumes knowledge of complete data, still it is not possible to reconstruct completely the current \( J^p \). The following important question remained open for a long time, in spite of extensive investigations by many authors: which part of \( J^p \) can be recovered from the knowledge of \( u_e \) on the scalp and of \( \hat{r} \cdot \mathbf{B} \) outside the head? The complete answer to this problem was finally given in [19] and summarized in this paper.

Regarding the results of [19], we note the following.

(i) The concrete non-uniqueness results of [19] may have useful implications for the interpretation of the current obtained via the existing imaging techniques.

(ii) A possible way to address the non-uniqueness question is to supplement the inversion algorithm with some particular constraint. Indeed, currently the so-called minimum-norm solution is extensively used, which assumes that the current \( J^p \) should have minimum \( L^2 \)-norm [27, 29]. The results of [19] suggest an alternative implementation of such constraints: first identify the part of the current that can be determined from the measurements, and then minimize the relative \( L^2 \)-norm. It is shown in [22] that in the case of spherical geometry this approach yields an explicit analytic formula for the unique current. The problem of obtaining an analogous formula for the more realistic ellipsoidal geometry is work in progress.
(iii) The remark in (ii) regarding the ‘minimum-norm solutions’ is also valid for other regularization strategies used in the literature, such as the minimizations used in FOCUSS [25], RWMN [33] and LORETA [39].

(iv) In the case of independent EEG measurements, the results of [19] for the case of spherical and ellipsoidal geometries provide the explicit analytical solutions for the so-called ELEKTRA model [21] (this model has the advantage that it can be compared directly with intra-cranial recordings).

(v) The question of implementing the reconstruction formulae of [19] supplemented with appropriate minimization constraints to real data and computing the relevant current with commercial software remains open.

Acknowledgments

Part of the research of ASF has been co-financed by the European Union (European Social Fund—ESF) and Greek national funds through the Operational Program ‘Education and Lifelong Learning’ of the National Strategic Reference Framework (NSRF)—Research Funding Program, ‘THALES’. Investigating in knowledge society through the European Social Fund. GD has been financed by the Secretariat General for Research and Technology of the Greek Ministry of Education through the research program, ‘ARISTEIA’.

References

[22] Fokas A S and Kurylev Y 2012 Electromagnetoencephalography for the three-shell model: minimal $l_2$-norm in spherical geometry Inverse Problems 28 035010
[29] Hauk O 2004 Keep it simple: a case for using classical minimum norm estimation in the analysis of EEG and MEG data Neuroimage 21 1612–21
[34] Morse P M and Feshbach H 1953 Methods of Theoretical Physics vol 2 1st edn (New York: McGraw-Hill)