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A generalized Prony method for reconstruction of sparse sums of eigenfunctions of linear operators

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Abstract

We derive a new generalization of Prony’s method to reconstruct $M$-sparse expansions of (generalized) eigenfunctions of linear operators from only $O(M)$ suitable values in a deterministic way. The proposed method covers the well-known reconstruction methods for $M$-sparse sums of exponentials as well as for the interpolation of $M$-sparse polynomials by using special linear operators in $C(\mathbb{R})$. Further, we can derive new reconstruction formulas for $M$-sparse expansions of orthogonal polynomials using the Sturm–Liouville operator. The method is also applied to the recovery of $M$-sparse vectors in finite-dimensional vector spaces.

1. Introduction

In signal analysis, we often have some a priori knowledge about the underlying structure of the wanted signal that we need to exploit suitably. Using this structure, we are faced with the problem of determining a certain number of parameters from the given signal measurements. Considering for example a structured function of the form

$$f(\omega) = \sum_{j=1}^{M} c_j e^{i\omega T_j}$$  \hspace{1cm} (1.1)

with (unknown) complex parameters $c_j$ and $T_j$, $j = 1, \ldots, M$, and assuming that $-\pi < \text{Im}T_1 < \ldots < \text{Im}T_M < \pi$, we aim to reconstruct $c_j$ and $T_j$ from a given small amount of (possibly noisy) measurement values $f(\ell)$. Using Prony’s method [30] or one of its stabilized variants, we are able to reconstruct $f$ with only $2M$ function values $f(\ell)$, $\ell = 0, \ldots, 2M - 1$, see [5, 18, 25, 26, 31]. The solution of this problem involves the determination of the so-called Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \lambda_j)$$
with \( \lambda_j := e^{T_j} \). Assuming that \( P(z) \) has the monomial representation
\[
P(z) = \sum_{k=0}^{M} p_k z^k,
\]
and using the structure of \( f \), a short computation yields for \( m = 0, \ldots, M - 1 \),
\[
\sum_{k=0}^{M} p_k f(k + m) = \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j e^{(k+m)T_j} = \sum_{j=1}^{M} c_j \lambda_j^m \left( \sum_{k=0}^{M} p_k \lambda_j^k \right) = \sum_{j=1}^{M} c_j \lambda_j^m P(\lambda_j) = 0.
\]

With \( p_M = 1 \) we obtain the linear Hankel system
\[
\sum_{k=0}^{M-1} p_k f(k + m) = -f(M + m), \quad m = 0, \ldots, M - 1,
\]
providing the coefficients \( p_k \) of the Prony polynomial \( P(z) \). Now, the unknown parameters \( T_j \) can be extracted from the zeros \( \lambda_j = e^{T_j} \) of \( P(z) \). Afterward, the coefficients \( c_j \) are obtained by solving the overdetermined linear system
\[
f(\ell) = \sum_{j=1}^{M} c_j e^{i\ell T_j}, \quad \ell = 0, \ldots, 2M - 1.
\]

In recent years, the Prony method has been successfully applied to different inverse problems such as e.g. for the approximation of Green functions in quantum chemistry [35] or fluid dynamics [4], for the localization of particles in inverse scattering [17], for the parameter estimation of dispersion curves of guided waves [32] and for the analysis of ultrasonic signals [7]. The renaissance of Prony’s method originates from some modifications of the algorithm described above that considerably stabilize the original approach, such as e.g. the ESPRIT method, the matrix pencil method or the approximate Prony method [31, 18, 26]. These techniques can also be applied if the number \( M \) of relevant frequencies is not known beforehand, provided that a sufficiently large number of measurements \( f(\ell) \) is given, and the applications in practice show that they work well even in the case of noisy measurements. Error estimates for the performance of Prony-like methods with noisy measurements are derived in [26, 11, 2]. Just recently, the reconstruction of functions of the form (1.1) (with \( \text{Re} T_j = 0 \)) using a total variation minimization has been proposed in [8]. To tackle this minimization problem, a semidefinite program is applied to solve the dual problem in a first step. The obtained vector is used to define a special polynomial that possesses exactly \( M \) zeros on the unit circle which are related to the wanted frequencies \( \text{Im} T_j \). The exact connections between the minimization approach in the context of super-resolution and the direct algorithms for the Prony method will be subject of further research.

Searching the literature, one finds different further reconstruction methods that are closely related to Prony’s method at second glance.

In spectral analysis, the annihilating filter method is frequently applied. This idea has already been used long ago for the construction of cyclic codes, [33]. For a given signal \( S[n] \), the FIR filter \( A[n] \) is called the annihilating filter of \( S[n] \), if
\[
(A * S)[n] = \sum_{j \in \mathbb{Z}} A[j]S[n - j] = \sum_{j \in \mathbb{Z}} A[-j]S[n + j] = 0.
\]
Using the $z$-transform $A(z) = \sum_{n=0}^{M} A[-n]z^{n}$ and comparing this convolution equation to (1.2), we observe that $A(z)$ undertakes the task of the Prony polynomial. Vetterli et al [34] introduced signals with a finite rate of innovation, i.e. signals with a special structure having only a finite number of degrees of freedom per time unit. Using the annihilating filter method, these signals can be completely reconstructed, see also [21, 6].

In some applications, one needs to apply an integral transform in order to obtain functions of the form (1.1); then the Prony method can be applied in the transform domain, see e.g. [14, 10, 23, 1, 24].

In computer algebra, one is faced with the computation and processing of multivariate polynomials of high order. But if the polynomial $f$ is $M$-sparse, i.e.

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{M} c_j x_{d_1 j_1} x_{d_2 j_2} \cdots x_{d_n j_n}$$

with $c_1, \ldots, c_M \in \mathbb{C}$ and with $M$ pairwise different vectors $(d_{j_1}, \ldots, d_{j_n}) \in \mathbb{N}^n$, then $f$ can be completely recovered using only $2M$ suitably chosen function values. Here again, the number of needed evaluations does not depend on the degree of the polynomial $f$ but on the number $M$ of active terms. A stochastic approach to the problem was given in [36]. A deterministic algorithm goes back to Ben-Or and Tiwari [3], and has been shown to be closely related to the Prony method, see e.g. [19, 13].

Only a few papers have been concerned with the question whether the above interpolation problem can also be solved efficiently if the polynomial is sparse in a different polynomial basis. This problem turned out to be difficult to solve and suitable methods were only given for the Pochhammer basis and the basis of Chebyshev polynomials of the first kind [20, 12, 29].

Recently, we considered the function reconstruction problem for sparse Legendre expansions of order $N$ of the form in [22]

$$f(x) = \sum_{j=1}^{M} c_j P_{n_j}(x)$$

with $0 \leq n_1 < n_2 \cdots < n_M = N$, where $M \ll N$, aiming at a generalization of Prony’s method for this case. We derived a reconstruction algorithm involving the function and derivative values $f^{(\ell)}(1), \ell = 0, \ldots, 2M - 1$. The reconstruction in [22] is based on special properties of Legendre polynomials, but it does not provide a direct approach for the further generalization of the method to other sparse orthogonal polynomial expansions or to other function systems apart from exponentials and monomials.

In [9], the idea of efficient sparse polynomial interpolation has been transferred to the more general case of $M$-term sums of characters of Abelian monoids. This approach has also been used in [16] for the reconstruction of functions being linear combinations of eigenfunctions of linear operators on suitable algebras on integral domains. This last paper can be seen as one starting point of our considerations.

In this paper, we want to present a new general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators. This new insight provides us with a tool to unify all Prony-like methods on the one hand and to essentially generalize the Prony approach on the other hand. Thus it will establish a much broader field of applications of the method. In particular, we will show that all well-known Prony-like reconstruction methods for exponentials and polynomials known so far can be seen as special cases of this approach. Moreover, the new insight into Prony-like methods enables us to derive new reconstruction algorithms for orthogonal polynomial expansions including Jacobi, Laguerre and Hermite
polynomials. The approach also applies to finite-dimensional vector spaces, and we derive a
deterministic reconstruction method for $M$-sparse vectors from only $2M$ measurements.

We point out that the provided generalized Prony method can also be applied to other
operators and eigenfunctions. In particular, one may apply it e.g. to Bessel or Hermite functions
that are eigenfunctions of special differential operators, to orthogonal polynomials that are
eigenfunctions of difference equations and to eigenfunctions of suitable integral operators.
However, in this paper we restrict ourselves to the examples described above.

We will concentrate on the representation of the generalized Prony method for recovering
expansions of eigenfunctions where the number $M$ of terms in the sum is known in advance.
Of course, our ideas can also be generalized to the case of an unknown number of terms
in the considered expansion. The construction and analysis of stabilized algorithms for the
generalized Prony approach will be subject of further research.

The paper is organized as follows. In section 2, we derive the general reconstruction
method for sparse expansions of eigenfunctions of suitable linear operators. We also consider
the case of generalized eigenfunctions. In section 3, we revisit the well-known reconstruction
methods for sparse sums of exponentials and for sparse polynomials, where the exponentials
can be regarded as eigenfunctions of the translation operator, and the monomials are
eigenfunctions of the dilation operator in $C(\mathbb{R})$. Moreover, we show that also sparse linear
combinations of $u_k(x) e^{T_j x}$ can be recovered using the general approach with generalized
eigenfunctions where $u_k(x)$ is a polynomial of order $k$. Further, exponentials as well as
monomials can also be seen as eigenfunctions of suitable differential operators yielding new
types of Prony-like reconstructions. Section 4 is dedicated to the reconstruction of sparse
expansions of orthogonal polynomials that can be seen as eigenfunctions of the Sturm–
Liouville operator. In particular, we show that the reconstruction algorithm for Legendre
expansions derived in [22] is a special example of this general method. Finally, we consider
the recovery of $M$-sparse vectors $\mathbf{x} \in \mathbb{C}^N$ with $N \ll M$ from $\mathbf{y} = \mathbf{F}_{N,2M} \mathbf{x}$, where $\mathbf{F}_{N,2M} \in \mathbb{C}^{2M \times N}$
contains the first $2M$ rows of the Fourier matrix $\mathbf{F}_N$.

2. Prony’s method for sparse expansions of eigenfunctions

We want to generalize the Prony method to sparse expansions of eigenfunctions of certain
linear operators. Let $V$ be a normed vector space over $\mathbb{C}$, and let $\mathcal{A} : V \to V$ be a linear
operator.

Assume that $\mathcal{A}$ possesses eigenvalues, and let $\Lambda := \{\lambda_j : j \in I\}$ be a (sub)set of
pairwise distinct eigenvalues of $\mathcal{A}$, where $I$ is a suitable index set. We consider the eigenspaces
$V_j = \{v : \mathcal{A}v = \lambda_j v\}$ to the eigenvalues $\lambda_j$, and for each $j \in I$, we predetermine a one-
dimensional subspace $V_j$ of $V_j$ that is spanned by the normalized eigenfunction $v_j$. In particular,
we assume that there is a unique correspondence between $\lambda_j$ and $v_j$, $j \in I$.

An expansion $f$ of eigenfunctions of the operator $\mathcal{A}$ is called $M$-sparse if its representation
consists of only $M$ non-vanishing terms, i.e. if

$$f = \sum_{j \in J} c_j v_j, \quad \text{with } J \subset I \text{ and } |J| = M. \quad (2.1)$$

Due to the linearity of the operator $\mathcal{A}$, the $k$-fold application of $\mathcal{A}$ to $f$ yields

$$\mathcal{A}^k f = \sum_{j \in J} c_j A^k v_j = \sum_{j \in J} c_j \lambda_j^k v_j. \quad (2.2)$$

Further, let $F : V \to \mathbb{C}$ be a linear functional with the property $F v_j \neq 0$ for all $j \in I$. We
show that the expansion $f$ in (2.1) can be reconstructed using only the $2M$ values $F(\mathcal{A}^k f)$,
$k = 0, \ldots, 2M - 1$. 4
Theorem 2.1. With the above assumptions, the expansion \( f \) in (2.1) of eigenfunctions \( v_j \in \tilde{V}_j \), \( j \in J \subseteq I \), of the linear operator \( A \), with \( c_j \neq 0 \) for all \( j \in J \), can be uniquely reconstructed from the values \( F(A^k f), k = 0, \ldots, 2M - 1 \), i.e. the ‘active’ eigenfunctions \( v_j \) as well as the coefficients \( c_j \in \mathbb{C}, j \in J \), in (2.1) can be determined uniquely.

Proof. We give a constructive proof.

1. We define the so-called Prony polynomial

\[
P(z) := \prod_{j \in J} (z - \lambda_j),
\]

where the roots \( \lambda_j, j \in J \), are the eigenvalues corresponding to the (unknown) active eigenfunctions \( v_j \) in the representation of \( f \). Further, let \( P(z) = \sum_{k=0}^{M} p_k z^k \), with \( p_M = 1 \), be the monomial representation of the Prony polynomial. Combining the unknown coefficients \( p_k \) with the given values \( F(A^k f), k = 0, \ldots, 2M - 1 \), and using (2.2) we observe the following relation for \( m = 0, 1, \ldots \),

\[
\sum_{k=0}^{M} p_k F(A^{k+m} f) = \sum_{k=0}^{M} p_k F \left( \sum_{j \in J} c_j \lambda_j^{k+m} v_j \right) = \sum_{j \in J} c_j \lambda_j^{m} \left( \sum_{k=0}^{M} p_k \lambda_j^{k} \right) F v_j = \sum_{j \in J} c_j \lambda_j^{m} P(\lambda_j) F v_j = 0.
\]

Together with \( p_M = 1 \), the coefficients \( p_k, k = 0, \ldots, M - 1 \), of the Prony polynomial can now be determined via the linear system

\[
\sum_{k=0}^{M-1} p_k F(A^{k+m} f) = -F(A^{M+m} f), \quad m = 0, \ldots, M - 1.
\]

Indeed, the coefficient matrix \( G := (F(A^{k+m} f))_{k,m=0}^{M-1,M-1} \) is an invertible Hankel matrix since (2.2) yields

\[
G = V_k \cdot \text{diag}(\mathbf{c}) \cdot \text{diag}(F \mathbf{v}) \cdot V_k^T
\]

with the Vandermonde matrix

\[
V_k := (\lambda_j^{k})_{k=0,j \in J}^{M-1}
\]

and with the diagonal matrices \( \text{diag}(\mathbf{c}) = \text{diag}(c_j)_{j \in J}, \text{diag}(F \mathbf{v}) = \text{diag}(F v_j)_{j \in J} \), where the indices \( j \in J \) are assumed to be given in a fixed order. By assumption, \( V_k \) as well as the diagonal matrices \( \text{diag}(\mathbf{c}) \) and \( \text{diag}(F \mathbf{v}) \) have full rank yielding the invertibility of \( G \).

2. Having determined the Prony polynomial

\[
P(z) = \sum_{k=0}^{M} p_k z^k = \prod_{j \in J} (z - \lambda_j),
\]

we can evaluate the eigenvalues \( \lambda_j, j \in J \), that are the zeros of \( P(z) \). Since the eigenspaces \( \tilde{V}_j \) are assumed to be one-dimensional we can uniquely determine the corresponding eigenfunctions \( v_j, j \in J \).

3. In the last step, we compute the coefficients \( c_j, j \in J \), of the expansion (2.1) by solving the overdetermined linear system

\[
F(A^k f) = \sum_{j \in J} c_j \lambda_j^k v_j, \quad k = 0, \ldots, 2M - 1,
\]

using the eigenvalues \( \lambda_j \) and eigenfunctions \( v_j \) found in the previous step. \( \square \)
This general approach to the Prony method enables us to derive reconstruction algorithms for a variety of different systems of eigenfunctions. We summarize the algorithm as follows.

**Algorithm 2.2** (Reconstruction of the sparse expansion (2.1)).

**Input:** $M, F(\lambda^f f), k = 0, \ldots, 2M - 1$.

1. Solve the linear system
   \[
   \sum_{k=0}^{M-1} p_k F(\lambda^{k+m} f) = - F(\lambda^{M+m} f), \quad m = 0, \ldots, M - 1. \tag{2.3}
   \]

2. Form the Prony polynomial $P(z) = \sum_{k=0}^{M} p_k z^k$ using the obtained values $p_k, k = 0, \ldots, M-1$ from step 1 and $p_M = 1$. Compute the zeros $\lambda_j, j \in J$, of $P(z)$ and determine the corresponding (normalized) eigenfunctions $v_j, j \in J$.

3. Compute the coefficients $c_j$ by solving the overdetermined system
   \[
   F(\lambda^f f) = \sum_{j \in J} c_j \lambda_j^k v_j, \quad k = 0, \ldots, 2M - 1.
   \]

**Output:** $c_j, v_j, j \in J$, determining $f$ in (2.1).

**Remark 2.3.** Just recently, Potts and Tasche [28] presented a unified approach to Prony-type algorithms for exponential sums that are numerically stable, such as e.g. ESPRIT [31] and the matrix pencil method [18]. The approach even applies if the parameter $M$ (the number of active basis functions) is unknown, supposed that we have a given set of $2N$ suitable values, where $N \geq M$ is an upper bound for $M$. This idea to derive a more stable algorithm can also be transferred to our general Prony method. Consider the Hankel matrix $G := (F(\lambda^{k+m} f))_{m=0}^{M-1} = (g_0, \ldots, g_{M-1})$, where $g_k, k = 0, \ldots, M - 1$, denote the columns of $G$. Further let $g_M := (F(\lambda^{M+m} f))_{m=0}^{M-1}$. Then (2.3) can be written as
   \[
   G p = -g_M.
   \]
   where $p := (p_0, \ldots, p_{M-1})^T$. Using the companion matrix $C(P)$ of the Prony polynomial $P(z)$, we obtain
   \[
   GC(P) = \bar{G},
   \]
   where $\bar{G} = (g_0, \ldots, g_M)$ is a shifted Hankel matrix. The eigenvalues of the companion matrix $C(P)$ are the wanted zeros $\lambda_j, j = 1, \ldots, M$, of the Prony polynomial $P(z)$. From the above relation, it follows that the $\lambda_j$ are also the eigenvalues of the matrix pencil, see [15, p 251],
   \[
   zG - \bar{G}, \quad z \in \mathbb{C}.
   \]
   Using this approach, the wanted values $\lambda_j, j = 1, \ldots, M$, can be obtained from a QR decomposition or a singular value decomposition of the rectangular Hankel matrix $(g_0, \ldots, g_M)$. Since the proofs in [28] are based only on rank arguments of Hankel matrices $G$ and $\bar{G}$ whose entries are the given input data, the case of unknown order $M$ can be applied directly to our approach too.

Let us also consider the case of generalized eigenfunctions. Let $r \geq 1$ be a fixed integer. Analogously as for linear operators in finite-dimensional vector spaces, we say that $v_\ell$, $\ell = 1, \ldots, r$, are generalized eigenfunctions of multiplicity $\ell$ of a linear operator $\mathcal{A} : V \to V$ to the eigenvalue $\lambda$, if
   \[
   (\mathcal{A} - \lambda I)^{\ell} v_\ell = 0, \quad \ell = 1, \ldots, r,
   \]
and

\[ \mathcal{A} \tilde{v}_\ell = \lambda \tilde{v}_\ell + \sum_{s=1}^{\ell-1} \alpha_{\ell,s} \tilde{v}_s, \quad \ell = 1, \ldots, r, \]  

(2.4)

with some constants \( \alpha_{\ell,s} \in \mathbb{C} \). Again, let \( \Lambda = \{ \lambda_j, \ j \in I \} \) be a (sub)set of pairwise distinct eigenvalues of \( \mathcal{A} \), and for each \( j \in I \), let \( \{ \tilde{v}_{j,\ell} : \ \ell = 1, \ldots, r \} \) be a predetermined set of linearly independent generalized eigenfunctions to the eigenvalue \( \lambda_j \). Further, let \( F : V \rightarrow \mathbb{C} \) be a functional with \( F(\tilde{v}_{j,\ell}) \neq 0 \) for \( j \in I, \ \ell = 1, \ldots, r \).

**Theorem 2.4.** With the above assumptions, the expansion

\[ f = \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \tilde{v}_{j,\ell}, \quad J \subset I, |J| = M, r \geq 1, \]

of generalized eigenfunctions of the linear operator \( \mathcal{A} \) to the eigenvalues \( \lambda_j, \ j \in J \subset I \), with \( \sum_{\ell=1}^{r} |c_{j,\ell}|^2 \neq 0 \) for all \( j \in J \), can be uniquely reconstructed from the values \( F(\mathcal{A}^k f), \ k = 0, \ldots, 2rM - 1 \), supposed that the matrix \( (F(\mathcal{A}^{k+m} f))_{k,m=0}^{rM-1} \) is invertible.

**Proof.** We only consider the special case, where, instead of (2.4), the generalized eigenfunctions \( \tilde{v}_{j,\ell} \) to the eigenvalue \( \lambda_j \) satisfy

\[ \mathcal{A} \tilde{v}_{j,\ell} = \lambda \tilde{v}_{j,\ell} + \alpha_{\ell-1} \tilde{v}_{j,\ell-1}, \quad \ell = 2, \ldots, r, \]

(2.5)

with constants \( \alpha_{\ell-1} \in \mathbb{C} \). Using an induction argument, equation (2.5) implies that

\[ \mathcal{A}^k \tilde{v}_{j,\ell} = \sum_{s=0}^{\ell-1} \begin{pmatrix} k \\ s \end{pmatrix} \lambda_j^{k-s} \prod_{v=1}^{s} \alpha_{j,\ell-v} \tilde{v}_{j,\ell-s}, \]

where we set \( \begin{pmatrix} k \\ s \end{pmatrix} := 0 \) if \( s > k \).

We consider now a generalized Prony polynomial of the form

\[ P(z) := \prod_{j \in J} (z - \lambda_j)^{y_j} = \sum_{k=0}^{Mr} p_k z^k, \]

(2.6)

where again \( \lambda_j \) denote the unknown eigenvalues of \( \mathcal{A} \) that determine the active sets of (generalized) eigenfunctions. Then we obtain the following relation for \( m = 0, \ldots, Mr - 1 \):

\[
\sum_{k=0}^{Mr} p_k F(\mathcal{A}^{k+m} f) = \sum_{k=0}^{Mr} p_k F \left( \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \mathcal{A}^{k+m} \tilde{v}_{j,\ell} \right)
= \sum_{k=0}^{Mr} p_k F \left( \sum_{j \in J} \sum_{\ell=1}^{r} \sum_{s=0}^{\ell-1} \begin{pmatrix} k + m \\ k + m - s \end{pmatrix} \lambda_j^{k+m-s} \prod_{v=1}^{s} \alpha_{j,\ell-v} \tilde{v}_{j,\ell-s} \right)
= \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \sum_{s=0}^{\ell-1} \left( \sum_{k=0}^{Mr} p_k \begin{pmatrix} k + m \\ k + m - s \end{pmatrix} \lambda_j^{k+m-s} \prod_{v=1}^{s} \alpha_{j,\ell-v} \right) F(\tilde{v}_{j,\ell-s}).
\]

Now, for each \( j \in J, \ \ell = 1, \ldots, r \) and \( s = 0, \ldots, \ell - 1 \), the term

\[
\sum_{k=0}^{Mr} p_k \begin{pmatrix} k + m \\ k + m - s \end{pmatrix} \lambda_j^{k+m-s} \prod_{v=1}^{s} \alpha_{j,\ell-v}
\]
can be written as a linear combination of the Prony polynomial \( P(z) \) and its first \( r-1 \) derivatives \( P^{(v)}(z) = \sum_{k=0}^{Mr} p_k \frac{k!}{(k-v)!} z^{-k} \) evaluated at \( \lambda_j \). For this purpose, we only have to show that there exist coefficients \( c_{j,k,v} \) such that
\[
\min_{k=0}^{r-1} c_{j,k,v} \frac{k!}{(k-v)!} = \left( k + m \right) \left( \frac{1}{\prod_{j=1}^{s} \alpha_{j,\ell - v}} \right)
\]
holds for each \( k = 0, \ldots, Mr \), where the coefficients \( c_{j,k,v} \) are independent of \( k \). This is obviously possible, since each polynomial in \( k \) of degree up to \( r-1 \) can be written in the form \( \sum_{v=0}^{Mr} c_{j,k,v} \frac{k!}{(k-v)!} \). Hence, by \( P^{(v)}(\lambda_j) = 0 \) for \( j \in J, v = 0, \ldots, r-1 \), it follows that
\[
\sum_{k=0}^{Mr} p_k F(A^{k+m} f) = 0
\]
for \( m = 0, \ldots, Mr - 1 \). In this way, we obtain again a linear Hankel system of the form
\[
\sum_{k=0}^{rM - 1} p_k F(A^{k+m} f) = -F(A^{rM+m} f), \quad m = 0, \ldots, rM - 1
\]
in order to determine the coefficients \( p_k \) of the Prony polynomial. Having found the zeros \( \lambda_j \) of the Prony polynomial, we obtain the corresponding eigenfunctions \( \tilde{v}_{j,1}, \ldots, \tilde{v}_{j,r} \), and afterward compute the complex coefficients \( c_{j,\ell} \) by solving the overdetermined system
\[
F(A^k f) = \sum_{j \in J} \sum_{\ell = 1}^{r} c_{j,\ell} A^k \tilde{v}_{j,\ell}, \quad k = 0, \ldots, 2rM - 1.
\]

**Remark 2.5.** Generalized Prony polynomials of the form (2.6) have already been used for the reconstruction of spline functions in [34] and [24].

### 3. Sparse sums of exponentials and monomials revisited

Using the general Prony approach introduced in section 2, we would like to revisit the well-known methods for reconstruction of sparse exponential sums and sparse monomial sums.

#### 3.1. Sparse exponential sums

Let us consider the vector space \( C(\mathbb{R}) \) of continuous functions, and let \( S_a : C(\mathbb{R}) \rightarrow C(\mathbb{R}) \) with
\[
S_a f(x) := f(x + a), \quad a \in \mathbb{R}\{0\}
\]
be the shift operator on \( C(\mathbb{R}) \). We observe that \( \{ e^{T a} : T \in \mathbb{C}, \text{Im} T \in [-\frac{\pi}{a}, \frac{\pi}{a}] \} \) is a set of pairwise distinct eigenvalues of \( S_a \), and by
\[
S_a e^{(T + i\frac{\pi}{a}) x} = e^{(x+a)(T + i\frac{\pi}{a})} = e^{T a} e^{x(T + i\frac{\pi}{a})}, \quad x \in \mathbb{R}, \; k \in \mathbb{Z},
\]
we find for each eigenvalue \( \lambda_T := e^{T a}, \; T \in I := \{ T \in \mathbb{C}, \text{Im} T \in [-\frac{\pi}{a}, \frac{\pi}{a}] \} \), the eigenspace \( V_T := \text{span} \{ e^{(T + i\frac{\pi}{a}) x} : k \in \mathbb{Z} \} \). In order to obtain a unique correspondence between \( \lambda_T \) and its eigenfunction, we consider only the subeigenspaces \( \hat{V}_T = \text{span} \{ e^{T x} \} \). Further, let the functional \( F : C(\mathbb{R}) \rightarrow \mathbb{C} \) be given by
\[
F(f) := f(x_0), \quad \forall f \in C(\mathbb{R}),
\]
with an arbitrarily fixed \( x_0 \in \mathbb{R} \). Hence, \( F(e^T) \equiv e^{Tx_0} \neq 0 \) for all \( T \in I \). Applying theorem 2.1 yields that the sparse sum of exponentials

\[
f(x) = \sum_{j=1}^{M} c_j e^{T_j x}
\]

with pairwise different \( T_j \in \mathbb{C} \) and \( \text{Im} \ T_j \in \left[ -\frac{n}{2}, \frac{n}{2} \right) \) can be uniquely reconstructed from the values

\[
F(S_k^m f) = F(f(\cdot + ka)) = f(x_0 + ka), \quad k = 0, \ldots, 2M - 1,
\]
e.g., from \( 2M \) equidistant sampling points with sampling distance \( a \), starting at point \( x_0 \).

Moreover, theorem 2.4 also admits efficient recovery of sums of the form

\[
f(x) = \sum_{j=1}^{M} \sum_{\ell=0}^{r} c_{j,\ell} w_\ell(x) e^{T_j x}
\]

with \( r \geq 0 \), where \( w_\ell(x), \ell = 0, \ldots, r \) denote algebraic polynomials of exact degree \( \ell \). Indeed we easily observe, that \( w_\ell(x) e^{T_j x}, \ell = 0, \ldots, r \) are linearly independent generalized eigenfunctions of multiplicity \( \ell + 1 \) of the shift operator \( S_k \) to the eigenvalue \( e^{T_j a} \). Therefore, we can apply theorem 2.4 for the reconstruction of \( f \) using the \( 2M(r + 1) \) function values \( f(x_0 + ka), k = 0, \ldots, 2M(r + 1) - 1 \).

**Remarks 3.1.**

1. There are other operators that also possess eigenfunctions of the form \( e^{T_j x} \). For example, the shift operator \( S_a \) in the above considerations can be replaced by the difference operator \( \Delta_a f(x) = f(x + a) - f(x) \) or even by an \( m \)-fold difference operator \( \Delta_a^m f := \Delta_{m-1}(\Delta_a f), (m \in \mathbb{N}) \). Using again \( F f := f(x_0) \), the reconstruction then involves the values

\[
F(\Delta_a^m f) = \Delta_a^m f(x_0) = \sum_{\ell=0}^{mk} \left( \frac{mk}{\ell} \right) (-1)\ell f(x_0 + \ell a).
\]

2. Instead of using the functional \( F f = f(x_0) \) for some fixed \( x_0 \in \mathbb{R} \), one can use also a different functional. The functional \( F f := \int_{x_0}^{x_0+ka} f(x) \, dx \) leads to a reconstruction method, where \( f \) in (3.1) can be reconstructed from the values

\[
\int_{x_0}^{x_0+ka} S_a f(x) \, dx = \int_{x_0}^{x_0+ka} f(x) \, dx = \int_{x_0+ka}^{x_0+(k+1)a} f(x) \, dx,
\]

\( k = 0, \ldots, 2M - 1 \).

3. The reconstruction method also applies to the multivariate case. Let \( S_a : C(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d) \) be the shift operator with

\[
S_a f(x_1, \ldots, x_d) = f(x_1 + a_1, x_2 + a_2, \ldots, x_d + a_d),
\]

with the set of eigenfunctions \( \{ e^{T \alpha} = e^{T_1 + \cdots + T_d \alpha} : T_\ell \in \mathbb{C}, \text{Im} T_\ell \in \left[ -\frac{n}{2}, \frac{n}{2} \right), \ell = 1, \ldots, d \} \). The corresponding eigenvalues \( e^{T \alpha} \) allow a unique conclusion to the corresponding eigenfunction \( e^{T \alpha} \) if there exists an injective linear mapping that maps \( T \) to \( T \cdot \alpha \). This condition can be satisfied by a suitable restriction of \( T \) and a special choice of \( \alpha \). Let \( f(x) \) be of the form

\[
f(x) = \sum_{j=1}^{M} c_j e^{T_j x}
\]

with \( T_j = (T_{j,1}, \ldots, T_{j,d})^T \), where \( T_{j,\ell} \in \mathbb{N} \) for \( j = 1, \ldots, M, \ell = 1, \ldots, d \). Choose now pairwise relatively prime numbers \( p_1, \ldots, p_d \) with \( p_\ell > \max_{j=1,\ldots,M} T_{j,\ell} \) for \( \ell = 1, \ldots, d \).
Further, let \( N = p_1 p_2 \cdots p_d \), and \( a := \left( \frac{b_1 N}{p_1}, \ldots, \frac{b_d N}{p_d} \right) \), where \( b_\ell \) is determined from \( b_\ell \mod p_\ell \) for \( \ell = 1, \ldots, d \). Then each variable \( T_j \in \mathbb{N}^d \) can be uniquely determined from \( \tau_j := T_j \cdot a \) using the reverse steps of the Chinese remainder theorem. We have
\[
\tau_j = \sum_{\ell=1}^d \frac{T_{j,\ell} b_\ell N}{p_\ell}.
\]
Hence, \( \tau_j \equiv T_{j,\ell} \mod p_\ell \) for \( 1 \leq \ell \leq d \), and we can recover \( T_j \) from \( \tau_j \) by \( T_{j,\ell} = \tau_j - p_\ell \left\lfloor \frac{\tau_j}{p_\ell} \right\rfloor \), see [13]. Unfortunately, this procedure is highly unstable.

Another procedure for recovery of multivariate exponential sums is based on the determination of \( T_j \in \mathbb{C}^d \), \( \text{Im} T_j \in [-\frac{\pi}{2}, \frac{\pi}{2})^d \), from different scalar products \( T_j \cdot a_1 \), \( T_j \cdot a_2 \), etc., see [27, 24].

Finally, we want to mention that the exponentials can also be seen as eigenfunctions of differential operators. Let us consider the vector space \( C^{\infty}(\mathbb{R}) \) of infinitely differentiable functions, and let \( \frac{d}{dx} : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}) \) be the differentiation operator. We observe that \( \{ T : T \in \mathbb{C} \} \) is a set of pairwise distinct eigenvalues of \( \frac{d}{dx} \) and by
\[
\frac{d}{dx} e^{Tx} = T e^{Tx}
\]
we can uniquely relate the eigenfunction \( e^{Tx} \) to the eigenvalue \( T \). Let \( x_0 \) be a fixed real number. Then, with \( F(f) := f(x_0), \forall f \in C^{\infty}(\mathbb{R}) \), we can again apply theorem 2.1 for recovering the sparse sum of exponentials
\[
f(x) = \sum_{j=1}^M c_j e^{T_j x}
\]
with pairwise different \( T_j \in \mathbb{C} \). The reconstruction of \( f \) can be uniquely performed using the values
\[
F \left( \frac{d^k}{dx^k} f \right) = f^{(k)}(x_0), \quad k = 0, \ldots, 2M - 1,
\]
where \( x_0 \in \mathbb{R} \) can be chosen arbitrarily.

Moreover, let \( \{ w_\ell \}_{\ell=0}^r \) be a basis of the space of polynomials of degree at most \( r \) and \( \deg(w_\ell) = \ell, \ell = 0, \ldots, r \). Then we easily check that the functions \( w_\ell(x) e^{Tx}, \ell = 0, \ldots, r \), form linearly independent generalized eigenfunctions of multiplicity \( \ell + 1 \) of the linear operator \( \frac{d}{dx} \), and theorem 2.4 applies for the recovery of sparse expansions of the form
\[
f(x) = \sum_{j=1}^M \sum_{\ell=0}^r c_{j,\ell} w_\ell(x) e^{T_j x}
\]
using the derivative values \( f^{(k)}(x_0), k = 0, \ldots, 2M(r + 1) - 1 \), for \( r \geq 0 \).

### 3.2. Sparse monomial sums

Let us consider the Banach space \( C(\mathbb{R}) \) of continuous functions and let \( D_a : C(\mathbb{R}) \to C(\mathbb{R}) \) with
\[
D_a f(x) := f(ax), \quad a \in \mathbb{R} \setminus \{1\}
\]
be the dilatation operator on \( C(\mathbb{R}) \). By
\[
D_a x^p = (ax)^p = a^p x^p
\]
we observe that \( \{x^p : p \in \mathbb{C} \} \) is a set of eigenfunctions to \( D_a \) with corresponding eigenvalues \( a^p \). In order to ensure that the eigenvalues \( a^p \) are pairwise distinct, we assume that \( \text{Im} \, p \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). Therefore, we consider only the ‘admissible’ set of eigenfunctions \( \{x^p : p \in \mathbb{C}, \text{Im} \, p \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \} \). Further, let the functional \( F : C(\mathbb{R}) \rightarrow \mathbb{C} \) be given by \( F(f) := f(x_0) \) for all \( f \in C(\mathbb{R}) \), where \( x_0 \in \mathbb{R} \setminus \{0\} \) is arbitrary but fixed, such that \( F(\cdot) = x_0 \neq 0 \).

With theorem 2.1, we can uniquely reconstruct the sparse sum of monomials

\[
f(x) = \sum_{j=1}^M c_j x^{p_j}
\]

with \( c_j \in \mathbb{C} \setminus \{0\} \) and pairwise different \( p_j \in \mathbb{C} \) which satisfy \( \text{Im} \, p_j \in \left[ -\frac{\pi}{2 \ln a}, \frac{\pi}{2 \ln a} \right] \), using the 2M values \( F(D^k_a f) = f(a^k x_0), k = 0, \ldots, 2M - 1 \).

**Remarks 3.2.**

1. The above approach generalizes the Ben-Or and Tiwari algorithm [3] for interpolating sparse polynomials in the sense that we are not restricted to integer exponents \( p_j \). It can be applied to multivariate sums, where some restrictions to the set of admissible eigenfunctions are needed in order to ensure an injective mapping from the eigenvalues to the eigenfunctions, see Remarks 3.1 for the similar case of multivariate exponential sums.

2. By using a different functional \( F \), a further generalization of the Ben-Or and Tiwari reconstruction method is possible. As before, for example \( F f := \int_0^1 f(x) \, dx \) leads to a reconstruction method using the values

\[
F(D^k_a f) = \int_0^1 f(a^k x) \, dx = \frac{1}{a^k} \int_0^{a^k} f(x) \, dx, \quad k = 0, \ldots, 2M - 1.
\]

We may admit also generalized eigenfunctions of the dilation operator \( D_a : C((0, \infty)) \rightarrow C((0, \infty)) \). Assuming that \( a > 0, a \neq 1 \), we observe that functions of the form \( (\ln x)^\ell x^p, \ell = 0, \ldots, r \) are generalized eigenfunctions of multiplicity \( \ell + 1 \) of \( D_a \) (as defined in section 2), since

\[
D_a ((\ln x)^\ell x^p) = (\ln ax)^\ell (ax)^p = (\ln a + \ln x)^\ell a^p x^p
\]

\[
= a^p (\ln x)^\ell x^p + \sum_{j=0}^{\ell-1} \binom{\ell}{j} (\ln a)^{\ell-j} a^p (\ln x)^j x^p.
\]

Thus, by theorem 2.4, we are able to recover also expansions of the form

\[
f(x) = \sum_{j=1}^M \sum_{\ell=0}^r c_{j,\ell} (\ln x)^\ell x^{p_j}
\]

from the measurements \( f(a^k x_0), k = 0, \ldots, 2(r + 1)M - 1 \) for \( r \geq 0 \), where \( x_0 > 0 \) is fixed.

Eigenfunctions of monomial form can also be obtained using suitable differential operators. Let \( d_x : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \) be the differential operator of the form

\[
d_x f(x) := \frac{d}{dx} (xf(x)) = f(x) + xf'(x).
\]

Then we have

\[
d_x (x^p) = \frac{d}{dx} (x^{p+1}) = (p + 1)x^p, \quad p \in \mathbb{R},
\]
where \(cnj\) argument shows that these values can be obtained recursively from the derivative values completely recovered from \(F_12\) with \(cj\) \(dx\) and the operator \((\alpha x - \text{Inverse Problems 29 (2013) 025001})\)

Let us again consider the Banach space \(C^4\). Sparse sums of orthogonal polynomials \(\lambda\) prominent orthogonal polynomials with their corresponding \(p\) polynomials, where we have \(Qnj\) suitably defined orthogonal polynomials \(Qn\) special sets of eigenvalues \(\lambda\) eigenvalues are pairwise different and well separated, i.e. \(\lambda_n \neq \lambda_m\) for \(n \neq m\). Further, we choose the functional \(F : C^\infty(\mathbb{R}) \to \mathbb{C}\) that returns \(f\) at a fixed value \(x_0 \in \mathbb{R}\), i.e. \(F(f) := f(x_0)\), with the condition that \(Q_n(x_0) \neq 0\) for all \(n \in \mathbb{N}_0\).

The polynomial \(f\) is now an \(M\)-sparse expansion of orthogonal polynomials \(Q_n, n \geq 0\), if it has the form

\[
f(x) = \sum_{j=0}^{M} c_{nj} Q_{nj}(x)
\]

where \(c_{nj} \in \mathbb{C} \setminus \{0\}, 0 \leq n_1 < \ldots < n_M = N\) are the indices of the ‘active’ basis polynomials \(Q_{nj}\) in the expansion, and \(n_M = N \gg M\) is the polynomial degree of \(f\).

<table>
<thead>
<tr>
<th>(p(x))</th>
<th>(q(x))</th>
<th>(\lambda)</th>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1-x^2))</td>
<td>((\beta - \alpha - (\alpha + \beta + 2)x))</td>
<td>(-n(n + \alpha + \beta + 1))</td>
<td>Jacobi</td>
<td>(P^{(\alpha,\beta)}_n)</td>
</tr>
<tr>
<td>((1-x^2))</td>
<td>(-(2\alpha + 1)x)</td>
<td>(-n(n + 2\alpha))</td>
<td>Gegenbauer</td>
<td>(C^{(\alpha)}_n)</td>
</tr>
<tr>
<td>((1-x^2))</td>
<td>(-2x)</td>
<td>(-n(n + 1))</td>
<td>Legendre</td>
<td>(P_n)</td>
</tr>
<tr>
<td>((1-x^2))</td>
<td>(-x)</td>
<td>(-n^2)</td>
<td>Chebyshev 1. kind</td>
<td>(T_n)</td>
</tr>
<tr>
<td>((1-x^2))</td>
<td>(-3x)</td>
<td>(-n(n + 2))</td>
<td>Chebyshev 2. kind</td>
<td>(U_n)</td>
</tr>
<tr>
<td>1</td>
<td>(-2x)</td>
<td>(-2n)</td>
<td>Hermite</td>
<td>(H_n)</td>
</tr>
<tr>
<td>((\alpha + 1 - x))</td>
<td>(-n)</td>
<td></td>
<td>Laguerre</td>
<td>(L^{(\alpha)}_n)</td>
</tr>
</tbody>
</table>

and the operator \(d_x\) possesses the set \(\{p + 1 : p \in \mathbb{R}\}\) of pairwise different eigenvalues with corresponding eigenfunctions \(x^p\). We consider now a sparse monomial expansion of the form

\[
f(x) = \sum_{j=0}^{M} c_j x^{p_j}
\]

with \(c_j \in \mathbb{C} \setminus \{0\}\) and pairwise different \(p_j \in \mathbb{R}\). Using theorem 2.1, this expansion can be completely recovered from \(F((d_x)^j f) = (d_x)^j f(x_0), k = 0, \ldots, 2M - 1\). A simple induction argument shows that these values can be obtained recursively from the derivative values \(f^{(k)}(x_0), \ell = 0, \ldots, 2M - 1\), where \(x_0 \in \mathbb{R} \setminus \{0\}\) can be arbitrarily chosen.

4. Sparse sums of orthogonal polynomials

Let us again consider the Banach space \(C^\infty(\mathbb{R})\) of infinitely differentiable functions. Let \(L_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})\) be the Sturm–Liouville differential operator of the form

\[
L_{p,q} f(x) := p(x)f''(x) + q(x)f'(x), \quad f \in C^\infty(\mathbb{R}),
\]

(4.1)

where \(p(x)\) and \(q(x)\) are polynomials of degree 2 and 1, respectively. It is well known, that suitably defined orthogonal polynomials \(Q_n\) are eigenfunctions of this differential operator for special sets of eigenvalues \(\lambda_n, n \in \mathbb{N}_0\), i.e. \(L_{p,q} Q_n = \lambda_n Q_n\). For convenience, we list the most prominent orthogonal polynomials with their corresponding \(p(x), q(x)\) and their eigenvalues \(\lambda_n, n \in \mathbb{N}\) in table 1.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jacobi</td>
<td>(P^{(\alpha,\beta)}_n)</td>
</tr>
<tr>
<td>Gegenbauer</td>
<td>(C^{(\alpha)}_n)</td>
</tr>
<tr>
<td>Legendre</td>
<td>(P_n)</td>
</tr>
<tr>
<td>Chebyshev 1. kind</td>
<td>(T_n)</td>
</tr>
<tr>
<td>Chebyshev 2. kind</td>
<td>(U_n)</td>
</tr>
<tr>
<td>Hermite</td>
<td>(H_n)</td>
</tr>
<tr>
<td>Laguerre</td>
<td>(L^{(\alpha)}_n)</td>
</tr>
</tbody>
</table>

**Table 1.** Polynomials \(p(x)\) and \(q(x)\) defining the Sturm–Liouville operator, corresponding eigenvalues \(\lambda_n\) and eigenfunctions.
Theorem 4.1. Let $f \in C^\infty(\mathbb{R})$ be an arbitrary polynomial of degree $N \in \mathbb{N}$ and let $L_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the Sturm–Liouville differential operator as given in (4.1). Then, for each fixed $x \in \mathbb{R}$, the values $L_{p,q}^k f(x), \ k = 0, \ldots, 2M - 1$, can be determined uniquely by the derivative values $f^{(m)}(x_0)$ for $m = 0, \ldots, 4M - 2$, and this assertion holds not only for sparse but for all expansions of orthogonal polynomials $f$.

Proof.

1. Let $\{g_n : n \in \mathbb{N}_0\}$ be the set of orthogonal polynomials that are eigenfunctions of the operator $L_{p,q}$ to the eigenvalues $\lambda_n$, i.e.

$$L_{p,q} g_n = \lambda_n g_n.$$

Since $\{g_n : n \in \mathbb{N}_0\}$ forms a basis of the space of polynomials we find a unique expansion of $f$,

$$f(x) = \sum_{n=0}^{N} c_n g_n(x).$$

2. For $k = 0$, we observe that $L_{p,q}^0 f(x) = f(x)$, i.e. $L_{p,q}^0$ is the identity operator. Since the operator $L_{p,q}$ is a differential operator of order 2, we can use for $k \geq 1$ the ansatz

$$L_{p,q}^k f(x) = \sum_{\ell=1}^{2k} g_{\ell,k}(x)f^{(\ell)}(x), \quad k \geq 1,$$

(4.4)

with polynomials $g_{\ell,k}, \ell, k \in \mathbb{N}, \ell = 1, \ldots, 2k$. In particular, for $k = 1$ we have

$$L_{p,q} f(x) = p(x)f'(x) + q(x)f(x),$$

i.e. $g_{1,1}(x) = q(x)$ and $g_{2,1}(x) = p(x)$. We now prove by induction on $k$ that the coefficients $g_{\ell,k}(x)$ in (4.4) satisfy the recursion (4.3) for $k \geq 2$ and $\ell = 1, \ldots, 2k$. Using (4.4) and the general Leibniz rule, we find for all $n \in \mathbb{N}_0$,

$$L_{p,q}^k g_n(x) = L_{p,q}^{k-1} (\lambda_n g_n(x)) = \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) \lambda_n Q_{n}^{(\ell)}(x)$$

$$= \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) [p(x)Q_{n}^{(\ell-1)}(x) + q(x)Q_{n}^{(\ell)}(x)].$$


\[ \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) \left[ p(x)Q_n^{(\ell+2)}(x) + \ell p'(x)Q_n^{(\ell+1)}(x) + \left( \frac{\ell}{2} \right)p''(x)Q_n^{(\ell)}(x) + q(x)Q_n^{(\ell+1)}(x) + \ell q'(x)Q_n^{(\ell)}(x) \right] \]

\[ = \sum_{\ell=1}^{2k} g_{\ell,k}(x) Q_n^{(\ell)}(x) \]

due to the vanishing higher derivatives of \( p(x), q(x) \). A comparison of coefficients leads to the recursion formula for \( g_{\ell,k} \) in (4.3). Hence, we finally obtain

\[ L_{p,q}^k f(x) = \sum_{n=0}^{N} \alpha_n L_{p,q}^k Q_n(x) = \sum_{n=0}^{N} \alpha_n \sum_{\ell=1}^{2k} g_{\ell,k}(x) Q_n^{(\ell)}(x) = \sum_{\ell=1}^{2k} g_{\ell,k}(x) f^{(\ell)}(x) \]

with \( g_{\ell,k}(x) \) as given in the theorem.

\[ \frac{\text{Corollary 4.2. If } x_0 \in \mathbb{R} \text{ is a zero of the polynomial } p(x) \text{ in the definition (4.1) of the Sturm–Liouville operator, i.e. if } p(x_0) = 0 \text{, then } L_{p,q} f(x_0) \text{ reduces to } L_{p,q} f(x_0) = q(x_0)f'(x_0), \text{ and the values } L_{p,q} f(x_0), k = 0, \ldots, 2M - 1 \text{ can be determined by } f^{(m)}(x_0), m = 0, \ldots, 2M - 1 \text{ only. More precisely, we have} \]

\[ L_{p,q}^k f(x_0) = \sum_{\ell=1}^{k} g_{\ell,k}(x_0) f^{(\ell)}(x_0) \]

with \( g_{1,1}(x_0) = q(x_0) \) and

\[ g_{\ell,k}(x_0) = \ell \left( \frac{\ell - 1}{2} p''(x_0) + q'(x_0) \right) g_{\ell-1,k}(x_0) + ((\ell - 1)p'(x_0) + q(x_0))g_{\ell-1,k-1}(x_0) \]

for \( k \geq 2, \ell \in \{1, \ldots, k\} \), where we assume that \( g_{\ell,k}(x_0) = 0 \) for \( k \geq 2, \ell \not\in \{1, \ldots, k\} \). In particular, for the Sturm–Liouville operator for Jacobi polynomials with \( p(x) = (1 - x^2) \), we need only the values \( f^{(m)}(1) \) (respectively \( f^{(m)}(-1) \)), \( m = 0, \ldots, 2M - 1 \), in order to reconstruct \( L_{p,q}^k f(1) \) (respectively \( L_{p,q}^k f(-1) \), \( k = 0, \ldots, 2M - 1 \).

The proof of corollary 4.2 is similar to that of theorem 4.1 and is therefore omitted.

We summarize the algorithm for reconstructing orthogonal polynomial expansions as follows.

**Algorithm 4.3 (Reconstruction of \( f \) in (4.2)).**

**Input:** Sturm–Liouville operator with \( p(x), q(x) \) and \( \lambda_n \) as well as the basis \( \{Q_n : n \in \mathbb{N}_0\} \), \( M, x_0 \in \mathbb{R}, f^{(m)}(x_0), m = 0, \ldots, 4M - 2 \).

**Preprocessing:** Construct \( G = (g_{\ell,k})_{\ell,k=1}^{2M-3,4M-2} \in \mathbb{R}^{(2M-1) \times (4M-2)} \) with \( g_{1,1} := q(x_0), g_{2,1} := p(x_0), g_{1,1} := 0 \) for \( \ell \not\in \{1, 2\} \), and

\[ g_{\ell,k} := \left\{ \begin{array}{ll}
\ell \left( \frac{\ell - 1}{2} p''(x_0) + q'(x_0) \right) g_{\ell-1,k} + ((\ell - 1)p'(x_0) + q(x_0))g_{\ell-1,k-1} + p(x_0)g_{\ell-2,k-1}, & k > 1, \ell \in \{1, \ldots, 2k\}, \\
0, & k > 1, \ell \not\in \{1, \ldots, 2k\}.
\end{array} \right. \]

Observe that the construction of \( G \) only depends on the chosen basis and not on the given data \( f^{(m)}(x_0) \).

1. Calculate \( h_1 := Gf_1 \), where \( f_1 := (f^{(m)}(x_0))_{m=1}^{4M-2} \). Put now

\[ h := \left( f(x_0) \right)_1 \]

such that \( h = (h_\ell)_{\ell=0}^{2M-1} = (L_{p,q}^\ell f(x_0))_{\ell=0}^{2M-1} \in \mathbb{C}^{2M} \).
2. Solve the Hankel system
\[
\begin{pmatrix}
h_0 & h_1 & \ldots & h_{M-1} \\
h_1 & h_2 & \ldots & h_M \\
\vdots & \vdots & \ddots & \vdots \\
h_{M-1} & h_M & \ldots & h_{2M-2}
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots \\
p_{M-1}
\end{pmatrix}
= -
\begin{pmatrix}
h_M \\
h_{M+1} \\
\vdots \\
h_{2M-1}
\end{pmatrix}.
\]

3. Put \( p_M = 1 \) and compute the zeros \( \lambda_{nj}, \ j = 1, \ldots, M \), of the Prony polynomial
\[
P(z) = \sum_{k=0}^{M} p_k z^k.
\]

4. Extract the indices \( n_j, \ j = 1, \ldots, M \), from the obtained eigenvalues \( \lambda_{nj} \) and compute the coefficients \( c_{nj} \) of the sparse orthogonal polynomial expansion in (4.2) by solving the overdetermined Vandermonde-type system
\[
\sum_{j=1}^{M} c_{nj} Q_{nj}^{(\ell)}(x_0) = f^{(\ell)}(x_0), \quad \ell = 0, \ldots, 2M - 1.
\]

**Output:** \( n_j, c_{nj}, \ j = 1, \ldots, M \), determining \( f \) in (4.2).

**Example 4.4. Sparse Laguerre expansions**

The Laguerre polynomials with parameter \( \alpha \) are solutions of the second-order differential equation
\[
x(L_n^{(\alpha)})''(x) + (\alpha + 1 - x)L_n^{(\alpha)}(x) = -nL_n^{(\alpha)}(x),
\]
with eigenvalues \( \lambda_n = -n \). Using theorems 2.1 and 4.1, a sparse Laguerre expansion of the form
\[
f(x) = \sum_{j=1}^{M} c_{nj} L_{nj}(x)
\]
with \( c_{nj} \in \mathbb{C}\setminus\{0\} \) and active indices \( 0 \leq n_1 < \ldots < n_M = N \) can be reconstructed from \( f^{(m)}(x_0), m = 0, \ldots, 4M - 2 \). Here, \( x_0 \) has to satisfy \( L_n^{(\alpha)}(x_0) \neq 0 \) for all \( n \in \mathbb{N}_0 \). If we choose \( x_0 = 0 \), formula (4.3) simplifies to
\[
\begin{align*}
g_{1,1}(0) &= \alpha + 1, \\
g_{\ell,k}(0) &= (\ell + \alpha)g_{\ell-1,k-1}(0) - \ell g_{\ell,k-1}(0), \quad k > 1, \ell = 1, \ldots, k, \\
g_{\ell,0}(0) &= 0, \quad k \geq 1, \ell \notin \{1, \ldots, k\}.
\end{align*}
\]

For example, for \( M = 2 \), this leads to the triangular matrix
\[
G = \begin{pmatrix}
(1 + \alpha) & 0 \\
-(1 + \alpha) & (1 + \alpha)(2 + \alpha) \\
(1 + \alpha) & -3(1 + \alpha)(2 + \alpha) & (1 + \alpha)(2 + \alpha)(3 + \alpha)
\end{pmatrix}
\]
in the preprocessing step of algorithm 4.3. Let us give a small numerical example. For \( \alpha = 0 \) and given values \( f(0), f'(0), \ldots, f^{(11)}(0) \) of the function
\[
f(x) = \sum_{j=1}^{6} c_{nj} L_{nj}(x),
\]
we use algorithm 4.3 to calculate the approximations \( \tilde{n}_j, \tilde{c}_{nj} \) of the original parameters \( n_j, c_{nj} \) for \( j = 1, \ldots, 6 \), as shown in Table 2.
yielding for $Mcnj$ with $\alpha$ constant 16.

Inverse Problems 29 (2013) 025001

Hence, a sparse Legendre expansion of the form

$$P_n(x) = \sum_{j=1}^{M} c_{nj} P_{nj}(x)$$

with $c_{nj} \in \mathbb{C} \setminus \{0\}$ and active indices $0 \leq n_1 < \ldots < n_M = N$ can be reconstructed from the values $f^{(m)}(x_0)$, $m = 0, \ldots, 4M - 2$, for arbitrarily chosen $x_0 \in \mathbb{R}$ satisfying $P_n(x_0) \neq 0$ for all $n \in \mathbb{N}_0$. In particular, for $x_0 = 1$ (or $x_0 = -1$) we need only the values from $f^{(m)}(1)$, $m = 0, \ldots, 2M - 1$, for the unique reconstruction of $f$. Multiplying (4.5) with a constant $\alpha \neq 0$ does not change the solutions. Thus, we can consider

$$L_{p,q,a}^k P_n(x) := \alpha(1-x^2)P_n^\prime(x) - 2qxP_n^\prime(x) = -n(n+1)\alpha P_n(x),$$

where $p_n(x) = \alpha(1-x^2)$, $q_n(x) = -2\alpha x$ and $\lambda_{n,a} = -n(n+1)\alpha$. Hence,

$$L_{p,q,a}^k f(1) = \sum_{\ell=1}^{k} g_{\ell,k}^a f^{(\ell)}(1)$$

with

$$g_{1,1}^a(1) = -2\alpha,$$

$$g_{\ell,k}^a(1) = -\ell(\ell+1)\alpha g_{\ell-1,k}^a(1) - 2\alpha \ell g_{\ell-1,k-1}^a(1), \quad k > 1, \ell = 1, \ldots, k,$$

$$g_{k,k}^a(1) = 0, \quad k \geq 1, \ell \not\in \{1, \ldots, k\}.$$

The constant $\alpha$ can be chosen suitably in order to improve the condition of the matrix $G$. In particular, for $\alpha = -\frac{1}{2}$, we obtain

$$g_{\ell,k}^{a-1/2}(1) = \frac{\ell(\ell+1)}{2} g_{\ell-1,k-1}^{a-1/2}(1) + \ell g_{\ell-1,k-1}^{a-1/2}(1)$$

yielding for $M = 3$,

$$G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
1 & 8 & 6 & 0 & 0 \\
1 & 26 & 60 & 24 & 0 \\
1 & 80 & 438 & 480 & 120
\end{pmatrix}.$$
Then, for Chebyshev polynomials of first kind we obtain indeed
\[ h:\]

\[ M \]

uniquely reconstructed from the values \( F \) is reconstructed from the function values \( \tilde{F} \approx \text{approximations} \)

We consider \( h \) instead we have to take the so-called 'Chebyshev-shift operator' algorithm is different from the Prony method that is based on the Sturm–Liouville operator.

be represented as linear combinations of the function values \( \tilde{F} \):

Indeed, one can simply prove by induction, that the values \( F(S_k f) \), \( k = 0, \ldots, 2M - 1 \), can be represented as linear combinations of the function values \( f(\cos\left(\frac{\pi}{2N-1}\right)) \), \( k = 0, \ldots, 2M - 1 \).

---

**Table 3.** Numerical evaluation of indices of active basis polynomials and coefficients of a sparse Legendre expansion using algorithm 4.3.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( n_j )</th>
<th>( c_{n_j} )</th>
<th>( \tilde{n}_j )</th>
<th>( \tilde{c}_{n_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54</td>
<td>2</td>
<td>53.983 951 125 658</td>
<td>2.000 000 000 000 048</td>
</tr>
<tr>
<td>2</td>
<td>465</td>
<td>-1</td>
<td>465.000 054 039 331</td>
<td>-1.000 000 000 000 048</td>
</tr>
<tr>
<td>3</td>
<td>5492</td>
<td>-3</td>
<td>5491.999 999 999 999</td>
<td>-3.000 000 000 000 000</td>
</tr>
</tbody>
</table>

see also [22]. We use algorithm 4.3 in order to recover the sparse Legendre expansion

\[ f(x) = -3P_{5492}(x) - P_{465}(x) + 2P_{54}(x) \]

of degree 5492 from the given values \( f(1), f'(1), \ldots, f^{(5)}(1) \). Table 3 contains the computed approximations \( \tilde{n}_j, \tilde{c}_{n_j} \) of the original parameters \( n_j, c_{n_j} \) for \( j = 1, 2, 3 \).

Here again, since we know that the orders \( n_j \) of the polynomials are integers, we have rounded the values \( \tilde{n}_j \) to the next integer before proceeding with the last step of algorithm 4.3.

**Remark 4.6.** In the recent preprint [29], the authors consider a Prony-like method for the reconstruction of \( M \)-sparse Chebyshev expansions with Chebyshev polynomials of first kind of the form \( T_n(x) = \cos n(\arccos x) \), for \( x \in [-1, 1] \) and \( n \in \mathbb{N}_0 \). In their approach, an expansion of the form

\[ f = \sum_{j=1}^{M} c_{n_j} T_{n_j} \]

is reconstructed from the function values \( f(\cos(\frac{k\pi}{2N-1})) \), \( k = 0, \ldots, 2M-1 \), where \( N > M \), and \( 2N-1 \) is an upper bound for the true degree of the polynomial \( f \). Obviously, this reconstruction algorithm is different from the Prony method that is based on the Sturm–Liouville operator. Instead we have to take the so-called ‘Chebyshev-shift operator’ \( S_h : C(\mathbb{R}) \to C(\mathbb{R}) \) with \( h \in [-1, 1] \).

\[ (S_h f)(x) := \frac{1}{2} f(hx - (1 - h^2)^{1/2}(1 - x^2)^{1/2}) + f(hx + (1 - h^2)^{1/2}(1 - x^2)^{1/2}) \]

We consider \( x \in [-1, 1] \) and set \( x = \cos t \) with \( t \in [0, \pi] \) as well as \( h = \cos \alpha \) with \( \alpha \in [0, \pi] \). Then, for Chebyshev polynomials of first kind we obtain indeed

\[ S_h T_n(x) = \frac{1}{2} (T_n(hx - (1 - h^2)^{1/2}(1 - x^2)^{1/2}) + T_n(hx + (1 - h^2)^{1/2}(1 - x^2)^{1/2})) \]

\[ = \frac{1}{2} (T_n(\cos(t + \alpha)) + T_n(\cos(t - \alpha))) \]

\[ = \cos(n\alpha) \cos(nt) = \cos(n\alpha)T_n(x). \]

Hence, the Chebyshev polynomials of first kind are eigenfunctions of the Chebyshev-shift operator \( S_{\cos\alpha} \) with the corresponding eigenvalues \( \cos(n\alpha) \). We define \( F(f) := f(1) \) and set \( h := \cos(\frac{\pi}{2N-1}) \). Applying theorem 2.1 yields that the sparse Chebyshev expansion can be uniquely reconstructed from the values \( F(S_k h f) \), \( k = 0, \ldots, 2M - 1 \), where

\[ F(S_k^0 f) = f(1), \]

\[ F(S_k^1 f) = f \left( \cos \left( \frac{\pi}{2N-1} \right) \right), \]

\[ F(S_k^2 f) = \frac{1}{2} \left( f(1) + f \left( \cos \left( \frac{2\pi}{2N-1} \right) \right) \right), \]

\[ F(S_k^3 f) = \frac{1}{4} \left( 3f \left( \cos \left( \frac{\pi}{2N-1} \right) \right) + f \left( \cos \left( \frac{3\pi}{2N-1} \right) \right) \right), \]

etc. Indeed, one can simply prove by induction, that the values \( F(S_k h f) \), \( k = 0, \ldots, 2M - 1 \), can be represented as linear combinations of the function values \( f(\cos(\frac{k\pi}{2N-1})) \), \( k = 0, \ldots, 2M - 1 \).
5. Recovery of sparse vectors

The generalized Prony method considered in section 2 can also be applied to finite-dimensional vector spaces. Let \( x \in \mathbb{C}^N \) be \( M \)-sparse, i.e. only \( M \) components of \( x = (x_0, \ldots, x_{N-1})^T \) are different from zero.

We want to recover \( x \) from only \( 2M \) samples \( y_k = a_k^T x \), \( k = 0, \ldots, 2M - 1 \), where the vectors \( a_k \in \mathbb{C}^N \) need to be chosen suitably. The problem of reconstructing sparse vectors using only a small amount of measurements has been heavily studied in the research field of compressed sensing, where the recovery algorithms are usually based on \( \ell^1 \)-minimization or greedy methods. In this regard, often a stochastic matrix \( A \in \mathbb{C}^{M \times N} \) is used in order to recover \( x \) from \( y = Ax \in \mathbb{C}^M \), \( (M \geq 2M) \) with high probability.

Here we want to derive a deterministic method to recover \( x \), where \( A \in \mathbb{C}^{2M \times N} \) with rows \( a_k^T \in \mathbb{C}^N \) is explicitly given and of minimal dimension. For this purpose, we use a linear operator \( D : \mathbb{C}^N \to \mathbb{C}^N \) that can be represented by a diagonal matrix \( D = \text{diag}(d_0, \ldots, d_{N-1}) \) with pairwise different entries \( d_j \), \( j = 0, \ldots, N-1 \). Obviously, the unit vectors \( e_j := (\delta_{j,0})_{j=0}^{N-1} \) form a system of eigenvectors of \( D \) with \( De_j = d_j e_j \) for \( j = 0, \ldots, N-1 \). Further, we choose a linear functional \( F : \mathbb{C}^N \to \mathbb{C} \) of the form \( Fx = \frac{1}{x^T x} := \sum_{j=0}^{N-1} x_j \). Hence, \( Fe_j = 1^T e_j = 1 \neq 0 \) holds.

Using theorem 2.1, we can now reconstruct a sparse vector \( x \) of the form

\[
x = \sum_{j=1}^{M} c_{n_j} e_{n_j}
\]

with \( 0 \leq n_1 < \ldots < n_M \leq N - 1 \) from the values

\[
F(D^T x) = 1^T \cdot D^T x = a_k^T x,
\]

where \( a_k^T = (d_0^k, \ldots, d_{N-1}^k) \), \( k = 0, \ldots, 2M - 1 \).

**Algorithm 5.1** (Reconstruction of a sparse vector).

**Input:** \( M, y_k = a_k^T x, k = 0, \ldots, 2M - 1 \).

1. Solve the Hankel system

\[
\begin{pmatrix}
y_0 & y_1 & \cdots & y_{M-1} \\
y_1 & y_2 & \cdots & y_M \\
\vdots & \vdots & \ddots & \vdots \\
y_{M-1} & y_M & \cdots & y_{2M-2}
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_M-1
\end{pmatrix}
= -
\begin{pmatrix}
y_M \\
y_{M+1} \\
\vdots \\
y_{2M-1}
\end{pmatrix}.
\]

2. Set \( p_M = 1 \) and compute the zeros \( \lambda_j, j = 1, \ldots, M \), of the Prony polynomial

\[
P(z) = \sum_{k=0}^{M} p_k z^k.
\]

3. The set of zeros \( \{\lambda_1, \ldots, \lambda_M\} \) of \( P(z) \) is a subset of the set of eigenvalues \( \{d_0, \ldots, d_{N-1}\} \) of \( D \). Determine the eigenvectors \( e_{n_j}, j = 1, \ldots, M \), respectively the indices \( n_j \), that correspond to the zeros/eigenvalues \( \lambda_1, \ldots, \lambda_M \).

4. Compute the coefficients \( c_{n_j}, j = 1, \ldots, M \), from the overdetermined system

\[
y_k = \sum_{j=1}^{M} c_{n_j} d_j^k, \quad k = 0, \ldots, 2M - 1.
\]

**Output:** \( n_j, c_{n_j}, j = 1, \ldots, M \), determining \( x \).
Table 4. Numerical evaluation of the indices and the coefficients of a sparse vector $x$ using algorithm 5.1.

<table>
<thead>
<tr>
<th>$n_j$</th>
<th>$x_{n_j}$</th>
<th>$\tilde{n}_j$</th>
<th>$\tilde{x}_{n_j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>3</td>
<td>27.999 999</td>
<td>999 999 999</td>
</tr>
<tr>
<td>71</td>
<td>−1</td>
<td>71.000 000</td>
<td>000 000 000</td>
</tr>
<tr>
<td>99</td>
<td>4</td>
<td>99.000 000</td>
<td>000 000 000</td>
</tr>
</tbody>
</table>

To demonstrate this approach, we want to present a small numerical example. Let $x \in \mathbb{R}^{128}$ be a 3-sparse vector with $x_{28} = 3$, $x_{71} = −1$, $x_{99} = 4$ and let $D = \text{diag}(k/32)_{k=-63}^{64} \in \mathbb{R}^{128 \times 128}$. For a given vector of values $f = (f_k)_{k=0}^{5}$ with $f_k = 1 \cdot D^k x$ we compute the approximations $\tilde{n}_j$ and $\tilde{x}_{n_j}$ according to algorithm 5.1. The results are shown in the table 4.

Remarks 5.2.

1. In order to obtain a stable algorithm, the operator $D$ may for example be chosen as

$$D = \text{diag}(\omega_0^N, \omega_1^N, \ldots, \omega_{N-1}^N),$$

where $\omega_N := e^{-2\pi i/N}$ denotes the $N$th root of unity. For this choice of $D$, the vector $y = (y_k)_{k=0}^{2M-1}$ of needed input values for algorithm 5.1 is given by

$$y = F_{N,2M} x,$$

where $F_{N,2M} = (\omega_k^N)^{2M-1-N} \in \mathbb{C}^{2M \times N}$ contains the first $2M$ rows of the Fourier matrix of order $N$.

2. In the above considerations, the canonical basis can be replaced by any other basis $B = \{b_1, \ldots, b_N\}$ of $\mathbb{C}^N$. Choose a diagonal matrix $D$ with pairwise different (complex) entries $\lambda_1, \ldots, \lambda_N$. Then the operator $A := BB^{-1} : \mathbb{C}^N \to \mathbb{C}^N$, where $B = (b_1, \ldots, b_N) \in \mathbb{C}^{N \times N}$ contains the columns $b_j$, possesses the eigenvalues $\lambda_1, \ldots, \lambda_N$ with corresponding eigenvectors $b_1, \ldots, b_N$ by construction. Further, we define a functional $F : \mathbb{C}^N \to \mathbb{C}$ satisfying $F b_\ell \neq 0$ for $\ell = 1, \ldots, N$. We can e.g., choose $F x := a^T x$ for all $x \in \mathbb{C}^N$, where $a$ is taken suitably. Hence, a sparse expansion

$$x = \sum_{j=1}^{M} c_{n_j} b_{n_j}$$

in the basis $B$ can by theorem 2.1 be recovered by

$$F(A^k x) = a^T A^k x, \quad k = 0, \ldots, 2M - 1.$$  

3. Using theorem 2.4, we can apply the recovery procedure also when the given operator possesses eigenvalues with higher multiplicity, where also generalized eigenvectors can be incorporated.

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