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On uniqueness of an inverse problem in electromagnetic obstacle scattering for an impedance cylinder

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Abstract

We consider an inverse problem for the scattering of an obliquely incident electromagnetic wave by a cylinder. In previous work, we have shown that the direct scattering problem is governed by a pair of Helmholtz equations subject to coupled oblique boundary conditions, where the wave number depends on the frequency and the incident angle with respect to the axis of the cylinder. In this paper, we are concerned with the inverse problem of uniquely identifying the cross-section of an unknown cylinder and the impedance function from the far-field patterns at a fixed frequency and a range of incident angles. A uniqueness result for such an inverse scattering problem is established. Our method is based on the analyticity of solution to the direct scattering problem, which is justified by using the Lax–Phillips method together with the perturbation theory of Fredholm operators.

(Some figures may appear in colour only in the online journal)

1. Introduction

The theory of inverse obstacle scattering has been an active research field in applied mathematics for a long time. The main objective of such inverse scattering problems is to detect the geometric information or physical properties of the scatterer from the scattering data, by using acoustic or electromagnetic waves. Uniqueness questions for these inverse problems are of central importance [3, 6]. In particular, when the scatterer $D$ is an impenetrable obstacle, there are two well-known inverse problems:

- \textbf{IP1}. Uniquely identify $D$ from the far-field data $u^\infty(\hat{x}, d, k)$ for all observation directions $\hat{x} \in \mathbb{S}$, all wave numbers $k \in \mathcal{I}$ and a fixed incident direction $d$, where $\mathbb{S}$ is the unit circle in $\mathbb{R}^2$ and $\mathcal{I}$ is an interval.
• IP2. Uniquely identify $D$ from the far-field data $u^\infty(\hat{x}, d, k)$ for all $\hat{x}, d \in \mathbb{S}$ and a fixed wave number $k$.

Many researchers have studied these two problems and established a variety of uniqueness results; see [6, 14, 15, 17, 21]. For uniqueness of inverse obstacle scattering problems under certain geometrical assumptions or restricted data, we refer to [1, 5, 8, 10, 18, 7, 11–13, 23, 24] and the references therein.

In this paper, we propose an inverse problem for the scattering of an obliquely incident electromagnetic wave by an impedance cylinder, and then prove a uniqueness result for this inverse problem. Although numerical methods for solving the direct scattering problem with oblique incidence have been developed, [4, 19, 22], analyzing the well posedness of the forward problem is not well studied. Moreover, the corresponding inverse scattering problem is largely open. In [20, 26], we investigated the solvability of the direct scattering problem from an impedance cylinder at oblique incidence. The problem is formulated in the following manner.

Let the cylinder be oriented parallel to the $z$-axis. Assume that the medium exterior to the cylinder is uniformly dielectric along the $z$ direction. We further suppose that the permittivity $\epsilon$ and the permeability $\mu$ of the host medium are positive constants. Let $(E', H') = (e'(x, y), h'(x, y)) e^{-i\omega t -i\beta z}$ be the time-harmonic incident electromagnetic plane wave, and $(E, H)$ be the total field which is the sum of the incident wave $(E', H')$ and the scattered wave $(E'', H'')$. Here, $\omega$ is the frequency and $\beta$ is a constant depending on the incident angle with respect to the $z$-axis. For more details about $\beta$, we refer readers to the explanation following equation (1.8). Denote by $D \subset \mathbb{R}^2$ the cross-section of the cylinder which is a domain with $C^2$ boundary $\partial D$. By expressing the $x$ and $y$ components of both the electric field $E$ and the magnetic field $H$ in terms of their $z$ components, the Maxwell equations for $(E, H)$ with the Leontovich impedance boundary condition $(v \times E) \times v = \lambda (v \times H)$ on $\partial D$ are reduced to a boundary value problem for a system of two Helmholtz equations with coupled oblique boundary conditions, where $v$ is the outward unit normal vector of $\partial D$ and $\lambda$ is the impedance function independent of $z$. Define $k_0 := \omega \sqrt{\mu \epsilon}$ and $k = \sqrt{k_0^2 - \beta^2}$ with non-negative imaginary part. Let $(e', h')$ be the $z$-component of the scattered wave where the time dependence and $z$ dependence have been suppressed. Then we have

\begin{align}
\Delta e' + (k_0^2 - \beta^2) e' & = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
\Delta h' + (k_0^2 - \beta^2) h' & = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},
\end{align}

(1.1)

\begin{align}
\frac{\partial e'}{\partial v} + i \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega} e' - \frac{\beta}{\epsilon \omega} \frac{\partial h'}{\partial \tau} & = f_1 \quad \text{on } \partial D, \\
\frac{\partial h'}{\partial v} + i \frac{\lambda (k_0^2 - \beta^2)}{\mu \omega} h' + \frac{\beta}{\mu \omega} \frac{\partial e'}{\partial \tau} & = f_2 \quad \text{on } \partial D,
\end{align}

(1.2)

\begin{align}
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial e'}{\partial r} - i k e' \right) & = 0, \quad r = |x|, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial h'}{\partial r} - i k h' \right) & = 0, \quad r = |x|,
\end{align}

(1.3)

where $v$ and $\tau$ are the unit outward normal vector and tangent vector to $\partial D$ such that $\tau$ is $\pi/2$ rotation of $v$ and in the plane of $D$, respectively. The boundary data $f_1$ and $f_2$ are determined by the incident wave $(E', H')$ as follows:

\begin{align}
f_1 & = -\left( \frac{\partial e'}{\partial v} + i \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega} e' - \frac{\beta}{\epsilon \omega} \frac{\partial h'}{\partial \tau} \right),
\end{align}

(1.4)
Figure 1. Incident angle $\theta$.

$$f_2 = - \left( \frac{\partial h'}{\partial \nu} + i \lambda \frac{(k_0^2 - \beta^2)}{\mu \omega} h' + \frac{\beta}{\mu \omega} \frac{\partial e'}{\partial \tau} \right),$$  \hspace{1cm} (1.8)$$

where $(e', h')$ is the $z$ component of the incident plane wave. Set $e := e' + e'$ and $h := h' + h'$. In addition, we assume that the impedance $\lambda(x)$ is positive and continuous on $\partial D$. Denote by $\theta \in (0, \pi)$ the incident angle with respect to the negative $z$-axis (figure 1). Then we have $\beta = k_0 \cos \theta$ and $k = \sqrt{k_0^2 - \beta^2} = k_0 \sin \theta$. We note that the wave number in (1.1) and (1.2) depends on the incident angle $\theta$. To emphasize the dependence of the far-field data on the angle $\theta$, we denote by $e^\infty(\hat{x}, \theta)$ and $h^\infty(\hat{x}, \theta)$ the far-field patterns of $e'$ and $h'$, respectively.

The unique solvability of the direct scattering problem with an oblique incidence in a more general setting is shown in [20]. We state the result for the forward problem (1.1)–(1.6) as follows.

**Lemma 1.1.** Assume that the cross-section $D$ is a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary $\partial D$ and the impedance $\lambda$ is positive. Then, for $(f_1, f_2) \in H^{1/2}(\partial D)$ the boundary value problem (1.1)–(1.6) has a unique solution in $H^2_{loc}(\mathbb{R}^2 \setminus \overline{D})$.

In this paper, we consider the corresponding inverse scattering problem:

- **Inverse problem.** Uniquely identify the cross-section of the cylinder $\partial D$ and the impedance $\lambda$ from the far-field data $e^\infty(\hat{x}, \theta)$ and $h^\infty(\hat{x}, \theta)$ for all $\hat{x} \in \hat{S}$, $\theta \in (\theta_1, \theta_2) \subset (0, \pi)$ and fixed frequency $\omega$.

This problem is closely related to the inverse problem $IP1$ which we mentioned before.

The main purpose of this paper is to establish the uniqueness of such a new inverse scattering problem. To this end, we apply the scheme proposed by Subbarayappa and Isakov in [2, 15]. The proof is based on the analyticity of the solution to the forward scattering problem (1.1)–(1.6) with respect to $\beta$. We emphasize that $\beta$ depends on the frequency $\omega$ and the incident angle $\theta$. In contrast to the inverse problem $IP1$ where the wave number is determined only by $\omega$, we fix the frequency $\omega$ but vary the incident angle $\theta$. By the Lax–Phillips method [14], the direct scattering problem is reduced to a boundary value problem defined in a bounded domain with oblique boundary condition. The unique solvability of this oblique derivative problem is proven in [20] by using the theory of M. Schechter. Following the idea in [2], we further show that the solution to the direct scattering problem (1.1)–(1.6) can be analytically extended into a small complex neighborhood of $(-k_0, k_0)$. As a consequence, a uniqueness result for our **inverse problem** is established by first using some real $\hat{\beta} \in (k_0 \cos \theta_2, k_0 \cos \theta_1) \subset (-k_0, k_0)$ and then using the complex $\beta$ with sufficiently small $|\beta - \hat{\beta}|$. 


The remainder of this paper is organized as follows. In section 2, we show that the direct scattering problem is uniquely solvable for \( \beta \in (-k_0, k_0) \) and the solution is holomorphic with respect to \( \beta \) in a small complex neighborhood of \((-k_0, k_0)\). Based on this fact, a uniqueness result for the proposed inverse problem is proven in section 3.

2. Analyticity of solution to direct scattering problem

In this section, we prove that the solution to the direct scattering problem (1.1)–(1.6) is holomorphic with respect to \( \beta \in N(-k_0, k_0)\)\(\setminus S\), where \(N(-k_0, k_0)\) is a complex neighborhood of \((-k_0, k_0)\) and \(S\) is a discrete set. For our later reference, in such a case we will say the solution is generically holomorphic in a complex neighborhood of \((-k_0, k_0)\). This result is the basis of our argument for the proposed inverse problem. Following the idea of [2], we use the Lax–Phillips method together with the perturbation theory of Fredholm operators. The following two lemmas (see [16, 2]) play a key role in this paper.

**Lemma 2.1.** Let \( T(x) \) be a family of compact operators in a Banach space \( X \) holomorphic for \( x \in \mathbb{C} \). We call \( x \) a singular point if 1 is an eigenvalue of \( T(x) \). Then either all \( x \in \mathbb{C} \) are singular points or there are only a finite number of singular points in each compact subset of \( \mathbb{C} \). We refer to the set of singular points in the latter case as ‘discrete’ in \( \mathbb{C} \).

**Lemma 2.2.** Let \( X, Y \) be Banach spaces. If \( T(x) \in \mathcal{L}(X, Y) \) is holomorphic with respect to \( x \) at \( x_0 \) and \( T^{-1}(x_0) \in \mathcal{L}(Y, X) \) exists, then \( T^{-1}(x) \) exists, belongs to \( \mathcal{L}(Y, X) \) and is holomorphic with respect to \( x \) for sufficiently small \( |x - x_0| \), where \( \mathcal{L}(Y, X) \) denotes the space of linear bounded operators from \( Y \) to \( X \).

Let \( \Omega \) be a bounded domain with \( C^2 \) boundary which contains \( D \) in its interior, and consider the associated boundary value problem in a bounded domain \( \Omega \setminus \overline{D} \):

\[
\Delta u + (k_0^2 - \beta^2)u = g_1 \quad \text{in } \Omega \setminus \overline{D},
\]

\[
\Delta v + (k_0^2 - \beta^2)v = g_2 \quad \text{in } \Omega \setminus \overline{D},
\]

\[
\frac{\partial u}{\partial v} + i \frac{(k_0^2 - \beta^2)}{\lambda \epsilon \omega} u - \frac{\beta}{\epsilon \omega} \frac{\partial v}{\partial \tau} = f_1 \quad \text{on } \partial D,
\]

\[
\frac{\partial v}{\partial v} + i \frac{(k_0^2 - \beta^2)}{\mu \omega} v + \frac{\beta}{\mu \omega} \frac{\partial u}{\partial \tau} = f_2 \quad \text{on } \partial D,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]

We denote by \( H^s(E) \) the Sobolev space of \( C^2 \)-valued functions with order \( s \) in a domain \( E \) which is a subset of either \( \mathbb{R}^2 \) or \( \partial D \). Then, for \((g_1, g_2) \in L^2(\Omega \setminus \overline{D}) := H^0(\Omega \setminus \overline{D})\) and \((f_1, f_2) \in H^{1/2}(\partial D)\), the solvability of the boundary value problem (2.1)–(2.6) in \( H^2(\Omega \setminus \overline{D}) \) and the analyticity of its solution with respect to \( \beta \) can be established. To this end, we will use the following result [20]:

**Lemma 2.3.** If \( k_0^2 - \beta^2 \in \mathbb{R} \) is not a Dirichlet eigenvalue for \( -\Delta \) in \( \Omega \setminus \overline{D} \). Then, for \((g_1, g_2) \in L^2(\Omega \setminus \overline{D})\) and \((f_1, f_2) \in H^{1/2}(\partial D)\) there is a unique solution to (2.1)–(2.6) in \( H^2(\Omega \setminus \overline{D}) \).
We remark that the assumption in lemma 2.3 is not an essential one because we can always take \( \Omega \) to satisfy it due to the monotonicity of the Dirichlet eigenvalue for \(-\Delta\) with respect to the domain \( \Omega \setminus \overline{D} \).

Based on this result, we can further prove:

**Theorem 2.4.** There is a discrete set \( S \subset \mathbb{C} \) such that for \( \beta \in \mathbb{C} \setminus S \) there exists a unique solution to the boundary value problem (2.1)–(2.6) in \( H^2(\Omega \setminus \overline{D}) \). Moreover, the solution is holomorphic with respect to \( \beta \in \mathbb{C} \setminus S \).

**Proof.** According to the above remark for lemma 2.3, we can find \( \beta_* \) in any interval \((\beta_1, \beta_2) \subset (-k_0, k_0)\) such that \( k_0^2 - \beta^2 \) is not a Dirichlet eigenvalue for \(-\Delta\) with respect to the domain \( \Omega \setminus \overline{D} \). Then, by lemma 2.3, for this \( \beta_* \), \((g_1, v_2) \in L^2(\Omega \setminus \overline{D})\) and \((f_1, f_2) \in H^{1/2}(\partial D)\) there is a unique solution to (2.1)–(2.6) in \( H^2(\Omega \setminus \overline{D}) \).

For any \( \beta \in \mathbb{C} \), we rewrite the boundary value problem (2.1)–(2.6) as a compact perturbation of (2.1)–(2.6) with \( \beta = \beta_* \):

\[
\begin{align*}
\Delta u + (k_0^2 - \beta_*^2)u + (\beta_*^2 - \beta^2)u &= g_1 \quad \text{in } \Omega \setminus \overline{D}, \\
\Delta v + (k_0^2 - \beta_*^2)v + (\beta_*^2 - \beta^2)v &= g_2 \quad \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial u}{\partial \nu} + \frac{k_0^2 - \beta_*^2}{\lambda \epsilon \omega} u - \frac{\beta_*}{\mu \omega} \frac{\partial v}{\partial \tau} + \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega} v - \frac{\beta}{\mu \omega} \frac{\partial u}{\partial \tau} &= f_1 \quad \text{on } \partial D, \\
\frac{\partial v}{\partial \nu} + \frac{\lambda (k_0^2 - \beta_*^2)}{\mu \omega} v + \frac{\beta_*}{\mu \omega} \frac{\partial u}{\partial \tau} + \frac{\lambda (k_0^2 - \beta^2)}{\mu \omega} u - \frac{\beta}{\mu \omega} \frac{\partial v}{\partial \tau} &= f_2 \quad \text{on } \partial D, \\
u &= 0 \quad \text{on } \partial \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Define the operator \( \mathcal{A}(\beta) : (u, v) \rightarrow (g_1, g_2) \times (f_1, f_2) \), where \((u, v)\) is the solution to (2.1)–(2.6). We now introduce the operator \( \mathcal{B}(\beta) = \mathcal{A}(\beta) - \mathcal{A}(\beta_*) \). Then the boundary value problem (2.1)–(2.6) can be rewritten as

\[
(\mathcal{A}(\beta) + \mathcal{B}(\beta))(u, v) = (g_1, g_2) \times (f_1, f_2),
\]

or

\[
(I + \mathcal{A}^{-1}(\beta_*) \mathcal{B}(\beta)) (u, v) = \mathcal{A}^{-1}(\beta_*) ((g_1, g_2) \times (f_1, f_2)),
\]

where

\[
\mathcal{B}(\beta)(u, v) = \left( (\beta_*^2 - \beta^2)u, (\beta_*^2 - \beta^2)v \right)
\]

\[
\times \left( 1 + \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega} u - \frac{\beta}{\epsilon \omega} \frac{\partial v}{\partial \tau}, 1 + \frac{\lambda (k_0^2 - \beta^2)}{\mu \omega} v + \frac{\beta}{\mu \omega} \frac{\partial u}{\partial \tau} \right)
\]

is a compact operator from \( H^2(\Omega \setminus \overline{D}) \) into \( L^2(\Omega \setminus \overline{D}) \) instead of \( H^2(\Omega \setminus \overline{D}) \) for simplicity. Whenever it is clear from the context, this kind of simplification will be used. Then, \( \mathcal{A}^{-1}(\beta_*) \mathcal{B}(\beta) : H^2(\Omega \setminus \overline{D}) \rightarrow H^2(\Omega \setminus \overline{D}) \) is compact and obviously holomorphic with respect to \( \beta \in \mathbb{C} \). Since \( \beta_* \) is not a singular point of \(-\mathcal{A}^{-1}(\beta_*) \mathcal{B}(\beta)\), the set \( S \) of its singular points \( \beta \in \mathbb{C} \) is discrete by lemma 2.1. Hence, the boundary value problem (2.1)–(2.6) is uniquely solvable for \( \tilde{\beta} \in \mathbb{C} \setminus S \). By lemma 2.2, the solution is holomorphic with respect to \( \beta \) in a small neighborhood of \( \tilde{\beta} \in \mathbb{C} \setminus S \). The proof is complete. \( \square \)

Next, based on theorem 2.4, we apply the Lax–Phillips method [14] to establish the unique existence and analyticity of the solution to the original scattering problem (1.1)–(1.6). In order to simplify the notation in the following arguments, we define

\[
H := \begin{pmatrix}
\Delta + (k_0^2 - \beta^2) & 0 \\
0 & \Delta + (k_0^2 - \beta^2)
\end{pmatrix}
\]
and
\[
B := \begin{pmatrix}
\frac{\partial_v + \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega}}{\epsilon \omega} & -\frac{\beta}{\epsilon \omega} \\
\frac{\beta}{\mu \omega} & \frac{\partial_v + \frac{k_0^2 - \beta^2}{\mu \omega}}{\mu \omega}
\end{pmatrix},
\]
where \(\partial_v\) and \(\partial_t\) denote the normal and tangential derivatives on \(\partial D\), respectively. Then the boundary value problem (1.1)–(1.6) takes the form:
\[
\begin{aligned}
Hu &= 0 & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\
Bu &= q & \text{on } \partial D, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} - i k \right) u &= 0 & r = |x|,
\end{aligned}
\]
(2.10)
where \(u = (e^i, h^i)^T\) and \(q = (f_1, f_2)^T\) and where the superscript \(T\) denotes the transposition.

Let \(u\) be the solution to (2.10) and \(u_0\) be the solution to (2.1)–(2.6) with \((g_1, g_2) = (0, 0)\). Assume that \(\tilde{\Omega}\) is a bounded domain satisfying \(\bar{D} \subset \tilde{\Omega} \subset \Omega\). Let \(\phi\) be a cutoff function such that \(\phi(x) = 1\) for \(x\) near \(D\) and \(\phi(x) = 0\) for \(x\) outside \(\tilde{\Omega}\). Then \(v = u - \phi u_0\) solves the following boundary value problem:
\[
\begin{aligned}
Hu &= f & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\
Bu &= 0 & \text{on } \partial D, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} - i k \right) v &= 0 & r = |x|,
\end{aligned}
\]
(2.11)
where
\[
f = -H(\phi u_0) \in L^2(\mathbb{R}^2 \setminus \bar{D}).
\]
Conversely, (2.11) implies (2.10).

The unique solvability and analyticity of the solution to (2.11) are shown in the following:

**Theorem 2.5.** For \(\beta \in (-k_0, k_0)\), the boundary value problem (2.11) has a unique solution in \(H^{2\infty}(\mathbb{R}^2 \setminus \bar{D})\). Moreover, this solution can be holomorphically extended with respect to \(\beta\) into a small complex neighborhood of \((-k_0, k_0)\).

**Proof.** The proof proceeds in three steps. The first two steps are devoted to the unique solvability of (2.11), and are essentially the same as we have shown in [20]. Note that they are the basis of the third step; so we still give the details here for the sake of completeness of the proof.

In the first step, we establish the uniqueness of solutions to (2.11) for \(\beta \in (-k_0, k_0)\). It is enough to prove that if \(v = (e^i, h^i)^T\) solves (2.11) for \(f = 0\), then \(e^i = h^i = 0\) in \(\mathbb{R}^2 \setminus \bar{D}\).

Let \(\Omega\) be a disk with radius \(r\) and center at the origin which contains \(D\) in its interior. Let \(D_r = \Omega \setminus \bar{D}\). Then, using Green’s identities and the boundary conditions for \(e^i\) and \(h^i\), we have
\[
\int_{\partial D} e^i \frac{\partial \bar{v}}{\partial v} \, ds = \int_{\partial D} e^i \frac{\partial \bar{e}}{\partial v} \, ds + \int_{D_r} \left[ |\nabla e^i|^2 + e^i \Delta \bar{e} \right] \, dx
\]
\[
= \int_{\partial D} e^i \left[ \frac{k_0^2 - \beta^2}{\lambda \epsilon \omega} e^i - \frac{\beta}{\epsilon \omega} \frac{\partial \bar{e}}{\partial \tau} \right] \, ds + \int_{D_r} \left[ |\nabla e^i|^2 - (k_0^2 - \beta^2)|e^i|^2 \right] \, dx
\]
\[
= \frac{i(k_0^2 - \beta^2)}{\epsilon \omega} \int_{\partial D} (1/\lambda)|e^i|^2 \, ds + \frac{\beta}{\epsilon \omega} \int_{\partial D} e^i \frac{\partial \bar{e}}{\partial \tau} \, ds
\]
\[
+ \int_{D_r} \left[ |\nabla e^i|^2 - (k_0^2 - \beta^2)|e^i|^2 \right] \, dx
\]
(2.13)
and
\[
\int_{\partialOmega} h^t \frac{\partial \overline{h^r}}{\partial v} \, ds = \int_{\partial D} h^t \frac{\partial \overline{h^r}}{\partial v} \, ds + \int_{D_r} \left[ |\nabla h^t|^2 + h^t \Delta \overline{h^r} \right] \, dx \\
= \int_{\partial D} h^t \left[ \frac{i}{\mu_\omega} \frac{\partial \overline{\mu_\omega}}{\partial \tau} - \frac{\partial \overline{\mu_\omega}}{\partial \tau} \right] + \int_{D_r} \left[ |\nabla h^t|^2 - (k_0^2 - \beta^2)|h^t|^2 \right] \, dx \\
= \frac{1}{i} \frac{(k_0^2 - \beta^2)}{\mu_\omega} \int_{\partial D} \lambda |h^t|^2 \, ds - \frac{\beta}{\mu_\omega} \int_{\partial D} h^t \frac{\partial \overline{\mu_\omega}}{\partial \tau} \, ds \\
+ \int_{D_r} \left[ |\nabla h^t|^2 - (k_0^2 - \beta^2)|h^t|^2 \right] \, dx. \tag{2.14}
\]

Noting that
\[
\int_{\partial D} e^t \frac{\partial \overline{h^r}}{\partial \tau} \, ds = - \int_{\partial D} h^t \frac{\partial \overline{\mu_\omega}}{\partial \tau} \, ds,
\]
we derive from (2.13) and (2.14) that for \( \beta \in (-k_0, k_0) \)
\[
\Im \left( \epsilon \int_{\partial \Omega_1} e^t \frac{\partial \overline{\epsilon^t}}{\partial v} \, ds + \mu \int_{\partial \Omega_1} h^t \frac{\partial \overline{h^t}}{\partial v} \, ds \right) \\
= \frac{k_0^2 - \beta^2}{\mu_\omega} \int_{\partial D} \left[ (1/\lambda) |\epsilon^t|^2 + \lambda |h^t|^2 \right] \, ds \geq 0, \tag{2.15}
\]
where \( \Im \) denotes the imaginary part of a complex number.

On the other hand, we deduce from the radiation condition that
\[
\int_{\partial \Omega_1} \left[ \frac{\partial \epsilon^t}{\partial v} \right]^2 + k^2 |\epsilon^t|^2 + 2k \Im \left( \epsilon^t \frac{\partial \overline{\epsilon^t}}{\partial v} \right) \, ds = \int_{\partial \Omega_1} \left| \frac{\partial \epsilon^t}{\partial v} - i k \epsilon^t \right|^2 \, ds \to 0,
\]
\[
\int_{\partial \Omega_1} \left[ \frac{\partial h^t}{\partial v} \right]^2 + k^2 |h^t|^2 + 2k \Im \left( h^t \frac{\partial \overline{h^t}}{\partial v} \right) \, ds = \int_{\partial \Omega_1} \left| \frac{\partial h^t}{\partial v} - i k h^t \right|^2 \, ds \to 0
\]
as \( r \to \infty \), which implies
\[
\lim_{r \to \infty} \int_{\partial \Omega_0} \left[ \epsilon \frac{\partial \epsilon^t}{\partial v} \right]^2 + \epsilon k^2 |\epsilon^t|^2 + \mu \left| \frac{\partial h^t}{\partial v} \right|^2 + \mu k^2 |h^t|^2 \, ds \\
= -2k \lim_{r \to \infty} \Re \left( \epsilon \int_{\partial \Omega_0} e^t \frac{\partial \overline{\epsilon^t}}{\partial v} \, ds + \mu \int_{\partial \Omega_1} h^t \frac{\partial \overline{h^t}}{\partial v} \, ds \right). \tag{2.16}
\]

Hence, we obtain from (2.15) and (2.16) that
\[
\lim_{r \to \infty} \int_{\partial \Omega_0} |\epsilon^t| \, ds = 0, \quad \lim_{r \to \infty} \int_{\partial \Omega_1} |h^t| \, ds = 0. \tag{2.17}
\]
Rellich's lemma immediately yields \( \epsilon^t = 0 \) and \( h^t = 0 \) in \( \mathbb{R}^2 \setminus \overline{D} \).

In the second step, we establish the existence of solutions to (2.11). Let \( \Omega_j \subset \mathbb{R}^2 \) \((j = 1, 2)\) be bounded domains satisfying \( \overline{D} \subset \subset \Omega_1 \subset \subset \Omega_2 \). Suppose that \( k_0^2 - \beta^2 \) is not a Dirichlet eigenvalue for \( -\Delta \) in each \( \Omega_j \). The boundary \( \partial \Omega_2 \) is assumed to be of class \( C^2 \) and \( \supp f \subset \Omega_2 \) with \( \supp f \) denoting the support of \( f \). Here the support of any function is defined as the closure in \( \mathbb{R}^2 \) of all the points at which the function is not zero.

For \( f^* \in L^2_1(\mathbb{R}^2 \setminus \overline{D}) := \{ f^* : f^* \in L^2(\mathbb{R}^2 \setminus \overline{D}), \supp f^* \subset \Omega_2 \} \),
let

$$\hat{f}^*(x) := \begin{cases} f^*(x) & (x \in \mathbb{R}^2 \setminus \overline{D}) \\ 0 & (x \in \overline{D}). \end{cases}$$

Then, we can find a radiating solution $w \in H^2_{\text{loc}}(\mathbb{R}^2)$ to the Helmholtz equation in the whole space, that is,

$$\begin{cases} Hw = \hat{f}^* & \text{in } \mathbb{R}^2, \\ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) = 0 & \text{with } r = |x|. \end{cases}$$  \tag{2.18}$$

By the trace theorem, there exists a function $\tilde{w} \in H^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ such that

$$\begin{cases} B\tilde{w} = -Bw & \text{on } \partial D, \\ \text{supp } \tilde{w} \subset \Omega_1. \end{cases}$$

Then we have

$$\hat{f} := f^* + H\tilde{w} \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D}).$$

It is easy to show that $W := w|_{\mathbb{R}^2 \setminus D} + \tilde{w} \in H^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ satisfies

$$\begin{cases} HW = \hat{f} & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ BW = 0 & \text{on } \partial D, \\ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial W}{\partial r} - ikW \right) = 0 & \text{with } r = |x|. \end{cases}$$

By theorem 2.4, there exists a unique solution $V \in H^2(\Omega_2 \setminus \overline{D})$ to

$$\begin{cases} HV = \hat{f} & \text{in } \Omega_2 \setminus \overline{D}, \\ BV = 0 & \text{on } \partial D, \\ V = 0 & \text{on } \partial \Omega_2. \end{cases}$$  \tag{2.19}$$

Now we look for the solution $v$ to (2.11) in the form

$$v = W - \chi(W - V)$$  \tag{2.20}$$

with $\chi \in C_0^\infty(\Omega_2)$ and $\chi = 1$ on $\overline{\Omega}_1$. By direct calculations, we obtain

$$f = Hv = HW - \chi H(W - V) + 2\nabla \chi \cdot \nabla (W - V) + (\Delta \chi)(W - V)$$

$$= \hat{f} - \chi(\hat{f} - \hat{f}) + K\hat{f}$$

$$= \hat{f} + K\hat{f} \quad \text{in } \Omega_2 \setminus \overline{D},$$  \tag{2.21}$$

where we have set

$$K\hat{f} = 2\nabla \chi \cdot \nabla (W - V) + (\Delta \chi)(W - V).$$

Since the operator

$$K : L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D}) \to H^1(\mathbb{R}^2 \setminus \overline{D}) \cap L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$$

is bounded and $H^1(\mathbb{R}^2 \setminus \overline{D}) \cap L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ is compactly embebbed into $L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$, $K$ is a compact operator from $L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ into itself. This implies that equation (2.21) is of Fredholm type and hence the solvability follows from its uniqueness.

Let $f = 0$ in (2.21). Then, we have

$$HV = 0 \quad \text{in } \Omega_2 \setminus \overline{D}.$$
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In addition, it follows from (2.20) that for $x \in \mathbb{R}^2 \setminus \Omega_2$ we have

$$Hv = HW = \hat{f} = 0,$$

and therefore

$$Hv = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}.$$  

Noting that

$$Bu = BV = 0 \quad \text{on } \partial D,$$

we obtain from the uniqueness of the forward scattering problem that

$$v = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

which leads to

$$W = \chi(W - V) \quad \text{in } \mathbb{R}^2 \setminus \overline{D}. \quad (2.22)$$

On the other hand, we also have

$$\begin{align*}
H(W - V) &= HW - HV = \hat{f} - \hat{f} = 0 \quad \text{in } \Omega_2 \setminus \overline{D}, \\
B(W - V) &= BW - BV = 0 - 0 = 0 \quad \text{on } \partial D.
\end{align*} \quad (2.23)$$

Since $\chi$ is zero on $\partial \Omega_2$, we know from (2.22) that

$$W = \chi(W - V) = 0 \quad \text{on } \partial \Omega_2. \quad (2.24)$$

From the definition of $V$, we have

$$V = 0 \quad \text{on } \partial \Omega_2. \quad (2.25)$$

Combining (2.24) with (2.25) yields

$$W - V = 0 \quad \text{on } \partial \Omega_2. \quad (2.26)$$

In terms of theorem 2.4, it follows from (2.23) and (2.26) that

$$W - V = 0 \quad \text{in } \Omega_2 \setminus \overline{D}. \quad (2.27)$$

Using (2.22) again, we have

$$W = 0 \quad \text{in } \Omega_2 \setminus \overline{D},$$

and therefore

$$\hat{f} = 0.$$  

This completes the proof for the uniqueness of solutions to (2.21). Hence, we conclude that there exists a unique solution to (2.11).

In the third step, we show the analyticity of the solution in a complex neighborhood of $(-k_0, k_0)$.

By theorem 2.4, we know that $u_0$ is holomorphic with respect to $\beta$ near each $\beta \in (-k_0, k_0)$, and hence $f$, given by (2.12), is holomorphic with respect to $\beta$. In addition, using the standard integral representation of the solution to (2.18), we can easily see that $w$ is holomorphic with respect to the wave number $k$. Noting that $k = \sqrt{k_0^2 - \beta^2}$ with $\Im k \geq 0$, we have that $k$ is holomorphic with respect to $\beta$ in a small complex neighborhood of $(-k_0, k_0)$. So $w$ is holomorphic with respect to $\beta$, and therefore $W$ is also holomorphic with respect to $\beta$ in a small complex neighborhood of $(-k_0, k_0)$. Using theorem 2.4 again, we have that $V$ is holomorphic with respect to $\beta$ in a small complex neighborhood of $(-k_0, k_0)$. Hence, the operator $K$ and the given data $f$ in (2.21) are holomorphic with respect to $\beta$ in a small complex neighborhood of $(-k_0, k_0)$. By lemma 2.2, the solution $\hat{f}$ to (2.21) is holomorphic with respect to $\beta$ in a small complex neighborhood of $(-k_0, k_0)$. This completes the proof of the theorem. □
3. Uniqueness of the inverse problem

This section is devoted to the uniqueness of the proposed inverse problem for the scattering of an obliquely incident electromagnetic wave by an impedance cylinder. We state our result as follows:

**Theorem 3.1.** The cross-section $D$ and the impedance function $\lambda$ can be uniquely determined by the far-field patterns $e^\infty(\hat{x}, \theta)$ and $h^\infty(\hat{x}, \theta)$ for all $\hat{x} \in \mathbb{S}$, all $\theta \in (\theta_1, \theta_2) \subset (0, \pi)$ with $\pi/2 \in (\theta_1, \theta_2)$.

**Proof.** Assume that there are two different cross-sections $D_1$ and $D_2$ with coefficients $\lambda_1$ and $\lambda_2$ which generate the same far-field data

$$e^\infty_1(\hat{x}, \theta) = e^\infty_2(\hat{x}, \theta), \quad h^\infty_1(\hat{x}, \theta) = h^\infty_2(\hat{x}, \theta).$$

Let $D^{12}$ be the unbounded connected component of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$ and let $D_0 = D_{12} \setminus \overline{D_1}$; see figure 2.

Let $(e^\infty_1, h^\infty_1)$ and $(e^\infty_2, h^\infty_2)$ be the solutions to (1.1)–(1.6) with $D_1, \lambda_1$ and $D_2, \lambda_2$, respectively. Since $D_1$ and $D_2$ have the same far-field data, we obtain from Rellich’s lemma that $(e^\infty_1, h^\infty_1) = (e^\infty_2, h^\infty_2)$ in $D^{12}$. Now set

$$\nu = -v_1, \tau = -\tau_1, \lambda = -\lambda_1 \quad \text{on } \partial D_0 \cap \partial D_1$$

and

$$\nu = v_2, \tau = \tau_2, \lambda = \lambda_2 \quad \text{on } \partial D_0 \cap \partial D_2,$$

where for each $j (j = 1, 2)$, $v_j$ and $\tau_j$ are the unit outer and normal vector of $\partial D_j$ such that $\tau_j$ is the $\pi/2$ rotation of $v_j$. Then we have

$$\frac{\partial e_1}{\partial v} + i \frac{k_0^2 - \beta^2}{\lambda \varepsilon \omega} e_1 = \frac{\beta}{\varepsilon \omega} \frac{\partial h_1}{\partial \tau} = 0 \quad \text{on } \partial D_0, \quad (3.1)$$

$$\frac{\partial h_1}{\partial v} + i \frac{\lambda(k_0^2 - \beta^2)}{\mu \varepsilon \omega} h_1 + \frac{\beta}{\mu \varepsilon \omega} \frac{\partial e_1}{\partial \tau} = 0 \quad \text{on } \partial D_0, \quad (3.2)$$

where $e_j = e_j^o + e'_j$, $h_j = h_j^o + h'_j$, $j = 1, 2$.

For further arguments, we need the following lemma on Green’s formula in domains without usual smoothness of their boundaries [9, 21, 25], which is stated as

**Lemma 3.2.** Green’s formula holds for the domains in $\mathbb{R}^n$ with finite perimeter and functions whose first derivatives are in the space $BV$, provided that their rough traces are summable on the reduced boundary of the domain with respect to the $(n - 1)$-dimensional Hausdorff measure.
We note that the boundary of $D_0$ may not have the usual smoothness. But it can be proven that the domain $D_0$ has finite perimeter. In addition, the functions $e_1$ and $h_1$ are smooth enough. So, according to lemma 3.2, Green’s formula can be applied for $e_1$ and $h_1$ in the domain $D_0$. Because the justification of the assumptions for the validity of Green’s formula is exactly the same as that given in section 2 of [21], we omit the proof here and refer the readers to [21].

Using Green’s formulas for $e_1$, $h_1$ and the boundary conditions above, we have

\[
0 = \int_{D_0} \left[ \Delta e_1 + (k_0^2 - \beta^2) e_1 \right] \frac{1}{e_1} \, dx \\
= \int_{\partial D_0} \frac{\partial e_1}{\partial n} \frac{1}{e_1} \, ds - \int_{D_0} |\nabla e_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |e_1|^2 \, dx \\
= - \int_{\partial D_0} \left( \frac{k_0^2 - \beta^2}{\lambda} e_1 - \frac{\beta}{\epsilon \omega} \frac{\partial h_1}{\partial \tau} \right) \frac{1}{e_1} \, ds - \int_{D_0} |\nabla e_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |e_1|^2 \, dx \\
= - \frac{k_0^2 - \beta^2}{\epsilon \omega} \int_{\partial D_0} (1/\lambda) |e_1|^2 \, ds + \frac{\beta}{\epsilon \omega} \int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \frac{1}{e_1} \, ds \\
- \int_{D_0} |\nabla e_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |e_1|^2 \, dx \tag{3.3}
\]

and

\[
0 = \int_{D_0} \left[ \Delta h_1 + (k_0^2 - \beta^2) h_1 \right] \frac{1}{h_1} \, dx \\
= \int_{\partial D_0} \frac{\partial h_1}{\partial n} \frac{1}{h_1} \, ds - \int_{D_0} |\nabla h_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |h_1|^2 \, dx \\
= - \int_{\partial D_0} \left( \frac{k_0^2 - \beta^2}{\mu \omega} h_1 + \frac{\beta}{\mu \omega} \frac{\partial e_1}{\partial \tau} \right) \frac{1}{h_1} \, ds - \int_{D_0} |\nabla h_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |h_1|^2 \, dx \\
= - \frac{k_0^2 - \beta^2}{\mu \omega} \int_{\partial D_0} \lambda |h_1|^2 \, ds - \frac{\beta}{\mu \omega} \int_{\partial D_0} \frac{\partial e_1}{\partial \tau} \frac{1}{h_1} \, ds - \int_{D_0} |\nabla h_1|^2 \, dx + \int_{D_0} (k_0^2 - \beta^2) |h_1|^2 \, dx. \tag{3.4}
\]

Multiplying (3.3) and (3.4) by $\epsilon$ and $\mu$, respectively, and then taking the sum of them, we have

\[
1 - \frac{k_0^2 - \beta^2}{\omega} = \int_{\partial D_0} \left[ (1/\lambda) |e_1|^2 + \lambda |h_1|^2 \right] \, ds \\
= \frac{\beta}{\omega} \int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \frac{1}{h_1} \, ds - \int_{\partial D_0} \frac{\partial e_1}{\partial \tau} \frac{1}{h_1} \, ds - \epsilon \int_{D_0} |\nabla e_1|^2 \, dx \\
+ \epsilon \int_{D_0} (k_0^2 - \beta^2) |e_1|^2 \, dx - \mu \int_{D_0} |\nabla h_1|^2 \, dx + \mu \int_{D_0} (k_0^2 - \beta^2) |h_1|^2 \, dx. \tag{3.5}
\]

Note that

\[
\int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \frac{1}{h_1} \, ds - \int_{\partial D_0} \frac{\partial e_1}{\partial \tau} \frac{1}{h_1} \, ds = \int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \frac{1}{h_1} \, ds + \int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \frac{1}{h_1} \, ds
\]

is real. Then, for real $\beta \in (k_0 \cos \theta_2, k_0 \cos \theta_1) \subset (-k_0, k_0)$, by taking the imaginary part of (3.5), we obtain

\[
\int_{\partial D_0} \left[ (1/\lambda) |e_1|^2 + \lambda |h_1|^2 \right] \, ds = 0. \tag{3.6}
\]
It follows from (3.5) that
\[
-\frac{\beta}{\omega} \left( \int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \, ds - \int_{\partial D_0} \frac{\partial e_1}{\partial \tau} \, ds \right) = - \int_{D_0} (|e_1|^2 + \mu |h_1|^2) \, dx + (k_0^2 - \beta^2) \int_{D_0} (|e_1|^2 + \mu |h_1|^2) \, dx. \tag{3.7}
\]

By theorem 2.5, \( e_1 \) and \( h_1 \) can be holomorphically extended with respect to \( \beta \) into a small complex neighborhood of \( \beta \in (k_0 \cos \theta_2, k_0 \cos \theta_1) \), which implies that (3.7) also holds for complex \( \beta \) near \((k_0 \cos \theta_2, k_0 \cos \theta_1)\). Particularly, under the assumption that \( \pi/2 \in (\theta_1, \theta_2) \), i.e. \( 0 \in (k_0 \cos \theta_2, k_0 \cos \theta_1) \), equality (3.7) is true for complex \( \beta \) in the neighborhood of 0 denoted by \( U(0) := \{ \beta \in \mathbb{C} : |\beta| < \varepsilon \} \) with sufficiently small \( \varepsilon > 0 \). For a pure imaginary number \( \beta \in U(0) \), the imaginary part of (3.7) implies
\[
\int_{\partial D_0} \frac{\partial h_1}{\partial \tau} \, ds - \int_{\partial D_0} \frac{\partial e_1}{\partial \tau} \, ds = 0
\]
and therefore
\[
- \int_{D_0} (|e_1|^2 + \mu |h_1|^2) \, dx + (k_0^2 - \beta^2) \int_{D_0} (|e_1|^2 + \mu |h_1|^2) \, dx = 0. \tag{3.8}
\]
Noting that \( e_1 \) and \( h_1 \) are holomorphic with respect to \( \beta \) in \( U(0) \), we have that (3.8) also holds for all complex \( \beta \in U(0) \). Now selecting a complex number \( \beta \in U(0) \) with \( \text{Im} \beta^2 > 0 \) and taking the imaginary part of (3.8) yields
\[
\int_{D_0} (|e_1|^2 + \mu |h_1|^2) \, dx = 0,
\]
which implies that
\[
e_1 = h_1 = 0 \text{ in } D_0.
\]
Using the unique continuation principle, we have that
\[
e_1 = h_1 = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D_1}.
\]
This contradicts the fact that \( e_1 = e_1' + e' \), \( h_1 = h_1' + h' \) where \( e_1' \) and \( h_1' \) satisfy the radiation condition but \( e' \) or \( h' \) does not. Hence, we have \( D_1 = D_2 \).

Next, we show that the impedance function \( \lambda(x) \) is also uniquely determined by the given far-field data.

We suppose that there are two functions \( \lambda_1 \) and \( \lambda_2 \) such that they generate the same far-field data. Let \((e_1, h_1)\) and \((e_2, h_2)\) be the solutions of the boundary value problem (1.1)–(1.6) corresponding to \( \lambda_1 \) and \( \lambda_2 \). Then we know that \((e_1, h_1) = (e_2, h_2)\) outside \( D \), and hence
\[
\frac{\partial e_1}{\partial v} = \frac{\partial e_2}{\partial v}, \quad \frac{\partial h_1}{\partial v} = \frac{\partial h_2}{\partial v}, \quad \frac{\partial e_1}{\partial \tau} = \frac{\partial e_2}{\partial \tau}, \quad \frac{\partial h_1}{\partial \tau} = \frac{\partial h_2}{\partial \tau} \text{ on } \partial D.
\]
From boundary conditions (1.3) and (1.4), we obtain
\[
(\lambda_1(x) - \lambda_2(x)) e_1(x) = (\lambda_1(x) - \lambda_2(x)) h_1(x) = 0 \tag{3.9}
\]
for \( x \in \partial D \). Note that if \( e_1 = h_1 = 0 \) on an open set \( \Gamma \) of \( \partial D \), then \( \frac{\partial e_1}{\partial v} = \frac{\partial h_1}{\partial v} = 0 \) on \( \Gamma \). By boundary conditions (1.3) and (1.4), we have \( \frac{\partial e_1}{\partial \tau} = \frac{\partial h_1}{\partial \tau} = 0 \) on \( \Gamma \) and hence \( e_1 = h_1 = 0 \) in \( \mathbb{R}^2 \setminus D \) due to Holmgren’s uniqueness theorem. This is a contradiction as we explained above. Hence, \( e_1 \) and \( h_1 \) cannot simultaneously vanish on any open set of \( \partial D \). So, for arbitrary \( x \in \partial D \) there exists a sequence of points \( x_n \to x \) such that \( e_1(x_n) \neq 0 \) or \( h_1(x_n) \neq 0 \). Then, from (3.9), we have \( \lambda_1(x_n) = \lambda_2(x_n) \), and hence \( \lambda_1(x) = \lambda_2(x) \). The proof is complete. \( \square \)
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