Inverse Problems

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On reconstruction in the inverse conductivity problem with one measurement

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Abstract. We consider an inverse problem for electrically conductive material occupying a domain Ω in $\mathbb{R}^2$. Let γ be the conductivity of Ω, and D a subdomain of Ω. We assume that γ is a positive constant k on D, $k \neq 1$ and is 1 on $\Omega \setminus D$; both D and k are unknown. The problem is to find a reconstruction formula of D from the Cauchy data on $\partial\Omega$ of a non-constant solution u of the equation $\nabla \cdot \gamma \nabla u = 0$ in Ω. We prove that if D is known to be a convex polygon such that diam $D <$ dist $(D, \partial\Omega)$, there are two formulae for calculating the support function of D from the Cauchy data.

1. Introduction

This paper is the sequel to [7] and, as predicted therein, we return to one of the problems treated by Friedman–Isakov [5]. They considered an inverse problem for electrically conductive material occupying a bounded domain Ω in $\mathbb{R}^2$. Let γ be the conductivity of Ω, and D a subdomain of Ω such that $\overline{D} \subset \Omega$. They assume that γ is a positive constant k on D with $k \neq 1$ and is 1 on $\Omega \setminus D$. Let u be a non-constant solution to the equation

$$\nabla \cdot \gamma \nabla u = 0 \qquad \text{in} \quad \Omega. \tag{1.1}$$

Let ν denote the unit outward normal vector field to $\Omega \setminus \overline{D}$.

They considered the following uniqueness problem.

Uniqueness problem

Assume that k is known and D is unknown. Can one determine D from the Cauchy data $u|_{\partial\Omega}$, $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$?

They proved that if D is known to be a convex polygon such that

$$\text{diam } D < \text{dist } (D, \partial\Omega), \tag{1.2}$$

the answer to the problem is yes.

A strong point of their result is that there is no additional assumption on the behaviour of $u|_{\partial\Omega}$ or $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ at the cost of (1.2). Barcelo *et al* [3] added such an assumption and dropped (1.2). Seo [9] proved a uniqueness theorem from two sets of the Cauchy data having an additional restriction on the behaviour and removed (1.2) and the convexity restriction on D. When ∂D has a special geometry, there are some results. For example, Kang–Seo [8] obtained a uniqueness result when D is a disc.

If both D and k are unknown, the problem becomes more difficult. Alessandrini–Isakov [1] considered this problem and obtained a uniqueness theorem of a convex polygon D and k without (1.2). Instead of this assumption they assume that $u|_{\partial\Omega}$ or $\frac{\partial u}{\partial\nu}|_{\partial\Omega}$ has a special property.

From these investigations one can say that the Cauchy data of a solution to (1.1) contain information about the location of D. However, their proofs do not tell us how to extract such information from the Cauchy data.

In this paper we consider the following reconstruction problem.

Reconstruction problem

Assume that both k and D are unknown. Find a formula for calculating information about the location of D from the Cauchy data of u.

This is a purely mathematical problem and remains open. In [6] we considered the extreme case $k = 0$, and obtained such formulae provided D was a convex polygon with the restriction (1.2). In this paper using the idea discovered therein we present such formulae under the same geometric assumption on D when $k > 0$, $k \neq 1$.

Now we describe the result more precisely. Let S^1 denote the set of all unit vectors of $\mathbb{R}^2$. Recall the definition of the support function:

$$h_D(\omega) = \sup_{x\in D} x\cdot\omega, \qquad \omega \in S^1.$$

From this function one can reconstruct the convex hull of general domain D.

We say that $\omega \in S^1$ is regular with respect to D if the set

$$\{x \in \mathbb{R}^2 \mid x\cdot\omega = h_D(\omega)\} \cap \partial D$$

consists of only one point.

Remark 1.1. Note that if D is a polygon, the counting number of the set of all unit vectors which are not regular with respect to D is finite. Therefore, it is very rare for us to choose a direction ω that is not regular with respect to D; $h_D(\cdot)$ is a continuous function. Therefore, the support function of D is uniquely determined by knowing its restriction to the set of all unit vectors which are regular with respect to D.

We merely assume that $\partial\Omega$ is Lipschitz and $u \in H^1(\Omega)$, and consequently we have to clarify what we mean by the symbol $\frac{\partial u}{\partial\nu}|_{\partial\Omega}$. It is defined as an element of the dual space of $H^{1/2}(\partial\Omega)$ by the formula

$$\left\langle \left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}, f\right\rangle = \int_\Omega \{1 + (k-1)\chi_D\}\nabla u\cdot\nabla\Psi\,\mathrm{d}x \tag{1.3}$$

where $f \in H^{1/2}(\partial\Omega)$, Ψ is in $H^1(\Omega)$ and satisfies $\Psi = f$ on $\partial\Omega$. From the definition of the weak solution we know that it is well defined and one may take Ψ such that $\Psi(x) = 0$ for x far from $\partial\Omega$. This means that $\frac{\partial u}{\partial\nu}|_{\partial\Omega}$ is uniquely determined by the value of u near $\partial\Omega$. We call $(u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}|_{\partial\Omega})$ the Cauchy data of u on $\partial\Omega$. It is a pair of the voltage potential and electric current distribution on $\partial\Omega$.

In this paper the following special harmonic functions are extremely important:

$$v = v(x) = \mathrm{e}^{\tau x\cdot(\omega + \mathrm{i}\omega^\perp)}, \qquad \tau > 0$$

where $\omega, \omega^\perp \in S^1$ and satisfy

$$\omega\cdot\omega^\perp = 0, \qquad \det\,(\omega \quad \omega^\perp) < 0.$$

Remark 1.2. Calderón [4] made use of these types of harmonic functions in the inverse conductivity problem with infinitely many measurements.

Using these functions and the Cauchy data of u on $\partial\Omega$ we give the following definition.

Definition 1.1 (Indicator function). *Let u be a weak solution to (1.1). Define*

$$I_\omega(\tau, t) = \mathrm{e}^{-\tau t}\left\{\left\langle \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}, v|_{\partial\Omega}\right\rangle - \left\langle \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega}, u|_{\partial\Omega}\right\rangle\right\}, \qquad \tau > 0, \qquad t \in \mathbb{R}.$$

Note that u is fixed. The result is the two following formulae.

Theorem 1.1. *Assume that D is a convex polygon satisfying (1.2) and that u is not a constant function. Let ω be regular with respect to D. The formulae*

$$\{t \in \mathbb{R} \mid \lim_{\tau \longrightarrow \infty} I_\omega(\tau, t) = 0\} = [h_D(\omega), \infty[\tag{1.4}$$

$$h_D(\omega) - t = \lim_{\tau \longrightarrow \infty} \frac{\log|I_\omega(\tau, t)|}{\tau}, \qquad \forall t \in \mathbb{R}, \tag{1.5}$$

are valid.

This is a direct corollary of the trivial identity

$$I_\omega(\tau, t) = \mathrm{e}^{\tau(h_D(\omega) - t)} I_\omega(\tau, h_D(\omega))$$

and the asymptotic behaviour of $I_\omega(\tau, h_D(\omega))$ as $\tau \longrightarrow \infty$ described below.

Key lemma. *Assume that D is a convex polygon satisfying (1.2) and that u is not a constant function. Let ω be regular with respect to D. There exist positive constants L and μ such that*

$$\lim_{\tau \longrightarrow \infty} \tau^\mu |I_\omega(\tau, h_D(\omega))| = L.$$

The proof of this lemma is delicate and the outline is as follows. From the regularity of ω we know that the line $x \cdot \omega = h_D(\omega)$ meets ∂D at a vertex x_0 of D. Using a well known expansion of u about x_0 (see proposition 2.1) and a formula which connects $I_\omega(\tau, h_D(\omega))$ with an integral on ∂D involving $u|_{\partial D}$ (see proposition 3.1), we obtain the asymptotic expansion of $I_\omega(\tau, h_D(\omega))$ as $\tau \longrightarrow \infty$ (see proposition 3.2):

$$I_\omega(\tau, h_D(\omega)) \sim \mathrm{e}^{\mathrm{i}\tau x_0 \cdot \omega^\perp} \sum_{j=1}^{\infty} \frac{L_j}{\tau^{\mu_j}}$$

where $0 < \mu_1 < \mu_2 < \cdots$. The problem is to show that $L_j \neq 0$ for some j. We see that if $L_j = 0$ for all j, u has a harmonic continuation in a neighbourhood of x_0 (see lemma 4.1). Then Friedman–Isakov's extension argument [5] tells us that u has to be a constant function and it is a contradiction. Restriction (1.2) is merely employed to make use of their argument.

It would be interesting to apply our method to the three-dimensional problem (see [3] for a uniqueness result) or a similar problem in the linear theory of elasticity. This will be considered in subsequent papers. The numerical testing of (1.4) and (1.5) remains open and we hope that someone performs this task in the future.

Finally, we note that in subsequent sections we always assume that ω is regular with respect to D.

2. Preliminaries

2.1. Notation

x_0 stands for the only one point of the set

$$\{x \in \mathbb{R}^2 \mid x \cdot \omega = h_D(\omega)\} \cap \partial D;$$

$$B_R(x_0) = \{x \in \mathbb{R}^2 \mid |x - x_0| < R\}, \qquad R > 0;$$

Θ stands for the outside angle at the vertex x_0 of D and thus $\pi < \Theta < 2\pi$.

2.2. Expansion of u about a vertex

Let u be a weak solution to (1.1). Define

$$u^e = u|_{\Omega \setminus \overline{D}}$$

$$u^i = u|_D.$$

We introduce polar coordinates. Let $\omega^\perp$ denote the unit-vector perpendicular to ω satisfying $\det(\omega \quad \omega^\perp) < 0$. Since x_0 is vertex of D and ω is regular with respect to D, one may write

$$B_{2\eta}(x_0) \cap (\Omega \setminus \overline{D}) = \{x_0 + r(\cos\theta \boldsymbol{a} + \sin\theta \boldsymbol{a}^\perp) \mid 0 < r < 2\eta, 0 < \theta < \Theta\}$$

$$B_{2\eta}(x_0) \cap \overline{D} = \{x_0 + r(\cos\theta \boldsymbol{a} + \sin\theta \boldsymbol{a}^\perp) \mid 0 < r < 2\eta, \Theta < \theta < 2\pi\}$$

$$B(x_0, \eta) \cap \partial D = \Gamma_p \cup \Gamma_q \cup \{x_0\}$$

$$\Gamma_p = \{x_0 + r(\cos p\omega^\perp + \sin p\omega) \mid 0 < r < \eta\}$$

$$\Gamma_q = \{x_0 + r(\cos q\omega^\perp + \sin q\omega) \mid 0 < r < \eta\}$$

where η is a small positive number,

$$-\pi < q < p < 0$$

$$p + \Theta = 2\pi + q$$

$$\boldsymbol{a} = \cos p\omega^\perp + \sin p\omega$$

$$\boldsymbol{a}^\perp = -\sin p\omega^\perp + \cos p\omega$$

$$\det(\boldsymbol{a} \quad \boldsymbol{a}^\perp) > 0.$$

Set

$$u(r, \theta) = u(x), \qquad x = x_0 + r(\cos\theta \boldsymbol{a} + \sin\theta \boldsymbol{a}^\perp).$$

The following proposition is only given for our purpose and the proof is well known. For example, the reader can find its outline in [2, section 2].

Proposition 2.1. *There exist a real number α, a monotone increasing sequence $(\mu_j)_{j=1,\ldots}$ of positive numbers and sequences $\{A_j^e\}$, $\{B_j^e\}$, $\{A_j^i\}$, $\{B_j^i\}$ of real numbers such that:*

$$(1 + k)^2 \sin^2 \pi\mu_j = (1 - k)^2 \sin^2(\pi - \Theta)\mu_j; \tag{2.1}$$

$$\begin{pmatrix} A_j^e \\ B_j^e \end{pmatrix} = \begin{pmatrix} \cos 2\pi\mu_j & \sin 2\pi\mu_j \\ -k\sin 2\pi\mu_j & k\cos 2\pi\mu_j \end{pmatrix} \begin{pmatrix} A_j^i \\ B_j^i \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} A_j^e \\ B_j^e \end{pmatrix} = \begin{pmatrix} \cos^2 \Theta\mu_j + k\sin^2 \Theta\mu_j & (1 - k)\cos\Theta\mu_j \sin\Theta\mu_j \\ (1 - k)\cos\Theta\mu_j \sin\Theta\mu_j & \sin^2 \Theta\mu_j + k\cos^2 \Theta\mu_j \end{pmatrix} \begin{pmatrix} A_j^i \\ B_j^i \end{pmatrix}, \tag{2.3}$$

$$\begin{aligned} u^e(r, \theta) - \alpha &= \sum_{j=1}^{\infty} r^{\mu_j}(A_j^e \cos\mu_j\theta + B_j^e \sin\mu_j\theta), \\ u^i(r, \theta) - \alpha &= \sum_{j=1}^{\infty} r^{\mu_j}(A_j^i \cos\mu_j\theta + B_j^i \sin\mu_j\theta); \end{aligned} \tag{2.4}$$

the series are absolutely convergent in $H^1(B_{s\eta}(x_0) \cap (\Omega \setminus \overline{D}))$ *and* $H^1(B_{s\eta}(x_0) \cap D)$, *respectively, and uniformly in* $B_{s\eta}(x_0)$ *for each* $0 < s < 2$*; moreover for each* $l = 1 \ldots,$

$$\begin{aligned} &\left| u(r, 0) - \alpha - \sum_{j=1}^{l} r^{\mu_j} A_j^e \right| \leqslant C_l r^{\mu_{l+1}} \\ &\left| u(r, \Theta) - \alpha - \sum_{j=1}^{l} r^{\mu_j} (A_j^i \cos \Theta\mu_j + B_j^i \sin \Theta\mu_j) \right| \leqslant C_l r^{\mu_{l+1}}, \qquad 0 < r < \eta. \end{aligned} \tag{2.5}$$

Note that from (2.2) and (2.3) we have

$$\begin{aligned} &A_j^i (\cos 2\pi\mu_j - \cos^2 \Theta\mu_j - k \sin^2 \Theta\mu_j) \\ &\qquad + B_j^i \{\sin 2\pi\mu_j + (k-1) \cos \Theta\mu_j \sin \Theta\mu_j\} = 0. \end{aligned} \tag{2.6}$$

3. Asymptotic expansion of the indicator function

Proposition 3.1. *Let* v *be a* $H^2(\Omega)$ *harmonic function. For any constant* λ *the formula*

$$\left\langle \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}, v|_{\partial\Omega} \right\rangle - \left\langle \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega}, u|_{\partial\Omega} \right\rangle = (1-k) \int_{\partial D} (u - \lambda) \frac{\partial v}{\partial \nu}, \tag{3.1}$$

is valid.

Proof. From (1.3) we have

$$\begin{aligned} \left\langle \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}, v|_{\partial\Omega} \right\rangle &= \int_\Omega \{1 + (k-1)\chi_D\} \nabla u \cdot \nabla v \, \mathrm{d}x \\ \left\langle \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega}, u|_{\partial\Omega} \right\rangle &= \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}x. \end{aligned} \tag{3.2}$$

Green's formula (see [6]) yields

$$\int_D \nabla u \cdot \nabla v \, \mathrm{d}x = -\int_{\partial D} (u - \lambda) \frac{\partial v}{\partial \nu}. \tag{3.3}$$

Note that ν is outward to $\Omega \setminus \overline{D}$. A combination of (3.2) and (3.3) gives (3.1). □

Proposition 3.2. *The asymptotic expansion*

$$I_\omega(\tau, h_D(\omega)) \sim (k-1)\mathrm{i}\mathrm{e}^{\mathrm{i}\tau x_0 \cdot \omega^\perp} \sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mu_j} \Gamma(1 + \mu_j) K_j \tau^{-\mu_j}, \tag{3.4}$$

is valid where

$$K_j = A_j^e \mathrm{e}^{\mathrm{i}p\mu_j} - (A_j^i \cos \Theta\mu_j + B_j^i \sin \Theta\mu_j) \mathrm{e}^{\mathrm{i}q\mu_j}.$$

Proof. For η in section 2 take a positive constant c in such a way that

$$\partial D \setminus B_\eta(x_0) \subset \{x \cdot \omega \leqslant h_D(\omega) - c\}.$$

It follows from (3.1) that

$$\begin{aligned} \frac{I_\omega(\tau, h_D(\omega))}{1-k} &= \mathrm{e}^{-\tau h_D(\omega)} \int_{\partial D} (u - \alpha) \frac{\partial v}{\partial \nu} \\ &= \mathrm{e}^{-\tau h_D(\omega)} \int_{\Gamma_p} (u - \alpha) \frac{\partial v}{\partial \nu} + \mathrm{e}^{-\tau h_D(\omega)} \int_{\Gamma_q} (u - \alpha) \frac{\partial v}{\partial \nu} + \mathrm{O}(\tau \mathrm{e}^{-c\tau}). \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}
&\nu = \sin p\omega^{\perp} - \cos p\omega \qquad \text{on} \quad \Gamma_p\\
&\nu = -\sin q\omega^{\perp} + \cos q\omega \qquad \text{on} \quad \Gamma_q\\
&x \cdot \omega = h_D(\omega) + r\sin(\theta + p)\\
&x \cdot \omega^{\perp} = x_0 \cdot \omega^{\perp} + r\cos(\theta + p)\\
&\nabla v = \tau(\omega + \mathrm{i}\omega^{\perp})\mathrm{e}^{\tau(x\cdot\omega + \mathrm{i}x\cdot\omega^{\perp})},
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{e}^{-\tau h_D(\omega)}\frac{\partial v}{\partial \nu} = -\tau \mathrm{e}^{-\mathrm{i}p}\mathrm{e}^{\mathrm{i}\tau x_0\cdot\omega^{\perp}}\mathrm{e}^{r\tau(\sin p + \mathrm{i}\cos p)} \qquad \text{on} \quad \Gamma_p\\
&\mathrm{e}^{-\tau h_D(\omega)}\frac{\partial v}{\partial \nu} = \tau \mathrm{e}^{-\mathrm{i}q}\mathrm{e}^{\mathrm{i}\tau x_0\cdot\omega^{\perp}}\mathrm{e}^{r\tau(\sin q + \mathrm{i}\cos q)} \qquad \text{on} \quad \Gamma_q.
\end{aligned} \tag{3.6}$$

From (2.5) and (3.6) we obtain

$$\begin{aligned}
&\mathrm{e}^{-\tau h_D(\omega)}\int_{\Gamma_p}\left(u - \alpha - \sum_{j=1}^{l} r^{\mu_j}A_j^e\right)\frac{\partial v}{\partial \nu} = \mathrm{O}\left(\frac{1}{\tau^{\mu_{l+1}}}\right),\\
&\mathrm{e}^{-\tau h_D(\omega)}\int_{\Gamma_q}\left\{u - \alpha - \sum_{j=1}^{l} r^{\mu_j}(A_j^i\cos\Theta\mu_j + B_j^i\sin\Theta\mu_j)\right\}\frac{\partial v}{\partial \nu} = \mathrm{O}\left(\frac{1}{\tau^{\mu_{l+1}}}\right).
\end{aligned} \tag{3.7}$$

A combination of (3.5)–(3.7) gives

$$\begin{aligned}
\frac{I_\omega(\tau, h_D(\omega))}{1-k} = &-\tau \mathrm{e}^{-\mathrm{i}p}\mathrm{e}^{\mathrm{i}\tau x_0\cdot\omega^{\perp}}\sum_{j=1}^{l}A_j^e\int_0^{\eta} r^{\mu_j}\mathrm{e}^{r\tau(\sin p + \mathrm{i}\cos p)}\,\mathrm{d}r\\
&+\tau \mathrm{e}^{-\mathrm{i}q}\mathrm{e}^{\mathrm{i}\tau x_0\cdot\omega^{\perp}}\sum_{j=1}^{l}(A_j^i\cos\Theta\mu_j + B_j^i\sin\Theta\mu_j)\int_0^{\eta} r^{\mu_j}\mathrm{e}^{r\tau(\sin q + \mathrm{i}\cos q)}\,\mathrm{d}r\\
&+\mathrm{O}\left(\frac{1}{\tau^{\mu_{l+1}}}\right).
\end{aligned} \tag{3.8}$$

We make use of the following formulae [7]:

$$\begin{aligned}
&\int_0^{\eta} r^{\mu_j}\mathrm{e}^{r\tau(\sin p + \mathrm{i}\cos p)}\,\mathrm{d}r = \tau^{-(1+\mu_j)}\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mu_j}\mathrm{e}^{\mathrm{i}p}\mathrm{e}^{\mathrm{i}p\mu_j}\Gamma(1+\mu_j) + \mathrm{O}\left(\frac{\mathrm{e}^{\eta\tau\sin p}}{\tau}\right),\\
&\int_0^{\eta} r^{\mu_j}\mathrm{e}^{r\tau(\sin q + \mathrm{i}\cos q)}\,\mathrm{d}r = \tau^{-(1+\mu_j)}\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mu_j}\mathrm{e}^{\mathrm{i}q}\mathrm{e}^{\mathrm{i}q\mu_j}\Gamma(1+\mu_j) + \mathrm{O}\left(\frac{\mathrm{e}^{\eta\tau\sin q}}{\tau}\right).
\end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we obtain (3.4). □

4. Proof of the key lemma

The problem is: what happens when

$$K_j = A_j^e\mathrm{e}^{\mathrm{i}p\mu_j} - (A_j^i\cos\Theta\mu_j + B_j^i\sin\Theta\mu_j)\mathrm{e}^{\mathrm{i}q\mu_j} = 0$$

for all $j = 1, \ldots$?

Since

$$p + \Theta = 2\pi + q,$$

we have

$$\mathrm{e}^{\mathrm{i}\Theta\mu_j}\mathrm{e}^{\mathrm{i}p\mu_j} = \mathrm{e}^{\mathrm{i}2\pi\mu_j}\mathrm{e}^{\mathrm{i}q\mu_j}.$$

So $K_j = 0$ if and only if

$$A_j^e e^{i2\pi\mu_j} = (A_j^i \cos\Theta\mu_j + B_j^i \sin\Theta\mu_j)e^{i\Theta\mu_j}. \tag{4.1}$$

Since A_j^e, B_j^e, A_j^i, B_j^i are all real, we know that (4.1) is equivalent to

$$A_j^i \cos\Theta\mu_j \cos(\Theta - 2\pi)\mu_j + B_j^i \sin\Theta\mu_j \cos(\Theta - 2\pi)\mu_j = A_j^e \tag{4.2}$$

and

$$A_j^i \cos\Theta\mu_j \sin(\Theta - 2\pi)\mu_j + B_j^i \sin\Theta\mu_j \sin(\Theta - 2\pi)\mu_j = 0. \tag{4.3}$$

In this section we only consider j satisfying

$$\begin{pmatrix} A_j^i \\ B_j^i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since A_j^i and B_j^i are non-trivial solutions of (2.6) and (4.3), we obtain

$$\begin{aligned} L \equiv\ & (\cos 2\pi\mu_j - \cos^2\Theta\mu_j - k\sin^2\Theta\mu_j)\sin\Theta\mu_j \sin(\Theta - 2\pi)\mu_j \\ & -\{\sin 2\pi\mu_j + (k-1)\cos\Theta\mu_j \sin\Theta\mu_j\}\cos\Theta\mu_j \sin(\Theta - 2\pi)\mu_j = 0. \end{aligned} \tag{4.4}$$

Since

$$\begin{aligned} L = \sin(\Theta - 2\pi)\mu_j \times \{&\cos 2\pi\mu_j \sin\Theta\mu_j - \cos^2\Theta\mu_j \sin\Theta\mu_j - k\sin^3\Theta\mu_j \\ &- \sin 2\pi\mu_j \cos\Theta\mu_j - (k-1)\cos^2\Theta\mu_j \sin\Theta\mu_j\} \\ = {}& \sin(\Theta - 2\pi)\mu_j\{\sin(\Theta - 2\pi)\mu_j - k\sin\Theta\mu_j\} \\ = {}& \sin(2\pi - \Theta)\mu_j\{\sin(2\pi - \Theta)\mu_j + k\sin\Theta\mu_j\}. \end{aligned}$$

Therefore, (4.4) becomes

$$\sin(2\pi - \Theta)\mu_j\{\sin(2\pi - \Theta)\mu_j + k\sin\Theta\mu_j\} = 0. \tag{4.5}$$

Moreover, from (4.2) and (4.3) it is easy to see that

$$A_j^e \sin\Theta\mu_j \sin(2\pi - \Theta)\mu_j = 0. \tag{4.6}$$

This is a compatibility condition of the system (4.2) and (4.3). Now we are ready to prove the central part of this paper.

Lemma 4.1. *Assume that $K_j = 0$ for all $j = 1, \ldots$. There exist an integer $a \geqslant 2$ independent of j and a harmonic continuation $\tilde{u}$ of u from $\Omega \setminus \overline{D}$ into $(\Omega \setminus \overline{D}) \cup B_\eta(x_0)$ such that*

$$\tilde{u}\left(r, \theta + \frac{2\pi}{a}\right) = \tilde{u}(r, \theta) \qquad \text{in} \quad B_\eta(x_0).$$

Proof. The proof is divided into three parts.

Step 1: $\sin(2\pi - \Theta)\mu_j = 0$.

To prove this we assume that $\sin(2\pi - \Theta)\mu_j \neq 0$. From (4.5) we get

$$\sin(2\pi - \Theta)\mu_j + k\sin\Theta\mu_j = 0 \tag{4.7}$$

and this thus yields $\sin\Theta\mu_j \neq 0$. From (4.6) we conclude that $A_j^e = 0$. Then taking the first components of (2.2) and (2.3), respectively, we get

$$\begin{pmatrix} \cos 2\pi\mu_j & \sin 2\pi\mu_j \\ \cos^2\Theta\mu_j + k\sin^2\Theta\mu_j & (1-k)\cos\Theta\mu_j \sin\Theta\mu_j \end{pmatrix} \begin{pmatrix} A_j^i \\ B_j^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since A^i_j, B^i_j are not trivial solutions for this system, we obtain

$$\begin{aligned}0 = \cos 2\pi\mu_j(1-k)\cos\Theta\mu_j\sin\Theta\mu_j - (\cos^2\Theta\mu_j + k\sin^2\Theta\mu_j)\sin 2\pi\mu_j \\ = \cos\Theta\mu_j(\cos 2\pi\mu_j\sin\Theta\mu_j - \cos\Theta\mu_j\sin 2\pi\mu_j) \\ -k\sin\Theta\mu_j(\cos 2\pi\mu_j\cos\Theta\mu_j + \sin\Theta\mu_j\sin 2\pi\mu_j) \\ = -(\cos\Theta\mu_j\sin(2\pi-\Theta)\mu_j + k\sin\Theta\mu_j\cos(2\pi-\Theta)\mu_j). \end{aligned} \tag{4.8}$$

A combination of (4.7) and (4.8) gives

$$\sin(2\pi-\Theta)\mu_j\{\cos\Theta\mu_j - \cos(2\pi-\Theta)\mu_j\} = 0$$

and this thus yields

$$\cos\Theta\mu_j = \cos(2\pi-\Theta)\mu_j.$$

Therefore, we obtain

$$|\sin\Theta\mu_j| = |\sin(2\pi-\Theta)\mu_j|. \tag{4.9}$$

A combination of (4.7) and (4.9) yields

$$\begin{aligned}|\sin\Theta\mu_j| &= |\sin(2\pi-\Theta)\mu_j| \\ &= k|\sin\Theta\mu_j|\end{aligned}$$

and hence $k = 1$. This is a contradiction.

Step 2: μ_j has to be an integer.

It follows from step 1 that $(2\pi-\Theta)\mu_j = n\pi$ for an integer n. Then $(\pi-\Theta)\mu_j = -\pi\mu_j + n\pi$. This gives

$$\sin(\pi-\Theta)\mu_j = (-1)^{n+1}\sin\pi\mu_j.$$

Combining this with (2.1), we obtain

$$\begin{aligned}(1+k)^2\sin^2\pi\mu_j &= (1-k)^2\sin^2(\pi-\Theta)\mu_j \\ &= (1-k)^2\sin^2\pi\mu_j.\end{aligned}$$

Since $k \neq 0$, we have the desired conclusion.

Step 3: From step 1 we know that there exits an integer n_j such that $(2\pi-\Theta)\mu_j = n_j\pi$. Since $\mu_j \neq 0$, we have

$$\frac{\Theta}{\pi} = 2 - \frac{n_j}{\mu_j}. \tag{4.10}$$

From step 2 one it concludes that $\frac{\Theta}{\pi}$ has to be a rational number. Since $\pi < \Theta < 2\pi$, one may write

$$\frac{\Theta}{\pi} = 1 + \frac{b}{a} \tag{4.11}$$

where $a = 2, \ldots$, $b = 1, \ldots$ with $(a, b) = 1$. Note that a and b are independent of j. From (4.10) and (4.11) we get

$$b\mu_j = a(\mu_j - n_j).$$

Since $(a, b) = 1$, there exists an integer l_j such that

$$\mu_j = l_j a.$$

Then

$$\left(\theta + \frac{2\pi}{a}\right)\mu_j = \theta\mu_j + 2l_j\pi, \tag{4.12}$$

and we have

$$u(r,\theta) = \alpha + \sum_{j=1}^{\infty} r^{\mu_j}(A_j^e \cos\theta\mu_j + B_j^e \sin\theta\mu_j) \qquad \text{in} \quad (\Omega \setminus \overline{D}) \cap B_\eta(x_0).$$

By virtue of (4.12), this right-hand side gives a desired harmonic continuation of u. □

Now we are ready to prove the key lemma. Assume that $K_j = 0$ for all $j = 1, \ldots$. From a combination of lemma 4.1 and Friedman–Isakov's extension argument (see [5, p 570, proof of theorem 1.1]) we obtain a harmonic extension of u into whole Ω. This yields that u has to be constant. This is a contradiction. So one can take

$$m = \min\{j | K_j \neq 0\}.$$

Then from (3.4) we have

$$I_\omega(\tau, h_D(\omega)) \sim (k-1)\mathrm{i}\mathrm{e}^{\mathrm{i}\tau x_0\cdot\omega^\perp}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mu_m}\Gamma(1+\mu_m)K_m\tau^{-\mu_m}.$$

This completes the proof.

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