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On null tratorial torsion in vacuum quadratic Poincaré gauge field theory

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Abstract. The field equations of vacuum quadratic Poincaré gauge field theory (QPGFT) are solved for purely null tratorial torsion. Expressing the contortion tensor for this case as $K_{\lambda\mu\nu} = -2(g_{\lambda\mu}a_\nu - g_{\lambda\nu}a_\mu)$, where a^μ is null, one can prove that any solution of vacuum QPGFT, whose V_4 part is also a solution of Einstein's vacuum field equations with a cosmological constant, is necessarily of algebraic type N with respect to the generalised Weyl tensor. This class of exact solutions has an expansion-free, shear-free and twist-free autoparallel repeated principal null congruence. The relationship between this class and a similar class with null axial vector torsion is also discussed.

1. Introduction

In a previous paper (Singh and Griffiths 1990), the field equations of vacuum quadratic Poincaré gauge field theory (QPGFT) (see Hehl 1980, Hehl *et al* 1980, Baekler *et al* 1982, 1983 and references cited therein) for the purely quadratic Lagrangian density first proposed by von der Heyde (1976)

$$\mathcal{V} = \frac{e}{4l^2}T^{\mu\nu}{}_i F_{\nu\mu}{}^i + \frac{e}{4k}F^{\mu\nu}{}_{ij}F_{\nu\mu}{}^{ij} \tag{1}$$

were expressed in the Newman–Penrose (N–P) formalism involving spin coefficients. The two gauge field strengths $F_{\mu\nu}{}^i$ (torsion tensor) and $F_{\mu\nu}{}^{ij}$ (curvature tensor) along with the so-called modified torsion tensor $T_{\mu\nu}{}^i$, e and the coupling constants k and l are defined in Singh and Griffiths (1990), as are the conventions and notation used in this paper. Arguments in favour of the choice given by equation (1) for the Lagrangian density may be found in Hehl (1980), von der Heyde (1976) and Hehl *et al* (1978), and will not be discussed here.

The field equations of vacuum QPGFT given in Singh and Griffiths (1990) must be used in conjunction with those given by Jogia and Griffiths (1980) in which the formalism of Newman and Penrose (1962) had previously been extended to include spacetimes with torsion in any theory.

The aim of this work is to use the technique given in Singh and Griffiths (1990) to obtain a class of exact solutions of the field equations of vacuum QPGFT with purely null tratorial torsion. The case of purely null axial torsion has already been considered in Singh (1990). The technique developed is designed to produce a more general class of exact solutions than would be possible via the double duality ansatz of Baekler *et al* (1982, 1983), Baekler and Mielke (1988) and Lenzen (1984). Such a class of exact solutions of algebraic type N with an expansion-free, shear-free and twist-free autoparallel repeated principal null congruence is presented.

First assume that the contortion tensor is tratorial, i.e. semi-symmetric. In which case one may describe it in terms of a vector a^μ defined by

$$a^\mu = \tfrac{1}{6} K_\alpha{}^{\mu\alpha} \tag{2}$$

then

$$K_{\lambda\mu\nu} = -2(g_{\lambda\mu} a_\nu - g_{\lambda\nu} a_\mu). \tag{3}$$

In addition, a^μ is assumed to be a continuously defined null vector. The tetrad vector l^μ is then aligned with a^μ. This requires that the individual components of the contortion tensor, in the notation of Jogia and Griffiths (1980), satisfy

$$\begin{gathered}\alpha_1 = \beta_1 = \epsilon_1 = \kappa_1 = \lambda_1 = \nu_1 = \pi_1 = \rho_1 = \sigma_1 = \tau_1 = 0 \\ \gamma_1 = -\tfrac{1}{2}\mu_1 \qquad \bar{\mu}_1 = \mu_1\end{gathered} \tag{4}$$

and the contortion is thus completely described by the real component μ_1.

As in Singh (1990), assume that the V_4 parts of the curvature tensor are solutions of Einstein's vacuum field equations with a cosmological constant, Λ_c, i.e.

$$R^0_{\mu\nu} - \tfrac{1}{2} R^0 g_{\mu\nu} = \Lambda_c g_{\mu\nu} \tag{5}$$

where $\Lambda_c = -6\Lambda^0 = -\frac{1}{4}R^0$. Of course, the Ricci identities and the Bianchi identities for the curvature in V_4 must also hold.

2. The class of exact solutions

Under the assumptions of section 1 the following theorem can easily be proved.

Theorem 1. Any solution of the field equations of vacuum QPGFT, with a^μ null, whose V_4 part is also a solution of Einstein's vacuum field equations with a cosmological constant, is necessarily of algebraic type N, with an expansion-free, shear-free and twist-free autoparallel repeated principal null congruence.

The class of exact solutions satisfying the assumptions of section 1 is given by the metric

$$ds^2 = 2du(-U\,du + dv + W\,dz + \bar{W}\,d\bar{z}) - \frac{1}{2P^2}\,dz\,d\bar{z} \tag{6}$$

where the metric function P is a real function of z and $\bar{z}$ only satisfying

$$4P^2(\ln P)_{,z\bar{z}} = \Lambda^0 \tag{7}$$

and the metric functions U and W are given by

$$U = (P^2\phi_{,z}\,\phi_{,\bar z} + \Lambda^0)v^2 - \tfrac{1}{2}\phi_{,u}\,v + U^0 \tag{8a}$$

$$W = \phi_{,z}\,v \tag{8b}$$

where $\phi = \phi(u, z, \bar z) = \bar\phi$ and $U^0 = U^0(u, z, \bar z) = \bar U^0$ satisfy,

$$\phi_{,zz} - \frac{1}{2}(\phi_{,z})^2 + \frac{2P_{,z}}{P}\phi_{,z} = 0 \tag{9a}$$

$$\phi_{,z\bar z} - \frac{1}{2}\phi_{,z}\,\phi_{,\bar z} - \frac{\Lambda^0}{P^2} = 0 \tag{9b}$$

$$U^0{}_{,z\bar z} + \frac{1}{2}(\phi_{,z}\,U^0{}_{,\bar z} + \phi_{,\bar z}\,U^0{}_{,z}) + \left(\frac{1}{2}\phi_{,z}\,\phi_{,\bar z} + \frac{\Lambda^0}{P^2}\right)U^0 = 0. \tag{9c}$$

A suitable complex null tetrad may be taken in the form

$$l^\mu = \delta^\mu_2 \qquad l_\mu = \delta^1_\mu \tag{10a}$$

$$m^\mu = P(-2\bar W\delta^\mu_2 + \delta^\mu_3 + \mathrm{i}\delta^\mu_4) \tag{10b}$$

$$n^\mu = \delta^\mu_1 + U\delta^\mu_2. \tag{10c}$$

It is mentioned in passing that a particular solution of (7) is

$$P = A + Bz + \bar B\bar z + Cz\bar z \tag{11}$$

where A and C are real constants and B is a complex constant, such that

$$AC - |B|^2 = \tfrac{1}{4}\Lambda^0. \tag{12}$$

The only non-zero components of the contortion tensor are

$$\mu_1 = n \qquad \gamma_1 = -\tfrac{1}{2}n \tag{13}$$

where n is an arbitrary real function of u and the non-zero spin coefficients are

$$\alpha = \tfrac{1}{2}(P\phi_{,z} + 2P_{,z}) \tag{14a}$$

$$\beta = \tfrac{1}{2}(P\phi_{,\bar z} - 2P_{,\bar z}) \tag{14b}$$

$$\gamma = -(P^2\phi_{,z}\,\phi_{,\bar z} + \Lambda^0)v + \tfrac{1}{4}\phi_{,u} - \tfrac{1}{2}n \tag{14c}$$

$$\mu = n \tag{14d}$$

$$\nu = -P\phi_{,uz}\,v - 2PU^0{}_{,z} - 2P\phi_{,z}\,U^0 \tag{14e}$$

$$\pi = -P\phi_{,z} \tag{14f}$$

$$\tau = P\phi_{,\bar z}\,. \tag{14g}$$

The only non-zero components of the curvature tensor are

$$\Psi_4 = -2P[2PU^0{}_{,zz} + 2(2P_{,z} + P\phi_{,z})U^0{}_{,z} + P(\phi_{,z})^2U^0] \tag{15a}$$

$$\Phi_{12} = -nP\phi_{,\bar z} \tag{15b}$$

$$\Phi_{22} = 2n(P^2\phi_{,z}\,\phi_{,\bar z} + \Lambda^0)v - \tfrac{1}{2}n\phi_{,u} + n^2 - n_{,u} \tag{15c}$$

$$\Lambda = \Lambda^0 = k/8l^2. \tag{15d}$$

One may note that when the torsion vanishes, these solutions reduce to the $K(\lambda)$ wave solutions of Diaz and Plebański (1981).

3. Discussion

The class of exact solutions obeying theorem 1 has

$$l_\mu = u_{,\mu} = \delta^1_\mu. \tag{16}$$

Thus, $a_\mu = -\frac{1}{2}nl_\mu$ can be written as the gradient of a real scalar function $\psi = \psi(u)$, namely,

$$a_\mu = \psi_{,\mu} \tag{17}$$

where ψ satisfies

$$\psi_{,u} = -\tfrac{1}{2}n. \tag{18}$$

The real scalar function ψ is sometimes known as the torsion (or contortion) potential function or the 'tlaplon' field (Hojman *et al* 1978). The contortion tensor now reads

$$K_{\lambda\mu\nu} = -2(g_{\lambda\mu}\psi_{,\nu} - g_{\lambda\nu}\psi_{,\mu}). \tag{19}$$

This is the type of contortion considered in the Brans–Dicke theory with torsion (Smalley 1978) and the theories with propagating torsion of Hojman *et al* (1978, 1979) and Mukku and Sayed (1979).

Ni (1979) has shown, however, that the implications of the theory with the propagating torsion of Hojman *et al* (1978, 1979) disagree with the experimental findings of Roll *et al* (1964) and Braginsky and Panov (1971, 1972).

Unlike the propagating torsion theory of Hojman *et al* (1978, 1979), the Poincaré gauge field theory does not allow the coupling of internal gauge bosons, like the electromagnetic potential A_μ, to the torsion fields. Otherwise gauge invariance, in the case of A_μ, $U(1)$ invariance, would be violated. This is by no means an *ad hoc* assumption of Poincaré gauge field theory, but a reflection of the fact that physical fields seem to divide naturally into matter fields and gauge potentials. Consequently, exempting internal gauge bosons from coupling to the set $(e^k{}_\mu, \gamma^{kl}{}_\mu)$ should not be viewed as a problem.

Consider the class of exact solutions of vacuum QPGFT obeying theorem 1. In this case the torsion of the spacetime, which is generated by the torsion potential function $\psi(u)$, represents a propagating plane torsion wave with retarded null time coordinate u. The gravitational field is represented by the so-called $K(\lambda)$ wave solutions of Diaz and Plebański (1981) with the metric given by equations (6) to (9).

The class of exact solutions obeying theorem 1 is remarkably similar to that obtained for null axial vector torsion by Singh (1990). It may be noted that the metric, and hence the V_4 parts, are identical. The solutions only differ in the type of torsion that they contain and their values of Λ^0. In view of this observation and in spite of the fact that the theory is non-linear, one may ask whether or not it is possible to combine these two types of torsion with the same metric to obtain a more general class of exact solutions.

Consider a contortion tensor which is a sum of axial and tratorial pieces, namely,

$$K_{\lambda\mu\nu} = n(g_{\lambda\mu}l_\nu - g_{\lambda\nu}l_\mu) - m\epsilon_{\lambda\mu\nu\alpha}l^\alpha \tag{20}$$

where m and n are arbitrary real functions of u. In which case the only non-zero components of the torsion tensor are

$$\mu_1 = n + im \qquad \gamma_1 = -\tfrac{1}{2}(n + im). \tag{21}$$

Suppose that the metric is given by equations (6) to (9). Using the same tetrad, the spin coefficients are exactly as in equation (14) except that now

$$\gamma = -(P^2\phi_{,z}\,\phi_{,\bar{z}} + \Lambda^0)v + \tfrac{1}{4}\phi_{,u} - \tfrac{1}{2}(n + im) \tag{22a}$$

$$\mu = n + im. \tag{22b}$$

The non-zero components of the curvature tensor are then easily calculated from the Ricci identities, the Bianchi identities for the torsion and the appendix in Jogia and Griffiths (1980), and read

$$\Psi_4 = -2P\big(2PU^0{}_{,zz} + 2(2P_{,z} + P\phi_{,z})U^0{}_{,z} + P(\phi_{,z})^2U^0\big) \tag{23a}$$

$$\Phi_{12} = -nP\phi_{,\bar{z}} \tag{23b}$$

$$\Phi_{22} = 2n(P^2\phi_{,z}\,\phi_{,\bar{z}} + \Lambda^0)v - \tfrac{1}{2}n\phi_{,u} - n_{,u} + n^2 - m^2 \tag{23c}$$

$$\Lambda = \Lambda^0 = \text{constant} \tag{23d}$$

$$\Theta_{12} = -mP\phi_{,\bar{z}} \tag{23e}$$

$$\Theta_{22} = 2m(P^2\phi_{,z}\,\phi_{,\bar{z}} + \Lambda^0)v - \tfrac{1}{2}m\phi_{,u} - m_{,u} + 2mn. \tag{23f}$$

Unfortunately, the two classes of exact solutions in question may not be combined in the context of vacuum QPGFT, as the following theorem will show.

Theorem 2. In vacuum QPGFT (for the purely quadratic Lagrangian density $\mathcal{V}$ of equation (1)), the contortion tensor (20) is not consistent with the metric given by equations (6)–(9), unless $K_{\lambda\mu\nu}$ is either purely axial ($n = 0$) or purely tratorial ($m = 0$).

Proof. Equations (29) and (30g) in Singh and Griffiths (1990) imply either

$$\Phi_{12} = 0 \tag{24}$$

or

$$\Lambda^0 = k/8l^2. \tag{25}$$

If $\Phi_{12} = 0$, then equation (23b) implies either $n = 0$ or $\phi_{,\bar{z}} = 0$. Suppose $\phi_{,\bar{z}} = 0$, then $\phi = \bar{\phi}$ requires $\phi_{,z} = 0$, in which case $\phi = \phi(u)$ and so equation (9b) gives

$$\Lambda^0 = 0. \tag{26}$$

Then from equation (31l) in Singh and Griffiths (1990)

$$\mu_1 = 0 \tag{27}$$

i.e. the solutions reside in V_4—which contradicts the fact that only solutions with non-zero torsion are being investigated here. Otherwise

$$n = 0. \tag{28}$$

If $\Lambda^0 = k/8l^2$, then equations (29) and (30j) in Singh and Griffiths (1990) imply

$$m = 0 \tag{29}$$

which completes the proof. □

Of course, that theorem 2 implies that the two classes of exact solutions in question may not be superposed, goes without saying.

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