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Neighborhoods of isolated horizons and their stationarity

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Abstract

A distinguished (invariant) Bondi-like coordinate system is defined in the spacetime neighborhood of a non-expanding horizon of arbitrary dimension via geometry invariants of the horizon. With its use, the radial expansion of a spacetime metric about the horizon is provided and the free data needed to specify it up to a given order are determined in spacetime dimension 4. For the case of an electro-vacuum horizon in four-dimensional spacetime, the necessary and sufficient conditions for the existence of a Killing field at its neighborhood are identified as differential conditions for the horizon data and data for the null surface transversal to the horizon.

Keywords: horizons, classical relativity, black holes

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1. Introduction

Systematic studies of black holes in various approaches to quantum gravity as well as an accurate description of the dynamical evolution of these exotic objects require a quasi-local description formalism—where a black hole can be treated as an ‘object in the lab’ and the global spacetime structure of the universe far away from it can be ignored. Among several approaches to constructing such a formalism [1, 2], one of considerable success is the theory of isolated horizons [3–5]. This approach was originally inspired by the ideas of Pejerski and Newman [1], next shaped into a solid formalism by Ashtekar [6], and subsequently developed by many researchers. Its main feature is the representation of a black hole in equilibrium through its surface—the non-expanding (or isolated) horizon—a null cylinder of codimension

1 and of compact spatial slices embedded in a Lorentzian spacetime. The black hole is characterized by the geometry (and possibly matter fields) data for this surface only. Both the geometry aspects [6, 7] and the mechanics [8, 9] have been systematically studied in spacetime dimension 4 and then extended to general dimensions, [10–12], the latter including in particular asymptotically anti-de Sitter (AdS) spacetimes [13]. Also, various matter content at the horizon has been considered [14–16] as well as the properties of symmetric horizons [17] and their relation to standard black hole solutions [18, 19]. The formalism has been further extended to non-equilibrium situations through the *dynamical horizons* [20, 21] (see also [4]). It is broadly applied in numerical relativity (see, for example, [22–25]) as well as in black hole descriptions in loop quantum gravity [26]—especially as the basis for entropy calculations (see, for example, [27–29]). The extension of this formalism has also found applications in supergravity [30, 31] and string theory-inspired gravity [32, 33].

The quasi-locality of the theory is a great advantage, as only the geometry objects at the horizon are relevant in the description; however, for this very reason one misses the information about the black hole neighborhood. The success of the formalism of *near horizon geometries* [34, 35] shows clearly that there is a strong demand for any black hole description method to be able to also ‘handle’ its neighborhood, a feature particularly relevant for the studies of black hole spacetimes in the context of AdS/CFT correspondence [36] and in numerical probing of the late stages of black hole mergers.

This article is dedicated to providing such an extension within the isolated horizon formalism. Its main ideas were originally published in [37]. The principal part of the presented work is providing a convenient way to describe the spacetime geometry in the neighborhood of the horizon through the *Bondi-like* coordinate system originally introduced in [38]. Here this construction is extended to an arbitrary spacetime dimension, and horizon spatial slice topology and its properties are studied. It is shown to be relatively convenient to use, providing, for example, a well-defined invariant radial spacetime metric expansion about the horizon. It provides a solid frame for addressing questions such as the conditions for the existence of a Killing vector field in the horizon neighborhood, a problem studied here in detail in the context of an electro-vacuum black hole in four-dimensional spacetime.

The paper is organized as follows. We start in section 2 with a short introduction of the geometric structure of general non-expanding horizons in an arbitrary dimension and discuss the definition and those properties of the horizon symmetries that will be needed in further studies of Killing horizons. Next, in section 3, still in the context of the general dimensions, we construct the Bondi-like coordinate system, which later provides the basis for describing the structures at the neighborhood. In the same section, we study the properties and the form of Killing fields possibly present there. The material of these two sections is then used in section 4, where we restrict the studies to horizons in four-dimensional electrovac spacetime, introducing in particular the convenient Newman–Penrose null frame (consistent with the Bondi-like coordinate system) for any matter type and studying the initial value problem for ‘radial’ metric expansion. Our focus is then restricted only to horizons embeddable in the electrovac spacetime. There the Maxwell evolution equations change the mentioned initial value problem, making it stronger (less initial data needed). This structure is then applied in section 5, where we formulate the necessary and sufficient conditions for the non-expanding horizon to be a Killing horizon. Analogous conditions for several classes of Killing fields are derived in section 6. We summarize the results in the concluding section 7. In order not to disrupt the line of reasoning, more complicated proofs as well as the definition of the Newman–Penrose frame have been moved to appendix A through appendix D.

2. Geometry of a non-expanding horizon

2.1. Non-expanding horizons

In this section we introduce the notation, review the definition, and outline the properties of non-expanding horizons in n dimensions [3, 7, 11, 17]. We often use abstract index notation and the following convention: the n -spacetime indexes are denoted by α, \dots, ν , every $n - 1$ dimensional vector spaces indexes are denoted by a, \dots, d , and every $n - 2$ vector space indexes are denoted by A, \dots, D .

2.1.1. Definition, the induced degenerate metric, 2-volume, and Hodge*. We start our consideration with an $(n - 1)$ -dimensional null surface Δ embedded in an n -dimensional time-oriented spacetime \mathcal{M} . In sections 4 and later $n = 4$. The spacetime metric tensor $g_{\mu\nu}$ of the signature $(-, +, \dots, +)$ is assumed to satisfy the Einstein field equations (possibly with a matter and cosmological constant). Throughout the paper, we assume that all the matter fields possibly present at the surface Δ satisfy the following:

Condition 2.1 (stronger energy condition). At every point of Δ , for every future-oriented null vector ℓ tangent to Δ , the vector $-T_{\mu\nu}\ell^\mu$ is causal and future oriented, where $T_{\mu\nu}$ is the energy-momentum tensor.

We denote the degenerate metric tensor induced at Δ by q_{ab} . The sub-bundle of the tangent bundle $T(\Delta)$ defined by the null vectors is denoted by L and referred to as the null direction bundle. Given a vector bundle P , the set of sections will be denoted by $\Gamma(P)$.

To recall the definition of a non-expanding horizon, consider the metric tensor q_{AB} induced by the tensor q_{ab} in the fibers of the quotient bundle $T(\Delta)/L$. Denote the inverse metric tensor defined in the fibers of the dual bundle $(T(\Delta)/L)^*$ by $q^{AB}(x)$ (that is, $q_{AB}q^{BC} = \delta_A^C$).

We will be assuming that Δ is a non-expanding horizon, in the following sense:

Definition 2.2. A null submanifold Δ of codimension 1 embedded in spacetime satisfying the Einstein field equations is called a *non-expanding horizon* (NEH) if:

- (i) for every point $x \in \Delta$ for every null vector ℓ^a tangent to Δ at x ,

$$q^{AB}\mathcal{L}_\ell q_{AB} = 0 \quad (2.1)$$

and

- (ii) Δ has the product structure $\hat{\Delta} \times \mathcal{I}$, that is, there is an embedding

$$\hat{\Delta} \times \mathcal{I} \rightarrow \mathcal{M} \quad (2.2)$$

such that:

- (i) Δ is the image
- (ii) $\hat{\Delta}$ is an $n - 2$ dimensional compact and connected³ manifold (referred to as the horizon base space)
- (iii) \mathcal{I} is an open interval
- (iv) for every maximal null curve in Δ there is $\hat{x} \in \hat{\Delta}$ such that the curve is the image of $\{\hat{x}\} \times \mathcal{I}$.

³ In the case where $\hat{\Delta}$ is not connected, all the otherwise global constants (such as surface gravity) remain constant only at maximal connected components of the horizon.

Each (non-vanishing in a generic set) null vector field ℓ defined in Δ (a section of L) determines a function $\kappa^{(\ell)}$ referred to as the surface gravity⁴ of ℓ , such that

$$\ell^\mu \nabla_\mu \ell^\nu = \kappa^{(\ell)} \ell^\nu. \quad (2.3)$$

In particular, given an NEH, there always exists a nowhere-vanishing null vector field ℓ_o^a of the identically vanishing surface gravity $\kappa^{(\ell_o)} = 0$. One can also choose a null vector field ℓ^a of $\kappa^{(\ell)}$ being an arbitrary constant⁵,

$$\kappa^{(\ell)} = \text{const.} \quad (2.4)$$

That vector field ℓ^a can vanish in a harmless (for our purposes) way in an $(n - 2)$ -dimensional section of Δ only.

From stronger energy condition 2.1, it follows in particular that $T_{\ell\ell} \geq 0$. This in turn implies via the generalized Raychaudhuri equation that the flow $[\ell]$ preserves the degenerate metric q

$$\mathcal{L}_\ell q_{ab} = 0 \quad (2.5)$$

and, using again the stronger energy condition, it can be shown that at every point p of Δ and for every $X \in T_p \Delta$

$$T_{a\beta} X^a \ell^\beta = {}^{(n)}\mathcal{R}_{a\beta} X^a \ell^\beta = 0. \quad (2.6)$$

The aforementioned property (2.5) combined with $\ell^a q_{ab} = 0$ means that q_{ab} is the pullback of a certain metric tensor field \hat{q}_{AB} defined for $\hat{\Delta}$. The horizon base space $\hat{\Delta}$ can be identified with the space of null curves tangent to Δ . Its manifold structure is unique. The pullback map is defined by the natural (and also unique) projection,

$$\Pi: \Delta \rightarrow \hat{\Delta}, \quad q_{ab} = \left(\Pi^* \hat{q} \right)_{ab}. \quad (2.7)$$

The pullback of the base space $\hat{\Delta}$ 2-volume form $\hat{\eta}_{AB}$,

$$\Pi^* \hat{\eta}_{ab} =: \eta_{ab} \quad (2.8)$$

defines the canonical area 2-form for Δ and its restriction η_{AB} to a 2-form in $T(\Delta)/L$. η_{AB} is used to define the horizon Hodge dualization $\star_\Delta: (T(\Delta)/L)^* \rightarrow (T(\Delta)/L)^*$,

$$\star_\Delta v_A := \eta_{AB} q^{BC} v_C. \quad (2.9)$$

2.1.2. The induced covariant derivative and the rotation 1-form. It can be shown by using (2.5) that the space time covariant derivative ∇_α determined by the metric tensor $g_{\alpha\beta}$ preserves the tangent bundle $T(\Delta)$. Indeed, for every pair of vector fields $X, Y \in \Gamma(T(\Delta))$,

$$\nabla_X Y \in \Gamma(T(\Delta)). \quad (2.10)$$

Therefore, there exists in $T(\Delta)$ an induced covariant derivative D_a such that for every pair of vector fields $X, Y \in \Gamma(T(\Delta))$ the following holds

$$D_X Y^a := \nabla_X Y^a. \quad (2.11)$$

⁴ We use dimensionless coordinates in spacetime; therefore, our surface gravity is also dimensionless.

⁵ The first one, ℓ_o , can be defined by fixing appropriately the affine parameter v at each null curve in Δ . Then the second vector field is simply $\ell = \kappa^{(\ell)} v \ell_o$.

The action of D_a on covectors, sections of the dual bundle $T^*\Delta$, is determined by the Leibniz rule. Together with the induced metric, the covariant derivative constitutes the *geometry of an NEH* (q_{ab}, D_a) .

The connection D_a preserves in particular the null direction bundle L ; thus, for every $\ell \in \Gamma(L)$, the derivative $D_a \ell^b$ is proportional to ℓ^b itself,

$$D_a \ell^b = \omega_a^{(\ell)} \ell^b \quad (2.12)$$

where $\omega_a^{(\ell)}$ is a 1-form defined uniquely for this subset of Δ for which $\ell \neq 0$ is defined. We call $\omega_a^{(\ell)}$ the *rotation 1-form potential* (see [7, 11]).

The evolution of $\omega_a^{(\ell)}$ along the null flow on Δ is responsible for the 0th Law of non-expanding horizon thermodynamics: the rotation 1-form potential $\omega^{(\ell)}$ and surface gravity of ℓ , related to $\omega_a^{(\ell)}$ via

$$\kappa^{(\ell)} = \ell^a \omega_a^{(\ell)} \quad (2.13)$$

satisfy the following constraint:

$$\mathcal{L}_\ell \omega_a^{(\ell)} = D_a \kappa^{(\ell)} \quad (2.14)$$

implied by (2.6). This tells us in particular, that there is always a choice of the section ℓ of the null direction bundle L such that $\omega^{(\ell)}$ is Lie dragged by ℓ . Indeed, we can always find a nontrivial section ℓ of L such that $\kappa^{(\ell)}$ is constant (0 for ℓ defined by an affine parameterization of the null geodesics tangent to Δ). Throughout the remainder of the article, we restrict our consideration to fields $\ell \in \Gamma(L)$ of such a class, or equivalently satisfying

$$\mathcal{L}_\ell \omega^{(\ell)} = 0. \quad (2.15)$$

Upon rescalings $\ell \mapsto \ell' = f\ell$ (where f is a real function defined at Δ) of a section ℓ^a of L , the rotation 1-form changes as follows:

$$\omega_a^{(\ell')} = \omega_a^{(\ell)} + D_a \ln f. \quad (2.16)$$

The requirement that both $\kappa^{(\ell)}$ and $\kappa^{(\ell')}$ be constants restricts the form of f to the following:

$$f = \begin{cases} B e^{-\kappa^{(\ell)} u} + \frac{\kappa^{(\ell')}}{\kappa^{(\ell)}} & \kappa^{(\ell)} \neq 0 \\ \kappa^{(\ell')} u - B & \kappa^{(\ell)} = 0 \end{cases} \quad (2.17)$$

where u is any function defined for Δ such that

$$\ell^a D_a u = 1 \quad (2.18)$$

and B is an arbitrary function constant along the null geodesics of Δ .

2.1.3. The constraints. The non-expanding horizon geometry (q_{ab}, D_a) is constrained by the Einstein equations. We have already used some of them in this paper. A *complete* set of the constraints on horizon geometry (q_{ab}, D_a) is encoded in the following identity

$$\begin{aligned} [\mathcal{L}_\ell, D_a] X^b &= \ell^b X^c \left[D_{(a} \omega_{c)}^{(\ell)} + \omega_a^{(\ell)} \omega_c^{(\ell)} + \frac{1}{2} \left(\binom{n}{n} \mathcal{R}_{ac} - \Pi^* \binom{n-2}{n} \mathcal{R}_{ac} \right) \right] \\ &=: \ell^b N_{ac}^{(\ell)} X^c, \end{aligned} \quad (2.19)$$

which holds for every $\ell \in \Gamma(L)$ and $X \in \Gamma(T(\Delta))$, where ${}^{(n-2)}\mathcal{R}_{AB}$ is the Ricci tensor of the metric tensor \hat{q}_{AB} induced in base space $\hat{\Delta}$, whereas ${}^{(n)}\mathcal{R}_{ac}$ is the pullback to Δ (by the embedding map $\Delta \rightarrow \mathcal{M}$) of the spacetime Ricci tensor. The constraints are given by replacing \mathcal{R}_{ac} with $8\pi G(T_{ac} - \frac{1}{2}Tq_{ac})$, where the pullback onto Δ of the stress energy tensor T_{ac} satisfies on Δ the condition

$$\ell^a T_{ab} = 0 \quad (2.20)$$

(see 2.6). In particular, the 0th law (2.14) is given by contracting (2.19) with a null vector ℓ . The remaining constraints determine the evolution of some other components of D_a along the null geodesics tangent to ℓ , provided the energy momentum tensor is given. In the Einstein vacuum or Einstein–Maxwell vacuum case, the constraints are solved explicitly [11].

Throughout this paper, we make a stronger assumption, namely, that for Δ , in addition to (2.20), the pullback onto Δ (by the embedding map $\Delta \rightarrow \mathcal{M}$) of the energy–momentum tensor is Lie dragged by (any)⁶ null vector field ℓ tangent to Δ :

$$\mathcal{L}_\ell T_{ab} = 0. \quad (2.21)$$

For the electromagnetic field considered later in this paper, this condition is a consequence of the Einstein–Maxwell equations; therefore, it is satisfied automatically. For the time being, however, we simply assume it is true.

2.1.4. The invariants. For every non-expanding horizon Δ that satisfies the assumptions made in the preceding subsection, the geometry (q, D) is analytic along the null geodesics in any affine coordinate. A given non-expanding horizon Δ can be incomplete in that coordinate; however, one can consider its (non-embedded) maximal analytic extension $\bar{\Delta}$ endowed with the analytic extension (\bar{q}, \bar{D}) of the geometry. This is what we do in this section. We use the same notation as previously; however, we mark all the symbols referring to $\bar{\Delta}$ by the extra $\bar{\cdot}$. Finally, all the invariant structures introduced into $\bar{\Delta}$ determine unique restriction to an original given unextended Δ .

Thus far, we have reduced the freedom in choice of a null vector field $\bar{\ell}$ tangent to $\bar{\Delta}$ to vector fields that satisfy (2.4) defined up to the transformations given by (2.17). This freedom can be further reduced by imposing a condition on the tensor $N_{ab}^{(\bar{\ell})}$ (2.19). We also notice that from (2.14, 2.15) it follows that

$$\bar{\ell}^a N_{ab}^{(\bar{\ell})} = 0 \quad (2.22)$$

hence this tensor defines a unique tensor $N_{AB}^{(\bar{\ell})}$ in the fibers of $T(\bar{\Delta})/\bar{L}$.

Let us now introduce a specific class of $\bar{\ell}$, namely:

Definition 2.3. A *natural vector field* $\bar{\ell}$ in $\bar{\Delta}$ is a tangent null vector field (that is, a section of \bar{L}) non-vanishing in a generic (open and dense) subset of $\bar{\Delta}$ that satisfies the following conditions

$$\kappa^{(\bar{\ell})} = 1, \quad \bar{q}^{AB} N_{AB}^{(\bar{\ell})} = 0. \quad (2.23)$$

The generic existence and uniqueness of the natural vector field in a given $\bar{\Delta}$ was shown in [11] (see equation (6.22) therein and the following paragraph). We review the following argumentation proving these properties further below. In the case where the natural vector field is unique, we will call it *the invariant vector field of the $\bar{\Delta}$ geometry*.

⁶ In fact, if this condition is satisfied by any given non-vanishing ℓ^a , then it is satisfied by every ℓ^a null and tangent to Δ .

Given a nonzero null vector field $\bar{\ell}$ in $\bar{\Delta}$ such that $\kappa^{(\bar{\ell})} = \text{const} \neq 0$, a unique foliation by spacelike $n - 2$ dimensional sections of Δ can be fixed by using the rotation 1-form potential $\omega^{(\bar{\ell})}$. In particular, one can choose a section $\bar{\sigma}: \hat{\Delta} \rightarrow \bar{\Delta}$ of $\bar{\pi}: \bar{\Delta} \rightarrow \hat{\Delta}$ such that the pullback $\bar{\sigma}^*\omega^{(\bar{\ell})}$ is divergence free:

$$d\star\bar{\sigma}^*\omega^{(\bar{\ell})} = \bar{q}^{AB}\bar{D}_A\bar{\sigma}^*\omega_B^{(\bar{\ell})} = 0. \quad (2.24)$$

Those sections of $\bar{\Delta}$ are called *good cuts* [7] and are defined uniquely modulo the action of the flow of the vector field $\bar{\ell}$. They set a foliation of $\bar{\Delta}$. If $\bar{\ell}$ is the invariant vector field of the $\bar{\Delta}$ geometry, then the corresponding good cut foliation is called the *invariant foliation of the $\bar{\Delta}$ geometry*. One has to be aware, though, that whereas the slices of the invariant foliation of $\bar{\Delta}$ are diffeomorphic to the base space $\hat{\Delta}$, the restriction of a slice of $\bar{\Delta}$ to Δ may be a proper subset of that slice. In other words, the slices of Δ may be nonglobal sections of $\Pi: \Delta \rightarrow \hat{\Delta}$. Given a null vector field $\bar{\ell}$ and a foliation, there is a unique differential 1-form $\bar{n}_a \in \Gamma(T^*(\bar{\Delta}))$ orthogonal to the leaves of the foliation and normalized by

$$\bar{n}_a\bar{\ell}^a = -1. \quad (2.25)$$

We will call \bar{n}_a the *invariant co-vector of the $\bar{\Delta}$ geometry* if $\bar{\ell}$ and the foliation are, respectively, the invariant vector field of the $\bar{\Delta}$ geometry and the invariant foliation of $\bar{\Delta}$. Finally, the invariant vector field, foliation, and covector field of $(\bar{\Delta}, \bar{q}_{ab}, \bar{D}_a)$ are restricted to a given NEH (Δ, q_{ab}, D_a) the geometry of which was the starting point of an extension $(\bar{\Delta}, \bar{q}_{ab}, \bar{D}_a)$. The uniqueness of this extension guarantees the uniqueness of the resulting invariant structures defined for (Δ, q_{ab}, D_a) .

The foregoing defined natural vector field exists and is unique for a certain class of NEH geometries, denoted as *generic* and defined as follows: Let (Δ, q_{ab}, D_a) be an NEH. Choose any null vector field ℓ' such that $\kappa^{(\ell')} = \text{const}$ and any global section $\bar{\sigma}: \hat{\Delta} \rightarrow \bar{\Delta}$. The existence of a natural vector field depends on the invertibility of a certain operator introduced in [11]. It involves the following ingredients defined for $\hat{\Delta}$: the induced metric tensor \hat{q}_{AB} (2.7), the corresponding covariant derivative \hat{D}_A and the Ricci scalar ${}^{(n-2)}\hat{\mathcal{R}}$, the pullback $\hat{\omega}_A$ of the rotation 1-form potential $\omega_a^{(\ell')}$ by the section $\bar{\sigma}$, the trace \hat{T} of the pullback \hat{T}_{AB} of the energy-momentum tensor $T_{\alpha\beta}$, and $T := T_\alpha^\alpha$. The operator is

$$\hat{M} = \hat{D}^A\hat{D}_A + 2\hat{\omega}^A\hat{D}_A + \hat{\omega}_A\hat{\omega}^A + \hat{D}_A\hat{\omega}^A + 4\pi G\left(\hat{T} - T - {}^{(n-2)}\hat{\mathcal{R}}\right). \quad (2.26)$$

If the kernel of this operator is trivial, then there exists exactly one natural vector field ℓ . Suppose, for given data, the operator has a nontrivial kernel. It follows that the operator corresponding to new, gently perturbed data, say, $T'_{\alpha\beta} = T_{\alpha\beta} + \delta\Lambda g_{\alpha\beta}$, has a trivial kernel for a nonzero perturbation $|\delta\Lambda| \in (0, \epsilon)$. That shows the genericity of the existence and uniqueness of the natural vector. It is important to point out that the operator \hat{M} previously defined was constructed with the use of more data than (q_{ab}, D_a) because the objects depending on the choice of a null vector field ℓ' and section of Δ are present on the right-hand side of (2.26). However, the dimension of the kernel of this operator is independent of those choices.

If the kernel is nontrivial, on the other hand, then Δ either admits no natural vector field or admits more than one. The latter happens for example if (Δ, q_{ab}, D_a) has a 2-dimensional group of null symmetries [17].

In conclusion:

Definition 2.4. An NEH (Δ, q_{ab}, D_a) is invariant-generic if it defines the invariant vector field.

For an invariant-generic case, we have defined the following unique structures of (Δ, q_{ab}, D_a) :

- An invariant tangent null vector field ℓ .
- An invariant foliation by good cuts—sections of $\pi: \Delta \rightarrow \hat{\Delta}$ —preserved (locally) by the flow of ℓ .
- A function $v: \Delta \rightarrow \mathbb{R}$, constant on the leaves of the foliation and such that

$$\ell^a D_a v = 1 \quad (2.27)$$

invariant up to $v \mapsto v + v_o$, $v_o \in \mathbb{R}$.

- An invariant covector field

$$n = -dv$$

orthogonal to the leaves of the invariant foliation.

The reason for the name ‘invariant’ is that given two NEHs (Δ, q_{ab}, D_a) and (Δ', q'_{ab}, D'_a) related by an isomorphism $\phi: \Delta \rightarrow \Delta'$, the corresponding invariants are mapped to each other by ϕ_* , ϕ , ϕ^* , and ϕ^* respectively. One has to remember, though, that the slices of Δ may be nonglobal sections of $\Pi: \Delta \rightarrow \hat{\Delta}$.

2.2. Symmetric NEH

2.2.1. Definitions, known results. Given a non-expanding horizon Δ , an *infinitesimal symmetry* of it is a non-trivial vector field $X \in \Gamma(T(\Delta))$ such that

$$\mathcal{L}_X q_{ab} = 0, \quad \text{and} \quad [\mathcal{L}_X, D_a] = 0. \quad (2.28)$$

Each Killing field defined in a spacetime neighborhood of an NEH Δ and tangent to Δ induces an infinitesimal symmetry of Δ . Therefore, reviewing the properties of symmetric NEHs (studied in [17]) is a natural starting point for the current paper. Hereafter we will briefly list those of their properties that are relevant to our studies.

Every infinitesimal symmetry X preserves the null direction; that is, for every null vector field $\ell \in \Gamma(T(\Delta))$,

$$[X, \ell] = f\ell, \quad f: \Delta \rightarrow \mathbb{R}. \quad (2.29)$$

Due to this property, the projection $\pi: \Delta \rightarrow \hat{\Delta}$ pushes X forward to a uniquely defined vector field on $\hat{\Delta}$; that is, there is a vector field $\hat{X} \in \Gamma(T(\hat{\Delta}))$ such that

$$\Pi_* X = \hat{X}. \quad (2.30)$$

The vector field \hat{X} is a Killing vector of the geometry $(\hat{\Delta}, \hat{q}_{ab})$.

Every infinitesimal symmetry X defines a unique analytic extension \bar{X} to the maximal analytic extension $\bar{\Delta}$ of Δ . The vector field \bar{X} defines a global flow for $\bar{\Delta}$, and the flow preserves the geometry (\bar{q}, \bar{D}) [17]. Using this property, in this subsection we consider symmetric maximal analytic extensions of the NEHs.

We distinguish several classes of infinitesimal symmetries. One of them is *null infinitesimal symmetry*, corresponding to X^a being a null vector field.

Corollary 2.5. *Let X be a null infinitesimal symmetry of Δ . Then*

- $d\kappa^{(X)} = 0$.
- If $\kappa^{(X)} \neq 0$, then there is a function $v: \Delta \rightarrow \mathbb{R}$ such that $dv \neq 0$ at every point of Δ and

$$X^a D_a v = \kappa^{(X)} v. \quad (2.31)$$

- If $\kappa^{(X)} = 0$, then there is a function $v: \Delta \rightarrow \mathbb{R}$ such that

$$X^b D_b (X^a D_a v) = 0 \quad (2.32)$$

for Δ and

$$d(X^a D_a v) \neq 0 \quad (2.33)$$

at each point such that $(X^a D_a v) = 0$.

That means that an infinitesimal null symmetry of nonzero surface gravity can vanish only on a two-dimensional slice of Δ , whereas in the case of zero surface gravity it may vanish along a finite set of the geodesics. The last item is proved in [17].

Furthermore, we distinguish the *cyclic* and *helical* infinitesimal symmetries, whose definitions are longer; therefore, we spell them out more carefully.

Definition 2.6. Given an NEH (Δ, q_{ab}, D_a) , a vector field $\Phi^a \in \Gamma(T(\Delta))$ is cyclic infinitesimal symmetry whenever the following holds:

- Φ^a is an infinitesimal symmetry of Δ (satisfies the equation (2.28)).
- The symmetry group of the maximal analytic extension $\bar{\Delta}$ it generates is diffeomorphic to $SO(2)$.
- Φ^a is spacelike at the points where it does not vanish.

Definition 2.7. An infinitesimal symmetry X^a of an NEH (Δ, q_{ab}, D_a) is called helical if

- The symmetry group generated by the projection \hat{X}^A of X^a onto the base space $\hat{\Delta}$ is diffeomorphic to $SO(2)$.
- In the maximal analytic extension $\bar{\Delta}$ there exists an orbit of the symmetry group generated by the extension \bar{X}^a that is not closed (i.e., it is diffeomorphic to a line).

An NEH admitting a helical infinitesimal symmetry will be called helical.

An important property of the latter is that by the local rigidity theorem, it induces into Δ also a null and cyclic symmetry, that is

Theorem 2.8. Suppose the energy–momentum tensor T_{ab} satisfies the condition (2.20) for a non-vanishing null vector field ℓ^a tangent to a non-expanding horizon Δ . If Δ admits a helical infinitesimal symmetry X^a , then it also admits a cyclic infinitesimal symmetry Φ^a and a null infinitesimal symmetry ℓ^a such that

$$X^a = \Phi^a + \ell^a. \quad (2.34)$$

Any cyclic symmetry admits in particular the choice of Δ foliation preserved by it, that is

Corollary 2.9. *Suppose a non-expanding horizon Δ admits a cyclic infinitesimal symmetry Φ^a . Then there exists $\ell \in \Gamma(T(\Delta))$ such that*

$$\kappa^{(\ell)} = \text{const}, \quad [\Phi, \ell] = 0. \quad (2.35)$$

Moreover, for the maximal analytic extension $\bar{\Delta}$ of Δ there exists a diffeomorphism

$$h: \bar{\Delta} \rightarrow \hat{\Delta} \times \mathbb{R} \quad (2.36)$$

such that

$$h_*\Phi = (\hat{\Phi}, 0), \quad h_*\ell = (0, \partial_v) \quad (2.37)$$

where $\hat{\Phi} = \Pi_\Phi$ and v is the coordinate for \mathbb{R} .*

2.2.2. The symmetries and invariants. An element new with respect to the situations studied in [17] is the presence and uniqueness of the invariants of NEHs introduced in section 2.1.4. It leads to the following results:

Theorem 2.10. *Suppose (Δ, q_{ab}, D_a) is an invariant-generic NEH (see definition 2.4) and X is its infinitesimal symmetry. Let ℓ^a and n_a be, respectively, the invariant vector field and the invariant covector field. Then*

- (a) $[X, \ell] = 0$, and $\mathcal{L}_X n_a = 0$.
- (b) *There exists a constant $a \in \mathbb{R}$ and a Killing vector field \hat{X}^A of the metric tensor \hat{q}_{AB} induced into the horizon base space $\hat{\Delta}$ such that*

$$X^a = a\ell^a + \tilde{X}^a \quad (2.38)$$

where \tilde{X}^a is tangent to the leaves of the invariant foliation of Δ and

$$(\Pi_*\tilde{X})^A = \hat{X}^A. \quad (2.39)$$

3. Spacetime neighborhoods of NEHs: induced structures

In the preceding section we defined invariant structures of generic, non-expanding horizons: the invariant vector ℓ^a , the invariant foliation, and the invariant covector n_a . This intricate structure provides a natural extension to the spacetime neighborhood of the horizon, defining in particular the coordinate system analogous to the Bondi system defined near the null scri. This construction was presented in [38] (and subsequently in [39]) in the context of horizons in four-dimensional spacetime and in [37] in a general dimension. Here (section 3.1) we perform the first step of this construction, extending the horizon invariants to the analogous invariant structures of its neighborhood. This extension is used next in section 3.3 to define a natural way of describing the (possibly present) Killing fields in the neighborhood of Δ .

3.1. The Bondi-like extension of a structure of an NEH

The construction of the coordinate system is graphically presented in figure 1. The detailed specification of the construction is as follows. Given a non-expanding horizon Δ , let us fix a null, nowhere-vanishing vector field $\ell^a \in \Gamma(T(\Delta))$ (not necessarily the invariant one), a foliation transversal to ℓ^a and preserved by its flow, and the corresponding covector n_a orthogonal to the foliation and normalized by the condition

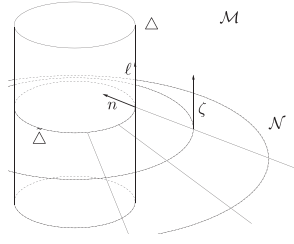


Figure 1. The construction of a Bondi-like coordinate system.

$$\ell^a n_a = -1 \quad (3.1)$$

as in the preceding section. The covector determines uniquely at Δ a null vector field \mathbf{n}^μ tangent to the spacetime \mathcal{M} such that the pullback of \mathbf{n}_μ onto Δ equals n_a . At each point of Δ , the vector \mathbf{n}^μ is transversal. We extend the vector field \mathbf{n}^μ to a null vector field defined in some spacetime neighborhood of the horizon by the parallel transport along the null geodesics tangent to \mathbf{n}^μ at Δ . In other words, we extend the vector field \mathbf{n}^μ from Δ to a vector field defined in a neighborhood of Δ such that

$$\nabla_{\mathbf{n}} \mathbf{n} = 0. \quad (3.2)$$

Assuming the vector field ℓ^a at the horizon is future (past) oriented, the vector \mathbf{n}^μ is also future (past) oriented. Due to the finiteness of the transversal expansion of the vector field \mathbf{n}^μ at the horizon, there exists some region \mathcal{M}' : $\mathcal{M} \supset \mathcal{M}' \supset \Delta$ of the spacetime such that the geodesics generated by \mathbf{n}^μ define the foliation of \mathcal{M}' .⁷ We will denote the maximal (for a given foliation of Δ) set of that property as the *domain of transversal null foliation*. In this region, the field \mathbf{n}^μ is defined uniquely.

Via the flow of this vector field, the vector field ℓ^a defined for the horizon is extended to a vector field ζ^μ defined for \mathcal{M}' such that

$$\mathcal{L}_{\mathbf{n}} \zeta^\mu = 0, \quad \zeta^\mu|_{\Delta} = \ell^\mu. \quad (3.3)$$

We will refer to this field as to the *Bondi-like extension* of the vector field ℓ .

Generically, away from the horizon, the vector field ζ^μ is no longer null. Indeed, the Lie derivative of $\zeta^\mu \zeta_\mu$ along \mathbf{n}^μ is, at the horizon, equal to

$$\mathcal{L}_{\mathbf{n}} \zeta^\mu \zeta_\mu|_{\Delta} = 2\kappa^{(\ell)}. \quad (3.4)$$

If $\kappa^{(\ell)} > 0$ (< 0), the vector ζ^μ becomes timelike near Δ on the side from (into) which the geodesics defined by the vector \mathbf{n}^μ are incoming (outgoing). Hence, it can be treated as a ‘time evolution’ vector co-rotating with the horizon⁸.

Finally, the foliation of the horizon Δ is mapped by the flow of \mathbf{n}^μ into a foliation with $n - 2$ surfaces diffeomorphic to the corresponding slices of Δ . The resulting foliation of the spacetime neighborhood of Δ is preserved by the flow of the vector field ζ^μ as well.

Also, there exists a uniquely defined function r in the neighborhood of Δ such that

$$\mathbf{n}^\mu r_{,\mu} = -1, \quad r|_{\Delta} = 0 \quad (3.5)$$

⁷ The range of a region \mathcal{M}' strongly depends on the choice of the foliation the field \mathbf{n}^μ is orthogonal to.

⁸ Exterior/interior of Δ is undefined at this point. One can define exterior to be that side of Δ at which near to Δ the vector field is timelike.

called ‘radial’ coordinate for \mathcal{M}' . Given a value of r , the flow of \mathbf{n} maps the horizon Δ into a cylinder $\Delta_{(r)}$. The radial coordinate r provides affine parameterization of the null geodesics tangent to \mathbf{n} . Each cylinder $\Delta_{(r)}$ is formed by the integral curves of the vector field ζ .

A parameterization of the integral curves of ζ can be fixed uniquely up to a constant, as a function v determined by its restriction to Δ and by

$$\zeta^\mu v_{,\mu} = 1, \quad \mathbf{n}^\mu v_{,\mu} = 0. \quad (3.6)$$

We assume that v is constant for the leaves of the foliation fixed for Δ . The condition $v = v_0$ defines an $n - 1$ dimensional surface \mathcal{N}_{v_0} in the neighborhood of Δ consisting of the null geodesics tangent to \mathbf{n}^μ .

Remaining $n - 2$ coordinates (x^A) can be defined in the neighborhood of Δ as the extension of any properly defined coordinate system \hat{x}^A given for the base space $\hat{\Delta}$ of the horizon

$$\forall p \in \Delta \quad x^A(p) := \hat{x}^A(\Pi^* p), \quad \zeta^\mu x_{,\mu}^A = \mathbf{n}^\mu x_{,\mu}^A = 0 \quad (3.7)$$

where Π is a projection onto base space defined via (2.7).

The set of n functions (x^A, v, r) previously defined forms a well-defined coordinate system at \mathcal{M}' . The coordinates v and r are defined globally for \mathcal{M}' . On the other hand, the coordinates \hat{x}^A are defined locally for elements of an open covering of $\hat{\Delta}$ next pulled back to Δ and finally extended to \mathcal{M}' . Due to the similarity with the coordinate system defined by Bondi at the null scri, it will be referred to as the *Bondi-like coordinate system* of the horizon spacetime neighborhood.

In these coordinates

$$\zeta^\mu = (\partial_v)^\mu, \quad \mathbf{n}^\mu = -(\partial_r)^\mu, \quad \mathbf{n}_\mu = -(\mathrm{d}v)_\mu. \quad (3.8)$$

3.2. Invariants of NEH neighborhoods

We use the Bondi extension to endow the neighborhood of an invariant-generic NEH Δ with invariant structures. Let the starting point for the construction of the preceding subsection be the invariant vector field ℓ^a of the geometry of Δ , the invariant foliation, and the invariant covector n_a . Then we get the following structures invariantly defined in the neighborhood of Δ : the vector fields ζ^μ and \mathbf{n}^μ , the foliation by the surfaces $\Delta_{(r)}$, and the foliation by the surfaces \mathcal{N}_v .

Definition 3.1. We will call ζ^μ the invariant vector field, \mathbf{n}_μ the invariant covector field, and the foliations the invariant foliations, respectively, of the neighborhood of Δ . We will refer the coordinates (x^A, v, r) as the Bondi invariant coordinates.

3.3. The invariants and the Killing vectors

The Bondi invariant coordinate system introduced in the preceding subsection in the neighborhood of Δ is well suited to identify and characterize the Killing vectors possibly existing in the neighborhood of Δ .

Lemma 3.2. Consider a non-expanding horizon Δ . Assume that it is invariant-generic. Let ζ^μ and \mathbf{n}_μ be the invariant vector field and covector field, respectively, of a neighborhood of Δ . Suppose K^μ is a Killing vector field defined in the spacetime \mathcal{M} and tangent to Δ . Then

- (i) the restriction of K^μ to Δ is an infinitesimal symmetry of the geometry of Δ , and
(ii)

$$[K, \zeta] = [K, \mathbf{n}] = 0. \quad (3.9)$$

The first item is obvious, and the second follows from the fact that there is an isometric flow of K^μ defined in a spacetime neighborhood of any cross-section of $\pi: \Delta \rightarrow \hat{\Delta}$.

The infinitesimal symmetries were characterized in terms of the invariant vector ℓ^a and the invariant foliation in the preceding section. Owing to lemma 3.2, we can characterize the Killing vectors by the invariant vector field ζ^μ and the invariant foliations, that is, by the Bondi invariant coordinates.

Theorem 3.3. *Suppose the assumptions of lemma (3.2) are satisfied. Then*

- (i) *if the restriction of K^μ to Δ is null everywhere for Δ , then there is a constant $a_0 \in \mathbb{R}$ such that*

$$K^\mu = a_0 \zeta^\mu; \quad (3.10)$$

- (ii) *if K^μ is not null at Δ , then it takes the following form in the Bondi-like invariant coordinate system (x^A, r, v) :*

$$K^\mu = a_0 \zeta^\mu + \left[\Phi^A (x^B) \partial_A \right]^\mu \quad (3.11)$$

where

$$k^a = a_0 \ell^a + \left[\Phi^A (\hat{x}^B) \partial_{\hat{x}^A} \right]^a, \quad a_0 \in \mathbb{R} \quad (3.12)$$

is an infinitesimal symmetry of Δ .

Proof. The conclusion follows from the following calculation

$$0 = [n, K] = -[\partial_r, K^\mu \partial_\mu] = -\partial_r(K^\mu) \partial_\mu$$

of direct consequence is

$$K^\mu(x^A, v, r) = K^\mu(x^A, v, 0) = k^\mu(x^A, v).$$

□

To see the strength of this result, let us note that, given the Bondi invariant coordinates (x^A, v, r) and the Killing vectors $\hat{k}_1^A (\hat{x}^B) \partial_{\hat{x}^A}, \dots, \hat{k}_m^A (\hat{x}^B) \partial_{\hat{x}^A}$ of the horizon base space $\hat{\Delta}$ geometry \hat{q}_{AB} , if we want to know whether there is a spacetime Killing vector field tangent to the horizon, all we have to do is check candidates K^μ of the form

$$K = a_0 \partial_v + \left(a^1 \hat{k}_1^A + \dots + a^m \hat{k}_m^A \right) \partial_{x^A} \quad (3.13)$$

for all $a_0, \dots, a_m \in \mathbb{R}$.

3.4. The Killing vectors in non-invariant Bondi coordinates

Even if we are not assuming that an NEH Δ is invariant-generic, a Killing vector field still takes a simple form in suitably chosen Bondi-like coordinates.

Theorem 3.4. *Consider a non-expanding horizon (Δ, q_{ab}, D_a) contained in a spacetime (M, g) ; suppose (M, g) admits a Killing field K^μ tangent to Δ ; suppose there is a nowhere-vanishing vector field $\ell^\mu \in \Gamma(T(\Delta))$ such that the restriction k^a of K^μ to Δ satisfies*

$$[\ell, k] = 0 \quad (3.14)$$

and a foliation of Δ by sections of $\pi: \Delta \rightarrow \hat{\Delta}$ preserved by the flow of k^a . Then, there are coordinates (\hat{x}^A, v) for Δ such that

$$k = a\partial_v + b\Phi^A(x^B)\partial_A, \quad a, b \in \mathbb{R} \quad (3.15)$$

and in the corresponding Bondi-like coordinates (x^A, r, v) , the Killing field K^μ is necessarily of the form:

$$K = a\partial_v + b\Phi^A(x^B)\partial_A \quad a, b = \text{const} . \quad (3.16)$$

4. 4D Einstein–Maxwell neighborhoods of 3d NEHs

In the remainder of the article, we restrict our studies to the spacetime neighborhood of the horizon that is of dimension 4. The focal point of this section is the analysis of the spacetime metric expansion in the radial (the vector field \mathbf{n}^μ) direction first without restrictions on the spacetime matter content and next assuming that the only matter admitted is the Maxwell field. In the latter case, we discuss the characteristic Cauchy problem for Δ .

We start with a short summary of the structure and evolution equations of the Maxwell field for an NEH. Next we provide a short comprehensive discussion of the Cauchy problem for an electro-vacuum NEH.

Although the description of an electrovac neighborhood of an NEH and the initial value problem corresponding to it can be (and are) formulated in the geometric formalism used so far in the article, it is particularly convenient (especially for the task of providing the metric expansion) to perform the analysis using an appropriately chosen Newman–Penrose null frame (see appendix A). Therefore, we introduce such a frame (in subsection 4.3) adapted to the Bondi-like coordinate system defined in section 3, further providing the lexicon between the formerly used geometric quantities and the new objects defined by the frame. This construction is then used to analyze the radial expansion of the spacetime metric and determine the initial data (at the horizon) necessary to uniquely specify the terms of expansion for any matter content and, next, with restriction to electrovac spacetime.

Next, in subsection 4.4 we restrict our interest solely to electrovac horizon neighborhoods. To do so, we reintroduce the lexicon between the two formalisms (geometric and null frame) provided earlier, this time adapting it to the context at hand. Using this lexicon, we discuss the properties of Maxwell field equations in the horizon neighborhood. These properties are next used to reexamine the set of initial data needed to specify the metric expansion previously studied in the general matter case.

The section concludes with a detailed discussion of the characteristic initial value problem for the electrovac case with initial surfaces defined by a Bondi-like coordinate extension

(subsection 4.4.2) and using the null frame introduced in section 4.3. In all the subsections, for the reader's convenience, we present only the results, showing the calculations in appendix B and appendix C.

4.1. Electromagnetic field on an NEH

Suppose the spacetime \mathcal{M} is four-dimensional and an electromagnetic field $F_{\mu\nu}$ is present in a neighborhood of Δ . We consider the Einstein–Maxwell equations in detail in section 4.4 in terms of the Newman–Penrose components. But before that, let us recall the equivalent geometric description of the constraints imposed on the components of $F_{\alpha\beta}$ by the Maxwell vacuum equations [19].

As in the earlier parts of this article, given a spacetime covector W_a , its pullback on Δ is denoted by W_a . Similarly for any covector W_a orthogonal at $p \in \Delta$ to ℓ^a , we denote the corresponding element of $(T_p\Delta/L)^*$ by W_A .

The electromagnetic field energy–momentum tensor $T_{\alpha\beta}$ satisfies stronger energy condition 2.1, hence T_{ab} satisfies (2.20). The tensor T_{ab} contributes to the constraints (2.19), and the (2.20) amounts to the vanishing of some components of $F_{\alpha\beta}$ at Δ , namely

$$\ell^a (F - i\star_{\mathcal{M}}F)_{ab} = 0 \quad (4.1)$$

where $\star_{\mathcal{M}}$ is the spacetime Hodge star.

We assume that $F_{\mu\nu}$ satisfies in a spacetime neighborhood of Δ the Maxwell vacuum equations

$$d(F - i\star_{\mathcal{M}}F) = 0. \quad (4.2)$$

The Maxwell equations constrain further the remaining components of $F_{\mu\nu}$ at Δ . In section 4.3.1, we derive these constraints by using a null frame adapted to Δ . Here we express the result in a frame-independent way.

The first of Maxwell constraints reads

$$\ell^a \star_{\mathcal{M}} d(F - i\star_{\mathcal{M}}F)_a = 0. \quad (4.3)$$

Therefore, the covector $\star_{\mathcal{M}} d(F - i\star_{\mathcal{M}}F)_a$ is orthogonal to ℓ and can be subject to the horizon Hodge dualization \star_{Δ} introduced in section 2.1.1. The second constraint following from the Maxwell equations takes the form of the horizon self-duality condition

$$\star_{\Delta} (\star_{\mathcal{M}} d(F - i\star_{\mathcal{M}}F))_A = i\star_{\mathcal{M}} d(F - i\star_{\mathcal{M}}F)_A. \quad (4.4)$$

4.2. Geometry of the Cauchy problem on Δ

As a null surface, an NEH Δ is not a part of a typical surface for formulating the initial value problem for the Einstein–matter field equations—a Cauchy problem for general relativity. A suitable formulation is known as the characteristic Cauchy problem [40–42]. We apply it to the NEHs in this subsection, using the null frame approach to gravity called the Newman–Penrose framework. Here we express the outline of the results in a geometric, frame-independent way. The details are presented in subsections 4.3 through 4.4 and in appendix C.

Consider an NEH (Δ, q_{ab}, D_a) in a spacetime $(\mathcal{M}, g_{\alpha\beta})$. The Bondi-like coordinate systems (introduced in section 3.1) adapted to Δ form an atlas on the neighbourhood of Δ . Let (x^A, v, r) be one of these coordinate systems.

At any point of Δ , the spacetime metric tensor can be written in the form

$$g_{\alpha\beta} dx^\alpha \otimes dx^\beta = q_{AB} dx^A \otimes dx^B - dv \otimes dr - dr \otimes dv. \quad (4.5)$$

Furthermore, the derivative $\partial_r g_{\alpha\beta}$ for all $\alpha, \beta = 1, 2, 3, 4$ is determined at Δ by the components of the connection D_a (see (4.32)). The higher derivatives $\partial_r^k g_{\alpha\beta}$ are determined by the Einstein-matter field equations and data defined for a 2-dimensional slice $\tilde{\Delta}$ of Δ [40, 41].

4.2.1. The Einstein vacuum case: pure gravitational field. Suppose, in a neighborhood of Δ , the vacuum Einstein equations hold. Then, the horizon geometry q_{ab}, D_a satisfies the constraints (2.19) with the arbitrary null vector field $\ell^a = (\partial_r)^a$ and

$$\mathcal{R}_{ab}^{(4)} = 0. \quad (4.6)$$

To determine the derivatives $(\mathcal{L}_\ell)^k g_{\alpha\beta}$, $k=2, \dots, n$ at Δ it is sufficient to know at a slice of Δ ,

$$\tilde{\Delta}_{v_0} = \{x \in \Delta: v(x) = v_0\} \quad (4.7)$$

(assuming it is a global section of $\Pi: \Delta \rightarrow \hat{\Delta}$), the shear (see 4.14c) λ of (transversal to Δ , orthogonal to the slice, null vector field) $n^\mu = (\partial_r)^\mu$ and all its derivatives:

$$\partial_r^k \lambda, \quad k = 0, \dots, n-2. \quad (4.8)$$

The preceding statement can be demonstrated explicitly ('exactly') by solving the hierarchy of the ordinary differential equations (ODEs); see the following subsections and appendix B.1⁹.

To determine the spacetime metric $g_{\alpha\beta}$ in a four-dimensional region containing Δ , we can use the three-dimensional surface in \mathcal{M} : $\mathcal{N}_{v_0} = \{x \in \mathcal{M}: v(x) = v_0\}$ spanned by the null geodesics tangent to the vector field n^μ and intersecting the slice $\tilde{\Delta}_{v_0}$ of Δ specified in (4.7). Then the following data:

- (1) for Δ : q_{ab}, D_a such that (2.19) with $\mathcal{R}_{ab}^{(4)} = 0$
- (2) for \mathcal{N}_{v_0} : λ with a certain boundary condition specified for $\lambda|_{\Delta \cap \mathcal{N}_{v_0}}$ induced by D_a (see (2.19))

determine the spacetime metric tensor (in terms of diffeomorphisms) in the domain of dependence of $\Delta \cup \tilde{\Delta}_{v_0}$. Moreover, as we vary the vacuum spacetime metric tensor in such a way that Δ is an NEH, and \mathcal{N}_{v_0} satisfies the definition, the preceding data (1–2) provide a range of all the possible NEH geometries for Δ such that (2.19, 4.6) and all possible functions $\lambda: \mathcal{N}_{v_0} \rightarrow \mathbb{C}$ that satisfy the boundary condition at $\Delta \cap \mathcal{N}_{v_0}$.

4.2.2. The Einstein–Maxwell vacuum case: pure gravitational and Maxwell field. Suppose now, in the neighborhood \mathcal{M}' , the vacuum Einstein–Maxwell equations hold. It follows that the horizon geometry (q_{ab}, D_a) and the pullbacks $F_{ab}, \star_{\mathcal{M}} F_{ab}$ satisfy the constraints ((4.1), (4.3), (4.4)). At every point of Δ , all the transversal derivatives $(\mathcal{L}_\ell)^k g_{\alpha\beta}$ as well as $(\mathcal{L}_\ell)^k F_{\alpha\beta}$ can be determined by certain components of the horizon geometry and the electromagnetic field on it (see the following subsections and appendix C.1). More precisely, the initial value problem is again the characteristic Cauchy problem with the initial data null surfaces $\Delta, \mathcal{N}_{v_0}$ specified exactly as in the preceding sub-subsection. Then, to determine the spacetime metric

⁹ Actually, the initial data that the geometry (q_{ab}, D_a) defines for the slice are also sufficient to determine (q_{ab}, D_a) for the entire Δ .

$g_{\alpha\beta}$ and the electromagnetic field $F_{\alpha\beta}$ in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$, it is sufficient to specify the following data:

- (1) for Δ : q_{ab} , D_a , F_{ab} and $\star_{\mathcal{M}} F_{ab}$ such that ((4.1), (4.3), (4.4)) hold
 - (2) for \mathcal{N}_{v_0} : λ with the boundary data $\lambda|_{\Delta \cap \mathcal{N}_{v_0}}$ induced by D_a and the pullbacks $\mathbf{n}^\mu F_{\mu a}$ and $\mathbf{n}^\mu \star_{\mathcal{M}} F_{\mu a}$ satisfying the boundary conditions induced by the Maxwell equations (4.2)
- Moreover, as we vary the vacuum solutions $g_{\mu\nu}$ and $F_{\mu\nu}$ to the Einstein–Maxwell equations such that Δ is an NEH and \mathcal{N}_{v_0} satisfies the definition, the preceding data (1-2) provide a range of: (1) all the NEH geometries and the electromagnetic fields for Δ such that ((4.1), (4.3), (4.4)) and (2) all the possible functions $\lambda: \mathcal{N}_{v_0} \rightarrow \mathbb{C}$ that satisfy the appropriate boundary condition at $\Delta \cap \mathcal{N}_0$ listed in point (1), and 1-forms $\mathbf{n}^\mu F_{\mu a}$ and $\mathbf{n}^\mu \star_{\mathcal{M}} F_{\mu a}$ defined for \mathcal{N}_{v_0} also satisfying the appropriate boundary condition at $\Delta \cap \mathcal{N}_0$ listed in point (2).

Following we present the expansion, specification of the data, and boundary conditions in detail using the Newman–Penrose framework.

4.3. The null frame, the metric expansion

Our starting point is the Bondi-like extension of the structures and coordinates defined in section 3.1 for an NEH Δ : a null nowhere-vanishing vector field ℓ^a tangent to Δ , a function $v: \Delta \rightarrow \mathbb{R}$ such that $\ell^a D_a v = 1$, and coordinates (x^A, v) for Δ such that $\ell^a D_a x^A = 0$. Whereas v is globally defined, the coordinates x^A are defined locally and form an atlas, with the pullback by Π^* of atlas \hat{x}^A defined for the base manifold $\hat{\Delta}$. The function v defines on Δ the covector $n = -dv$, which, in the neighborhood \mathcal{M}' of Δ , defines in particular the null vector field \mathbf{n}^μ . Using this structure, we define a null frame (see appendix A for the basic properties of null frames) $(e_1, e_2, e_3, e_4) = (m, \bar{m}, \mathbf{n}, \ell)$ such that

$$(e_3)^\mu = \mathbf{n}^\mu \text{ in } \mathcal{M}', \quad (e_4)^\mu = \ell^\mu \text{ at } \Delta, \quad (4.9)$$

and e_1, e_2 are tangent to the leaves of the foliation of Δ . Whereas the vector fields e_3 and e_4 are defined for the entire neighborhood \mathcal{M}' , the domains of the vector fields e_1 and e_2 coincide with those of the coordinates (x^A) . By (e^1, e^2, e^3, e^4) we denote the dual coframe. This construction of the frame has been already discussed in [37, 38] and subsequently presented in [39].

The spacetime metric tensor $g_{\mu\nu}$ for \mathcal{M}' and the degenerate metric tensor q_{ab} induced into Δ take in that frame the following form:

$$g_{\mu\nu} = \left(e^1 \otimes e^2 + e^2 \otimes e^1 - e^3 \otimes e^4 - e^4 \otimes e^3 \right)_{\mu\nu}, \quad (4.10a)$$

$$q_{ab} := g_{ab} = \left(e^1 \otimes e^2 + e^2 \otimes e^1 \right)_{ab} \quad (4.10b)$$

where $(\cdot)_{ab}$ stands for the pullback to Δ of a tensor originally defined onto \mathcal{M} .

4.3.1. Geometry and constraints at the horizon, the invariants. Let us now focus on the properties of the frame at Δ itself. For this purpose, through this sub-subsection we will adopt shortened notation, using ‘=’ for ‘= $|_\Delta$ ’. For the Bondi-like coordinates (x^A, v, r) defined in section 3.1, $r = 0$ on Δ , and at Δ the vector field ∂_v is null:

$$\ell^a = (\partial_v)^a \quad (4.11)$$

and has constant surface gravity. The real vectors $\Re(m)^\mu, \Im(m)^\mu$ are (automatically) tangent to Δ . To adapt the frame further, we assume that the vector fields $\Re(m)^a, \Im(m)^a$ tangent to the constancy surfaces $\hat{\Delta}_v$ of the coordinate v (3.6) are Lie dragged by the flow $[\ell]$:

$$\mathcal{L}_\ell m^a = 0. \quad (4.12)$$

The immediate implication is that the projection of m^a onto $\hat{\Delta}$ uniquely defines, for a horizon base space $\hat{\Delta}$, a null vector frame $(\hat{m}, \bar{\hat{m}})$ and the differential operators $\delta, \bar{\delta}$

$$(\Pi_* m)^A =: \hat{m}^A, \quad \delta := \hat{m}^A (x^B) \partial_A \quad (4.13)$$

corresponding to the frame vectors.

The specified frame is adapted to the vector field ℓ^a , the flow of ℓ^a invariant foliation of Δ , and the null complex-valued frame \hat{m}^A defined for the manifold $\hat{\Delta}$. Spacetime frames constructed in this way on Δ will be called *adapted*.

Because all the frame elements are Lie dragged by ℓ^a , the connection D induced into Δ can be decomposed as follows¹⁰:

$$m^\nu D\bar{m}_\nu = \pi^* (\hat{m}^A \hat{D}\bar{\hat{m}}_A) =: \Pi^* \hat{\Gamma} \quad (4.14a)$$

$$-n_\nu D\ell^\nu = \omega^{(\ell)} = \pi e_{(\Delta)}^2 + \bar{\pi} e_{(\Delta)}^1 + \kappa^{(\ell)} e_{(\Delta)}^3 \quad (4.14b)$$

$$-\bar{m}^\nu Dn_\nu = \mu e_{(\Delta)}^1 + \lambda e_{(\Delta)}^2 + \pi e_{(\Delta)}^4 \quad (4.14c)$$

$$m_\mu D\ell^\mu = 0 \quad (4.14d)$$

where $\hat{\Gamma}$ is the Levi-Civita connection 1-form corresponding to the covariant derivative \hat{D} defined by \hat{q} and to the null frame \hat{m}^A defined for $\hat{\Delta}$

$$\hat{\Gamma} =: 2\bar{a}\hat{e}^1 + 2a\hat{e}^2. \quad (4.15)$$

The rotation 1-form potential $\omega^{(\ell)}$ in the chosen frame takes the form

$$\omega^{(\ell)} = \pi e_{(\Delta)}^2 + \bar{\pi} e_{(\Delta)}^1 - \kappa^{(\ell)} e_{(\Delta)}^4. \quad (4.16)$$

In terms of the coordinates (x^A, v) for Δ , the functions a and π satisfy

$$\partial_v a = \partial_v \pi = 0. \quad (4.17)$$

The Ricci tensor is represented by the set of Newman–Penrose coefficients Φ_{ij} , $i, j = 0, 1, 2$ (A.7). In terms of these coefficients, the constraints induced on the horizon geometry (q_{ab}, Da) by the Einstein field equations described in section 2.1.3 are, by the identity (2.19), equivalent to the following set of equations:

$$\begin{aligned} 8\pi G \left(T_{m\bar{m}} - \frac{1}{2} T q_{m\bar{m}} \right) &= -2 \left(\Phi_{11} + 3 {}^{(4)}\mathcal{R} \right) \\ &= 2D\mu + 2\kappa^{(\ell)}\mu - \text{div}\tilde{\omega}^{(\ell)} - \left| \tilde{\omega}^{(\ell)} \right|_{\hat{q}}^2 + {}^{(2)}\mathcal{R}_{m\bar{m}} \end{aligned} \quad (4.18a)$$

$$8\pi G T_{\bar{m}\bar{m}} = -2\Phi_{20} = 2D\lambda + 2\kappa^{(\ell)}\lambda - 2\bar{\delta}\pi - 4a\pi - 2\pi^2 \quad (4.18b)$$

¹⁰ The decomposition is consistent with the definition of connection coefficients presented in appendix A.

where $D := \ell^a \partial_a$, $\delta := m^a \partial_a$, ${}^{(2)}\mathcal{R}_{m\bar{m}} := \left(\Pi^\star {}^{(2)}\mathcal{R} \right)_{ab} m^a \bar{m}^b$, and $(\hat{\text{div}} \hat{\omega}^{(\ell)})$ is the divergence of projected rotation (2.12). As functions of the variables x^A (3.7), by (2.5) and (2.15) the latter two objects equal their counterparts $\left({}^{(2)}\mathcal{R}_{m\bar{m}}, \hat{\text{div}} \hat{\omega}^{(\ell)} \right)$ defined on the horizon base space:

$${}^{(2)}\mathcal{R}_{m\bar{m}} := 2\delta a + 2\bar{\delta}\bar{a} - 8a\bar{a} \quad (4.19a)$$

$$\hat{\text{div}}\hat{\omega}^{(\ell)} = \delta\pi + \bar{\delta}\bar{\pi} - 2a\bar{\pi} - 2\bar{a}\pi. \quad (4.19b)$$

Remark 4.1. In terms of the Newman–Penrose coefficients, Definition 2.3 of the natural vector field ℓ^a of an NEH geometry reads: ℓ^a is tangent to Δ , null, $\kappa^{(\ell)} = 1$ and

$$\ell^a \mu_a = 0. \quad (4.20)$$

The invariant foliation listed in definition 2.4 and the corresponding invariant variable v are defined by the following condition:

$$\delta\pi + \bar{\delta}\bar{\pi} - 2a\bar{\pi} - 2\bar{a}\pi = 0. \quad (4.21)$$

Furthermore, condition (2.21) reads

$$D\Phi_{20} = D\left(\Phi_{11} + 3 {}^{(4)}\mathcal{R}\right) = 0 \quad (4.22)$$

whereas the functions Φ_{11} , Φ_{20} , and ${}^{(4)}\mathcal{R}$ are determined by the electromagnetic field (which we see hereafter).

Some components of the energy momentum and Weyl tensor vanish due to Stronger Energy condition 2.1. Indeed, the following Ricci tensor components (listed in (4.23a)) vanish on Δ due to (2.6), and the Weyl tensor components (listed in (4.23b)) vanish due to the definition of NEH and the Bianchi equalities (see (A.7) for the definition of components):

$$\Phi_{00} = \Phi_{01} = \Phi_{10} = 0 \quad (4.23a)$$

$$\Psi_0 = \Psi_1 = 0. \quad (4.23b)$$

Moreover, the horizon geometry and the matter fields at Δ determine the values at the horizon of the Weyl tensor components Ψ_2 and Ψ_3 (via the NP equations (A.8b) and (A.8c) respectively):

$$\Psi_2 = -\frac{1}{4} {}^{(2)}\mathcal{R} - \frac{1}{2}(\delta\pi - \bar{\delta}\bar{\pi}) - a\bar{\pi} + \bar{a}\pi + \Phi_{11} + \frac{1}{24} {}^{(4)}\mathcal{R} \quad (4.24a)$$

$$\Psi_3 = \bar{\delta}\mu - \delta\lambda + \pi\mu + (4\bar{a} - \bar{\pi})\lambda + \Phi_{12}. \quad (4.24b)$$

The remaining Weyl tensor component, Ψ_4 , is constrained by Bianchi identity (A.11d), which at the horizon reads:

$$\begin{aligned} D\Psi_4 = & -2\kappa^{(\ell)}\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 \\ & - \bar{\mu}\Phi_{20} + (\pi + 2a)\Phi_{21} - 2\lambda\Phi_{11} - \Phi_{20,r}. \end{aligned} \quad (4.25)$$

In this way, Ψ_4 at Δ is uniquely determined by the value of Ψ_4 for the chosen section, the horizon geometry, and the matter fields.

4.3.2. Extension of the spacetime neighborhood. Given the coframe (e^1, \dots, e^4) dual to the frame (m, \bar{m}, n, ℓ) defined previously at Δ , the condition

$$\nabla_n e^\mu = 0 \quad (4.26)$$

defines its unique extension of the spacetime neighborhood \mathcal{M}' . The corresponding connection coefficients $g(e_\gamma, \nabla_\alpha e_\beta)$ are defined in (A.4). In the Bondi-like coordinate system, this adapted (co) frame extended by (4.26) takes the form

$$e_1 = m = \bar{e}_2 = m^A (\partial_A + Z_A \partial_r), \quad e^1 = \bar{e}^2 = \bar{m}_A dx^A + X dv, \quad (4.27a)$$

$$e_3 = n = -\partial_r, \quad e^3 = -dr + Z_A dx^A + H dv, \quad (4.27b)$$

$$e_4 = \ell = \partial_v - \bar{X} e_1 - X e_2 + H \partial_r, \quad e^4 = dv, \quad (4.27c)$$

where Z_A, H are real functions and m^A, X are complex. At the horizon, these functions take the following values:

$$X|_\Delta = H|_\Delta = Z_A|_\Delta = 0 \quad m_A|_\Delta = \Pi^* \hat{m}_A. \quad (4.28)$$

The condition (4.26) (consistent with $\nabla_n n^\mu = 0$ and $n^\mu n_\mu = \text{const}$) imposes on the connection coefficients (defined via (A.4a)) corresponding to it the following constraints that are true for \mathcal{M}' ,

$$\tau = \gamma = \nu = \mu - \bar{\mu} = \pi - (\alpha + \bar{\beta}) = 0. \quad (4.29)$$

In particular, the last constraint allows us to express the coefficients (α, β) in terms of π and

$$a := \frac{1}{2}(\alpha - \bar{\beta}). \quad (4.30)$$

The commutators of the differential operators corresponding to the frame vectors can be expressed in terms of the functions (H, X, m^A, Z_A) and their derivatives. On the other hand, they are determined by the connection coefficients via (A.10). That correspondence leads to the constraints on the frame coefficients that determine their evolution of the functions (X, H, m_A, Z_A) along the transversal to Δ null geodesics:

$$-\partial_r X = \bar{\pi} + \mu X + \bar{\lambda} \bar{X} \quad (4.31a)$$

$$\partial_r H = (\epsilon + \bar{\epsilon}) + \pi X + \bar{\pi} \bar{X} \quad (4.31b)$$

$$\partial_r m_A = \bar{\lambda} \bar{m}_A + \mu m_A \quad (4.31c)$$

$$\partial_r Z_A = \pi m_A + \bar{\pi} \bar{m}_A. \quad (4.31d)$$

This set is supplemented by analogous evolution equations for the spin (connection) coefficients (C.2c)–(C.2f), (C.3a), (C.3b), (C.4) and Weyl tensor components (C.2h), (C.3d), (C.6), (C.5).

The global structure of the resulting frame is as follows: The neighborhood \mathcal{M}' of a given NEH Δ is covered by open sets $\mathcal{U}_I, I = 1, \dots, K$ obtained from a covering $\hat{\mathcal{U}}_I, I = 1, \dots, K$ of the base $\hat{\Delta}$. Each open set \mathcal{U}_I is the union of the null geodesics tangent to n^μ or to ℓ^μ , and intersecting the set $\hat{\mathcal{U}}_I$, for every $I = 1, \dots, K$.

4.3.3. Metric expansion at the horizon. Because in this sub-subsection we consider objects on Δ only, we again adopt the notation $' = ' \equiv ' = |_\Delta'$.

It is a straightforward observation that the horizon geometry (q_{ab}, D_a) already determines the frame components at Δ (through (4.28)) as well as their 1-order radial derivative ∂_r (via

(4.31)):

$$X_{,r} = -\bar{\pi} \quad (4.32a)$$

$$H_{,r} = \kappa^{(\ell)} \quad (4.32b)$$

$$Z_{A,r} = \pi m_A + \bar{\pi} \bar{m}_A \quad (4.32c)$$

$$m_{A,r} = \bar{\lambda} \bar{m}_A + \mu m_A. \quad (4.32d)$$

The second order of the frame expansion, following directly from (A.8j), (A.8n), (A.8m), (A.8l), is

$$X_{,rr} = -\bar{\Psi}_3 - \Phi_{12} \quad (4.33a)$$

$$H_{,rr} = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (4.33b)$$

$$Z_{A,rr} = (\Psi_3 + \Phi_{21})m_A + (\bar{\Psi}_3 + \Phi_{12})\bar{m}_A \quad (4.33c)$$

$$m_{A,rr} = -\Phi_{22}m_A - \bar{\Psi}_4\bar{m}_A. \quad (4.33d)$$

Note that the derivatives $H_{,rr}$, $X_{,rr}$, $Z_{A,rr}$ for Δ are determined directly by (q_{ab}, D_a) and the Ricci tensor (see (4.24a), (4.24b), (4.25)). The last derivative, $m_{A,rr}$, involves a solution Ψ_4 to the equation (4.25) uniquely determined by the initial value of Ψ_4 for the chosen section and the horizon geometry.

To summarize, by direct inspection of the system of equations used here, we see that the data which are not determined and thus must be specified, consist of the following components:

- (i) Φ_{21} , Φ_{22} , \mathcal{R} , $\Phi_{20,r}$ given for the entire Δ , and
- (ii) Ψ_4 given for an initial slice $\tilde{\Delta}$.

Remark 4.2. As is pointed out at the end of section 4.3.2, the elements e_1 and e_2 of the frame are defined locally for the sets $\Pi^{-1}(\hat{U}_I)$, $I = 1, \dots, K$ covering the horizon Δ . For each intersection between two sets, say, $\Pi^{-1}\mathcal{U}_I$ and $\Pi^{-1}\mathcal{U}_J$, there is an obvious transformation law,

$$e_1^{(I)} = u \left(x^A \right)^{(IJ)} e_1^{(J)} \quad (4.34)$$

where $u \left(x^A \right)^{(IJ)} \in \text{U}(1)$. On the other hand, e_3 and e_4 are defined globally at every point of Δ . Now the functions Φ_{21} , Φ_{22} , and Φ_{20} , as components of a tensor, are also defined locally for each set $\Pi^{-1}(\hat{U}_I)$ and satisfy the corresponding transformation laws for the intersections. On the other hand, the Weyl tensor component Ψ_4 is not sensitive to the frame transformations preserving e_3 and e_4 , and ${}^{(4)}\mathcal{R}$ is just a scalar. The derivative $\partial_r = e_4^\mu \partial_\mu$, hence it is defined globally and commutes with the transformations.

Finally, we address the question, what data are required to determine the g derivatives $\partial_r^n(e^\mu)_\nu$? It turns out that the general case is described by the following:

Corollary 4.3. *Given an NEH Δ in a four-dimensional spacetime satisfying the Einstein field equations with a general kind of matter, the Bondi-like coordinates (x^A, v, r) defined in*

section 3.1, and a null frame (e_1, e_2, e_3, e_4) defined in sections 4.3.1 and 4.3.2, the following data

- (i) the value of the constant $\kappa^{(\ell)}$
- (ii) for the initial slice $\tilde{\Delta}: m^a$ (which is tangent to the slice by construction), π , μ , λ , and $\partial_r^k \Psi_4 \forall k \in \{0, \dots, n-2\}$
- (iii) for Δ : $\partial_r^k \Phi_{11}, \partial_r^k \Phi_{21}, \partial_r^k \Phi_{22}, \partial_r^k \mathcal{R}, \partial_r^{k+1} \Phi_{20} \forall k \in \{0, \dots, n-2\}$

determine uniquely all the radial derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ (at Δ) of the frame components up to the order $k = n$. The data are free, that is, not subject to any extra constraints, in accordance with remark 4.2. Also, in the current work we make the additional assumption (2.21), which in terms of the Newman–Penrose coefficients reads: $\partial_v \Phi_{02} = \partial_v \Phi_{20} = \partial_v (\Phi_{11} + 3 \mathcal{R}) = 0$.

For the detailed proof of the preceding corollary, the reader is referred to appendix B. At this point, one must remember, though, that corollary 4.3 is not an existence or a uniqueness statement. For that, the data for Δ must be completed by suitable data defined for another null surface. Also, the Einstein equations for $g_{\mu\nu}$ have to be completed by equations satisfied by the matter that contributes to the energy–momentum tensor (see section 4.2).

The preceding expansion was discussed in [37] and subsequently presented up to a second order (also specifically in the Einstein–Maxwell case) in [39].

4.4. Four-dimensional electrovac NEH

Let us now restrict our studies to the case, where \mathcal{M}' admits an electromagnetic field as the sole matter content. The geometry of a non-expanding horizon in that case was analyzed already in [7]. Here we extend these studies by analysis of the properties of an electrovac NEH's spacetime neighborhood. First, in section 4.4.1, we introduce the necessary geometric objects used for the description of the Maxwell fields, discuss their properties, and describe how the Maxwell evolution equations influence the set of data necessary to determine the metric expansion at the horizon. Next, in section 4.4.2, we discuss the characteristic initial value problem for the system under consideration in context of Bondi-like coordinate system introduced in section 3.

Given an NEH Δ of a geometry (q_{ab}, D_a) , we use throughout this subsection the following objects:

- The Bondi-like coordinates (x^A, v, r) adapted to Δ and such that $\ell^a = (\partial_v)^a$ at Δ is a null vector of a constant surface gravity $\kappa^{(\ell)}$.
- The null tangent frame $(e_1, \dots, e_4) = (m^\mu, \bar{m}^\mu, n^\mu, \ell^\mu)$ of the form (4.27) and the dual coframe (e^1, \dots, e^4) .

4.4.1. Constraints and metric expansion. Given a null frame as previously specified, the electromagnetic field can be represented by the field coefficients defined in the following (equivalent to (A.12)) way¹¹:

$$F := \frac{1}{2} F_{\mu\nu} e^\mu \wedge e^\nu = -\Phi_0 e^4 \wedge e^1 + \Phi_1 (e^4 \wedge e^3 + e^2 \wedge e^1) - \Phi_2 e^3 \wedge e^2 + c. c. \quad (4.35)$$

¹¹ The decomposition is valid for a general Newman–Penrose null frame.

The components of the energy–momentum tensor corresponding to the field are simply products of the respective field coefficients (A.14) via (A.7). The immediate implication of their structure is that $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$. Because, from the Raychaudhuri equation it follows that Φ_{00} vanishes on Δ , so does

$$\Phi_0|_\Delta = 0 \quad (4.36)$$

and so do all the components of $T_{\mu\nu}$ containing Φ_0 :

$$\Phi_{01}|_\Delta = \Phi_{02}|_\Delta = \Phi_{10}|_\Delta = \Phi_{20}|_\Delta = 0. \quad (4.37)$$

As a consequence, stronger energy condition 2.1 holds for this kind of matter and (2.20) is satisfied at Δ automatically.

The component Φ_1 is encoded into the pullback onto Δ of $F_{a\beta} - i^*_{\mathcal{M}}F_{a\beta}$:

$$F_{ab} - i^*_{\mathcal{M}}F_{ab} = \Phi_1 (e^2 \wedge e^1)_{ab}. \quad (4.38)$$

The electromagnetic field $F_{\mu\nu}$ is subject to the Maxwell equations, which in the null frame can be written in the form (A.13). For Δ , these equations reduce to

$$\Phi_0|_\Delta = 0, \quad D\Phi_1|_\Delta = 0, \quad (4.39a)$$

$$D\Phi_2|_\Delta = -\kappa^{(\ell)}\Phi_2 + (\bar{\delta} + 2\pi)\Phi_1. \quad (4.39b)$$

The values of Φ_1, Φ_2 given for the chosen initial slice $\tilde{\Delta}$ are then sufficient to determine the field $F_{a\beta}$ at Δ (provided all the necessary frame and connection components are given). Also, the pullback F_{ab} of F to Δ is determined only by Φ_1 , which furthermore can be represented as a pull-back $\Phi_1 = \Pi^*\hat{\Phi}_1$ of the scalar $\hat{\Phi}_1$ defined for $\hat{\Delta}$.

The contribution of the Maxwell field to the frame expansion derived in section 4.3.3 can be summarized as follows: because the Ricci tensor components (and thus the Maxwell field tensor) do not contribute to the zeroth and first orders of expansion, the set of data required to determine the expansions will be modified only for $n \geq 2$. In this case, the modification can be summarized as:

Corollary 4.4. *Suppose Δ is a non-expanding horizon embedded in four-dimensional electrovac spacetime. Let (x^A, v, r) be the Bondi-like coordinates defined in section 3.1, (e_1, e_2, e_3, e_4) a null frame defined in 4.3.1 and 4.3.2, and $\Phi_I, I = 0, 1, 2$ the electromagnetic field coefficients previously defined. Then the value of the constant $\kappa^{(\ell)}$ and the following data defined for the initial slice $\tilde{\Delta}$*

- horizon geometry: $m^\alpha, \pi, \mu, \lambda$
- electromagnetic field: Φ_1 , and $\partial_r^k \Phi_2, \forall k \in \{0, \dots, n-2\}$
- the Weyl tensor component: $\partial_r^k \Psi_4 \forall k \in \{0, \dots, n-2\}$

determine for Δ uniquely all the radial derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ of the frame components up to the order $k = n$. These data are free; they are not subject to any constraints, in accordance with remark 4.2.

The proof of this corollary, being a modification to the proof of corollary 4.3, is presented in appendix C. As in the case of corollary 4.3, the existence or uniqueness is not guaranteed (see the discussion following corollary 4.3).

Corollary 4.4 and the form that the Maxwell–Einstein equations take for an NEH imply a rather interesting property of the spacetime metric at the horizon. There is a well-defined limit

in which a given spacetime metric tensor $g_{\mu\nu}$ defines perturbatively—in terms of the expansion at an NEH Δ —a new ‘would be’ (that is, provided it exists) stationary solution to the Einstein questions in *all* the orders in the transversal variable r . Indeed, we can easily determine the dependence on v of all the data listed in corollary 4.4. In particular, the frame and rotation components are (by definition) v independent, whereas μ, λ are (due to the reduction to the horizon of (A.8i), (A.8h)) exponential (when $\kappa^{(\ell)} \neq 0$) or linear (otherwise) in v respectively. Acting with ∂_r^n on the transversal evolution equations (A.8k)–(A.8r) (expressed in more convenient form as (C.2a)–(C.4), one can show that the n th transversal derivative of the metric (represented by the respective derivative of the frame components) behaves like

$$\partial_r^n g_{\alpha\beta} \Big|_{\Delta} \sim g_{\alpha\beta}^{(n)} e^{-n\kappa^{(\ell)}v} + \dots + g_{\alpha\beta}^{(0)} \quad (4.40)$$

[38] if $\kappa^{(\ell)} \neq 0$. Because the Bondi-like variable can always be chosen in such a way that $\kappa^{(\ell)} \neq 0$, this result can always be interpreted in the way that the horizon neighborhood geometry settles down to the geometry representing a Killing horizon with ∂_v as a Killing vector. Note, however, that the result does not mean that the horizon neighborhood approaches the symmetric spacetime because (i) we do not know whether there is a metric tensor, solution to the Einstein equations, that satisfies the limit expansion and (ii) solutions to the characteristic initial Cauchy problem do not need to be analytic.

4.4.2. Characteristic Cauchy problem. Now we can complete the data of corollary 4.4 for characteristic Cauchy data. This will be the null frame version of the Cauchy data introduced in a geometric manner in section 4.2. Here we provide a formulation in terms of the null tangent frame $(m^\mu, \bar{m}^\mu, \mathbf{n}^\mu, \ell^\mu)$, the corresponding Newman–Penrose coefficients of the connection, curvature, and the electromagnetic field. We apply the results of sections 3.1 and 4.4.1 to specify the class of the reduced Friedrich data [40–42] corresponding to the case at hand and the NEHs in question.

As in section 4.2, in addition to a given NEH Δ , we use another null surface \mathcal{N}_{v_0} orthogonal to a slice $\tilde{\Delta}$ of Δ such that $v|_{\tilde{\Delta}} = v_0$. Now, for the Bondi-like coordinates (x^A, v, r) , the null surfaces Δ and \mathcal{N}_{v_0} satisfy:

$$r|_{\Delta} = 0, \quad v|_{\mathcal{N}_{v_0}} = v_0. \quad (4.41)$$

The NEH horizon geometry and the component Φ_1 of the electromagnetic field defined for Δ , coupled with the component Ψ_4 of the Weyl tensor and with Φ_2 set freely for the entire \mathcal{N}_{v_0} , provide at the slice $\tilde{\Delta}$ the data of Corollary 4.4. Furthermore, they determine uniquely all the (spacetime) frame, connection, Maxwell field, and Riemann tensor components at \mathcal{N}_{v_0} . The key idea of the proof is the observation that the Einstein–Maxwell equations and Bianchi identities form for \mathcal{N}_{v_0} a hierarchy of the ordinary differential equations. For reading convenience the proof of this fact is presented in appendix C.2. The consequence of the foregoing observations is:

Corollary 4.5. *Given a non-expanding horizon Δ and the transversal null surface \mathcal{N}_{v_0} , the following data are the Friedrich reduced data [43, 44]:*

- (i) *the surface gravity $\kappa^{(\ell)} \in \mathbb{R}$ ($=0$ or $=1$);*
- (ii) *for $\tilde{\Delta}_{v_0} = \Delta \cap \mathcal{N}_{v_0}$: m^α (by construction, tangent to $\tilde{\Delta}$), $\pi, \mu, \lambda, \Phi_1$;*
- (iii) *for \mathcal{N}_{v_0} : Φ_2, Ψ_4 .*

The data are freely defined in accordance with remark 4.2. Given these data, in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$, the NP equations (A.8d)–(A.8i), (A.8j)–(A.8r) coupled with the Einstein–Maxwell field equations (A.7), (A.12), Maxwell evolution equations (A.13), Bianchi identities (A.11a)–(A.11h), frame component evolution equations (4.31), and, with the gauge choice equations ((4.9), (4.11), (4.15), (4.17), (4.20), (4.27), (4.28), (4.29), (4.30)) define a unique electrovac spacetime $(e_1, \dots, e_4, \Phi_0\text{--}\Phi_2, \Psi_0\text{--}\Psi_4, X, H, Z_A, m_A)$. In the spacetime defined in \mathcal{M}^\pm (the future/past to Δ part of the domain of dependence) by the resulting solution, Δ is a non-expanding horizon, (x^A, v, r) is an adapted Bondi-like coordinate system, and $(m^\mu, \bar{m}^\mu, \mathbf{n}^\mu, \ell^\mu)$ is a null frame of the properties of the frame (4.9)–(4.25). In particular, the vector field

$$\mathbf{n}^\mu := -(\partial_r)^\mu \quad (4.42)$$

is null, satisfies

$$\nabla_n n = 0 \quad (4.43)$$

and is orthogonal to the slices

$$v = \text{const} \quad (4.44)$$

of the horizon. Also, the vector field

$$\xi^\mu := (\partial_v)^\mu \quad (4.45)$$

satisfies

$$\xi^\mu|_\Delta = \ell^\mu, \quad \mathcal{L}_n \xi = 0. \quad (4.46)$$

Therefore, the current vector fields \mathbf{n}^μ and ξ^μ coincide with the vector fields n^μ and ξ^μ induced in a neighborhood of Δ in section 3.1.

This structure will be applied in the next section to identify and characterize the necessary and sufficient conditions for the existence of a timelike Killing field for the domain \mathcal{M}^\pm .

5. Electrovac Killing horizon

5.1. The induced structures, the Bondi-like coordinates, and the adapted null frame

We now restrict our interest to the situation where an electrovac spacetime \mathcal{M} admits a Killing vector field K^μ tangent to and null at a horizon Δ . We also assume that K^μ is an infinitesimal symmetry of the electromagnetic field, that is,

$$\mathcal{L}_K F_{\mu\nu} = 0. \quad (5.1)$$

On the horizon,

$$(K|_\Delta)^a = \ell^a \quad (5.2)$$

is an infinitesimal symmetry. We also assume that the surface gravity of ℓ^a (necessarily constant) is not zero:

$$\kappa^{(\ell)} \neq 0. \quad (5.3)$$

We will apply the general results of sections 2.2.1, 3.3, and 3.4 as well as employ the adapted null frames introduced in section 4.

If the NEH Δ geometry is invariant-generic (see section 3.3) and ξ^μ is the Δ neighborhood invariant vector field, then due to theorem 3.3 the Killing vector necessarily coincides

with ξ^μ taking into account a rescaling by a constant factor

$$K^\mu = \xi^\mu. \quad (5.4)$$

Otherwise, the results of section 3.4 apply.

In either case, for Δ there are coordinates (x^A, v) such that

$$\ell^a = (\partial_v)^a. \quad (5.5)$$

We assume that they are given and use the corresponding Bondi-like extension and the related Bondi-like coordinates (x^A, v, r) . Then, according to theorem 3.4, the Killing vector K in the neighborhood \mathcal{M}' necessarily is

$$K^\mu = (\partial_v)^\mu. \quad (5.6)$$

It turns out that the null frame (e_1, \dots, e_4) (4.26) adapted to the structures introduced on Δ and the adapted Newman–Penrose framework defined in section 4 are surprisingly compatible with the Killing vector fields:

Lemma 5.1. *Suppose K^μ is a Killing vector field tangent to an NEH (Δ, q_{ab}, D_c) . The components of the null frame (e_1, e_2, e_3, e_4) and the dual coframe (e^1, \dots, e^4) introduced in section 4.3 are Lie dragged by K^μ , that is,*

$$\mathcal{L}_K e^1 = \dots = \mathcal{L}_K e^4 = 0 = \mathcal{L}_K e_1 = \dots = \mathcal{L}_K e_4 \quad (5.7)$$

provided

$$\mathcal{L}_K e^i|_\Delta = \dots = \mathcal{L}_K e_4|_\Delta = 0. \quad (5.8)$$

Indeed, it follows from the following calculation, true in the neighborhood \mathcal{M}' of Δ for every value of $\mu = 1, \dots, 4$ (no abstract index):

$$0 = \mathcal{L}_K (\nabla_{\mathbf{n}} e_\mu) = \nabla_{\mathbf{n}} \mathcal{L}_K e_\mu \quad (5.9)$$

where the first equality follows from $\nabla_{\mathbf{n}} e_\mu = 0$, whereas the second follows from

$$[\mathbf{n}, K] = 0 \quad (5.10)$$

which for a Killing vector K^μ implies that the parallel transport along the integral lines of \mathbf{n}^μ commutes with the flow of K^μ . The second equation in (5.9) combined with the initial condition

$$\mathcal{L}_K e_\mu|_\Delta = 0 \quad (5.11)$$

completes the proof.

Corollary 5.2. *Consider an NEH Δ such that its neighborhood admits a Killing vector field tangent to Δ and null thereon. If Δ is invariant-generic, introduce into Δ coordinates (x^A, v) such that ∂_v is the invariant vector on Δ . Otherwise, assume that K^μ is not zero restricted to any null generator of Δ and introduce into Δ coordinates (x^A, v) such that*

$$K^\mu|_\Delta = (\partial_v)^\mu. \quad (5.12)$$

Extend (x^A, v) to the Bondi-like coordinates in the neighborhood \mathcal{M}' . Introduce the null frame of sections 4.3.1 and 4.3.2. Then, in the entire \mathcal{M}' , the Killing vector K^μ is of the form

$$K^\mu = (\partial_v)^\mu. \quad (5.13)$$

Furthermore, as a consequence of lemma 5.1, all the frame coefficients e_μ^A , e_μ^v , e_μ^r as well as all the Newman–Penrose coefficients of the Levi–Civita connection, the Maxwell field, and the Weyl tensor are constant along the orbits of K^μ ; that is, they are independent of the variable v .

5.2. Necessary conditions: data at Δ , and metric expansion

The expansion in the radial coordinate¹² r at the given NEH Δ of the coefficients of the frame e_1, \dots, e_4 in the general case assuming the vacuum Einstein–Maxwell equations was developed in section 4.4.1. The presence of the Killing field imposes new constraints on the data considered for the horizon Δ that follow from the substitution of the symmetry condition

$$\ell^\alpha \partial_\alpha (n^\beta \partial_\beta)^k f = 0 \quad (5.14)$$

where f is any component of the frame $e_1^\alpha, \dots, e_4^\alpha$ in the Bondi-like coordinates (x^A, v, r) and any component (in that frame) of the connection, the Weyl tensor, and the Maxwell field; and where $k \in \mathbb{N}$. Indeed, the equation (4.18) with 0 substituted for $\partial_v \mu$ and $\partial_v \lambda$ determines the expansion and shear (μ, λ) of n as functionals of the remaining elements of the horizon geometry—the complex vector m^a , the surface gravity $\kappa^{(\ell)}$, and the component π of the rotation 1-form potential—and the component Φ_1 of the electromagnetic field,

$$\mu = \frac{1}{\kappa^{(\ell)}} \left[m^a \partial_a + |\pi|^2 - 2\pi\bar{a} + \Psi_2 \right] \quad (5.15a)$$

$$\lambda = \frac{1}{\kappa^{(\ell)}} \left[\bar{m}^a \partial_a + |\pi|^2 + 2a\pi \right]. \quad (5.15b)$$

Also, the Maxwell field equation (4.39b) upon the assumption $\partial_v \Phi_2 = 0$ determines the value of Φ_2 at Δ for known (Φ_0, Φ_1) and connection coefficients,

$$\Phi_2 = \frac{1}{\kappa^{(\ell)}} \left[\bar{m}^a \partial_a + 2\pi\Phi_1 \right]. \quad (5.16)$$

Hence the value of an entire energy–momentum tensor at Δ is known. Therefore, given $(m, \kappa^{(\ell)}, \pi, \Phi_1)$, one can calculate all the connection coefficients as well as the Weyl tensor components $\Psi_0, \Psi_1, \Psi_2, \Psi_3$ (see the analysis in section 4.4.1). In an analogous manner, the last component Ψ_4 is determined as a functional of $(m, \kappa^{(\ell)}, \pi, \Phi_1)$ via the substitution of 0 for $\partial_v \Psi_4$ in (4.25) and expressing $\Phi_{20,r}$ therein by a suitable functional of $(m, \kappa^{(\ell)}, \pi, \Phi_1)$ following from $\Phi_0 = 0$ and the Maxwell equation (A.13c):

$$\begin{aligned} \Psi_4 = & \frac{1}{2\kappa^{(\ell)}} \left[\bar{m}^a \partial_a \Psi_3 - 3\lambda \Psi_3 + (5\pi + 2a) \Psi_3 - \Phi_{20,r} \right] \\ & + \frac{1}{2\kappa^{(\ell)}} \left[\bar{m}^a \partial_a \Phi_{21} + (\pi + 2a) \Phi_{21} - 2\lambda \Phi_{11} \right]. \end{aligned} \quad (5.17)$$

In this way, the free degrees of freedom (represented by the triple (m^a, π, Φ_1) defined for the horizon Δ) determine then the frame expansion up to the second order. Furthermore, given the frame e_1^μ, \dots, e_4^μ at Δ and the derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ for $k = 1, \dots, n$, the values of $\partial_r^{n-1} \Phi_2$ and $\partial_r^{n-1} \Psi_4$ necessary for the $n+1$ th order of expansion can be derived by differentiating the equations (A.13b) and (A.11d), respectively, in the radial direction (see appendix B.1 and appendix C.1 for details). Finally, the following is true:

¹² The affine parameter of transversal null geodesics.

Corollary 5.3. *Suppose (Δ, ℓ) is a Killing horizon in a four-dimensional electrovac spacetime and the surface gravity $\kappa^{(\ell)} \neq 0$ (5.3). Then the following data defined on the horizon: the complex vector m^a (4.27a), the rotation 1-form potential ω_a (2.12), and the pullback onto Δ of $(F - i^*_{\mathcal{M}}F)_{ab}$ uniquely determine all the derivatives $\partial_r^n e_\nu^\mu$ of the frame (4.27) coefficients as well as all the derivatives $\partial_r^n \Phi_I$, $I = 0, 1, 2$ of the components of the electromagnetic field (A.12) at the horizon Δ , for all $n \in \mathbb{N}$.*

In particular, in the Einstein vacuum case, that is, in the absence of the Maxwell field, if an NEH Δ is a Killing horizon and the Killing vector field is given at the horizon as ℓ , then all the transversal derivatives $(\mathcal{L}_n)^k g_{\mu\nu}$, $k \in \mathbb{N}$, where n is a transversal to the Δ vector field, are determined by the degenerate metric tensor q_{ab} induced in Δ and the rotation 1-form potential ω_a (2.12) provided $\ell^a \omega_a \neq 0$.

From corollary 5.3, it follows immediately that, provided the spacetime metric $g_{\mu\nu}$ and the electromagnetic field tensor $F_{\mu\nu}$ are analytic, they are uniquely determined in the spacetime by the data $(m, \kappa^{(\ell)}, \pi, \Phi_1)$. In a vacuum case, the analyticity is ensured in the region where the Killing vector field (KVF) is timelike [45, 46] and (3.4). This is the case outside (in the direction against \mathbf{n}^μ) (when $\kappa^{(k)} > 0$) or inside (for $\kappa^{(k)} < 0$) the horizon in the connected region¹³. However, we still do not know whether the metric is analytic up to the horizon Δ .

This requirement is satisfied in particular by non-degenerate Killing horizons in the static vacuum spacetime [47]. The listed data representing a non-rotating horizon¹⁴ uniquely determine, then, the metric and Maxwell field of a static spacetime in the connected region in which the Killing field is timelike.

5.3. Necessary conditions: data for the transversal surface \mathcal{N}_{v_0}

In subsection 5.2, we characterized the Δ part of the characteristic Cauchy data of section 4.4.2. Now we turn to the data defined for the null surface \mathcal{N}_{v_0} transversal to the horizon Δ , which consist of the functions (Φ_2, Ψ_4) . The presence of the Killing field inducing the null symmetry at the horizon imposes some constraints on these (otherwise free) data. The specifics of the construction of the Bondi-like coordinates imply that (see section 3.3), given coordinates (x^A, v, r) such that the Killing field (if present) null at the horizon Δ has the form $K|_\Delta = \partial_v$, it takes this form (i.e., $K^\mu = (\partial_v)^\mu$) in the neighborhood \mathcal{M}' covered by the coordinates (see theorem 3.3 (i)). If the horizon is invariant-generic, then the coordinate v is *a priori* given as the invariant coordinate and $(\partial_v)^\mu = \zeta^\mu$ is the invariant vector field of the neighborhood of Δ . Otherwise, the problem is reduced to finding a suitable v for Δ . Therefore, probing for a Killing vector field comes down to the question of whether the field ∂_v is a symmetry of $g_{\mu\nu}$ and $F_{\mu\nu}$ at the horizon neighborhood. We also remember from corollary 5.2 that if ∂_v is a KVF, then it preserves all the frame coefficients. The converse statement is straightforward: the independence of v of all the frame, Maxwell field, connection, and Weyl tensor coefficients is equivalent to the fact that ∂_v is a KVF. The condition

$$\partial_v \Psi_4|_{\mathcal{N}_{v_0}} = 0, \quad \partial_v \Phi_2|_{\mathcal{N}_{v_0}} = 0 \quad (5.18)$$

is then a necessary condition for ∂_v to be the KVF.

Using the Bianchi identity (A.11d) and the Maxwell field equation (A.13b) (and expressing the operator $D := \ell^\mu \partial_\mu$ in the Bondi-like coordinate system via (4.27)), one can rewrite these conditions as the differential constraints involving the derivatives in the directions tangent to \mathcal{N}_{v_0} only:

¹³ We assume here that the edge of this region has a non-empty intersection with the horizon.

¹⁴ The rotation of a static Killing horizon necessarily vanishes.

$$\begin{aligned} (H\partial_r - \bar{X}\delta - X\bar{\delta})\Psi_4 = & -(4\epsilon - \rho)\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 - \kappa_0\bar{\mu}\Phi_2\bar{\Phi}_0 \\ & + \kappa_0((\pi + 2a)\Phi_2\bar{\Phi}_1 - 2\lambda\Phi_1\bar{\Phi}_1 + \partial_r\Phi_2\bar{\Phi}_0 + \bar{\sigma}\Psi_2\bar{\Phi}_2) \end{aligned} \quad (5.19a)$$

$$(H\partial_r - \bar{X}\delta - X\bar{\delta})\Phi_2 = \bar{\delta}\Phi_1 - \lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2. \quad (5.19b)$$

The conditions (5.18) become, then, well defined for the \mathcal{N}_{v_0} constraint (5.19) for the geometry components. Note that this system involves the transversal to Δ derivatives ($\partial_r\Psi_4, \partial_r\Phi_2$). It can then be treated as the completion of the ‘evolution’ equations (C.2a)–(C.6). However, we do not know whether the resulting system of equations has a well-defined Cauchy problem for Δ . The first difficulty is that the r -dependent coefficient H vanishes at Δ . Second, the completed system now constitutes a system of partial differential equations (PDEs) instead of ordinary ones. The structure of this system is not manifested; however, the action of the Killing flow makes it possible to recast it into the system defined for the Cauchy surface where an equivalent system is elliptic in the region where the KVF is timelike [46]. In contrast with the static case [47], the question about the ellipticity of the system at the horizon remains open.

5.4. The necessary and sufficient conditions

In this subsection we formulate the set of necessary and sufficient conditions for the existence of a Killing vector tangent to and null at the horizon in the four-dimensional, electrovacuum case. As previously, we will use the transversal null surface \mathcal{N}_{v_0} .

Theorem 5.4. *Suppose (Δ, q_{ab}, D_a) is an invariant-generic non-expanding horizon contained in four-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ that satisfies the vacuum Einstein–Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let ζ^μ be the invariant vector field of the neighborhood of Δ and (x^A, v, r) be the invariant Bondi-like coordinate system (see section 3.1). Each of the following conditions (i) and (ii) is equivalent to the local existence (in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$ in point (ii)) of a Killing vector K^μ tangent to and null at Δ , and such that $\mathcal{L}_K T_{\mu\nu} = 0$:*

(i)

$$\mathcal{L}_\zeta g_{\mu\nu} = 0, \quad \mathcal{L}_\zeta T_{\mu\nu} = 0. \quad (5.20)$$

(ii) *The Cauchy data defined for Δ and \mathcal{N}_{v_0} satisfy:*

(a) *for Δ : ζ^a is an infinitesimal symmetry of (q_{ab}, D_a) and $\mathcal{L}_\zeta F_{\mu\nu} = 0$*

(b) *for \mathcal{N}_{v_0} : the conditions (5.19) are satisfied, provided (for the necessary condition) the null frame (4.27) is constructed such that $\mathcal{L}_\zeta e_1|_\Delta = \dots = \mathcal{L}_\zeta e_4|_\Delta = 0$.*

Remark 5.5. The preceding condition (ii)(a) is equivalent to (4.12), (4.14), (4.17), (4.23), (4.39a), (5.15a).

The necessary conditions follow from the fact that in the invariant Bondi-like coordinate system the invariant vector ζ of the neighborhood of Δ has the form $\zeta^\mu = (\partial_v)^\mu$, from theorem 3.3 and from the preceding section. To complete the proof of sufficiency, it is enough to show that at $\Delta \cup \mathcal{N}_{v_0}$ the vector field ζ^μ satisfies the Racz conditions [48, 49]:

$$\mathcal{L}_\zeta g_{\mu\nu} = \nabla_\alpha \mathcal{L}_\zeta g_{\mu\nu} = \mathcal{L}_\zeta T_{\mu\nu} = 0 \quad (5.21a)$$

$$\nabla^\mu \nabla_\mu \zeta_\nu + \mathcal{R}_\nu^\mu \zeta_\mu = 0. \quad (5.21b)$$

These conditions are ensured by lemma 5.6 (cond. (5.21)) and lemma 5.7 (cond. (5.21b)) (following) Once they hold, ζ is necessarily a Killing field according to the Racz theorem [48, 49] (which we quote in appendix D.1).

Lemma 5.6. *Suppose (Δ, q_{ab}, D_a) is a non-expanding horizon contained in four-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ that satisfies the vacuum Einstein–Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let (x^A, v, r) be the Bondi-like coordinate system of section 3.1. Suppose the conditions (ii)(a), (b) of theorem 5.4 are satisfied (however, ∂_v is not assumed here to be the invariant vector field). Then the vector field ∂_v satisfies at $\Delta \cup \mathcal{N}_{v_0}$ the following condition for arbitrary $n \in \mathbb{N}$,*

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \mathcal{L}_{\partial_v} g_{\mu\nu} \Big|_\Sigma = 0. \quad (5.22)$$

Lemma 5.7. *Suppose an NEH (Δ, q_{ab}, D_c) and a vector field ∂_v satisfy all the assumptions of lemma 5.6. Then the solution to the initial value problem*

$$\nabla^\mu \nabla_\mu K'_\alpha + \mathcal{R}_\alpha^\mu K'_\mu = 0, \quad K'_\mu \Big|_{\Delta \cup \mathcal{N}} = \zeta_\mu \quad (5.23)$$

agrees at $\Sigma = \Delta \cup \mathcal{N}$ with the vector field ∂_v up to the arbitrary order

$$\forall n \in \mathbb{N} \quad \nabla_{\alpha_1, \dots, \alpha_n}^{(n)} K'_\mu = \nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \zeta_\mu. \quad (5.24)$$

The proofs of the foregoing lemmas 5.6 and 5.7 are presented in the appendix D.2 and D.3 respectively.

In theorem 5.4, we assumed that Δ was invariant-generic. According to that assumption and to theorem 3.3, the only candidate for the Killing vector field was the invariant vector ζ^μ , which, in the invariant Bondi-like coordinates, equals ∂_v . On the other hand, if we relax the invariant-genericity assumption, we still have theorem 3.4. Combined with lemma 5.6 and lemma 5.7, it leads to the following non-invariant version of theorem 5.4:

Theorem 5.8. *Suppose (Δ, q_{ab}, D_a) is an NEH contained in four-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ that satisfies the vacuum Einstein–Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let (x^A, v, r) be the Bondi-like coordinate system (see section 3.1) such that $\kappa^{(\partial_v)} = \text{const} \neq 0$. Each of the following conditions (i) and (ii) is equivalent to the local existence (in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$ in point (ii)) of a Killing vector K^μ tangent to Δ such that $K|_\Delta = \partial_v$ and $\mathcal{L}_K T_{\mu\nu} = 0$:*

(i)

$$\mathcal{L}_{\partial_v} g_{\mu\nu} = 0, \quad \mathcal{L}_{\partial_v} T_{\mu\nu} = 0. \quad (5.25)$$

(ii) *The Cauchy data defined for Δ and \mathcal{N}_{v_0} satisfy:*

(a) *for Δ : ∂_v is an infinitesimal symmetry of (q_{ab}, D_a) and $\mathcal{L}_{\partial_v} F_{\mu\nu}|_\Delta = 0$*

(b) for \mathcal{N}_{v_0} : the (5.19) are satisfied, provided (for the necessary condition) the null frame (4.27) is constructed such that $\mathcal{L}_\zeta e_1|_\Delta = \dots = \mathcal{L}_\zeta e_4|_\Delta = 0$.

6. Axial and helical Killing fields in 4D electrovac spacetime

The Killing fields null at the horizon are not the only possible type of spacetime symmetries. Two additional classes are possible: axial and helical KVFs. In this section, we formulate the set of necessary and sufficient conditions for their existence analogous to theorem 5.4. We still assume that the studied horizons are NEHs embedded in a four-dimensional electrovac spacetime.

6.1. Axial KVF

If the spacetime neighborhood of an NEH (Δ, q_{ab}, D_c) admits a rotational Killing field Φ^μ tangent to Δ , then one can choose at Δ a null vector field ℓ^a such that

$$[\Phi, \ell] = 0, \quad \kappa^{(\ell)} = \text{const} \neq 0 \quad (6.1)$$

and a foliation of Δ by spacelike slices, each preserved by the symmetry generated by $\Phi|_\Delta$ [17]. In the case of an invariant-generic NEH Δ , the invariant vector ℓ^a and the invariant foliation have this property. Otherwise, we will assume that ℓ^a and the foliation are given. In the corresponding Bondi-like coordinates $(x^A, v, r) = (\theta, \phi, v, r)$, such that (θ, ϕ) are the spherical coordinates defined for the spheres $v = \text{const}$, $r = \text{const}$, and such that

$$\Phi^a|_\Delta = (\partial_\phi)^a \quad (6.2)$$

owing to theorem 3.3 in the invariant-generic case and theorem 3.4 otherwise, in the entire domain of the Bondi-like extension, it is true that

$$\Phi^\mu = (\partial_\phi)^\mu. \quad (6.3)$$

The Bondi-like coordinates are determined by the coordinates (θ, ϕ, v) defined for Δ by using only $\Phi|_\Delta$. We also assume that the KVF Φ is a symmetry of the Maxwell field. Due to lemma 5.1, in the adapted null frame (4.27) such that

$$\mathcal{L}_\Phi e^1|_\Delta = \dots = \mathcal{L}_\Phi e^4|_\Delta = \mathcal{L}_\Phi e_1|_\Delta = \dots = \mathcal{L}_\Phi e_4|_\Delta = 0 \quad (6.4)$$

the Cauchy data defined for the surfaces $\Delta \cup \mathcal{N}_{v_0}$ (\mathcal{N}_{v_0} such that $v = v_0$) as specified in corollary 4.5 is invariant with respect to $\Phi = \partial_\phi$.

The opposite statement is obviously true. Suppose the Cauchy data defined for $\Delta \cup \mathcal{N}_{v_0}$ has an infinitesimal axial symmetry ∂_ϕ . Then the vector field Φ^μ defined in the corresponding Bondi-like coordinates as

$$\Phi^\mu = (\partial_\phi)^\mu \quad (6.5)$$

is a Killing vector field.

6.2. Helical KVF

If the spacetime neighborhood of an IH admits a helical Killing vector field X (see theorem 3.4 for the definition), a result analogous to theorem 5.4 and theorem 5.8 can be established. In the current case, however, the structure of spacetime symmetries is much richer. If present, the KVF X induces (see theorem 2.8) at Δ both, null and axial symmetry. Then, if it exists, by

theorem 3.4 we construct for \mathcal{M}' the Bondi-like coordinate system, using as the boundary condition at Δ the assumption, that restriction to $T(\Delta)$ of X is a linear combination of a null ∂_v and axial ∂_φ symmetry generators.

Knowing the expected form of KVF, one can repeat the steps performed in the proof of theorem 5.4, simply inserting as a candidate for a KVF the field $X^\mu := a\zeta^\mu + b(\partial_\varphi)^\mu$ (where a, b are constants) instead of ζ^μ . The frame coefficients must then satisfy an additional condition, namely, that at the horizon they are invariant with respect to the axial symmetry induced at the horizon:

$$\partial_\varphi e^\mu \Big|_\Delta = 0. \quad (6.6)$$

Upon that assumption, the generalization of theorem 5.4 to the case of a helical KVF is almost straightforward. The only step that requires some attention is the proof of (5.22) (part of the proof of lemma 5.6) at Δ because the evolution of higher-order derivatives of $g_{\mu\nu}$ along the orbits of X is not known *a priori*. We know, however, that the components of the Δ internal geometry (i.e., all the elements of the set of (external) geometry data $\bar{\chi}$ except Ψ_4, Φ_2) are preserved by the flow of ∂_v and ∂_φ as well as the flow of X . Therefore, applying the method used in the proof of lemma 5.6, we can show, that additionally all the transversal derivatives (i.e., the derivatives over radial coordinate r) of all the elements of $\bar{\chi}$ are preserved by the flow of $\partial_v, \partial_\varphi$, and X .

Note that if the spacetime metric is analytic at $\Delta \cup \mathcal{N}$, the immediate implication of the invariance of all the transversal metric (and Maxwell field) derivatives with respect to ∂_φ is, that the Bondi-like extensions ζ, ∂_φ of the null and axial symmetry induced at Δ are Killing fields at the horizon neighborhood. This is also true in a higher dimension and for any compact topology of the horizon base space [50]. It is *a priori* not known whether this statement will remain true if we drop the analyticity assumption.

The necessary condition for X to be a KVF:

$$X^\mu \partial_\mu \Psi_4 \Big|_{\mathcal{N}} = X^\mu \partial_\mu \Phi_2 \Big|_{\mathcal{N}} = 0 \quad (6.7)$$

can be expressed in the form similar to (5.19):

$$\begin{aligned} (H\partial_r - \bar{X}\delta - X\bar{\delta} - C\partial_\varphi)\Psi_4 = & -(4\epsilon - \rho)\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 \\ & -\kappa_0(\bar{\mu}\Phi_2\bar{\Phi}_0 - (\pi + 2a)\Phi_2\bar{\Phi}_1) \\ & +\kappa_0(-2\lambda\Phi_1\bar{\Phi}_1 + \partial_r\Phi_2\bar{\Phi}_0 + \bar{\sigma}\Psi_2\bar{\Phi}_2), \end{aligned} \quad (6.8a)$$

$$(H\partial_r - \bar{X}\delta - X\bar{\delta} - C\partial_\varphi)\Phi_2 = \bar{\delta}\Phi_1 - \lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \quad (6.8b)$$

where C is a constant on \mathcal{N} such that $X = c_1(\partial_v + C\partial_\varphi)$ (with c_1 being a constant) at $\Delta \cap \mathcal{N}$. As a consequence, one can formulate necessary and sufficient conditions for the existence of a helical HVF in the neighborhood of the NEH in the form of the analogs of theorems 5.4 and 5.8. The only difference with respect to those theorems is (i) a different differential condition for data at \mathcal{N}_v , namely, (5.19) is replaced with (6.8a) and (ii) an additional condition at Δ : the adapted null frame (4.27) is preserved by the flow of a cyclic symmetry of Δ induced into it owing to theorem 2.8.

7. Conclusions

In the article, we explored the possibility of constructing a well defined, convenient-to-use description of a non-expanding horizon spacetime neighborhood. This goal has been achieved

by a construction of a preferred coordinate system, built with the use of the geometric invariant of horizon geometry: invariant null vector field and the foliation compatible with its flow. For the class of horizons named ‘generic-invariant’ here (and covering all horizon geometries except for special non-generic cases), such a structure is unique, whereas the small class of non-generic horizons may allow for several such structures on one horizon. This structure made it possible to arrive at the following results, which are true for the neighborhood of a horizon of arbitrary dimension and arbitrary compact spatial slice (or equivalent base space) topology.

- (i) The distinguished coordinate system defined at the non-expanding horizon has been extended to the horizon neighborhood analogously to the construction of the Bondi coordinate system near the null scri: the horizon coordinates are transported along the (defined uniquely for the generic-invariant horizon) geodesics generated by the null field transversal to the slices of distinguished foliation of the horizon. These coordinates are then supplemented by the affine parameter along the aforementioned geodesics. Specifically, the coordinates are determined uniquely by the distinguished null flow and foliation of the horizon via (3.2), (3.5), (3.6), (3.7).
- (ii) The specific construction of the foregoing coordinate system proves to be very convenient in the case where the non-expanding horizon is a Killing horizon; that is, there exists at the horizon neighborhood a Killing vector field tangent to the horizon and null at the horizon. The Killing vector field takes in a specified coordinate system a particular very simple form, given by (3.10).

This makes it possible in particular to test immediately whether the non-expanding horizon is a Killing horizon because one needs only to determine whether the fields of the form (3.10) are Killing fields.

- (iii) The quasi-local version of the Hawking rigidity theorem [17] (see also [50]) makes it possible to generalize the foregoing results to the Killing fields, which are not necessarily null at the horizon. Then again, such fields take, in a Bondi-like coordinate system, a specific form given by (3.11).

In the case of non-expanding horizons in four-dimensional spacetime (but still for the arbitrary compact topology of the horizon base space) and arbitrary (up to energy condition 2.1) matter content, the known Newman–Penrose formalism has made it possible to construct on the horizon neighborhood an invariant null frame compatible with the horizon invariant structure and the Bondi-like coordinate system. In the case of a generic-invariant horizon, such a frame is again defined uniquely. An application of this frame to describe the spacetime metric near the horizon has led to the following result.

- (v) The invariant Bondi-like coordinate system made it possible to define in an invariant way a radial expansion of the spacetime metric about the horizon. This and representation of the geometric data in a distinguished null frame made it possible in turn to identify free data needed to determine the expansion of the spacetime metric at the horizon up to desired order. These data are specified by corollary 4.3. The coordinate system does not require knowledge of the evolution equations of the matter fields present at the horizon neighborhood.

However, if one considers specific matter content, the matter field equations may induce additional constraints on the foregoing (otherwise free) data.

The subject matter was subsequently restricted to horizons in four-dimensional spacetime, allowing for the Maxwell field only. In this case, the distinguished null frame introduced before allows for a convenient representation of the Maxwell field equations. This in turn made it possible to improve the result of point (v) as well as make additional points. In particular:

- (vi) The Maxwell field equations coupled with Einstein–Maxwell equations made it possible to reduce the set of data at the horizon required to determine the expansion of the spacetime metric up to the desired order. Such expansion along the entire horizon is now determined by the appropriate data (specified in corollary 4.4) on a single spatial slice of the horizon.
- (vii) Knowing the formulation of the characteristic initial value problem for Einstein–Maxwell field equations (together with Maxwell evolution equations) and knowing the specification of the free data (so-called Friedrich reduced data regarding the boundary surfaces: in our case, the horizon and the null surface transversal to it) in terms of the components of the geometry and matter field in the distinguished null frame made it possible in turn to determine the necessary and sufficient conditions for the existence on the non-expanding horizon neighborhood of the Killing vector field tangent to and null at the horizon. This condition takes the form of a differential constraint involving the (otherwise free) data for the null surface transversal to the horizon. In the general electro-vacuum case, it is the set of constraints (5.19); whereas in the vacuum case, it is a single constraint (5.19a) with Φ_0 , Φ_1 , Φ_2 set to zero.

These constraints make it possible in particular to probe the spacetime geometries determined numerically near the horizon for stationarity. Also, the discovered constraints make it possible to construct in a straightforward way an invariant quantity that vanishes when a non-expanding horizon is a Killing horizon and that constitutes the measure of departure from stationarity otherwise. This quantity may then prove useful in describing spacetime near the black hole horizon in its final stages of evolution (such as the final stages of the black hole merging process).

The developments presented in this paper, although of considerable potential use in and of themselves, are essentially an exploration of possibilities to build and use the description of a black hole neighborhood via natural expansion of the distinguished geometry structures of its horizon. Therefore, they are meant to be mainly a methodology example rather than a complete study. For this reason, only the Maxwell field as matter content was considered, for example, when developing the results listed in the preceding points (vi) and (vii). These results can be easily extended to other types of matter provided the appropriate formulation of the characteristic initial value problem for any such type of matter exists and the free initial data for such a problem are identified. In particular, the theorems and construction of the entire article can be extended in a straightforward way to admit a nonvanishing cosmological constant, thus paving the way for applications describing black holes in asymptotically anti-de Sitter spacetimes.

Similarly, the results presented in point (v) rely heavily on the Newman–Penrose complex tetrad formalism, tailored specifically to four-dimensional spacetimes. One can, however, introduce into an arbitrary dimension a real orthonormal vielbein (see, for example, [13]) playing the same role. Rewriting the Einstein equations for such a vielbein, while more involved than for dimension 4, does not pose any new qualitative challenges. As a consequence, at least the description of radial metric expansion at the horizon can be extended in a systematic way to a higher dimension.

The result of point (i) can be extended in two ways. On the one hand, the invariant geometric structure of the horizon (although a bit modified) is still present on the dynamical horizon. Because it is possible to build the Bondi-like coordinate system on the horizon neighborhood with only this structure and the congruence of the null geodesics transversal to the horizon, it is straightforward to introduce such a description regarding the neighborhood of a dynamical black hole. A similar construction to ours (based on general foliations by marginally trapped tubes) has been proposed in [51], for example.

On the other hand, the specified coordinate system construction does not employ the Einstein field equations outside energy condition 2.1, which in turn can be formulated in terms of purely geometric quantities (instead of the matter stress-energy tensor). As a consequence, this construction can also be extended to modified theories of gravity, although for such an extension certain care is needed to probe how the distinguished geometry structure at the horizon changes.

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Appendix A. The Newman–Penrose formalism

Here we briefly summarize the Newman–Penrose formalism, as specified in [52].

A.1. The NP null frame

The complex null vector frame $(e_1, \dots, e_4) = (m, \bar{m}, n, \ell)$ (where n, ℓ are real vectors and m is complex) of a four-dimensional spacetime makes up the Newman–Penrose null tetrad if the following scalar products

$$g^{\mu\nu} m^\mu \bar{m}^\nu = 1 \quad g^{\mu\nu} n^\mu \ell^\nu = -1 \quad (\text{A.1})$$

are the only nonvanishing products of the frame components. The dual frame corresponding to the tetrad will be denoted as (e^1, \dots, e^4) . In terms of these coframe components, the metric tensor takes the form:

$$g_{\mu\nu} = e_\mu^1 e_\nu^2 + e_\mu^2 e_\nu^1 - e_\mu^3 e_\nu^4 - e_\mu^4 e_\nu^3. \quad (\text{A.2})$$

The torsion-free spacetime connection is determined by the 1-forms defined as follows:

$$\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} \quad de^\alpha + \Gamma_{\beta}^{\alpha} \wedge e^\beta = 0 \quad (\text{A.3})$$

which can be decomposed into the following complex coefficients:

$$-\Gamma_{14} = \sigma e^1 + \rho e^2 + \tau e^3 + \chi e^4 \quad (\text{A.4a})$$

$$-\frac{1}{2}(\Gamma_{12} + \Gamma_{34}) = \beta e^1 + \alpha e^2 + \gamma e^3 + \epsilon e^4 \quad (\text{A.4b})$$

$$\Gamma_{23} = \mu e^1 + \lambda e^2 + \nu e^3 + \pi e^4 \quad (\text{A.4c})$$

called spin coefficients.

The Riemann tensor is given by the equation

$$\frac{1}{2} {}^{(4)}R_{\beta\gamma\delta}^\alpha e^\gamma \wedge e^\delta = d\Gamma_\beta^\alpha + \Gamma_\gamma^\alpha \wedge \Gamma_\beta^\gamma. \quad (\text{A.5})$$

The Ricci and Weyl tensor

$${}^{(4)}\mathcal{R}_{\alpha\beta} := {}^{(4)}R_{\alpha\gamma\beta}^\gamma := {}^{(4)}\mathcal{R}_\gamma^\gamma \quad (\text{A.6a})$$

$$\begin{aligned} {}^{(4)}C_{\alpha\beta\gamma\delta} &:= {}^{(4)}R_{\alpha\beta\gamma\delta} + \frac{1}{6} {}^{(4)}\mathcal{R} (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \\ &\quad - \frac{1}{4} \left(g_{\alpha\gamma} {}^{(4)}\mathcal{R}_{\beta\delta} - g_{\beta\gamma} {}^{(4)}\mathcal{R}_{\alpha\delta} + g_{\beta\delta} {}^{(4)}\mathcal{R}_{\alpha\gamma} - g_{\alpha\delta} {}^{(4)}\mathcal{R}_{\beta\gamma} \right) \end{aligned} \quad (\text{A.6b})$$

is described in the formalism by the complex coefficients defined as:

$$\Psi_0 = - {}^{(4)}C_{4141}\Phi_{00} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{44}\Phi_{12} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{31} \quad (\text{A.7a})$$

$$\Psi_1 = - {}^{(4)}C_{4341}\Phi_{01} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{41}\Phi_{20} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{22} \quad (\text{A.7b})$$

$$\Psi_2 = - {}^{(4)}C_{4123}\Phi_{02} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{11}\Phi_{21} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{32} \quad (\text{A.7c})$$

$$\Psi_3 = - {}^{(4)}C_{4323}\Phi_{10} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{42}\Phi_{22} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{33} \quad (\text{A.7d})$$

$$\Psi_4 = - {}^{(4)}C_{3232}\Phi_{11} = -\frac{1}{4} \left({}^{(4)}\mathcal{R}_{43} + {}^{(4)}\mathcal{R}_{12} \right) \frac{1}{24} {}^{(4)}\mathcal{R} = \frac{1}{12} \left({}^{(4)}\mathcal{R}_{43} - {}^{(4)}\mathcal{R}_{12} \right). \quad (\text{A.7e})$$

The equation (A.5) written in terms of the components (A.4, A.7a) takes the form

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01} \quad (\text{A.8a})$$

$$\begin{aligned} \delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} \\ &\quad + \frac{1}{24} {}^{(4)}\mathcal{R} \end{aligned} \quad (\text{A.8b})$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \quad (\text{A.8c})$$

$$D\rho - \bar{\delta}\chi = (\rho^2 + \sigma\bar{\sigma}) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \quad (\text{A.8d})$$

$$D\sigma - \delta\chi = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0 \quad (\text{A.8e})$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10} \quad (\text{A.8f})$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \quad (\text{A.8g})$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \bar{\beta}) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20} \quad (\text{A.8h})$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2 + \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{A.8i})$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \frac{1}{24} {}^{(4)}\mathcal{R} \quad (\text{A.8j})$$

$$D\tau - \Delta\chi = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \chi(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01} \quad (\text{A.8k})$$

$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21} \quad (\text{A.8l})$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4 \quad (\text{A.8m})$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22} \quad (\text{A.8n})$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \quad (\text{A.8o})$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \chi\bar{\nu} + \Phi_{02} \quad (\text{A.8p})$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{A.8q})$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3 \quad (\text{A.8r})$$

where the differential operators δ , D , Δ correspond to the null vectors:

$$\delta := m^\mu \partial_\mu, \quad \Delta := n^\mu \partial_\mu, \quad D := \ell^\mu \partial_\mu. \quad (\text{A.9})$$

The equalities (A.8) are called the Newman–Penrose equations.

The commutators of the operators (A.9) can be expressed by the spin coefficients

$$[\Delta D - D\Delta] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta \quad (\text{A.10a})$$

$$[\delta D - D\delta] = (\bar{\alpha} + \beta - \bar{\pi})D + \chi\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \quad (\text{A.10b})$$

$$[\delta\Delta - \Delta\delta] = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta \quad (\text{A.10c})$$

$$[\bar{\delta}\delta - \delta\bar{\delta}] = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta. \quad (\text{A.10d})$$

The Bianchi identity written in terms of the coefficients defined by (A.4), (A.7a) consists of the system of complex PDEs:

$$0 = -\bar{\delta}\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\chi\Psi_2 - D\Phi_{01} + \delta\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\chi\Phi_{11} - \bar{\chi}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \quad (\text{A.11a})$$

$$0 = \bar{\delta}\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\chi\Psi_3 + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} + 2\rho\Phi_{11} + \bar{\rho}\Phi_{02} - (\bar{\mu} - 2\gamma - 2\bar{\gamma})\Phi_{00} - 2\tau\Phi_{10} - \frac{1}{12} D {}^{(4)}\mathcal{R} \quad (\text{A.11b})$$

$$0 = -\bar{\delta}\Psi_2 + D\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \chi\Psi_4 - D\Phi_{21} + \delta\Phi_{20} + 2(\bar{\rho} - \epsilon)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \bar{\chi}\Phi_{22} - (2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - \frac{1}{12} \bar{\delta} {}^{(4)}\mathcal{R} \quad (\text{A.11c})$$

$$0 = \bar{\delta}\Psi_3 - D\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 - \Delta\Phi_{20} + \bar{\delta}\Phi_{21} \\ + 2(\alpha - \bar{\tau})\Phi_{21} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} - 2\lambda\Phi_{11} - (\bar{\mu} + 2\gamma - 2\bar{\gamma})\Phi_{20} \quad (\text{A.11d})$$

$$0 = -\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - D\Phi_{02} + \delta\Phi_{01} \\ + 2(\bar{\pi} - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} \quad (\text{A.11e})$$

$$0 = -\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} \\ + 2(\bar{\mu} - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} + (\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + \frac{1}{12}\delta^{(4)}\mathcal{R} \quad (\text{A.11f})$$

$$0 = -\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 - D\Phi_{22} + \delta\Phi_{21} \\ + 2(\bar{\pi} + \beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - \frac{1}{12}\Delta^{(4)}\mathcal{R} \quad (\text{A.11g})$$

$$0 = -\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 + \Delta\Phi_{21} - \bar{\delta}\Phi_{22} \\ + 2(\bar{\mu} + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22} \quad (\text{A.11h})$$

$$0 = -D\Phi_{11} + \delta\Phi_{10} + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - \frac{1}{8}D^{(4)}\mathcal{R} + (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} \\ + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} \\ - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} \quad (\text{A.11i})$$

$$0 = -D\Phi_{12} + \delta\Phi_{11} + \bar{\delta}\Phi_{02} - \Delta\Phi_{01} - \frac{1}{8}\delta^{(4)}\mathcal{R} + (-2\alpha + 2\bar{\beta} + \pi - \bar{\tau})\Phi_{02} \\ + (\bar{\rho} + 2\rho - 2\bar{\epsilon})\Phi_{12} + 2(\bar{\pi} - \tau)\Phi_{11} + (2\gamma - 2\bar{\mu} - \mu)\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{21} \\ - \kappa\Phi_{22} \quad (\text{A.11j})$$

$$0 = -D\Phi_{22} + \delta\Phi_{21} + \bar{\delta}\Phi_{12} - \Delta\Phi_{11} - \frac{1}{8}\Delta^{(4)}\mathcal{R} + (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} \\ + (2\bar{\beta} + 2\pi - \bar{\tau})\Phi_{12} + (2\beta + 2\bar{\pi} - \tau)\Phi_{21} - 2(\mu + \bar{\mu})\Phi_{11} + \nu\Phi_{01} + \bar{\nu}\Phi_{10} \\ - \bar{\lambda}\Phi_{20} - \lambda\Phi_{02}. \quad (\text{A.11k})$$

A.2. The Einstein–Maxwell field equations

Given an electromagnetic field 2-form $F_{\mu\nu}$, we define the following complex coefficients:

$$\Phi_0 := F_{41} \quad \Phi_1 := \frac{1}{2}(F_{43} + F_{21}) \quad \Phi_2 := F_{23} \quad (\text{A.12})$$

which completely determine it.

The Maxwell field equations expressed in terms of the field and spin coefficients take the form:

$$D\Phi_1 - \bar{\delta}\Phi_0 = (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \quad (\text{A.13a})$$

$$D\Phi_2 - \bar{\delta}\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \quad (\text{A.13b})$$

$$\delta\Phi_1 - \Delta\Phi_0 = (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \quad (\text{A.13c})$$

$$\delta\Phi_2 - \Delta\Phi_1 = -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2. \quad (\text{A.13d})$$

In an electrovac spacetime with vanishing cosmological constant, the Ricci tensor is related to the field energy momentum via the Einstein field equations of the following form:

$$\Phi_{\tilde{\alpha}\tilde{\beta}} = 16\pi G\Phi_{\tilde{\alpha}}\tilde{\Phi}_{\tilde{\beta}} \quad \tilde{\alpha}, \tilde{\beta} \in \{0, 1, 2\}. \quad (\text{A.14})$$

Appendix B. Four-dimensional neighborhood of the horizon

In this section, we present the detailed proof of the corollary 4.3 of section 4.3.3.

B.1. Proof of corollary 4.3

The proof is based on an explicit construction of the algorithm that makes it possible to calculate the derivatives $\partial^n e^\mu$ of the frame components m_A, Z_A, H , provided the results for all the lower orders are given. This algorithm thus makes it possible to establish the conclusion via induction.

In general, the higher radial derivatives of the frame components at the horizon can be obtained by differentiation (of appropriate order) over r of the identity (4.25) and transversal parts (i.e., the contraction of the considered equations with n) of (A.3b), (A.5). In particular, the zeroth order is given directly by (4.28), whereas the first radial derivatives of e^μ_ν are determined via (4.32) by (q, D) . To demonstrate the method for $n > 1$, we start with an explicit calculation of the second order before presenting the general derivation of the $n + 1$ th order.

The presented method is applicable to any kind of matter field; however, here we assume that for each order of the expansion the required Ricci tensor components and their radial derivatives are given for Δ . That assumption is true, for example, in the Maxwell and/or scalar and/or dilaton case where the necessary Ricci tensor components are determined via the matter field equations by the respective data defined for the initial slice $\tilde{\Delta}$. The explicit expansion in the Einstein–Maxwell case is provided in section 4.4.1.

B.1.1. The second order. Assume now that the geometry (q, D) and the results of the first order evaluation are at our disposal. Then, acting with ∂_r on (4.31), one can derive $H_{,rr}, X_{,rr}, Z_{A,rr}, m_{A,rr}$ in terms of the first radial derivatives of the connection coefficients, which in turn are given by the equations (A.8j), (A.8n), (A.8m), (A.8l). The resulting formula for the second radial derivatives of the frame coefficients for \mathcal{M}' reads

$$H_{,rr} = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} + (\Psi_3 + \Phi_{21})X + (\bar{\Psi}_3 + \Phi_{12})\bar{X} \quad (\text{B.1a})$$

$$X_{,rr} = -\bar{\Psi}_3 - \Phi_{12} - \Phi_{22}X - \Psi_4\bar{X} \quad (\text{B.1b})$$

$$Z_{A,rr} = (\Psi_3 + \Phi_{21})m_A + (\bar{\Psi}_3 + \Phi_{12})\bar{m}_A \quad (\text{B.1c})$$

$$m_{A,rr} = -\Phi_{22}m_A - \bar{\Psi}_4\bar{m}_A. \quad (\text{B.1d})$$

The values of these derivatives at the horizon are given by replacing the frame coefficients with their values for Δ . In particular:

$$H_{,rr} \big|_{\Delta} = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{B.2a})$$

$$X_{,rr} \big|_{\epsilon} = -\bar{\Psi}_3 - \Phi_{12}. \quad (\text{B.2b})$$

Note that the derivatives $H_{,rr}$, $X_{,rr}$, $Z_{A,rr}$ for Δ are determined directly by (q, D) and the Ricci tensor. The last derivative, $m_{A,rr}$, involves the solution Ψ_4 to the equation

$$D\Psi_4 \big|_{\Delta} = -2\kappa^{(\ell)}\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 - \bar{\mu}\Phi_{20} + 2\alpha\Phi_{21} - 2\lambda\Phi_{11} - \Phi_{20,r} \quad (\text{B.3})$$

(which is the restriction on Δ of equation (4.25)). The value of this solution is uniquely determined by the value of Ψ_4 for the chosen section and the horizon geometry.

To summarize, by direct inspection of the system of equations used here, we see that the data that are not determined and thus must be specified consist of the following components:

- (i) Φ_{21} , Φ_{22} , \mathcal{R} , $\Phi_{20,r}$ given for the entire Δ
- (ii) Ψ_4 given for an initial slice $\tilde{\Delta}$.

B.1.2. The $n + 1$ th order. In this step, we assume that at our disposal are the results of the derivation up to the n th order, that is:

- (i) the components of the frame and their radial derivatives up to the n th order
- (ii) the components of the connection and their radial derivatives up to the $n - 1$ th order,
- (iii) the components of the Ricci tensor, and their derivatives up to the $n - 2$ th order, as well as the following higher derivatives $\partial_r^{n-1}\Phi_{00}$, $\partial_r^{n-1}\Phi_{01}$, $\partial_r^{n-1}\Phi_{20}$, $\partial_r^{n-1}(\Phi_{11} - \frac{1}{8} {}^{(4)}\mathcal{R})$
- (iv) all the components of the Weyl tensor and their derivatives up to the $n - 2$ th order.

The following higher derivatives: $\partial_r^{n-1}\Psi_0$, $\partial_r^{n-1}\Psi_1$, $\partial_r^{n-1}(\Psi_2 + \frac{1}{12} {}^{(4)}\mathcal{R})$, $\partial_r^{n-1}(\Psi_3 - \Phi_{21})$ are then determined by the Bianchi identities (A.11e-h). Furthermore, the $n + 1$ th radial derivatives of the frame components are given by differentiating the equations (B.1), whereas the n th derivatives of the connection coefficients are given by differentiating the equations (A.8k-r). These data are also sufficient to derive the n th derivatives of Ψ_0 , ..., Ψ_3 , Φ_{00} , Φ_{10} , Φ_{20} via the equations obtained by differentiating (A.8a-c), (A.8d-i)¹⁵ sufficiently many times.

The remaining Weyl tensor component $\partial_r^{n-1}\Psi_4$ is given by the equation derived by the differentiation of the Bianchi identity (A.11d). The equation has the following form

$$D \partial_r^{n-1}\Psi_4 = -(n+1)\kappa^{(\ell)}\partial_r^{n-1}\Psi_4 + \mathcal{P}_n \left(e, \Gamma^{(n-1)}, \Psi^{(n-1)}, \partial_r^n\Phi_{20}, \Phi^{(n-1)}, \partial_r^n {}^{(4)}\mathcal{R} \right), \quad (\text{B.4})$$

where e represents the components of the frame, whereas $\Gamma^{(n-1)}$, $\Phi^{(n-1)}$, and $\Psi^{(n-1)}$ stands for the components and their derivatives up to the order of $n - 1$ of, respectively, the connection, the traceless part of the Ricci tensor, and all of the Weyl tensor except for Ψ_4 .

In summary, given the results up to the n th order, we need to specify the following data:

- (i) $\partial_r^{n-1}\Phi_{11}$, $\partial_r^{n-1}\Phi_{21}$, $\partial_r^{n-1}\Phi_{22}$, $\partial_r^{n-1} {}^{(4)}\mathcal{R}$, $\partial_r^{n-1}\Phi_{02}$ given on the entire Δ
- (ii) $\partial_r^{n-1}\Psi_4$ given on the initial slice $\tilde{\Delta}$ to determine the $n + 1$ th order of the expansion.

¹⁵ Provided the rest of the Ricci components appearing in the equations are given.

Appendix C. Four-dimensional electrovac NEH

This appendix contains the derivation of technical results used in section 4.4: the proof of corollary 4.4 and the detailed description of the derivation of Friedrich reduced data at the transversal surface \mathcal{N} used in section 4.4.2.

C.1. Proof of corollary 4.4

The structure of the proof is analogous to the one presented in B.1; that is, we again construct an algorithm of derivation of $n + 1$ th order expansion for all given lower orders and use mathematical induction. The only difference is that part of the previously undetermined data can be now determined by Maxwell field equations. The modifications to the proof of section B.1 are as follows.

- The Ricci tensor components (thus the Maxwell field tensor) do not contribute to the zeroth and first order of expansion, so for these orders we can directly apply the analogous part of the proof in B.1.
- In the second order, the components Φ_{11} , Φ_{12} , Φ_{22} are determined by Φ_1 , Φ_2 given for the distinguished initial slice $\tilde{\Delta}$. The value of $\partial_r \Phi_{02}$ (needed to derive Ψ_4 via (B.3)) at the horizon is determined by (A.13c):

$$\partial_r \Phi_{20}|_{\Delta} = -8\pi G \Phi_2 \bar{\delta} \bar{\Phi}_1. \quad (\text{C.1})$$

- Finally, given the frame and the Maxwell field expanded to an n th order and $\partial_r^{n+1} \Phi_2|_{\tilde{\Delta}}$ (where $\tilde{\Delta}$ is a slice from the previous point), the derivative $\partial_r^{n+1} \Phi_2$ for Δ is the solution to the equation obtained by the action of ∂_r^{n+1} on (A.13b). This completes the set of data needed for calculation of the $n + 1$ th order of expansion.

C.2. Characteristic IVP: data derivation on \mathcal{N}

Let \mathcal{N} be a transversal null surface defined as in section 3 and intersecting a non-expanding horizon Δ at the slice $\tilde{\Delta}$. The part of the Friedrich reduced data in the characteristic initial value problem that corresponds to \mathcal{N} consists of the following components: Newman–Penrose frame and connection coefficients, Weyl tensor components Ψ_1, \dots, Ψ_4 , and Maxwell field tensor components Φ_1, Φ_2 . We show here that as soon as we specify Ψ_4 and Φ_2 for \mathcal{N} , all the remaining data are determined by the data at $\tilde{\Delta}$.

Indeed, provided the coefficients (Ψ_4, Φ_2) are known, the evolution equations (4.31c), (4.31d), the Newman–Penrose equations (combined with the appropriate Einstein field equations (A.8n), (A.8m), (A.8l), (A.8r), (A.8o)), the Maxwell equation (A.13d), and the Bianchi identity (A.11h) form on \mathcal{N} the following ODE system:

$$\partial_r m^A = \bar{\lambda} \bar{m}^A + \mu m^A \quad (\text{C.2a})$$

$$\partial_r (m^A Z_A) = \pi + \mu (m^A Z_A) + \bar{\lambda} (\bar{m}^A Z_A) \quad (\text{C.2b})$$

$$\partial_r \mu = (\mu^2 + \lambda \bar{\lambda}) + \kappa_0 \Phi_2 \bar{\Phi}_2 \quad (\text{C.2c})$$

$$\partial_r \lambda = 2\mu \lambda + \Psi_4 \quad (\text{C.2d})$$

$$\partial_r \pi = \pi \mu + \bar{\pi} \lambda + (\Psi_3 + \kappa_0 \Phi_2 \bar{\Phi}_1) \quad (\text{C.2e})$$

$$\partial_r \varpi = \mu \varpi - \lambda \bar{\varpi} + (\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) \quad (\text{C.2f})$$

$$-\partial_r \Phi_1 = \delta \Phi_2 - 2\mu \Phi_1 + (\bar{\pi} - \bar{\varpi}) \Phi_2 \quad (\text{C.2g})$$

$$\begin{aligned} -\partial_r (\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) &= \delta \Psi_4 - \kappa_0 \bar{\delta} \Phi_2 \bar{\Phi}_2 - 4\mu (\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) \\ &\quad + 2(\bar{\pi} - \bar{\varpi}) \Psi_4 - 2\kappa_0 (\pi \Phi_{22} + \mu \Phi_{21} - \lambda \Phi_{12}) \end{aligned} \quad (\text{C.2h})$$

for the coefficients $(m^A, m^A Z_A, \mu, \lambda, \pi, a, \phi_1, \Psi_3)$.¹⁶ This system has a unique solution for the initial data given for $\tilde{\Delta}$ (which in turn are already determined by the horizon geometry; see the preceding section).

The known solution of (C.2a) can next be applied to the system formed by (A.8q), (A.8p), (A.13c), (A.11g) (where the Bianchi identity (A.11k) was used to determine the value of $D\Phi_{22}$ in (A.11g)):

$$\partial_r \rho = (\rho \mu + \sigma \lambda) + \left(\Psi_2 - \frac{1}{12} \Lambda \right) \quad (\text{C.3a})$$

$$\partial_r \sigma = (\mu \sigma + \bar{\lambda} \rho) + \Phi_{02} \quad (\text{C.3b})$$

$$\partial_r \Phi_0 = \delta \Phi_1 - \mu \Phi_0 + \sigma \Phi_2 \quad (\text{C.3c})$$

$$\begin{aligned} -\partial_r (\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1) &= \delta \Psi_3 - \kappa_0 \bar{\delta} \Phi_1 \bar{\Phi}_2 + (\bar{\pi} - \bar{\varpi}) + \sigma \Psi_4 - 3\mu (\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1) \\ &\quad - \kappa_0 (\rho \Phi_2 \bar{\Phi}_2 + (\pi - \varpi) \Phi_1 \bar{\Phi}_2 - 5\mu \Phi_1 \bar{\Phi}_1 - \lambda \Phi_0 \bar{\Phi}_2) \end{aligned} \quad (\text{C.3d})$$

which is then the system of ODEs for $(\rho, \sigma, \Phi_0, \Psi_2)$. The solution to this system (also unique for the initial data given for $\tilde{\Delta}$) determines, furthermore, the value of ϵ via (A.8j):

$$\partial_r \epsilon = \pi \bar{\pi} + \frac{1}{2} (\bar{\pi} \varpi - \pi \bar{\varpi}) + \left(\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1 + \frac{1}{12} \Lambda \right) \quad (\text{C.4})$$

and determines the pair (X, H) via (4.31a, 4.31b).

This is the last remaining part of the reduced data: Ψ_1 is determined via equation (A.11f):

$$\partial_r \Psi_1 = \delta \Psi_2 - 2\mu \Psi_1 + 2\sigma \Psi_3 + \kappa_0 (\partial_r \bar{\Phi}_0 \Phi_1 - \bar{\delta} \bar{\Phi}_0 \Phi_2) + 2\kappa_0 (\bar{\mu} \bar{\Phi}_0 \Phi_1 - \rho \bar{\Phi}_1 \Phi_2) \quad (\text{C.5})$$

where the radial derivatives of Φ_0, Φ_1 are determined via equations (C.2g), (C.3c).

The Weyl tensor Ψ_0 component is not a part of the Friedrich reduced data; however, we also describe its evolution because it (as well as the evolution of $D\Phi_0$) is needed in section 5. The analysis of it requires a little more effort than the other components because we do not have a direct analog of ‘radial evolution’ equations (C.2a) for this component. We overcome this problem by applying the Bianchi identity (A.11e), which implies that the transversal derivative of Ψ_0 is equal to:

$$\begin{aligned} -\partial_r \Psi_0 - \delta \Psi_1 + \kappa_0 (D\Phi_0 \bar{\Phi}_2 - \delta \Phi_0 \bar{\Phi}_1) &= -\mu \Psi_0 - (\bar{\pi} - \bar{\varpi}) \Psi_1 + 3\sigma \Psi_2 \\ &\quad + \kappa_0 (2(\epsilon - \bar{\epsilon}) + \bar{\rho}) \Phi_0 \bar{\Phi}_2 + \kappa_0 ((\bar{\pi} - 2\bar{a}) \Phi_0 \bar{\Phi}_1) \\ &\quad + 2\sigma \Phi_1 \bar{\Phi}_1 - 2\chi \Phi_1 \bar{\Phi}_2 - \bar{\lambda} \Phi_0 \bar{\Phi}_0, \end{aligned} \quad (\text{C.6})$$

where the component $D\Phi_2$ is determined by the Maxwell equation (A.13b).

¹⁶ Note that the functions $(m^A, m^A Z_A)$ appear in the system as coefficients of $\delta, \bar{\delta}$.

To compute the value of $D\Phi_0$, one can apply the equations that involve the derivatives of the connection components along ℓ . Acting via the operator D on the equation (A.13c) and substituting the values $D\Phi_2$, $D\Phi_1$, $D\mu$, $D\sigma$, $D\pi$ via the equations (A.13b), (A.13a), (A.8i), (A.8e) and the combination of (A.8f), (A.8g) (to extract $D\pi$), one obtains the following equation:

$$-\partial_r D\Phi_0 = -\mu D\Phi_0 + \Phi_2 \Psi_0 + \mathcal{F} \quad (\text{C.7})$$

where \mathcal{F} is a functional of the Friedrich reduced data (frame, connection, field, and Riemann except Ψ_0) and their derivatives along the directions tangent to \mathcal{N} up to the second order. Both the equations (C.6), (C.7) form the system of ODEs that (similar to the systems previously considered) has a unique solution for the initial data given $(\Psi_0, D\Phi_0)|_{\Delta}$.¹⁷

Appendix D. Four-dimensional electrovac Killing horizon

Here we present the original form of the Racz theorem of [48, 49] and the proofs of lemma 5.7 and 5.6 needed to prove that the necessary conditions for the vector field ζ to be a Killing field (null at the horizon) are also sufficient.

D.1. Racz theorem

Consider the field K'_μ defined α the solution to the following initial value problem

$$\nabla^\mu \nabla_\mu K'_\alpha + {}^{(4)}\mathcal{R}_\alpha^\mu K'_\mu = 0 \quad K'_\mu \Big|_{\Delta \cup \mathcal{N}} = K'_{0\mu}. \quad (\text{D.1})$$

Theorem D.1 [48, 49]. *Suppose $(M, g_{\mu\nu})$ is a spacetime equipped with a metric tensor and admitting matter fields represented by the set of tensor fields $\mathcal{T}_{(I)\alpha_1, \dots, \alpha_k}$ satisfying a quasi-linear hyperbolic system*

$$\nabla^\mu \nabla_\mu \mathcal{T}_{(I)} = \mathcal{F}_{(I)}(\mathcal{T}_{(J)}, \nabla_\nu \mathcal{T}_{(J)}, g_{\alpha\beta}) \quad (\text{D.2})$$

such that the energy–momentum tensor is a smooth function of the fields, their covariant derivatives, and the metric; thus

$${}^{(4)}\mathcal{R}_{\mu\nu} = {}^{(4)}\mathcal{R}_{\mu\nu}(\mathcal{T}_{(J)}, \nabla_\nu \mathcal{T}_{(J)}, g_{\alpha\beta}). \quad (\text{D.3})$$

Then, on the domain of dependence of some initial hypersurface Σ (within an appropriate initial value problem) there exists a nontrivial Killing vector field K'^μ (such that $\mathcal{L}_{K'}\mathcal{T}_{(I)} = 0$) if and only if there exists a nontrivial initial data set $K'^\mu|_\Sigma$ for (D.1a) that satisfies

$$0 = \mathcal{L}_{K'} g_{\mu\nu} \Big|_\Sigma = \nabla_\alpha \mathcal{L}_{K'} g_{\mu\nu} \Big|_\Sigma = \mathcal{L}_{K'} \mathcal{T}_{(I)} \Big|_\Sigma. \quad (\text{D.4})$$

D.2. Proof of lemma 5.6

Let us start with the zeroth order first. In the chosen null frame, the symmetric tensor $A_{\mu\nu}$ such that

¹⁷ Because Δ is an NEH, both components vanish on it.

$$A_{\mu\nu} := \zeta_{\mu;\nu} + \zeta_{\nu;\mu} \quad (\text{D.5})$$

can be expressed at $\Delta \cup \mathcal{N}$ directly by the frame coefficients (H, X) and their first derivatives. Due to (4.32), the components A_{33} , A_{34} , A_{31} vanish identically. The remaining components are equal to

$$A_{44} = -DH + (\epsilon + \bar{\epsilon})H + \bar{\kappa}X + \kappa\bar{X} \quad (\text{D.6a})$$

$$A_{14} = DX - \delta H + 2\pi H - \kappa + (\bar{\rho} - \epsilon + \bar{\epsilon})X + \sigma\bar{X} \quad (\text{D.6b})$$

$$A_{11} = \delta X + 2\bar{a}X + \bar{\lambda}H - \bar{\sigma} \quad (\text{D.6c})$$

$$A_{12} = \delta\bar{X} + \bar{\delta}X - 2\bar{a}\bar{X} - 2aX + 2\mu H - (\rho - \bar{\rho}). \quad (\text{D.6d})$$

At the horizon, all the components of $A_{\mu\nu}$ are identically zero, as $H = X = 0$ there. Furthermore, the constraint (5.19b) implies that $D\Phi_2 = 0$ at $\Delta \cap \mathcal{N}$; thus (due to (4.39b)), Φ_2 is Lie-dragged by ζ at the entire horizon. Because the frame components are preserved by the flow of ℓ at Δ , the field $\ell^\mu = \zeta^\mu|_\Delta$ at the horizon is a symmetry of $F_{\mu\nu}$.

An analysis of the behavior of ζ at the transversal surface \mathcal{N} requires a little more effort. To proceed, let us re-express the components of $A_{\mu\nu}$ listed in (D.6a) in terms of the commutators $[\delta, \partial_v]$. Decomposing ζ in the null frame and applying the equations (A.10), one can express the commutator $[\delta, \partial_v]$ in terms of the connection coefficients and the derivatives of (X, H) . On the other hand, the same commutator is determined by the derivatives of the coefficients (m^A, Z_A) with respect to v :

$$\begin{aligned} [\delta, \partial_v] &= \left((\delta\bar{X}) - X\varpi + H\mu - \bar{\rho} - (\epsilon - \bar{\epsilon}) \right) \delta + \left((\delta X) + X\bar{\varpi} + \bar{\lambda}H - \sigma \right) \bar{\delta} \\ &\quad + \left((\delta H) + X(\rho - \bar{\rho}) - \bar{\pi}H + \kappa \right) \Delta \\ &= \left(m^A Z_A \right)_{,v} \partial_r - m_{,v}^A \partial_A. \end{aligned} \quad (\text{D.7})$$

Comparing the foregoing expressions with (D.6a), one realizes that $A_{\mu\nu}$ can be written in terms of the components of $[\delta, \partial_v]$:

$$A_{44} = -\partial_v H + X \overline{[\delta, \partial_v]_\Delta} + \bar{X} [\delta, \partial_v]_\Delta \quad (\text{D.8a})$$

$$A_{14} = -X [\delta, \partial_v]_{\bar{\delta}} - \bar{X} \overline{[\delta, \partial_v]_{\delta}} - \partial_v X - [\delta, \partial_v]_\Delta \quad (\text{D.8b})$$

$$A_{11} = [\delta, \partial_v]_{\bar{\delta}} \quad (\text{D.8c})$$

$$A_{21} = [\delta, \partial_v]_{\delta} + \overline{[\delta, \partial_v]_{\delta}} \quad (\text{D.8d})$$

where

$$[\delta, \partial_v] = [\delta, \partial_v]_{\delta} \delta + [\delta, \partial_v]_{\bar{\delta}} \bar{\delta} + [\delta, \partial_v]_\Delta \Delta. \quad (\text{D.9})$$

Due to the second equality in (D.7), tensor $A_{\mu\nu}$ vanishes at \mathcal{N} if and only if the derivatives over v of the frame coefficients vanish there.

To determine whether the latter holds, let us consider on \mathcal{N} the set $\dot{\chi}_{(I)}$ formed by the derivatives over v of the following data: the frame coefficients, the connection coefficients $\{\mu, \lambda, \pi, a, \rho, \sigma, \epsilon\}$, the Maxwell field, and the Weyl tensor components except Ψ_0 . At the intersection $\Delta \cap \mathcal{N}$, all these data vanish. On the other hand, one can build a system of the ‘evolution equations’ for $\dot{\chi}_{(I)}$, making it possible to evolve the initial values at $\Delta \cap \mathcal{N}$ along the null geodesics spanning \mathcal{N} . Such system can be obtained by action of the operator ∂_v on the system (C.2a)–(C.4), (4.31b). Like the original system (C.2a)–(C.4), (4.31b), it constitutes

a hierarchy of quasilinear ODEs that are polynomial in terms of the data involved. As the operator ∂_v commutes with ∂_r , ∂_A , the resultant system for $\dot{\chi}_{(I)}$ inherits the properties of the original (C.2a)–(C.4), (4.31b). In particular, for known values of the frame, connection, Maxwell field, and Weyl tensor coefficients and known $\hat{\Psi}_4$, $\hat{\Phi}_2$, the new system forms the hierarchy of ODEs analogous to the hierarchy represented by (C.2a)–(C.4), (4.31b). As a consequence, to solve our system one can apply the same algorithm as the one applied to (C.2a)–(C.4), (4.31b). If the geometry data is known at \mathcal{N} , the system describing the evolution of $\dot{\chi}_{(I)}$ has a unique solution for the given $(\hat{\Psi}_4, \hat{\Phi}_2)$. In particular, in the case where $\hat{\Psi}_4 = \hat{\Phi}_2 = 0$ at \mathcal{N} , the equations are linear and homogeneous in $\dot{\chi}_{(I)}$; thus $\dot{\chi}_{(I)} = 0$ is a unique solution. It means that $\dot{H} = \dot{X} = \dot{m}^A = \dot{Z}_A = 0$ and $\dot{\Phi}_0 = \dot{\Phi}_1 = \dot{\Phi}_2 = 0$, which implies vanishing of $A_{\mu\nu}$ and $\mathcal{L}_\zeta F_{\mu\nu}$ at \mathcal{N} .

To show (5.22) for arbitrary n , it is sufficient to show vanishing of the derivatives in directions transversal to Δ and \mathcal{N} respectively. This requirement is equivalent to the following condition:

$$\partial_r^n \mathcal{L}_\zeta g_{\mu\nu} \Big|_\Delta = 0, \quad \partial_v^n \mathcal{L}_\zeta g_{\mu\nu} \Big|_\mathcal{N} = 0. \quad (\text{D.10})$$

Because (5.22) holds at Σ for $n = 0$, the foregoing condition is automatically satisfied at the intersection $\Delta \cap \mathcal{N}$. At Δ , the satisfaction of the condition (D.10) can be shown by direct inspection of the equations analogous to the ones used in the development of the metric expansion in section 4.4.1 (and earlier in section 4.3.3). Indeed, the only coefficients in the set $\bar{\chi}$ of geometry data¹⁸ that are not automatically constant along the horizon null geodesics are Ψ_4 , Φ_2 . Their derivatives over v are constant (exponential)¹⁹ along the null geodesics for $\kappa^{(\ell)} \neq 0$ ($\kappa^{(\ell)} = 0$) respectively due to (B.3, 4.39b) (as all the data on the right-hand side except Ψ_4 , Φ_2 are constant along integral curves of ℓ). Thus, they vanish because (5.19) holds in particular at $\Delta \cap \mathcal{N}$.

Provided all the derivatives $\partial_v \partial_r^k \bar{\chi}_{(I)}$ up to the n th order vanish, the derivatives of the next order of all the components of $\bar{\chi}_{(I)}$ except Ψ_4 , Φ_2 also vanish because they are determined by $\partial_v^k \bar{\chi}_{(I)}$ (where $k \in \{0, \dots, n\}$) via the respective derivatives (namely, $\partial_v \partial_r^n$) of equations (C.2a)–(C.7). The similar derivatives of (B.3), (4.39b) imply then the constancy (exponentiality)²⁰ of $\partial_v \partial_r^{n+1} \Psi_4$, $\partial_v \partial_r^{n+1} \Phi_2$. They satisfy then $\partial_v \partial_r^{n+1} \Psi_4|_\Delta = \partial_v \partial_r^{n+1} \Phi_2|_\Delta = 0$ due to satisfaction of these conditions at $\Delta \cap \mathcal{N}$. Finally, by induction, $\partial_v \partial_r^n \bar{\chi}_{(I)} = 0$ for arbitrary n .

At the horizon, the n th order transversal derivatives of $A_{\mu\nu}$ can be expressed as functionals homogeneous in the elements of $\dot{\chi}$ and its transversal derivatives up to the n th order. It was shown previously that these components vanish at Δ . Therefore,

$$\partial_v \partial_r^n A_{\mu\nu} \Big|_\Delta = 0. \quad (\text{D.11})$$

The condition (D.10) holds then at Δ , as it is satisfied at $\Delta \cap \mathcal{N}$.

To obtain an analogous result at \mathcal{N} , we can apply the method used previously to verify the condition $A_{\mu\nu}|_\mathcal{N} = 0$. Now, instead of the initial value problem for $\dot{\chi}_{(I)}$, we need to consider IVP for $\partial_v^n \chi_{(I)}$ (where $\chi := \bar{\chi} \setminus \{\Psi_0\}$) analogous to it. This IVP has, however, the same properties as the one for $\dot{\chi}_{(I)}$. This statement completes the proof.

¹⁸ The set consists of all the components of frame, connection, Weyl tensor, and Maxwell field.

¹⁹ I.e., Ψ_4 , Φ_2 are exactly of the form $\Psi_4 = e^{-2\kappa^{(\ell)}v} \hat{\Psi}_4$, $\Phi_2 = e^{-\kappa^{(\ell)}v} \hat{\Phi}_2$, where $\hat{\Psi}_4$, $\hat{\Phi}_2$ are constant along the generators of Δ .

²⁰ In the same strict sense as for Ψ_4 , Φ_2 .

D.3. Proof of lemma 5.7

Equation (5.24) is already satisfied for $n=0$ by (5.23). Suppose now it is satisfied up to the n th order. Lemma 5.6 implies, then, that

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \left(\nabla^\mu \nabla_\mu \zeta_\nu + {}^{(4)}\mathcal{R}_\nu^\mu \zeta_\mu \right) = 0 \quad (\text{D.12})$$

at Σ . As a consequence, due to (D.1) and the inductive assumption, the following holds:

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \nabla^\mu \nabla_\mu (K'_\nu - \zeta_\nu) \Big|_\Sigma = 0. \quad (\text{D.13})$$

Let us consider the preceding equation at Δ first. Its contraction (in the entire α_i) with \mathbf{n}^μ produces the following condition:

$$(\mathcal{L}_\mathbf{n})^n \mathcal{L}_\ell \mathcal{L}_\mathbf{n} (K'_\mu - \zeta_\mu) + (\mathcal{L}_\mathbf{n})^n \mathcal{L}_\mathbf{n} \mathcal{L}_\ell (K'_\mu - \zeta_\mu) = \mathcal{F}(\mathcal{L}_\mathbf{n})^n \mathcal{L}_\mathbf{n} (K'_\mu - \zeta_\mu) \quad (\text{D.14})$$

where we decomposed $g^{\mu\nu}$ in $\nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu$ via (A.2) and applied the inductive assumption to express covariant derivatives as Lie ones. \mathcal{F} is a functional of metric derivatives up to the $n+2$ th order.

Due to the Jacobi identity and $\mathcal{L}_\ell \mathbf{n} = 0$, this formula can be re-expressed in the following way:

$$\mathcal{L}_\ell (\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu) = \frac{1}{2} \mathcal{F}(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu). \quad (\text{D.15})$$

It constitutes a linear homogeneous ODE for $(\mathcal{L}_\mathbf{n})^{n+1} K'_\mu$. As at $\Delta \cap \mathcal{N}$ $(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu) = 0$, its only solution at Δ is

$$(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu) \Big|_\Delta = 0. \quad (\text{D.16})$$

The similar initial value problem can be formulated for $(\mathcal{L}_\ell)^{n+1} (K'_\mu - \zeta_\mu)$ at \mathcal{N} ; thus

$$(\mathcal{L}_\ell)^{n+1} (K'_\mu - \zeta_\mu) \Big|_\mathcal{N} = 0. \quad (\text{D.17})$$

As a consequence, provided (5.24) is satisfied for $k \in \{0, \dots, n\}$, it holds also for $n+1$. Thus, by induction (5.24) holds for arbitrary $n \in \mathbb{N}$.

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