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# Hawking radiation of Dirac particles via tunneling from the Kerr black hole

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**Abstract**

We investigated Dirac particles' Hawking radiation from the event horizon of the Kerr black hole in terms of the tunneling formalism. Applying the WKB approximation to the general covariant Dirac equation in the Kerr spacetime background, we obtain the tunneling probability for fermions and Hawking temperature of the Kerr black hole. The result obtained by taking the fermion tunneling into account is consistent with the previous literature.

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Hawking [1] discovered the thermal radiation of a collapsing black hole using the techniques of quantum field theory in curved spacetime. Since the Hawking radiation relates the theory of general relativity to quantum field theory and statistical thermodynamics, it is generally believed that a deeper understanding of the Hawking radiation may shed some light on seeking the underlying quantum gravity. Since then, several derivations of the Hawking radiation have been proposed. The original method presented by Hawking is direct but complicated to be generalized to other spacetime backgrounds. In recent years, a semi-classical derivation of the Hawking radiation as a tunneling process [2] has been developed and has already attracted a lot of attention. In this method, the imaginary part of the action is calculated using the null geodesic equation. Zhang and Zhao extended this method to the Reissner–Nordström black hole [3] and Kerr–Newman black hole [4]. Angheben *et al* [5] also proposed a derivation of the Hawking radiation by calculating the particles' classical action from the Hamilton–Jacobi equation, which is an extension of the complex path analysis of Padmanabhan *et al* [6]. All of these approaches to tunneling used the fact that the tunneling probability for the classically forbidden trajectory from inside to outside the horizon is given by

$$\Gamma = \exp\left(-\frac{2}{\hbar}\mathrm{Im}\, I\right), \tag{1}$$

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where $I$ is the classical action of the trajectory. The crucial thing in the tunneling formalism is to calculate the imaginary part of the classical action. The difference between these two methods consists in how the classical action is calculated. For a detailed comparison of the Hamilton–Jacobi method and the null-geodesic method, one can refer to [7].

However, most of the authors only considered the scalar particles' radiation. Very recently, a new calculation concerning fermions' radiation from the stationary spherical symmetric black hole was done by Kerner and Mann [8]. This method has been generalized to the Banados–Teitelboim–Zanelli (BTZ) black hole by us in [9] and to the dynamical black hole in [10]. In this paper, we will generalize the tunneling method presented in [8] to calculate the Dirac particles' Hawking radiation from the Kerr black hole. Starting with the general covariant Dirac equation in a curved background, we calculate the tunneling probability and Hawking temperature by using the WKB approximation. The result obtained by taking the fermion tunneling into account is consistent with the previous literature.

The Kerr black hole solution in Boyer–Linquist coordinates is given by

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\,\mathrm{d}t^2 - 2a\sin^2\theta\cdot\frac{r^2+a^2-\Delta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi \\ &+\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\cdot\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2Mr + a^2 = (r-r_+)(r-r_-), \\ r_\pm &= M \pm \sqrt{M^2-a^2}. \end{aligned}$$

We have assumed the non-extremal condition $M > a$, so that $r_+$ and $r_-$ correspond to the outer event horizon and the inner event horizon, respectively. The determinant of the metric is

$$\sqrt{-g} = \Sigma\sin\theta,$$

and the inverse of the metric of the $(t, \phi)$ parts is

$$\begin{aligned} g^{tt} &= -\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma\Delta}, \\ g^{\phi\phi} &= \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}, \\ g^{t\phi} &= -\frac{a(r^2+a^2-\Delta)}{\Sigma\Delta}. \end{aligned}$$

Now we calculate the Dirac particles' Hawking radiation from the outer event horizon of the Kerr black hole via the tunneling formalism. For simplicity, we only consider that the massless spinor field $\Psi$ obeys the general covariant Dirac equation

$$-\mathrm{i}\hbar\gamma^a e_a^\mu \nabla_\mu \Psi = 0, \tag{3}$$

where $\nabla_\mu$ is the spinor-covariant derivative defined by $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{[a}\gamma_{b]}$ and $\omega_\mu^{ab}$ is the spin connection, which can be given in terms of the tetrad $e_a^\mu$. The $\gamma$ matrices are selected as

$$\gamma^0 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$

where the matrices $\sigma^k (k = 1, 2, 3)$ are the Pauli matrices. According to the line element (2), the tetrad fields $e_a^\mu$ can be selected to be

$$
\begin{aligned}
e_0^\mu &= \left(\sqrt{-g^{tt}}, 0, 0, \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\right),\\
e_1^\mu &= \left(0, \sqrt{\frac{\Delta}{\Sigma}}, 0, 0\right),\\
e_2^\mu &= \left(0, 0, \frac{1}{\sqrt{\Sigma}}, 0\right),\\
e_3^\mu &= \left(0, 0, 0, \frac{1}{\sqrt{g_{\phi\phi}}}\right).
\end{aligned}
$$

We employ the ansatz for the spin-up spinor field $\Psi$ as follows [8, 10]:

$$
\Psi = \begin{pmatrix} A(t, r, \theta, \phi) \\ 0 \\ B(t, r, \theta, \phi) \\ 0 \end{pmatrix} \exp\left[\frac{\mathrm{i}}{\hbar} I(t, r, \theta, \phi)\right]. \tag{4}
$$

Note that we will only analyze the spin-up case since the spin-down case is just analogous. In order to apply the WKB approximation, we can insert the ansatz for a spinor field $\Psi$ into the general covariant Dirac equation. Dividing by the exponential term and neglecting the terms with $\hbar$, one can arrive at the following four equations:

$$
\begin{cases}
\mathrm{i}A\left(\sqrt{-g^{tt}}\partial_t I - \dfrac{g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I\right) + B\sqrt{\dfrac{\Delta}{\Sigma}}\partial_r I = 0,\\
\left(\dfrac{1}{\sqrt{\Sigma}}\partial_\theta I + \mathrm{i}\dfrac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi I\right) B = 0,\\
A\sqrt{\dfrac{\Delta}{\Sigma}}\partial_r I - \mathrm{i}B\left(\sqrt{-g^{tt}}\partial_t I - \dfrac{g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I\right) = 0,\\
\left(\dfrac{1}{\sqrt{\Sigma}}\partial_\theta I + \mathrm{i}\dfrac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi I\right) A = 0.
\end{cases} \tag{5}
$$

Note that although $A$ and $B$ are not constant, their derivatives and the components $\omega_\mu$ are all of the factor $\hbar$, so they can be neglected to the lowest order in the WKB approximation. Because we only consider the Dirac field outside the event horizon, the condition $\Delta \geqslant 0$ is always satisfied in the above equations. The second and fourth equations indicate that

$$
\frac{1}{\sqrt{\Sigma}}\partial_\theta I + \mathrm{i}\frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi I = 0. \tag{6}
$$

From the first and third equations, one can see that these two equations have a non-trivial solution for $A$ and $B$ if and only if the determinant of the coefficient matrix vanishes. Then we can get

$$
\left(\sqrt{-g^{tt}}\partial_t I - \frac{g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I\right)^2 - \frac{\Delta}{\Sigma}(\partial_r I)^2 = 0. \tag{7}
$$

Because there are two Killing vectors $(\partial/\partial t)^\mu$ and $(\partial/\partial\phi)^\mu$ in the Kerr–Newman spacetime, we can separate the variables for $I(t, r, \theta, \phi)$ as follows:

$$
I = -\omega t + j\phi + R(r, \theta) + K, \tag{8}
$$

where $\omega$ and $j$ are the Dirac particle's energy and angular momentum, respectively, and $K$ is a complex constant. Inserting it into equation (7), one can arrive at

$$\left(\sqrt{-g^{tt}}\omega + \frac{g^{t\phi}}{\sqrt{-g^{tt}}}j\right)^2 - \frac{\Delta}{\Sigma}(\partial_r R)^2 = 0. \tag{9}$$

Equation (6) indicates that $R(r,\theta)$ is a complex function. Now solving equation (9) for definite $\theta_0$ yields [5, 7]

$$\begin{aligned} R_\pm(r,\theta_0) &= \pm\int \mathrm{d}r\sqrt{\frac{\Sigma(\theta_0)}{\Delta}}\left(\sqrt{-g^{tt}(\theta_0)}\omega + \frac{g^{t\phi}(\theta_0)}{\sqrt{-g^{tt}(\theta_0)}}j\right), \\ &= \pm\int\frac{\mathrm{d}r}{\Delta}\left(\omega\sqrt{(r^2+a^2)^2-\Delta a^2\sin^2\theta_0} - j\frac{a(r^2+a^2-\Delta)}{\sqrt{(r^2+a^2)^2-\Delta a^2\sin^2\theta_0}}\right). \end{aligned}$$

The imaginary part of $R_+$ can be calculated using the above equation. Integrating the pole at the horizon leads to the result (see [7, 12] for a detailed similar process)

$$\mathrm{Im}\, R_\pm = \pm\frac{\pi(r^2+a^2)}{r_+-r_-}(\omega - j\Omega_H), \tag{10}$$

where $\Omega_H = \frac{a}{r_+^2+a^2}$ is the angular velocity of the event horizon. One can see that this result is independent of $\theta$.

As discussed in the Hamilton–Jacobi method [11, 12], one solution corresponds to Dirac particles moving away from the outer event horizon and the other solution corresponds to the particles moving toward the outer event horizon. The probabilities of crossing the outer horizon each way are, respectively, given by

$$\begin{aligned} P_{\mathrm{out}} &= \exp\left[-\frac{2}{\hbar}\mathrm{Im}\, I\right] = \exp\left[-\frac{2}{\hbar}(\mathrm{Im}\, R_+ + \mathrm{Im}\, K)\right], \\ P_{\mathrm{in}} &= \exp\left[-\frac{2}{\hbar}\mathrm{Im}\, I\right] = \exp\left[-\frac{2}{\hbar}(\mathrm{Im}\, R_- + \mathrm{Im}\, K)\right]. \end{aligned} \tag{11}$$

To ensure that the probability is normalized, we should note that the probability of any incoming wave crossing the outer horizon is unity [12]. So we get $\mathrm{Im}\, K = -\mathrm{Im}\, R_-$. Since $\mathrm{Im}\, R_+ = -\mathrm{Im}\, R_-$, this implies that the probability of a Dirac particle tunneling from inside to outside the event horizon is given by

$$\begin{aligned} \Gamma &= \exp\left[-\frac{4}{\hbar}\mathrm{Im}\, R_+\right], \\ &= \exp\left[\frac{4\pi(r_+^2+a^2)}{(r_+-r_-)}(\omega - j\Omega_H)\right], \end{aligned} \tag{12}$$

where in the last step we set $\hbar$ to unity. It should be noted that the higher terms about $\omega$ and $j$ are neglected in our derivation and expression (12) for tunneling probability implies pure thermal radiation.

From the tunneling probability, the fermionic spectrum of the Hawking radiation of Dirac particles from the Kerr black hole can be deduced following the standard arguments [13, 14]

$$N(\omega, j) = \frac{1}{\mathrm{e}^{2\pi(\omega - j\Omega_H)/\kappa} + 1}, \tag{13}$$

where $\kappa = \frac{(r_+-r_-)}{2(r_+^2+a^2)}$ is the surface gravity of the event horizon. From the tunneling probability and radiant spectrum, the Hawking temperature of the Kerr black hole can be determined as

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2-a^2}}{4\pi M(M+\sqrt{M^2-a^2})}. \tag{14}$$

In summary, we have calculated the Dirac particles' Hawking radiation from the Kerr black hole using the tunneling formalism. Starting with the Dirac equation, we obtained the radiation spectrum and Hawking temperature of the Kerr black hole by using the WKB approximation. The results coincide with the previous literature [4, 5, 7, 15, 16].

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After completing this paper, we noted that some related works were done by other authors. In [17], charged Fermions tunneling from the Kerr–Newman black holes were investigated. In [18], the authors considered Fermions tunneling from the more general and complicated spacetime background. This work was supported by the National Natural Science Foundation of China and Cuiying Project of Lanzhou University.

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