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The physical basis for infra-red divergences in inflationary quantum gravity

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Abstract. We consider tree-order graviton scattering amplitudes on a (3 + 1)-dimensional de Sitter background in conformally flat coordinates. Infra-red divergences which cannot be absorbed using conventional techniques are shown to arise because conformal factors from the vertices are not compensated either by propagators or by external wavefunctions. The physical problem at tree order is therefore late interaction times rather than small spatial coordinate momenta, despite a mathematical problem at small momenta in the naive mode expansion of the propagator. Even in loops, concern over small spatial momenta is physically irrelevant because the chaotic conditions likely to prevail after the Big Bang could not have resulted in the simultaneous onset of inflation over a patch extending much beyond the Hubble radius. This motivates our proposal for a propagator which can be used to compute expectation values well inside the de Sitter patch of a plausible initial state.

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1. Introduction

By 'quantum cosmological gravity'—or QCG for short—we mean the theory of quantum gravity with a non-zero cosmological constant. QCG is of enormous interest to inflationary cosmology because infra-red processes in this theory tend, over time, to screen the exponential expansion induced by a positive cosmological constant [1, 2]. As long as the effect remains small it can be followed reliably using perturbation theory [2]. Should it persist beyond the breakdown of perturbation theory QCG would provide a mechanism which naturally extinguishes inflation slowly enough to solve the smoothness problem [2].

To study how QCG affects the expansion of spacetime one should really follow the causal time evolution of the expectation value of the metric in the presence of a plausible initial state. We have done this, and explicit calculations at two loops show that screening does occur in perturbation theory [2]. However, we emphasize that there is still significant information to be gleaned from 'in'–'out' matrix elements and scattering amplitudes which are computed under the incorrect assumption that the geometry of the far future is locally de Sitter. In particular, the fact that causal time evolution does not show infra-red divergences, whereas 'in'–'out' matrix elements and scattering amplitudes do¶ means that the true

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¶ It is worth noting that the 'in'–'out' matrix elements of QCG are even infra-red divergent off-shell, and that the infra-red divergences of QCG scattering amplitudes cannot be avoided by the traditional device of summing over degenerate ensembles of states.

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background must suffer non-perturbatively large corrections at late times. For if the 'out' vacuum was the same as the 'in' vacuum, then there would be no difference between 'in'–'out' matrix elements and expectation values—and if the 'out' vacuum suffered only perturbatively small corrections then the difference would have to be perturbatively small. The fact that infra-red divergences in interacting QCG break time translation invariance is what protects the relaxation effect against being absorbed into an allowed counterterm. The fact that they originate from interactions at late times is what provides the crucial lag that permits a long period of inflation. And the fact that they become arbitrarily strong in the absence of relaxation is what ensures that the process can never cease.

In this paper we wish to establish the reality of infra-red divergences in interacting QCG—given the unrealistic assumption that the background remains locally de Sitter—and to elucidate the physical basis of the effect. We shall also see that neither the behaviour of the propagator nor problems with its definition explain why there are tree-order infra-red divergences in the non-resonant scattering amplitudes of gravitons with non-zero momenta. And although this is not a paper about the non-existence of a causal, de Sitter invariant propagator for free QCG on the full de Sitter manifold, we should like to emphasize that such a propagator is inconsistent with the causal structure of de Sitter space and the nature of the constraint equations of QCG [3]. However, this issue has no relevance for the physics of a realistic inflating universe and, in any case, we are not concerned with it here.

After a notational and historical review in section 2 we explain the mathematical problem with the graviton propagator in section 3. Section 4 demonstrates the physical problem with QCG scattering amplitudes. We show in section 5 that this problem cannot be avoided using conventional techniques, and that the propagator has nothing to do with it. The real culprit is the intense inflationary red-shift of the graviton's physical energy and momenta. At late times *all* gravitons tend towards having the same (zero) physical energy and momentum, which makes the interaction overlap grow without bound. This turns out to imply the background's decay [2], but the mathematical problem with the propagator might still obscure the study of this decay at the loop level. In section 6 we argue that this problem does not occur when a de Sitter phase is reached by causal evolution from the chaotic conditions likely to prevail after the Big Bang. We also propose a propagator which can be used to study relaxation in this situation. A brief summary of our conclusions comprises section 7.

2. Free QCG in conformally flat coordinates

It was recognized early on that the physical mode solutions of free QCG on a de Sitter background are two transverse, traceless and purely spatial polarizations, each of which obeys the equation of motion for a massless, minimally coupled scalar [4]. Although general scalar mode solutions have been known since the sixties [5] the problem of finding an acceptable vacuum long delayed the construction of a graviton propagator, and hence the development of QCG. The resolution lay in resisting the natural tendency to assume that the background's maximal symmetry implies the existence of de Sitter invariant states. Early researchers were attracted to de Sitter space in the expectation that its isometries could be exploited to organize and simplify perturbative quantum field theory in the same way that Poincaré invariance serves in flat space. However, it turns out that there are no normalizable, de Sitter invariant states for the massless, minimally coupled scalar [6], and the same thing has recently been proven for free QCG [7].

The situation for QCG was further complicated by the fact that in certain completely valid gauges there *is* a de Sitter invariant vacuum [8, 9]. In this case the problem is that the gauge-fixed field equations fail to agree with the invariant ones, even for conserved sources [9]. In fact, no causal propagator can reproduce the invariant field equations on the full manifold owing to a subtle conspiracy between the causal structure of de Sitter space and its linearization instability [3]. (This problem does not arise for the massless, minimally coupled scalar because it lacks the non-dynamical constraint equations of QCG.) For a closely related reason there is an open submanifold on which one can reconcile the invariant field equations with manifest causality. Although this patch covers only half of the full de Sitter manifold it has the two crucial properties that causal, forward-directed time evolution never carries one off the submanifold and that information from outside can only enter via the past where it can be subsumed into initial value data. This open submanifold is therefore an acceptable arena on which to study QCG. In describing the conjectured de Sitter phase of inflationary cosmology we shall presently come to understand that, far from being too small, the open submanifold is actually implausibly too large.

Although the maximal symmetry of de Sitter space only served to confuse the development of QCG, the background's conformal flatness does provide a powerful organizing principle. To take maximum advantage of this we shall employ a coordinate system for which the invariant line element is

$$\mathrm{d}s^2 = \Omega^2\big(-\mathrm{d}u^2 + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}\big)\,. \tag{2.1}$$

Here $\Omega \equiv (1/Hu)$ is the conformal factor and H is the Hubble constant. We shall refer to the surface at $u = +\infty$ as the 'infinite past', even though this is not $\mathcal{I}^-$ on the full de Sitter manifold. The conformal time u runs from the infinite past at $u = +\infty$ to the infinite future are $u = 0$; its relation to the time of co-moving coordinates is, $\exp(Ht) = (1/Hu)$. The flat space limit is obtained by setting $u = (1/H) - x^0$ and then taking H to zero.

It is simplest to formulate perturbative QCG in terms of fluctuations of the conformally rescaled metric,

$$g_{\mu\nu} \equiv \Omega^2\widetilde{g}_{\mu\nu} \equiv \Omega^2(\eta_{\mu\nu} + \kappa\psi_{\mu\nu})\,. \tag{2.2}$$

Here $\kappa^2 \equiv 16\pi G$ is the usual loop counting parameter of quantum gravity and we call $\psi_{\mu\nu}$ the 'pseudo-graviton' field. The full inverse of $\widetilde{g}_{\mu\nu}$ is denoted by $\widetilde{g}^{\mu\nu}$—i.e. $\widetilde{g}_{\mu\nu}\widetilde{g}^{\nu\rho} = \delta_\mu^{\ \rho}$—while its determinant by $\widetilde{g}$. Pseudo-graviton indices are raised and lowered with the Lorentz metric.

After some judicious partial integrations the invariant Lagrangian of QCG can be cast in a form not too different from that of $H = 0$ quantum gravity [3]:

$$\begin{aligned}\mathcal{L}_{\text{inv}} - S^{\nu}{}_{,\nu} = &-\tfrac{1}{2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\rho\sigma}\,\widetilde{g}^{\mu\nu}\,\psi_{\rho\sigma,\mu}\psi_{\nu}{}^{\alpha}\,\big(\Omega^2\big)_{,\alpha} + \sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\,\widetilde{g}^{\rho\sigma}\,\widetilde{g}^{\mu\nu} \\ &\times\big[\tfrac{1}{2}\psi_{\alpha\rho,\mu}\psi_{\nu\sigma,\beta} - \tfrac{1}{2}\psi_{\alpha\beta,\rho}\psi_{\sigma\mu,\nu} + \tfrac{1}{4}\psi_{\alpha\beta,\rho}\psi_{\mu\nu,\sigma} - \tfrac{1}{4}\psi_{\alpha\rho,\mu}\psi_{\beta\sigma,\nu}\big]\Omega^2\,.\end{aligned} \tag{2.3}$$

(Commas denote ordinary differentiation in this and all subsequent formulae.) Except for the first term, and the factor of Ω^2 on the second, this form for the Lagrangian is the same as that for perturbation theory around flat space in $H = 0$ quantum gravity! The surface term $S^{\nu}{}_{,\nu}$ can be found in [3]. Although $S^{\nu}{}_{,\nu}$ was neglected in the interest of obtaining a more tractable formalism we should reassure those who feel nervous about this that it *improves* whatever chance QCG has of avoiding problems at $u = 0$. The invariant Lagrangian is far worse behaved; it includes terms that go like $1/u^4$.

The simplest gauge fixing term is $-\frac{1}{2}F_\mu F_\nu \eta^{\mu\nu}$, where

$$F_\mu = \Omega\big(\psi^{\nu}{}_{\mu,\nu} - \tfrac{1}{2}\psi_{,\mu} + 2\,\psi^{\nu}{}_{\mu}\,\Omega_{,\nu}\,\Omega^{-1}\big)\,. \tag{2.4}$$

Adding this term to the invariant action and dropping some surface terms permits us to write the gauge-fixed, quadratic Lagrangian as, $\mathcal{L}^2_{\mathrm{GF}} = \frac{1}{2}\psi^{\mu\nu} D_{\mu\nu}{}^{\rho\sigma} \psi_{\rho\sigma}$, where the pseudo-graviton kinetic operator is [3]

$$D_{\mu\nu}{}^{\rho\sigma} \equiv \left[\tfrac{1}{2}\overline{\delta}_{\mu}{}^{(\rho}\overline{\delta}_{\nu}{}^{\sigma)} - \tfrac{1}{4}\eta_{\mu\nu}\eta^{\rho\sigma} - \tfrac{1}{2}\delta_{\mu}{}^{0}\delta_{\nu}{}^{0}\delta_{0}{}^{\rho}\delta_{0}{}^{\sigma}\right]\mathrm{D}_A + \delta_{(\mu}{}^{0}\overline{\delta}_{\nu)}{}^{(\rho}\delta_{0}{}^{\sigma)}\mathrm{D}_B + \delta_{\mu}{}^{0}\delta_{\nu}{}^{0}\delta_{0}{}^{\rho}\delta_{0}{}^{\sigma}\mathrm{D}_C\,. \tag{2.5}$$

(Parenthesized indices are symmetrized.) The symbol $\mathrm{D}_A \equiv \Omega(\partial^2 + 2/u^2)\Omega$ is the kinetic operator for a massless, minimally coupled scalar; $\mathrm{D}_B = \mathrm{D}_C \equiv \Omega\partial^2\Omega$ is the kinetic operator for a conformally coupled scalar. A bar over a standard tensor such as the Kronecker delta means to suppress all zero components, for example:

$$\overline{\delta}_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} - \delta_{\mu}{}^{0}\delta_{0}{}^{\nu}\,. \tag{2.6}$$

Two important simplifications of this gauge are that it makes the kinetic operator invariant under spatial translations and rotations, and that all tensors in the kinetic operator are spacetime constants. Note, however, that our gauge is not de Sitter invariant. Since *physical* de Sitter invariance is necessarily broken in free QCG there is no point in burdening the propagator with the non-constant tensor factors that would be needed in a de Sitter invariant gauge.

It is perhaps worthwhile to point out here that the choice of gauge is incapable of physically breaking a true global symmetry. For example, axial gauge does not break the Lorentz invariance of QED, it merely causes the true Lorentz generators to contain a compensating gauge transformation. Similarly, the use of only a portion of what a mathematician might consider 'the full manifold', cannot affect local physics. For example, no experiment whose spatial extent and temporal duration are sufficiently limited can detect the difference between flat $R^3 \times R$ and flat $T^3 \times R$. In fact, our current universe may well possess the latter topology rather than the former. Earthbound experiments performed over human lifetimes still show the full Lorentz group, despite the mathematician's insistence that topological obstructions prevent its realization on $T^3 \times R$.

Manifest spatial translation invariance allows us to write the general linearized solution as a superposition of plane waves [10]

$$\psi_{\mu\nu}(u, \boldsymbol{x}) = \sum_{\lambda} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \{\Psi(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) a_{\mu\nu}(\boldsymbol{k}, \lambda) + \Psi^*(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) a^{\dagger}_{\mu\nu}(\boldsymbol{k}, \lambda)\}\,. \tag{2.7}$$

The polarization index λ belongs to sets 'A', 'B' or 'C' depending upon which part of the kinetic operator fails to annihilate the associated tensor structure. The six A modes have purely spatial polarizations and the following functional dependence:

$$\Psi(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) = \frac{Hu}{\sqrt{2k}}\left(1 + \frac{\mathrm{i}}{ku}\right)\exp\left[\mathrm{i}k\left(u - \frac{1}{H}\right) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\right] \qquad \forall\lambda \in A\,. \tag{2.8}$$

Canonical quantization of $\mathcal{L}^2_{\mathrm{GF}}$ reveals the polarization sum for the A modes to be

$$\sum_{\lambda,\lambda' \in A} \left[a_{\mu\nu}(\boldsymbol{k}, \lambda), a^{\dagger}_{\rho\sigma}(\boldsymbol{k}', \lambda')\right] = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')2\left(\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}\right). \tag{2.9}$$

We shall denote the tensor factor to the right of the delta function as, $\left[{}_{\mu\nu}T^A_{\rho\sigma}\right]$. The three B modes have polarization tensors with one time and one space index; the associated spacetime dependence is

$$\Psi(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) = \frac{Hu}{\sqrt{2k}} \exp\left[ik\left(u - \frac{1}{H}\right) + i\boldsymbol{k}\cdot\boldsymbol{x}\right] \qquad \forall\lambda \in B\,. \tag{2.10}$$

Canonical quantization reveals that these modes have negative norm and that they give the following polarization sum:

$$\sum_{\lambda,\lambda'\in B} \left[a_{\mu\nu}(\boldsymbol{k},\lambda), a^\dagger_{\rho\sigma}(\boldsymbol{k}',\lambda')\right] = (2\pi)^3\,\delta^3(\boldsymbol{k}-\boldsymbol{k}')\,4\delta_{(\mu}{}^0\,\overline{\eta}_{\nu)(\rho}\eta_{\sigma)0}\,. \tag{2.11}$$

We denote the tensor factor to the right as $\left[{}_{\mu\nu}T^B_{\rho\sigma}\right]$. The single C mode is proportional to the Euclidean metric and has the same functional dependence as the B modes. Canonical quantization determines its polarization sum to be

$$\left[a_{\mu\nu}(\boldsymbol{k}, C), a^\dagger_{\rho\sigma}(\boldsymbol{k}', C)\right] = (2\pi)^3\,\delta^3(\boldsymbol{k}-\boldsymbol{k}')\left(\delta_\mu{}^0\delta_\nu{}^0 + \overline{\eta}_{\mu\nu}\right)\left(\delta_\rho{}^0\delta_\sigma{}^0 + \overline{\eta}_{\rho\sigma}\right) \tag{2.12}$$

and we denote the tensor factor as $\left[{}_{\mu\nu}T^C_{\rho\sigma}\right]$.

The two physical polarizations are transverse-traceless A modes. They form a closed set under transformations of the de Sitter group [7], in spite of the fact that our gauge is not de Sitter invariant. (The situation here is not qualitatively different from that of QED in lightcone gauge where one of the Lorentz generators requires the addition of a field dependent gauge transformation to restore the gauge condition.) The three (positive norm) longitudinal A modes and the single (positive norm) C mode pair with the three (negative norm) B modes and the (negative norm) trace A mode to decouple from the theory [10].

3. The spurious infra-red problem at zero spatial momentum

If we recall that the conformal time u is inverted with respect to physical time we see from the mode functions that, at least in what we are calling the 'asymptotic past', the $a_{\mu\nu}$s should behave like annihilation operators and the $a^\dagger_{\mu\nu}$s like creation operators. It is therefore almost irresistible to suppose the existence of a free vacuum which obeys

$$a_{\mu\nu}(\boldsymbol{k},\lambda)|0\rangle = 0\,. \tag{3.1}$$

Because de Sitter transformations carry $a_{\mu\nu}$s onto $a_{\mu\nu}$s for the physical polarizations [7], this condition defines a de Sitter invariant wavefunctional up to the gauge structure. For this wavefunctional to represent a valid quantum state it must be normalizable. Of course, in the way Fock space is usually treated one would simply *define* $\langle 0|0\rangle \equiv 1$ and then use the commutation relations to compute all other norms. But since this could be done for *any* wavefunctional, and since not all wavefunctionals are normalizable, it has to be expected that any problems would merely appear somewhere else. In fact, they show up in the second moment, better known as the propagator.

In view of the temporal inversion the propagator should be the expectation value in the presence of $|0\rangle$ of the *anti*-time-ordered product of two free fields. From the polarization sums we obtain the following expression for this propagator:

$$\mathrm{i}[{}_{\mu\nu}\Delta_{\rho\sigma}](x;x') = \langle 0|\overline{T}\{\psi_{\mu\nu}(x)\psi_{\rho\sigma}(x')\}|0\rangle \tag{3.2a}$$

$$= \mathrm{i}\Delta_A(x;x')\big[{}_{\mu\nu}T^A_{\rho\sigma}\big] + \mathrm{i}\Delta_B(x,x')\big\{\big[{}_{\mu\nu}T^B_{\rho\sigma}\big] + \big[{}_{\mu\nu}T^C_{\rho\sigma}\big]\big\} \tag{3.2b}$$

$$\mathrm{i}\Delta_A(x;x') \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\epsilon k}\{\theta(u'-u)\Psi(u,\boldsymbol{x};\boldsymbol{k},A)\Psi^*(u',\boldsymbol{x}';\boldsymbol{k},A) + \theta(u-u')\Psi^*(u,\boldsymbol{x};\boldsymbol{k},A)\Psi(u',\boldsymbol{x}';\boldsymbol{k},A)\} \tag{3.2c}$$

$$\mathrm{i}\Delta_B(x;x') \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\epsilon k}\{\theta(u'-u)\Psi(u,\boldsymbol{x};\boldsymbol{k},B)\Psi^*(u',\boldsymbol{x}';\boldsymbol{k},B) + \theta(u-u')\Psi^*(u,\boldsymbol{x};\boldsymbol{k},B)\Psi(u',\boldsymbol{x}';\boldsymbol{k},B)\}\,. \tag{3.2d}$$

Note that in (3.2c) and (3.2d) we have included the ultraviolet convergence factors traditionally used to promote mode sums from distributions into well defined functions.

Of course $\mathrm{i}\Delta_B(x;x')$ is conformal to the propagator of a massless scalar in flat space:

$$\mathrm{i}\Delta_B(x;x') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{H^2uu'}{2k}\exp[-\mathrm{i}k|\Delta u| + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x}) - \epsilon k] \tag{3.3a}$$

$$= \frac{H^2uu'}{8\pi^2\Delta x}\left[\frac{1}{\Delta x - |\Delta u| + \mathrm{i}\epsilon} + \frac{1}{\Delta x + |\Delta u| - \mathrm{i}\epsilon}\right] \tag{3.3b}$$

$$= \frac{1}{4\pi^2}\,\frac{H^2uu'}{(x-x')^2 + \mathrm{i}\epsilon} \tag{3.3c}$$

where $\Delta x \equiv \|\boldsymbol{x}'-\boldsymbol{x}\|$, $\Delta u \equiv u'-u$ and $(x-x')^2 \equiv \Delta x^2 - \Delta u^2$. This propagator also turns out to be a de Sitter invariant. The problem comes from the massless, minimally coupled scalar

$$\mathrm{i}\Delta_A(x;x') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{H^2uu'}{2k}\left[1 + \frac{1+\mathrm{i}k|\Delta u|}{k^2uu'}\right]\exp[-\mathrm{i}k|\Delta u| + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x}) - \epsilon k] \tag{3.4a}$$

$$= \frac{1}{4\pi^2}\frac{H^2uu'}{(x-x')^2+\mathrm{i}\epsilon} + \frac{H^2}{4\pi^2\Delta x}\int_0^\infty \mathrm{d}k\,\sin(k\Delta x)\frac{\partial}{\partial k}\left\{-\frac{1}{k}\exp[-\mathrm{i}k|\Delta u| - \epsilon k]\right\} \tag{3.4b}$$

$$= \frac{1}{4\pi^2}\,\frac{H^2uu'}{(x-x')^2+\mathrm{i}\epsilon} + \frac{H^2}{4\pi^2} + \frac{H^2}{4\pi^2}\int_0^\infty \frac{\mathrm{d}k}{k}\cos(k\Delta x)\,\exp[-\mathrm{i}k|\Delta u| - \epsilon k]\,. \tag{3.4c}$$

The final integral contains a logarithmic divergence at $k=0$ which it is useful to regulate with a cut-off

$$\frac{H^2}{4\pi^2}\int_{k_0}^\infty \frac{\mathrm{d}k}{k}\cos(k\Delta x)\,\exp[-\mathrm{i}k|\Delta u| - \epsilon k] = \frac{H^2}{8\pi^2}\{-\mathrm{Ei}[-k_0(\epsilon - \mathrm{i}\Delta x + \mathrm{i}|\Delta u|)] - \mathrm{Ei}[-k_0(\epsilon + \mathrm{i}\Delta x + \mathrm{i}|\Delta u|)]\} \tag{3.5a}$$

$$= \frac{H^2}{8\pi^2}\big\{-\ln\big[k_0^2(\Delta x^2 - \Delta u^2 + \mathrm{i}\epsilon)\big] - 2\gamma + \mathrm{O}(k_0)\big\}\,. \tag{3.5b}$$

Here $\mathrm{Ei}(x)$ is the exponential integral function and γ stands for Euler's constant [11].

This infra-red divergence was first noticed in the context of the massless, minimally coupled scalar by Ford and Parker [12]. Both the infra-red divergence and the apparent logarithmic growth outside the lightcone have inspired pronouncements concerning the instability of de Sitter space as a background for QCG [13, 14]. In fact, de Sitter space *is* unstable, but not because of any problem with defining the graviton propagator or with its behaviour once a suitable definition is given. Allen and Folacci showed that the infra-red divergence derives from the invalid assumption of de Sitter invariance [6], which was implicitly made in our definition (3.1) of the vacuum. We shall have more to say about this after making two points about the behaviour of the propagator. First, one can see from expressions (3.3a) and (3.4a) that its spatial Fourier transform is well behaved for any non-zero spatial momentum. This suffices for non-resonant tree-order scattering amplitudes. Second, if it was somehow correct to retain a non-zero cut-off k_0—as we shall argue it is—then the integral (3.5) would *fall off* outside the lightcone, rather than growing.

The infra-red divergence in $\mathrm{i}\Delta_A$ is what comes of having incorrectly quantized the zero mode as a harmonic oscillator when it should be a free particle. This error was forced by the assumption of de Sitter invariance and correcting it necessarily entails the breaking of de Sitter invariance. Because a further complication results from the continuum normalization of the plane wave modes of our conformal coordinate system, Allen and Folacci [6] worked in a closed coordinate system for which the modes are discrete. We can understand their result by taking the zero frequency limit of a one-dimensional harmonic oscillator of mass m and frequency ω, in which limit it might be thought that the ground state should be time translation invariant.

The Hamiltonian is

$$\mathrm{H} = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 q^2 \tag{3.6}$$

which implies the following time development:

$$q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t)\,. \tag{3.7}$$

The ground-state wavefunction is just

$$G(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}q^2\right] \tag{3.8}$$

from which we derive the following propagator:

$$\langle G|T[q(t)\,q(t')]|G\rangle = \frac{\hbar}{2m\omega}\exp[-\mathrm{i}\omega|t-t'|]\,. \tag{3.9}$$

Now consider the limit $\omega \longrightarrow 0$. Even though the norm of $|G\rangle$ is formally unity in this limit, we see from (3.8) that this is achieved in an illegitimate fashion by making the wavefunction tend towards the constant zero. Had we simply defined $\langle G|G\rangle \equiv 1$ and computed other norms from the raising and lowering operators we would be made aware of the problem by the divergence in the real part of the propagator. Note that $|G\rangle$ is the ket of lowest energy—zero—and highest symmetry—time translation invariance—but it is unsuitable as a state on account of its non-normalizability.

What we should really do in this situation is to use a normalizable state, a typical example of which would be

$$N(q) = \left(\frac{m}{\pi\hbar T}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{m}{T\hbar}q^2\right] \tag{3.10}$$

where T is a constant with the dimensions of time. The resulting propagator

$$\langle N|T[q(t)q(t')]|N\rangle = \frac{\hbar T}{2m}\left\{1 - \mathrm{i}\frac{|t-t'|}{T} + \frac{tt'}{T^2}\right\} \tag{3.11}$$

has the same imaginary part as (3.9)—for $\omega = 0$—but possesses a finite real part as well. Note that the imposition of normalizability has cost us time translation invariance and the uniqueness of the ground state. There is no normalizable state of minimum energy, in fact, there are no normalizable energy eigenstates at all.

Each mode of the quantum field theory consists of such a harmonic oscillator with the role of the frequency being played by $\omega = \|\boldsymbol{k}\|$. Whether or not the divergence at $\boldsymbol{k} = 0$ has significance depends upon the weight carried by the zero mode and, in the continuum normalization, by its nearby neighbours, which are infinitesimally close to being free particles. For flat space in more than two dimensions the volume of phase space cancels the divergence. In two-dimensional flat space there is no such cancellation and the resulting infra-red divergence is the basis of Coleman's famous proof concerning the impossibility of spontaneous symmetry breaking in two dimensions [15]. In the spirit of our current work it would be fairer to say that any two-dimensional symmetry breaking must entail the loss of Poincaré invariance. (Note that one can isolate Coleman's zero mode by formulating the theory on $S \times R$. This manifold admits the Lorentz metric and, for sufficiently large radius, it should not be locally distinguishable from R^2.)

In de Sitter space the effect occurs for any dimension. Although their argument was different, this is what Allen and Folacci [6] proved for the massless, minimally coupled scalar. The same result has recently been obtained for QCG by Kleppe [7]. Our comments serve merely to resolve the apparent paradox implied by these proofs that neither theory possesses normalizable, de Sitter invariant states in spite of the fact that both have de Sitter invariant sets of mode solutions.

4. The physical infra-red problem at late times

Having understood that peculiarities of the graviton propagator are not responsible for any breakdown of the de Sitter background in QCG let us see what is. Note first that the de Sitter non-invariance of QCG *states* says nothing about the stability of de Sitter space as a *background* for QCG. To pursue the previous quantum mechanical analogy, there are no normalizable, time translation invariant states for the free particle, but the fact that $\langle N|q(t)|N\rangle = 0$ for all t implies that $\widehat{q} = 0$ is a stable background.

We can postpone, for the moment, the issue of what propagator to use if we note that the problem with the naive choice (3.2) only occurs for the modes near $\boldsymbol{k} = 0$. It seems reasonable to conclude that only they require revision, in which case we can probe tree-order scattering amplitudes with the spatial Fourier transform of $\mathrm{i}[_{\mu\nu}\Delta_{\rho\sigma}]$ if we simply arrange that no intermediate momentum becomes small. Combining (3.3a) and (3.4a) gives the following very simple expression for this propagator:

$$\int \mathrm{d}^3x\ \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{i}[_{\mu\nu}\Delta_{\rho\sigma}](u,\boldsymbol{x};u',0) = \frac{H^2uu'}{2k}\ \exp[-\mathrm{i}k|\Delta u| - \epsilon k]\,[2\,\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}]$$
$$+ \frac{H^2}{2k^3}\,(1 + \mathrm{i}k|\Delta u|)\ \exp[-\mathrm{i}k|\Delta u| - \epsilon k]\,[2\,\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - 2\,\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}]\,. \tag{4.1}$$

The final tensor factor is, of course, $[{}_{\mu\nu}T^A_{\rho\sigma}]$; the initial one is the sum of T^A, T^B and T^C. Except for the initial factor of H^2uu', the first term in this propagator is precisely the same as that of the full de Donder gauge propagator of $H = 0$ quantum gravity!

The physical graviton degrees of freedom consist of the two A modes which have transverse-traceless polarizations. We can extract them from the free-field expansion using an external wavefunction of the following form:

$$\Psi_{\mu\nu}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) = \frac{Hu}{\sqrt{2k}}\left(1 + \frac{\mathrm{i}}{ku}\right) \exp\left[\mathrm{i}k\left(u - \frac{1}{H}\right) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\right] \epsilon_{\mu\nu}(\boldsymbol{k}, \lambda) \tag{4.2}$$

where the polarization tensor obeys the conditions

$$\epsilon_{\mu 0}(\boldsymbol{k}, \lambda) = k_i\, \epsilon_{ij}(\boldsymbol{k}, \lambda) = \epsilon_{ii}(\boldsymbol{k}, \lambda) = 0 \tag{4.3a}$$

$$\epsilon^*_{\mu\nu}(\boldsymbol{k}, \lambda) = \epsilon_{\mu\nu}(-\boldsymbol{k}, \lambda) \tag{4.3b}$$

$$\epsilon^*_{ij}(\boldsymbol{k}, \lambda)\, \epsilon_{ij}(\boldsymbol{k}, \lambda') = 2\,\delta_{\lambda\lambda'}\,. \tag{4.3c}$$

Because $\boldsymbol{k}$ is the Fourier conjugate to the coordinate $\boldsymbol{x}$ it represents the *coordinate momentum*. Consideration of the invariant interval (2.1) suggests that the *physical momentum* is $\Omega^{-1}\boldsymbol{k}$. *The red-shift as u approaches zero turns out to have profound* consequences.

Examination of the kinetic operator (2.5) reveals the Wronskian to be

$$\left[{}^{\alpha\beta}\overleftrightarrow{W}{}^{\rho\sigma}_{\ \ \mu}\right] = \Omega\,(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\Omega\left[\tfrac{1}{2}\eta^{\alpha(\rho}\eta^{\sigma)\beta} - \tfrac{1}{4}\eta^{\alpha\beta}\eta^{\rho\sigma}\right]. \tag{4.4}$$

Since the kinetic operator is of the Stürm–Liouville-type we can use the Wronskian to form a conserved inner product

$$-\,\mathrm{i}\int \mathrm{d}^3x\, \Psi^*_{\alpha\beta}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda)\left[{}^{\alpha\beta}\overleftrightarrow{W}{}^{\rho\sigma}_{\ \ 0}\right]\Psi_{\rho\sigma}(u, \boldsymbol{x}; \boldsymbol{k}', \lambda') = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')\,\delta_{\lambda\lambda'}\,. \tag{4.5}$$

We can therefore use $\Psi^*_{\alpha\beta}$ to extract the annihilation operator for a physical graviton from the free-field expansion

$$a(\boldsymbol{k}, \lambda) = -\mathrm{i}\int \mathrm{d}^3x\, \Psi^*_{\alpha\beta}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda)\left[{}^{\alpha\beta}\overleftrightarrow{W}{}^{\rho\sigma}_{\ \ 0}\right]\psi_{\rho\sigma}(u, \boldsymbol{x})\,. \tag{4.6}$$

The creation operators follow from Hermitian conjugation. If we assume the usual asymptotic conditions in what we are calling the 'far past' (i.e. at $u = +\infty$) and in the far future (i.e. at $u = 0$) then the standard reduction formalism relates amplitudes to the spacetime integrals of amputated Green functions against external wavefunctions. In particular, integrating against (and contracting into) $\Psi_{\alpha\beta}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda)$ inserts an 'in' graviton of momentum $\boldsymbol{k}$ and polarization λ through the field at $(u, \boldsymbol{x})$; the conjugate wavefunction would remove an 'out' graviton.

One obtains the 3-graviton scattering amplitude by expanding (2.3) to cubic order in the pseudo-graviton field, replacing the three ψs of each interaction term with external wavefunctions and then permuting. Sixteen distinct cubic interaction terms emerge from the expansion, of which only four make non-zero contributions to 3-graviton scattering at tree order. The others give zero owing to one or more of the identities (4.3) obeyed by the polarization tensors. In particular, none of the three 'new' interactions from the first term of (2.3) survive because the polarization tensors are purely spatial.

Each non-zero contribution has the general form

$$i\kappa H\,(2\pi)^3\,\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,\frac{C(\boldsymbol{k}_1,\lambda_1;\boldsymbol{k}_2,\lambda_2;\boldsymbol{k}_3,\lambda_3)}{\sqrt{2k_1}\,\sqrt{2k_2}\,\sqrt{2k_3}}\times\int_0^\infty du\,\exp\left[i(k_1+k_2+k_3)\left(u-\frac{1}{H}\right)\right]I(u,k_1,k_2,k_3) \tag{4.7}$$

where the Cs are contractions of the polarization tensors and possibly also momenta, and the Is are functions of the conformal time u and the norms of the momenta. To economize on space and to facilitate comparison with standard results we shall adopt the notation of Sannan [16] whereby polarization tensors and momenta enclosed in parentheses are contracted into one another in the order they appear. The following examples should clarify the notation:

$$(\epsilon_1\epsilon_2\epsilon_3)\equiv\epsilon_{ij}(\boldsymbol{k}_1,\lambda_1)\,\epsilon_{jk}(\boldsymbol{k}_2,\lambda_2)\,\epsilon_{ki}(\boldsymbol{k}_3,\lambda_3) \tag{4.8a}$$

$$(k_3\epsilon_1k_2)\equiv k_{3i}\,\epsilon_{ij}(\boldsymbol{k}_1,\lambda_1)\,k_{2j}\,. \tag{4.8b}$$

Only three distinct contributions emerge. The one from $\frac{1}{2}\kappa\,\psi^{\rho\sigma}\,\psi_{\rho\mu,\nu}\,\psi_\sigma{}^{\mu,\nu}\,\Omega^2$ is characterized by

$$C=(\epsilon_1\epsilon_2\epsilon_3) \tag{4.9a}$$

$$I=(k_1k_2+k_2k_3+k_3k_1)u+i\left(\frac{k_1k_2}{k_3}+\frac{k_2k_3}{k_1}+\frac{k_3k_1}{k_2}\right)+\tfrac{1}{2}(k_1^2+k_2^2+k_3^2)\left(1+\frac{i}{k_1u}\right)\left(1+\frac{i}{k_2u}\right)\left(1+\frac{i}{k_3u}\right). \tag{4.9b}$$

The contribution from $\frac{1}{4}\kappa\,\psi^{\rho\sigma}\,\psi_{\mu\nu,\rho}\,\psi^{\mu\nu}{}_{,\sigma}\,\Omega^2$ is characterized by

$$C=(k_3\epsilon_1k_2)(\epsilon_2\epsilon_3)+(k_1\epsilon_2k_3)(\epsilon_3\epsilon_1)+(k_2\epsilon_3k_1)(\epsilon_1\epsilon_2) \tag{4.10a}$$

$$I=-\tfrac{1}{2}u\left(1+\frac{i}{k_1u}\right)\left(1+\frac{i}{k_2u}\right)\left(1+\frac{i}{k_3u}\right). \tag{4.10b}$$

The contributions from $-\kappa\,\psi^{\rho\sigma}\,\psi_\rho{}^{\mu,\nu}\,\psi_{\mu\nu,\sigma}\,\Omega^2$ and from $-\frac{1}{2}\kappa\,\psi^{\rho\sigma}\,\psi_\rho{}^{\mu,\nu}\,\psi_{\sigma\nu,\mu}\,\Omega^2$ together give

$$C=(k_3\epsilon_2\epsilon_1\epsilon_3k_2)+(k_1\epsilon_3\epsilon_2\epsilon_1k_3)+(k_2\epsilon_1\epsilon_3\epsilon_2k_1) \tag{4.11a}$$

$$I=-u\left(1+\frac{i}{k_1u}\right)\left(1+\frac{i}{k_2u}\right)\left(1+\frac{i}{k_3u}\right). \tag{4.11b}$$

Note that there can be no cancellation between the three contributions because their C factors differ. Substitution into (4.7) and summation of the three contributions gives the amplitude for three 'in' gravitons to go into the 'out' vacuum. Any of the particles can be changed from an 'in' graviton to an 'out' one by the replacements: $k\mapsto -k$, $\boldsymbol{k}\mapsto-\boldsymbol{k}$ and $\epsilon_{ij}(\boldsymbol{k},\lambda)\mapsto\epsilon^*_{ij}(\boldsymbol{k},\lambda)$.

5. What it all means

There are three troublesome points about the amplitude we have just derived.

(i) It is not zero. The fact that a single graviton can decay into a pair of gravitons implies that the one graviton state is not stable. Worse still, the fact that *the vacuum* can decay into three gravitons implies it is not stable either. In flat space both of these processes would be forbidden by the energy-conserving delta function which is absent here.

(ii) It is not purely imaginary. This implies a *tree-order* breakdown of unitarity.

(iii) It is not finite. We shall argue that this implies an instability in the background. Also note that an infra-red divergence in the lowest possible order of perturbation theory can not be avoided by the Lee–Nauenberg [17] procedure of considering only transition amplitudes between ensembles of degenerate states. This method can only work when the first infra-red divergence in a given process is not also the lowest-order contribution. Then the cross term, in the square of the amplitude, between the finite lowest contribution and the infra-red divergent one can sometimes be cancelled by the square of the lowest-order contribution from a nearly degenerate process containing an extra soft quantum.

The first two problems could be resolved acceptably in the context of asymptotic quantum field theory were it not for the third. It is instructive to sketch how this resolution goes for a conformally invariant theory such as QCD, the Lagrangian for which is

$$\mathcal{L}_{\text{QCD}} = -\tfrac{1}{4}\, F_{a\mu\nu}\, F_{a\rho\sigma}\, \widehat{g}^{\mu\rho}\, \widehat{g}^{\nu\sigma}\, \sqrt{-\widehat{g}}\,. \tag{5.1}$$

Here $\widehat{g}_{\mu\nu}$ stands for the de Sitter metric, the field strength tensor is

$$F_{a\mu\nu} \equiv A_{a\nu,\mu} - A_{a\mu,\nu} - g f_{abc}\, A_{b\mu}\, A_{c\nu} \tag{5.2}$$

where f_{abc} stands for the structure constants, and g is the coupling constant. Because the de Sitter geometry is conformally flat the position space integrands for QCD on this background are identical to those of flat space. This is obvious from expressing the Lagrangian in conformal coördinates,

$$\mathcal{L}_{\text{QCD}} = -\tfrac{1}{4}\, F_{a\mu\nu}\, F_{a\rho\sigma}\, \Omega^{-2}\, \eta^{\mu\rho}\, \Omega^{-2}\, \eta^{\nu\sigma}\, \Omega^{4}\,. \tag{5.3}$$

(In conformally invariant theories for which the dynamical fields have a non-trivial weight the Lagrangian depends upon Ω, however, all factors of Ω cancel out in the position space integrands of amplitudes.) We insert an 'in' gluon with the following external wavefunction:

$$\mathcal{A}_{a\mu}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) = \frac{1}{\sqrt{2k}} \exp\left[\mathrm{i}k\left(u - \frac{1}{H}\right) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\right] \epsilon_{a\mu}(\boldsymbol{k}, \lambda) \tag{5.4}$$

whose polarization tensor obeys the identities

$$\epsilon_{a0}(\boldsymbol{k}, \lambda) = k_i\, \epsilon_{ai}(\boldsymbol{k}\lambda) = 0 \tag{5.5a}$$

$$\epsilon^{*}_{a\mu}(\boldsymbol{k}, \lambda) = \epsilon_{a\mu}(-\boldsymbol{k}, \lambda) \tag{5.5b}$$

$$\epsilon^{*}_{ai}(\boldsymbol{k}, \lambda)\, \epsilon_{ai}(\boldsymbol{k}\,.\lambda') = \delta_{\lambda\lambda'}\,. \tag{5.5c}$$

If we agree to represent contractions thus,

$$[\epsilon_1\, \epsilon_2\, \epsilon_3\, k_1] \equiv f_{abc}\, \epsilon_{ai}(\boldsymbol{k}_1, \lambda_1)\, \epsilon_{bi}(\boldsymbol{k}_2, \lambda_2)\, \epsilon_{cj}(\boldsymbol{k}_3, \lambda_3)\, k_{1j} \tag{5.6}$$

then we obtain the following result for 3-gluon scattering:

$$g\,(2\pi)^3\,\delta^3(k_1+k_2+k_3)\,\frac{C(k_1,\lambda_1;k_2,\lambda_2;k_3,\lambda_3)}{\sqrt{2k_1}\,\sqrt{2k_2}\,\sqrt{2k_3}}\int_0^\infty \mathrm{d}u\,\exp\left[\mathrm{i}(k_1+k_2+k_3)\left(u-\frac{1}{H}\right)\right] \tag{5.7a}$$

$$C=[\epsilon_1\,\epsilon_2\,\epsilon_3\,k_1]+[\epsilon_1\,\epsilon_3\,\epsilon_2\,k_1]+[\epsilon_2\,\epsilon_3\,\epsilon_1\,k_2]+[\epsilon_2\,\epsilon_1\,\epsilon_3\,k_2]+[\epsilon_3\,\epsilon_1\,\epsilon_2\,k_3]+[\epsilon_3\,\epsilon_2\,\epsilon_1\,k_3]\,. \tag{5.7b}$$

Although the position space integrand is the same as for flat space, the QCD amplitude possesses properties (i) and (ii) because the conformal time u is only integrated from 0 to ∞. To promote the distributional integral into a well defined function we should include the standard convergence factor at $u=\infty$. This gives the following result:

$$\int_0^\infty \mathrm{d}u\,\exp\left[\mathrm{i}(k_1+k_2+k_3)\left(u-\frac{1}{H}\right)\right]\exp[-\epsilon\,u]=\frac{\mathrm{i}\,\exp[-\mathrm{i}(k_1+k_2+k_3)/H]}{k_1+k_2+k_3+\mathrm{i}\epsilon} \tag{5.8a}$$

$$=\mathrm{P}\left\{\frac{\mathrm{i}\exp[-\mathrm{i}(k_1+k_2+k_3)/H]}{k_1+k_2+k_3}\right\}+\pi\,\delta(k_1+k_2+k_3) \tag{5.8b}$$

where 'P' denotes the principal value. Of course the energy-conserving delta function contributes nothing to the amplitude because it cannot be satisfied for non-zero momenta. If we conjugate one of the 'in' gluons into an 'out' gluon the energy-conserving delta forces the three 4-momenta to be collinear, in which case there is always a polarization tensor contracted into its own momentum, and we still get zero by (5.5a). The entire contribution comes therefore from the principal value term.

The origin of the non-zero QCD result is the failure of the asymptotic 'out' condition at $u=0$. This was already obvious from (5.4) in the wavefunction's lack of oscillatory behaviour as one approaches the infinite future. The reason asymptotic quantum field theory is so useful is that we can extract the true vacuum and the true single particle state from the free ones merely by infinite time evolution. The procedure works because the free states have a non-zero overlap with the full ones, and we can tune the oscillatory factors they carry so that, against normalizable states and in the weak sense, only this overlap survives in the limit that x^0 goes to infinity. But without oscillations the limit accomplishes nothing. It is as if we were to turn the interaction of flat space field theory off at $x^0=0$, in which case we would also encounter properties (i) and (ii).

The preceding discussion suggests a simple resolution: merely extend the range of integration for u to cover the full real axis. This may seem gratuitous but it is, in fact, the standard procedure [18]. There is also an excellent geometrical justification for it in that negative values of u correspond to those portions of the full de Sitter manifold which are not covered by conformal coordinates. The only strange feature about the extended manifold is that the connection comes on our original submanifold not through what we are calling the 'infinite past' but rather through the infinite future as u passes from 0^+ to 0^-. (It is worth commenting that the peculiar connection circumvents the argument [3] that precludes the use of a causal propagator on the conventionally connected, full manifold.) A mathematical purist would object that neither the 'in' region nor the new 'out' region really corresponds to the asymptotic past or future of a geodesically complete manifold. However, the extension works because the relevant point for getting free states to interpolate interacting ones is not what the 'asymptotic' coordinate regions mean geometrically but rather how the fields behave in them. The new 'out' region is at $u=-\infty$, where the wavefunctions show fine oscillatory behaviour. It should be clear from (5.3) that scattering on the extended de Sitter manifold is identical, for tree-order QCD, to that on flat space.

Now consider the situation in QCG. Since positive powers of u can always be removed as derivatives with respect to k, we see from (4.2) that the wavefunctions of QCG show fine oscillatory behaviour as u approaches $-\infty$. However, extending the manifold does not result in an acceptable asymptotic field theory because of the infra-red divergence at $u = 0$. The *mathematical origin of the instabilities (in the graviton and in the vacuum) is the breakdown* of energy conservation. It cannot be repaired by extending the manifold because there are *inverse* powers of u. This is also why there are infra-red divergences. The physical origin of the instabilities seems to be that the time-dependent background is serving as a source of radiation. The infra-red problem seems to originate in the terrific red-shift QCG gravitons experience owing to their masslessness and their lack of conformal invariance. This red-shift tends to drive all *coordinate* momenta down to *physical* momentum zero, making the overlap between plane waves infinitely strong at $u = 0$. Though we eventually reach a 'quiet' regime at $u = -\infty$, the asymptotic 'in' and 'out' vacua are infinitely different. The reason we encounter the infra-red divergence is that we have tried to ignore this.

The preceding discussion establishes the instability of the QCG vacuum, and of the single graviton state, but it does not quite show that the *background* changes. To see this *it suffices to consider what the aforementioned instabilities mean to the stress tensor which* is the source of the background metric. Though the details require a finite-time formalism we have presented elsewhere [2], two things should be obvious on physical grounds. First, the universe is being filled at late times with a sea of soft gravitons. Since the graviton is unstable—and most strongly so for the softest gravitons—this sea carries negative energy. Second, the instability of the vacuum means that the vacuum energy must also be negative. The first effect occurs at one loop while the second starts at two loops, and both begin *affecting the background at order* $\kappa^4 H^4$. *Together the two effects act as a break on the* expansion of spacetime and cause the effective cosmological constant to relax [1, 2].

There are four tasks yet to accomplish. The first is debunking the argument that infra-red divergences can be avoided by using different wavefunctions. It is certainly true that the average of $\Psi_{\mu\nu}$ and its conjugate is much better behaved than either one as u approaches zero. In fact, it is easy to check that this average vanishes as u^3:

$$\begin{aligned} &\tfrac{1}{2}\Psi_{\mu\nu}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) + \tfrac{1}{2}\Psi^*_{\mu\nu}(u, \boldsymbol{x}; -\boldsymbol{k}, \lambda) \\ &\qquad = \frac{Hu}{\sqrt{2k}}\left[\cos(ku) - \frac{\sin(ku)}{ku}\right] \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\, \epsilon_{\mu\nu}(\boldsymbol{k}, \lambda) &(5.9a) \\ &\qquad \underset{u\to 0}{\longrightarrow} -\frac{Hk^2u^3}{3\sqrt{2k}} \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\, \epsilon_{\mu\nu}(\boldsymbol{k}, \lambda)\,. &(5.9b) \end{aligned}$$

The inclusion of even one such wavefunction on a vertex would be enough to cancel the factor of Ω^2 which is the origin of the infra-red problem. The difficulty is that the average spans only half the space of the mode functions. The orthogonal complement is spanned by wavefunctions of the form

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{i}\,\Psi_{\mu\nu}(u, \boldsymbol{x}; \boldsymbol{k}, \lambda) + \tfrac{1}{2}\mathrm{i}\,\Psi^*_{\mu\nu}(u, \boldsymbol{x}; -\boldsymbol{k}, \lambda) \\ &\qquad = \frac{Hu}{\sqrt{2k}}\left[\sin(ku) + \frac{\cos(ku)}{ku}\right] \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\, \epsilon_{\mu\nu}(\boldsymbol{k}, \lambda) &(5.10a) \\ &\qquad \underset{u\to 0}{\longrightarrow} \frac{H}{\sqrt{2k}\,k} \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\, \epsilon_{\mu\nu}(\boldsymbol{k}, \lambda)\,. &(5.10b) \end{aligned}$$

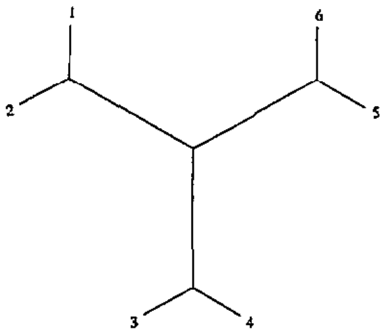

Figure 1. Unambiguously infra-red divergent contribution to the tree order 6-graviton scattering amplitude.

The 3-particle amplitude formed with three of these wavefunctions diverges as badly as ever.

The possibility of simply excluding the bad wavefunctions was studied in the context of light scalars by Börner and Dürr [5] who concluded that it was incompatible with causality. The reason is fairly obvious: purging some subset of the modes can be thought of as imposing a non-local constraint upon the field which disrupts microcausality. No doubt there would also be a problem with unitarity since there is no unexploited gauge symmetry which can drive decoupling. A further problem is that all the modes must be present in the flat space limit and it is difficult to accept that making a continuous change in one of the parameters of QCG can effect a discontinuous change in the space of states.

It is worth pointing out as well that, even if it was allowed, truncating the space of states could only solve the problem on vertices which possess an external leg. There would still be divergences from purely internal vertices such as that of the 6-graviton tree depicted in figure 1. It might be objected that this conclusion ignores the role of the propagators but a little thought reveals that no acceptable choice can rescue an internal vertex. Whatever we do to the space of states had better not change the propagator's imaginary part because this gives the classical retarded Green function. That the imaginary part of (4.1) is correct as it is can be seen directly from the linearized field equations [3]. (A particularly nice demonstration of this is that the linearized response to a point mass agrees with the well known de Sitter–Schwarzchild solution [3].) But the imaginary part of (4.1) fails to vanish at $u = 0$ so the infra-red divergence still occurs and is independent of the choice of vacuum.

A related idea is that the problem might have to do with using plane waves. It is certainly true that the causal structure of de Sitter space differs radically from that of flat space. One of the chief differences is that spacetime expands so rapidly as to preclude any overlap between the future light cones of events whose invariant length separation is more than about a Hubble radius. It follows that global constructs such as plane waves cannot really be observed, even if an infinite amount of time is allowed. Might we not avoid the infra-red problem by averaging over the unobservable portions of the asymptotic wavefront?

We can average over momenta to simulate the necessarily finite resolution in bandwidth, but examination of the amplitude reveals that this will not do any good unless the averaging functions also contain a very strong time dependence. The physics is fairly transparent: because the breakneck expansion red-shifts any finite *physical* momentum down to zero the wavefunctions must prevent observation of *coordinate* momenta below an ever increasing limit. The problem is that plane waves in the free theory really do evolve onto plane waves, and also in the full theory by virtue of its spatial translation invariance. This means that we must use constant wavepackets as in flat space. Failing to do so amounts to the introduction of a new interaction term into the QCG Lagrangian. One can get any answer this way but it is not QCG. If we are prepared to perpetrate such a mutilation it would be simpler and no less erroneous to drop the interaction Lagrangian altogether. Note as well that time-dependent momentum screening is not necessary to make sense of de Sitter QCD; in fact it would produce incorrect results.

Our second task is to debunk the notion that the infra-red divergence we have exhibited is merely some sort of compensation for the missing energy-conserving delta function of flat space. We have already explained that this is not so; in a conformally invariant theory this delta function is already (half) present, and the extra terms disappear when the range of u is extended to the entire real axis. Then too, there is only one energy conserving delta function in flat space, no matter how many particles are scattered. In contrast, the tree amplitudes of QCG show increasing problems as more particles are scattered. To illustrate this, and to give a concrete example of the total irrelevance of the propagator's bad behaviour at $k = 0$, we have computed the s-channel amplitude for 4-graviton scattering at tree order. The evidence of infra-red divergences on the pole terms of this suffices to make the point because neither the other channels nor the contribution from the 4-point vertex can cancel them.

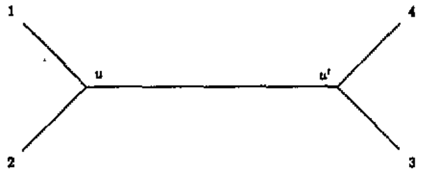

Figure 2. s-channel contribution to the tree order 4-graviton scattering amplitude.

The relevant diagram is shown in figure 2. We performed the calculation by first contracting two external wavefunctions onto the 3-point vertex and then sequentially contracting the resulting 25 distinct terms into one another through an exchange propagator (4.1). The tensor algebra was carried out by computer using Mertig's program, FeynCalc [19], and the result was checked against that of Sannan [16] by taking the flat space limit. Each distinct contribution has the form

$$- \kappa^2 H^2 (2\pi)^3 \delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4) \frac{C(\boldsymbol{k}_1, \lambda_1; \boldsymbol{k}_2, \lambda_2; \boldsymbol{k}_3, \lambda_3, \boldsymbol{k}_4, \lambda_4)}{\sqrt{2k_1}\,\sqrt{2k_2}\,\sqrt{2k_3}\,\sqrt{2k_4}}$$
$$\times \int_0^\infty \mathrm{d}u \, \exp\left[\mathrm{i}(k_1 + k_2)\left(u - \frac{1}{H}\right)\right] \int_0^\infty \mathrm{d}u' \, \exp\left[\mathrm{i}(k_3 + k_4)\left(u' - \frac{1}{H}\right)\right]$$
$$\times I(u, u', k_1, k_2, k_3, k_4) \qquad (5.11)$$

where the Cs are contractions of the polarization tensors and also possibly momenta, and the Is are functions of the two conformal interaction times and the norms of the momenta.

There are many, many terms. It suffices for the point we wish to make to consider the contraction, $C = (\epsilon_1 \epsilon_2)(\epsilon_3 \epsilon_4)$, and the coefficient of $\|\boldsymbol{k}_1 + \boldsymbol{k}_2\|^{-3} u^{-2} u'^{-2}$:

$$I = \left\{\tfrac{1}{4}[\boldsymbol{k}_1 \cdot \boldsymbol{k}_3 \, \boldsymbol{k}_1 \cdot \boldsymbol{k}_4 + \boldsymbol{k}_2 \cdot \boldsymbol{k}_3 \, \boldsymbol{k}_2 \cdot \boldsymbol{k}_4 + \boldsymbol{k}_1 \cdot \boldsymbol{k}_3 \, \boldsymbol{k}_2 \cdot \boldsymbol{k}_3 \boldsymbol{k}_1 \cdot \boldsymbol{k}_4 \, \boldsymbol{k}_2 \cdot \boldsymbol{k}_4] \right.$$
$$\left. + \tfrac{9}{32}\left(k_1^2 + k_2^2\right)\left(k_3^2 + k_4^2\right) + \tfrac{3}{8}(\boldsymbol{k}_1 \cdot \boldsymbol{k}_3 \boldsymbol{k}_2 \cdot \boldsymbol{k}_4 + \boldsymbol{k}_1 \cdot \boldsymbol{k}_4 \boldsymbol{k}_2 \cdot \boldsymbol{k}_3)\right\}$$
$$\times \frac{1}{\|\boldsymbol{k}_1 + \boldsymbol{k}_2\|^3 u^2 u'^2} + \text{(lower poles)}. \qquad (5.12)$$

This term is distinguished by its pole structure and its contraction of polarizations, hence the order of divergence increases as more particles are scattered, as was claimed. Note as well that there is no problem with the propagator for $k_1 + k_2 \neq 0$. As before, the problem is the factors of Ω^2 carried by each vertex and the fact that they are not compensated by either the external wavefunctions or the propagator.

So much for the second task; the third is to explain what the problems we have discovered in tree-order QCG scattering mean to the classical theory. The answer is, 'not much'. Although tree-order phenomena are completely determined by the classical action, they are not classical. To give one example, the quantization of allowed energies in any non-zero mode is surely a quantum effect, and this occurs in the free theory, before even tree-order interactions have been included. Quantization results because we impose the requirement that only normalizable energy eigenvectors correspond to states—otherwise one can solve the equation $H\,|V\rangle = E\,|V\rangle$ for *any* real number E. Another inherently quantum concept is the notion of transition probabilities.

Any disease of the classical theory should manifest itself in the solutions. The most direct connection between tree-order scattering amplitudes and classical solutions has been described by DeWitt [20]. (See section 4 of [21] for a detailed review of this formalism.) One starts with a general linearized solution, like our expansion (2.7) but with the creation and annihilation operators considered as arbitrary C-number fields. The classical field equations are then solved perturbatively, subject to Feynman boundary conditions, to produce what DeWitt calls the general 'scattering solution'. One obtains a generating functional for the connected, tree-order S-matrix by evaluating the classical action at this solution, adding an asymptotic surface term, and taking the exponential of i times the whole thing. S-matrix elements can be found by differentiating this generating functional with respect to the fields $a_{\mu\nu}(\boldsymbol{k}, \lambda)$ and $a^{\dagger}_{\mu\nu}(\boldsymbol{k}, \lambda)$, and then setting $a_{\mu\nu} = a^{\dagger}_{\mu\nu} = 0$.

The problem with QCG tree amplitudes implies similar infra-red divergences in DeWitt's scattering solutions. However, this is not a breakdown of the classical theory so much as it is a failure of Feynman boundary conditions. (As with the tree amplitudes, DeWitt's scattering solutions show infra-red divergences whether the 'asymptotic future' is defined as $u = 0$ on the original manifold or as $u = -\infty$ on the extended manifold.) One encounters no infra-red divergences in causal evolution over a finite period using retarded boundary conditions. A de Sitter energy functional can be defined on the open submanifold [22] which shows stability against perturbations whose wavelength is within the de Sitter horizon, and independent, semiclassical stability tests confirm this result [23]. We know less about the non-perturbative regime but it is worth noting that there are de Sitter generalizations to the black hole solutions of $H = 0$ gravity. So if there is a problem with the stability of the classical theory it seems very well disguised.

Our fourth task is to comment on previous work. Similar divergences were noted long ago by Tagirov [24] in the context of light, minimally coupled scalars. The analogous problem with tree-order Green functions has recently been studied by Sasaki *et al* [25]. While this work is very suggestive of a problem with the classical backgrounds of QCG—which all show exponential expansion locally, and which all tend to de Sitter over finite invariant distances—it might be dismissed instead as ruling out scalars which are either too light or not conformally coupled. (Indeed, this was precisely the conclusion reached by Tagirov.) Such a view is arguable because we lack direct evidence for *any* light, fundamental scalars, and because we have no reason to suppose that the various conjectured light scalars are not conformally coupled. No one can level this criticism on the infra-red divergences we have exposed in *graviton* scattering amplitudes. Acceptance of the de Sitter geometry as a background implies acceptance of the QCG Lagrangian from whence it came. The appearance of infra-red divergences in tree-order scattering amplitudes is an unavoidable consequence.

There has also been work in QCG by Floratos *et al* [14] who argued for an infra-red divergence in the graviton exchange contribution to two-body scattering for a conformally coupled scalar. This result is based upon an early form for the graviton propagator which was in use before the previously discussed problem was either recognized [9] or understood [3]. This propagator contains a term which shows logarithmic growth for increasing space-like separation, and the problem it gives in the aforementioned amplitude is that the spatial integral fails to converge. In fact we have seen that the naive mode sum for the true graviton propagator is well defined in momentum space; it is the position space expression which fails to converge. One might wonder about the conformal time integrations—which was not the problem discussed in [14]—but they are also finite for this amplitude. This is as it should be because it is only through the pseudo-graviton propagator that this process can differ from the finite result of flat space. Since the pseudo-graviton propagator is free of problems at $u = 0$, and since the positive powers of u it contains can be extracted as derivatives, there is no problem.

6. Free QCG on $T^3 \times R$

It remains to describe our proposal for a graviton propagator which can be used to any order in perturbation theory. Owing to the previously described infra-red problem the appropriate formalism is no longer 'in'–'out' *matrix elements* of asymptotic scattering theory but rather *expectation values* in the presence of a prepared initial state. A formalism for computing 'in'–'in' expectation values was worked out long ago by Schwinger [26], and has been studied more recently by Jordan [27]. In [2] we describe how it can be modified to give expectation values in the presence of a prepared state of free vacuum at $Hu = 1$ by merely dropping all interactions which occur for $u > (1/H)$. One important property of the formalism is its causality: no interactions from outside the past lightcone can influence an observation. It is immediately obvious that the region of conformal coordinate space over which we require the propagator is very small.

Now consider the means by which a de Sitter phase can arise in the early universe. We suppose, as in [1], that the cosmological constant is positive and not unreasonably small. However, this cosmological constant is little in evidence at first because the initial temperatures are so much greater than $(H/\kappa)^{1/2}$. There is no inflation since the thermal stress–energy far exceeds the cosmological contribution. Except for inhomogeneities, the scale factor expands in the characteristic fashion of a radiation dominated universe, as the square root of the co-moving time. There are no strong infra-red effects since thermal fluctuations disrupt the long-range correlations necessary to support them. For example, the wavefronts of graviton modes are rapidly broken up by scattering off of thermal fluctuations, so they cannot remain in phase long enough to experience an enhanced interaction overlap due to the red-shift.

The temperature of this early universe falls as it expands, however, the cooling cannot be uniform. Because of thermal fluctuations and initial inhomogeneities some regions will be cooler than others. It is unreasonable to expect uniformity over distances much larger than the inverse effective Hubble constant. (By $H_{\rm eff}(t)$ we mean the logarithmic time derivative of the co-moving scale factor, which would greatly exceed the constant H right after the big bang.) Eventually the temperature in a patch of space will fall below $(H/\kappa)^{1/2}$ and the cosmological constant begins to dominate the stress tensor. This sets off inflation.

The resulting red-shift very quickly makes the temperature so small that physics in the inflating patch becomes indistinguishable from that of zero-temperature quantum field theory starting from a prepared state. For simplicity we have taken this state to be 'free de Sitter vacuum' [2]—by which we mean minimum uncertainty wavefunctions for the zero modes and condition (13) for the non-zero modes—but different choices might be considered. The key point is that the patch has finite size; in fact its radius should be on the order of $1/H$. Of course the rest of the universe is not far behind in cooling and then inflating, but its quantum fluctuations are not correlated with those of the patch, and after even a little inflation they fall out of causal contact. We might equally well focus on a patch which is not the first to begin inflation; the point about finite size is equally valid. *It follows that we can ignore problems associated with arbitrarily long coordinate wavelengths because the inflating patch contains no such wavelengths.*

Formulating QCG on a patch is not difficult since the theory is still diagonal in Fourier space. The biggest change is that the modes become discrete. The issue of boundary conditions arises but only in a formal way because points in the interior very rapidly lose causal contact with the boundary. For simplicity we shall use periodic boundary conditions. Each component of the spatial coordinate vector runs over the range

$$-\frac{1}{2H} < x_i \leqslant \frac{1}{2H} \tag{6.1}$$

with identification between $x_i = -(1/2H)$ and $x_i = +(1/2H)$. This amounts to formulating QCG on the manifold $T^3 \times R$, which does admit the de Sitter background.

The allowed spatial momenta on the patch are

$$\boldsymbol{k} = 2\pi\, H\, \boldsymbol{n} \tag{6.2}$$

where $\boldsymbol{n}$ is any triplet of integers. The mode functions are unchanged from (2.8) and (2.10) for $\boldsymbol{k} \neq 0$. The free-field expansion (2.7) changes only to the extent that the integral is replaced by a sum, and the zero modes require special treatment as free particles:

$$\begin{aligned}\psi_{\mu\nu}(u,\boldsymbol{x}) = H^3 \sum_{\lambda}\sum_{\boldsymbol{k}\neq 0} &\{\Psi(u,\boldsymbol{x};\boldsymbol{k},\lambda)\, a_{\mu\nu}(\boldsymbol{k},\lambda) + \Psi^*(u,\boldsymbol{x};\boldsymbol{k},\lambda)\, a^{\dagger}_{\mu\nu}(\boldsymbol{k},\lambda)\}\\ &+\sum_{\lambda\in A}\left\{1\, q_{\mu\nu}(\lambda) + \tfrac{1}{3}H^2\left(H^3u^3-1\right) p_{\mu\nu}(\lambda)\right\}\\ &+\sum_{\lambda\in B,C}\left\{Hu\,(2-Hu)\, q_{\mu\nu}(\lambda) + H^2u\,(-1+Hu)\, p_{\mu\nu}(\lambda)\right\}.\end{aligned} \tag{6.3}$$

The zero-mode coordinates and momenta are defined at $u_0 = (1/H)$, which we take to the onset of inflation. The zero-mode variables commute as their symbols indicate

$$\sum_{\lambda,\lambda'\in A} [q_{\mu\nu}(\lambda),\, p_{\rho\sigma}(\lambda')] = \mathrm{i}\left[{}_{\mu\nu}T^{A}_{\rho\sigma}\right] \tag{6.4a}$$

$$\sum_{\lambda,\lambda'\in B} [q_{\mu\nu}(\lambda),\, p_{\rho\sigma}(\lambda')] = \mathrm{i}\left[{}_{\mu\nu}T^{B}_{\rho\sigma}\right] \tag{6.4b}$$

$$[q_{\mu\nu}(C),\, p_{\rho\sigma}(C)] = \mathrm{i}\left[{}_{\mu\nu}T^{C}_{\rho\sigma}\right]. \tag{6.4c}$$

The A, B and C polarization sums are changed from (2.9), (2.11) and (2.12) only by the replacement

$$(2\pi)^3\,\delta^3(\boldsymbol{k}-\boldsymbol{k}') \mapsto H^{-3}\,\delta_{\boldsymbol{k},\boldsymbol{k}'}\,. \tag{6.5}$$

Our 'vacuum'—which is really just the initial state at the onset of inflation—is defined by the conditions

$$a_{\mu\nu}(k,\lambda)\,|0\rangle = 0 \qquad \forall k \neq 0 \tag{6.6a}$$

$$p_{\mu\nu}(\lambda)\,|0\rangle = \mathrm{i}\frac{q_{\mu\nu}(\lambda)}{\Delta^2\psi(\lambda)}\,|0\rangle \tag{6.6b}$$

where the spreads—$\Delta\psi = (H/\sqrt{3})$ for the A modes and $\Delta\psi = (H/\sqrt{2})$ for the B–C modes—were chosen to minimize the dispersion in the infinite future. It is now straightforward to obtain the following result for the propagator:

$$\mathrm{i}[_{\mu\nu}\Delta_{\rho\sigma}](x;x') = \langle 0|\,\overline{T}\{\psi_{\mu\nu}(x)\,\psi_{\rho\sigma}(x')\}\,|0\rangle \tag{6.7a}$$

$$= \mathrm{i}\Delta_A(x;x')\left[_{\mu\nu}T^A_{\rho\sigma}\right] + \mathrm{i}\Delta_B(x,x')\left\{\left[_{\mu\nu}T^B_{\rho\sigma}\right] + \left[_{\mu\nu}T^C_{\rho\sigma}\right]\right\} \tag{6.7b}$$

$$\mathrm{i}\Delta_A(x;x') \equiv H^3\sum_{k\neq 0}\frac{H^2uu'}{2k}\left[1+\frac{1+\mathrm{i}k|\Delta u|}{k^2uu'}\right]\exp[-\mathrm{i}k|\Delta u| + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x}) - \epsilon k]$$
$$+\tfrac{1}{6}H^2\left[2 - H^3u^3 - H^3u'^3 + H^6u^3u'^3\right] + \tfrac{1}{6}\mathrm{i}H^2\left|H^3u'^3 - H^3u^3\right| \tag{6.7c}$$

$$\mathrm{i}\Delta_B(x;x') \equiv H^3\sum_{k\neq 0}\frac{H^2uu'}{2k}\exp[-\mathrm{i}k|\Delta u| + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x}) - \epsilon k]$$
$$+\tfrac{1}{4}H^4u^2u'^2\left[8 - 6Hu - 6Hu' + 5H^2uu'\right] + \tfrac{1}{2}\mathrm{i}H^4u^2u'^2\left|Hu' - Hu\right|. \tag{6.7d}$$

For Hu, Hu' and $H\Delta x$ all much less than unity the integral approximation to the sum ought to be excellent. For $\mathrm{i}\Delta_B$ we can extend the integration over the zero mode and the result is just (3.3c). For $\mathrm{i}\Delta_A$ we must cut the integral off at $k_0 \sim H$ and the result is

$$\mathrm{i}\Delta_A(x;x') \longrightarrow \frac{H^2uu'}{8\pi^2\Delta x}\left\{\frac{\exp(\mathrm{i}H\Delta x)}{\Delta x - |\Delta u| + \mathrm{i}\epsilon} + \frac{\exp(-\mathrm{i}H\Delta x)}{\Delta x + |\Delta u| - \mathrm{i}\epsilon}\right\}\exp[-\mathrm{i}H(|\Delta u| - \mathrm{i}\epsilon)]$$
$$+\frac{H^2}{4\pi^2}\frac{\sin(H\Delta x)}{H\Delta x}\exp[-\mathrm{i}H(|\Delta u| - \mathrm{i}\epsilon)]$$
$$-\frac{H^2}{8\pi^2}\{\mathrm{Ei}[\mathrm{i}H(\Delta x - |\Delta u| + \mathrm{i}\epsilon)] + \mathrm{Ei}[-\mathrm{i}H(\Delta x + |\Delta u| - \mathrm{i}\epsilon)]\}$$
$$+\tfrac{1}{6}H^2\left[2 - H^3u^3 - H^3u'^3 + H^6u^3u'^3\right] + \tfrac{1}{6}\mathrm{i}H^2\left|H^3u'^3 - H^3u^3\right| \tag{6.8a}$$
$$= \frac{1}{4\pi^2}\frac{H^2uu'}{\Delta x^2 - \Delta u^2 + \mathrm{i}\epsilon} - \frac{H^2}{8\pi^2}\ln\left[H^2(\Delta x^2 - \Delta u^2 + \mathrm{i}\epsilon)\right]$$
$$+\frac{H^2}{4\pi^2}\left\{1 - \gamma + \tfrac{4}{3}\pi^2\right\} + \mathrm{O}(Hu, Hu', H\Delta x). \tag{6.8b}$$

(This technique of representing the sum of all higher modes as an integral ought to be familiar from the approximations traditionally made to infer the statistical mechanics of a free Bose particle below the condensation temperature.) It is important to recognize that expression (6.8b) is only valid for small separations, which implies at most small extensions beyond the lightcone, that the apparent logarithmic growth outside the lightcone is a fiction can be seen from (6.8a); for large space-like separations $\mathrm{i}\Delta_A$ falls off as $1/(\Delta x)^2$ plus the zero-mode contribution.

Combining (6.7) and (6.8) gives the following approximate form for the pseudo-graviton propagator:

$$\begin{aligned} \mathrm{i}[_{\mu\nu}\Delta_{\rho\sigma}](u, x; u', x') \approx \frac{1}{4\pi^2} \frac{H^2 u u'}{\Delta x^2 - \Delta u^2 + \mathrm{i}\epsilon} \left[2\, \eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}\right] \\ - \frac{H^2}{8\pi^2} \ln\left[H^2\left(\Delta x^2 - \Delta u^2 + \mathrm{i}\epsilon\right)\right] \left[2\, \overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - 2\, \overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}\right]. \end{aligned} \tag{6.9}$$

We stress the familiarity and simplicity of this result. Except for the factor of $H^2 u u'$, the first term is just the *flat space* graviton propagator in the De Donder gauge. Further, the decoupling between tensor indices and functions of spacetime means that the tedious tensor algebra of loop diagrams can be isolated from the integrations. This was crucial in the two-loop computation we recently performed to establish that relaxation occurs in causal time evolution [2].

7. Conclusions

We have shown that the asymptotic scattering theory of QCG on a de Sitter background is very sick. Problems appear even in the lowest-order amplitudes, 3-graviton trees. The decay of a single graviton into two means that the one graviton state is not stable; the decay of the vacuum into three gravitons means that it is not stable either. These instabilites arise because energy is not conserved in the time-dependent background. The instabilites become arbitrarily large due to the presence of infra-red divergences. Since these divergences afflict even the lowest-order scattering amplitudes they cannot be avoided by the Lee–Nauenberg technique of restricting to transition probabilities between degenerate ensembles of states. The physical origin of the infra-red problem seems to be that inflation red-shifts the physical momenta of all gravitons to the same value—zero—thereby making the interaction overlap diverge. The problem has nothing to do with either the behaviour of the graviton propagator or mathematical problems with its definition, as witness the fact that the 3-point tree contains no propagators.

Our analysis proves that neither the vacuum nor the single graviton state are stable in QCG. Consideration of what this means for the stress tensor strongly suggests that one consequence is to slow the expansion of spacetime, thereby reducing the effective cosmological constant. We have elsewhere made the detailed case for this relaxation scenario [1, 2]. Although we did not need a definite form here for the graviton propagator our study of relaxation does of course require one. Our proposal (6.9) was motivated by consideration of the manner in which a de Sitter phase would arise in the early universe. In particular, it is very unlikely that an initial, correlated patch of de Sitter background could extend much beyond the Hubble radius of $1/H$. On such a finite patch the spatial momenta are discrete and the zero mode can be isolated for proper treatment as a free particle. The result is a well defined propagator which shows no growth outside the lightcone and which is almost as simple as the De Donder gauge propagator of flat space. This simplicity, coupled with our analogously simple expression (2.3) for the interaction Lagrangian, is what has made it possible to analyse high-order processes in QCG [2].

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