Communications in Theoretical Physics

GENERAL

Klein Gordon Oscillators in Commutative and Noncommutative Phase Space with Psudoharmonic Potential in the Presence and Absence Magnetic Field

To cite this article: Hassanabadi H. *et al* 2013 *Commun. Theor. Phys.* **60** 9

Commun. Theor. Phys. **60** (2013) 9–18 Vol. 60, No. 1, July 15, 2013

Klein Gordon Oscillators in Commutative and Noncommutative Phase Space with Psudoharmonic Potential in the Presence and Absence Magnetic Field

H. Hassanabadi,[1] S.S. Hosseini,[1,*] and Z. Molaee[2]

[1]Physics Department, Shahrood University of Technology, Shahrood, Iran P.O. Box 3619995161-316, Shahrood, Iran

[2]Physics Department, Semnan University, Semnan, Iran

(Received November 14, 2012; revised manuscript received March 4, 2013)

Abstract *We study the Klein–Gordon oscillator in commutative, noncommutative space, and phase space with psudoharmonic potential under magnetic field hence the other choice is studying the Klein–Gordon equation oscillator in the absence of magnetic field. In this work, we obtain energy spectrum and wave function in different situations by NU method so we show our results in tables.*

PACS numbers: 02.40.Gh, 03.65.-w, 03.65.Ge, 03.65.Pm
Key words: Klein–Gordon oscillator equation, noncommutative space, noncommutative phase space, psudoharmonic potential, NU method

1 Introduction

The Klein–Gordon equation provides us to study spin-zero particles. Solutions of the Klein–Gordon equation have received great attention for some interactions. Hence this equation has been also studied for different types of potentials such as Aharonov–Bohm (AB) potential[1] under generalized Morse potential,[2] with Hulthen-type potential,[3] studying of the Klein–Gordon–Yukawa problem.[4] Recently, noncommutative space theories were developed from string theory[5−6] so an extensive research has been done in several areas of physics[7] such as M-theory[8] quantum Hall-effect[9] quantum gravity.[10] Furthermore there are a lot of research papers about commutative space[11] and noncommutative phase space, for example Aharonov–Bohm effect,[12] Aharonov–Casher effect,[13] Landau levels.[14] The goal of this work is to study Klein–Gordon oscillator with Pseudoharmonic potential in the presence and absence of magnetic field in commutative, noncommutative space and noncommutative phase space, so we derive energy spectrum and corresponding wavefunctions by Nikiforov–Uvarov method. Hence Pseudoharmonic interaction was first investigated by Goldman *et al.* 1960.[15] This potential has been considered by many authors due to its importance in Chemical Physics, Molecular Physics, and other areas of Physics by [16–19]. This work is organized as follows. Section 2 includes introducing of noncommutative quantum mechanics, and also we have investigated the Klein–Gordon oscillator in a commutative space in the presence of the psudoharmonic potential in (1+2) dimensions in Sec. 3. In Sec. 4 we have studied Klein–Gordon Oscillator in commutative space in the presence of the psudoharmonic potential in a uniform magnetic field. Furthermore we have considered Klein–Gordon equation in the presence of the psudoharmonic potential in uniform magnetic field in noncommutative space in Sec. 5. Finally in Sec. 6 we have studied the Klein–Gordon equation in the presence of the psudoharmonic potential in the noncommutative phase space with magnetic field. Moreover we introduce the NU method in Appendix.

2 Noncommutative Quantum Mechanics

In quantum mechanics, several problems have been considered in noncommutative space. Some interesting results are dependent on geometric phases,[20−21] so the other effects which involve the dynamics of dipoles.[22] Noncommutative space maps onto commutative space by replacing the coordinates x^i and the momentum p^i by operators $\hat{x}^i$ and $\hat{p}^i$, which obey the relations,

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}, \quad [\hat{p}^i, \hat{p}^j] = 0, \quad [\hat{x}^i, \hat{p}^j] = \mathrm{i}\delta^{ij}, \tag{1}$$

where $\theta^{ij} = \theta\varepsilon^{ij}$ and ε^{ij} is the anti-symmetric tensor. Thus, we must change the x^i in the Klein–Gordon equation as,

$$\hat{x}^i \to x^i - \frac{1}{2}\theta\varepsilon^{ij}p^j. \tag{2}$$

And the Moyal product is defined by $(f * g)(x) = \exp(\mathrm{i}\theta^{ij}\partial_{x^i}\partial_{x^j})f(x^i)g(x^j)$, where f and g are arbitrary functions.

Some physical systems were analyzed in noncommutative phase space. In this case the operators x^i and p^i obey the commutation relations,[23]

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}, \quad [\hat{p}^i, \hat{p}^j] = \mathrm{i}\bar{\theta}^{ij}, \quad [\hat{x}^i, \hat{p}^j] = \mathrm{i}\delta^{ij}, \tag{3}$$

*Corresponding author, E-mail: seyedesamira.hosseini@gmail.com

where $\bar{\theta}^{ij}$ is also an anti-symmetric constant tensor, $\bar{\theta}^{ij} = \bar{\theta}\varepsilon^{ij}$. Thus, to map non-commutative phase space on commutative space, we must change $\hat{x}$ and $\hat{p}$ as

$$\hat{x}^i \to \lambda x^i - \frac{1}{2\lambda}\theta\varepsilon^{ij}p^j, \quad \hat{p}^i \to \lambda p^i + \frac{1}{2\lambda}\bar{\theta}\varepsilon^{ij}x^j, \tag{4}$$

where the constant λ is a scaling factor. The parameter λ, $\bar{\theta}$, and θ represent the noncommutative of the phase space, being related by the constraint[24−25]

$$\bar{\theta}^{ij}\theta^{ij} = \theta^{ij}\bar{\theta}^{ij} = \theta\bar{\theta}\mathbf{1} = 4\lambda^2(\lambda^2-1)\mathbf{1}\,, \tag{5}$$

where $\mathbf{1}$ is the identity matrix.

3 Klein–Gordon Oscillator in a Commutative Space in Presence of Pseudoharmonic Potential

The Klein–Gordon oscillator in two dimensions in commutative space can be defined by the following equation,

$$[-\nabla^2 + (m+S(r))^2]\psi_{n,l}(r) = [\varepsilon - V(r)]^2\psi_{n,l}(r)\,, \tag{6}$$

where the vector potential V is equal to scalar potential.

So the Klein–Gordon oscillator equation is,

$$((\boldsymbol{p} + \mathrm{i}m\omega\boldsymbol{r})\cdot(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r}))\psi_{n,l}(\boldsymbol{r}) = (\varepsilon_{n,l}^2 - m^2 - 2V(\varepsilon_{n,l}+m))\psi_{n,l}(\boldsymbol{r})\,, \tag{7}$$

In this situation the Pseudoharmonic potential has the form

$$V(r) = \frac{kr_e^2}{8}\Big(\frac{r}{r_e} - \frac{r_e}{r}\Big)^2.$$

The Pseudoharmonic explains roto-vibrational States of diatomic molecules, features of quantum dots and anti-dots, and nuclear rotations and vibrations,[26−29]

$$V(r) = V_0\Big(\frac{r}{r_0} - \frac{r_0}{r}\Big)^2. \tag{8}$$

The general form is

$$V(r) = \alpha r^2 + \frac{\beta}{r^2} + \gamma\,, \tag{9}$$

where

$$\alpha = \frac{k}{8}, \quad \beta = \frac{kr_e^4}{8}, \quad \gamma = \frac{-kr_e^2}{4}\,. \tag{10}$$

And we know where $V_0 = kr_e^2/8$ is the dissociation energy between two atoms in a solid with force constant k and r_e is the equilibrium bond length and β is a positive real constant. This proposed potential reduces to the Pseudoharmonic potential in the limiting case of $\beta = 0$.

$$\Big\{p^2 + r^2(m^2\omega^2 + 2\alpha\varepsilon_{n,l} + 2\alpha m) + \frac{1}{r^2}(2\beta\varepsilon_{n,l} + 2\beta m) - 2m\omega\hbar - \varepsilon_{n,l}^2 + m^2 + 2\gamma\varepsilon_{n,l} + 2\gamma m\Big\}\psi_{n,l}(r) = 0\,, \tag{11}$$

The Klein–Gordon oscillator equation in cylindrical coordinates is written in the form,

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \Big(\frac{1}{r}\Big)\frac{\mathrm{d}}{\mathrm{d}r} + \Big(\frac{1}{r^2}\Big)\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Big]\psi_{n,l}(r) + \Big\{-r^2(m^2\omega^2 + 2\alpha\varepsilon_{n,l} + 2\alpha m) - \frac{1}{r^2}(2\beta\varepsilon_{n,l} + 2\beta m) + 2m\omega\hbar + \varepsilon_{n,l}^2 - m^2 - 2\gamma\varepsilon_{n,l} - 2\gamma m\Big\}\psi_{n,l}(r) = 0\,, \tag{12}$$

by using the following ansatz

$$\psi_{n,l}(r) = \mathrm{e}^{\mathrm{i}l\phi}R_{n,l}(r)\,, \tag{13}$$

where l is an integer number. Thus,

$$\frac{\mathrm{d}^2R_{n,l}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_{n,l}(r)}{\mathrm{d}r} - \frac{l^2}{r^2}R_{n,l}(r) + \Big\{-r^2(m^2\omega^2 + 2\alpha\varepsilon_{n,l} + 2\alpha m) - \frac{1}{r^2}(2\beta\varepsilon_{n,l} + 2\beta m) + 2m\omega\hbar + \varepsilon_{n,l}^2 - m^2 - 2\gamma\varepsilon_{n,l} - 2\gamma m\Big\}R_{n,l}(r) = 0\,, \tag{14}$$

Using the following change of variables,

$$R_{n,l}(r) = r^{-1/2}U_{n,l}(r)\,, \tag{15}$$

Thus,

$$\frac{\mathrm{d}^2U_{n,l}(r)}{\mathrm{d}r^2} + \Big\{\frac{1}{r^2}\Big(\frac{1}{4} - (2\beta\varepsilon_{n,l} + 2\beta m + l^2)\Big) - r^2(m^2\omega^2 + 2\alpha\varepsilon_{n,l} + 2\alpha m) + (2m\omega\hbar + \varepsilon_{n,l}^2 - m^2 - 2\gamma\varepsilon_{n,l} - 2\gamma m)\Big\}U_{n,l}(r) = 0\,. \tag{16}$$

By using the following change of variables $S(r) = r^2$, one has

$$\frac{\mathrm{d}^2U_{n,l}}{\mathrm{d}s^2} + \frac{(1/2)}{s}\frac{\mathrm{d}U_{n,l}}{\mathrm{d}s} + \Big\{\frac{1}{s^2}\Big(\frac{1}{16} - \frac{\beta\varepsilon_{n,l}}{2} - \frac{\beta m}{2} - \frac{l^2}{4}\Big) - \Big(\frac{m^2\omega^2}{4} + \frac{\alpha\varepsilon_{n,l}}{2} + \frac{\alpha m}{2}\Big) + \frac{1}{s}\Big(\frac{m\omega\hbar}{2} + \frac{\varepsilon_{n,l}^2}{4} - \frac{m^2}{4} - \frac{\gamma\varepsilon_{n,l}}{2} - \frac{\gamma m}{2}\Big)\Big\}U_{n,l} = 0\,, \tag{17}$$

With the change of variables

$$U_{n,l}'' + \frac{(1/2)}{s}U_{n,l}' + \Big(\frac{1}{s^2}\Big)[-Cs^2 + As - B]U_{n,l} = 0\,, \tag{18}$$

where

$$A=\frac{m\omega\hbar}{2}+\frac{\varepsilon_{n,l}^2}{4}-\frac{m^2}{4}-\frac{\gamma\varepsilon_{n,l}}{2}-\frac{\gamma m}{2},\quad B=-\Big(\frac{1}{16}-\frac{\beta\varepsilon_{n,l}}{2}-\frac{\beta m}{2}-\frac{l^2}{4}\Big),\quad C=\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}\,. \tag{19}$$

So the energy levels are given by

$$\begin{aligned}&(2n+1)\sqrt{\Big(\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}\Big)}-\frac{m\omega\hbar}{2}-\frac{\varepsilon_{n,l}^2}{4}+\frac{m^2}{4}+\frac{\gamma\varepsilon_{n,l}}{2}+\frac{\gamma m}{2}\\&+2\sqrt{\Big(\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}\Big)\Big(\frac{\beta\varepsilon_{n,l}}{2}+\frac{\beta m}{2}+\frac{l^2}{4}\Big)}=0\,.\end{aligned} \tag{20}$$

Table 1 Energy eigenvalues in commutative space. ($m=\alpha=\beta=\gamma=1$, $\omega=0.1$).

State $\lvert n,l\rangle$	$\varepsilon_{n,l}$
$\lvert 1,0\rangle$	9.590 122 686
$\lvert 2,0\rangle$	11.057 776 96
$\lvert 2,1\rangle$	11.127 736 47
$\lvert 3,0\rangle$	12.396 254 25
$\lvert 3,1\rangle$	12.457 610 04
$\lvert 3,2\rangle$	12.632 769 74

We determine the energy Eigen values relation as Eq. (20) then we show the energy vs. n,l in Table 1 and wavefunctions

$$\begin{aligned}\psi_n(s)&=s^{\alpha_{12}}\,\mathrm{e}^{\alpha_{13}s}\mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}s)\\&=s^{\frac{1}{4}+\sqrt{\frac{\beta\varepsilon}{2}+\frac{\beta m}{2}+\frac{l^2}{4}}}\,\mathrm{e}^{-(\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon}{2}+\frac{\alpha m}{2})^{1/2}s}\mathrm{L}_n^{2\sqrt{\frac{\beta\varepsilon}{2}+\frac{\beta m}{2}+\frac{l^2}{4}}}\Big(2\sqrt{\Big(\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon}{2}+\frac{\alpha m}{2}\Big)}s\Big)\,,\end{aligned} \tag{21}$$

We have, for $\alpha=\beta=\gamma=m=l=1$,

$$\psi_n(s)=s^{\frac{1}{4}+\sqrt{\frac{\varepsilon_{n,l}}{2}+\frac{3}{4}}}\,\mathrm{e}^{-(\frac{\omega^2}{4}+\frac{\varepsilon_{n,l}}{2}+\frac{1}{2})^{1/2}s}\mathrm{L}_n^{2\sqrt{\frac{\varepsilon_{n,l}}{2}+\frac{3}{4}}}\left(2\sqrt{\Big(\frac{\omega^2}{4}+\frac{\varepsilon_{n,l}}{2}+\frac{1}{2}\Big)}s\right)\,. \tag{22}$$

4 Klein-Gordon Oscillator Equation in the Presence of the Psudoharmonic Potentials in Uniform Magnetic Field in Commutative Space

In the same way as for the study of the Klein–Gordon equation in the presence of normal space-like $\boldsymbol{A}=(\boldsymbol{B}\times\boldsymbol{r})/2$ and vector potential $\boldsymbol{p}\to\boldsymbol{p}-e\boldsymbol{A}/c$ Oscillator under the magnetic field

$$\{(\boldsymbol{p}-\mathrm{i}m\omega\boldsymbol{r})\cdot(\boldsymbol{p}+\mathrm{i}m\omega\boldsymbol{r})\}\psi_{n,l}(r)=\Big\{\varepsilon_{n,l}^2-m^2-2\Big(\alpha r^2+\frac{\beta}{r^2}+\gamma\Big)(\varepsilon_{n,l}+m)\Big\}\psi_{n,l}(r)\,, \tag{23}$$

The Klein–Gordon equation in commutative space constant and uniform magnetic field in the presence of quasi-oscillatory potentials

$$\Big(\Big(\boldsymbol{p}-e\frac{\boldsymbol{B}\times\boldsymbol{r}}{2c}\Big)-\mathrm{i}m\omega\boldsymbol{r}\Big)\Big(\Big(\boldsymbol{p}-e\frac{\boldsymbol{B}\times\boldsymbol{r}}{2c}\Big)+\mathrm{i}m\omega\boldsymbol{r}\Big)\psi_{n,l}(r)=\Big\{\varepsilon_{n,l}^2-m^2-2\Big(\alpha r^2+\frac{\beta}{r^2}+\gamma\Big)(\varepsilon_{n,l}+m)\Big\}\psi_{n,l}(r)\,. \tag{24}$$

With the Physical and mathematical calculations

$$\Big\{p^2+r^2\Big(m^2\omega^2+\frac{e^2B^2}{4c^2}+2\alpha\varepsilon_{n,l}+2\alpha m\Big)+\frac{1}{r^2}(2\beta\varepsilon_{n,l}+2\beta m)-\frac{eBL_z}{c}-2m\omega\hbar-\varepsilon_{n,l}^2+m^2+2\gamma\varepsilon_{n,l}+2\gamma m\Big\}\psi_{n,l}(r)=0\,, \tag{25}$$

we used wave function $\psi_{n,l}(r)=\mathrm{e}^{\mathrm{i}l\varphi}R_{n,l}(r)$, and with representation $R_{n,l}(r)=r^{-1/2}U_{n,l}(r)$ and $S(r)=r^2$

$$U_{n,l}''+\frac{(1/2)}{s}U_{n,l}'+\Big(\frac{1}{s^2}\Big)[-Cs^2+As-B]U_{n,l}=0\,. \tag{26}$$

In the same way with the change of variables,

$$\begin{aligned}&A=\frac{m\omega\hbar}{2}+\frac{eBL_z}{4c}+\frac{\varepsilon_{n,l}^2}{4}-\frac{m^2}{4}-\frac{\gamma\varepsilon_{n,l}}{2}-\frac{\gamma m}{2},\quad B=-\Big(\frac{1}{16}-\frac{\beta\varepsilon_{n,l}}{2}-\frac{\beta m}{2}-\frac{l^2}{4}\Big),\\&C=\frac{m^2\omega^2}{4}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}+\frac{e^2B^2}{16c^2}\,.\end{aligned} \tag{27}$$

The energy eigenvalues relation as follows

$$(2n+1)\sqrt{\frac{m^2\omega^2}{4}+\frac{e^2B^2}{16c^2}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}}-\frac{m\omega\hbar}{2}-\frac{eBL_z}{4c}-\frac{\varepsilon^2}{4}+\frac{m^2}{4}+\frac{\gamma\varepsilon_{n,l}}{2}+\frac{\gamma m}{2}$$
$$+2\sqrt{\Big(\frac{m^2\omega^2}{4}+\frac{e^2B^2}{16c^2}+\frac{\alpha\varepsilon_{n,l}}{2}+\frac{\alpha m}{2}\Big)\Big(\frac{\beta\varepsilon_{n,l}}{2}+\frac{\beta m}{2}+\frac{l^2}{4}\Big)}=0\,. \tag{28}$$

We determine the energy Eigen values relation as Eq. (28) then we show the energy vs. n, l in Table 2 from which the results of the previous section can be derived.

Table 2 Energy eigenvalues in commutative space in uniform magnetic field. ($e = m = \alpha = \beta = \gamma = 1$, $B = 0.079\ 884\ 046\ 5$, $\omega = 0.1$).

State $\lvert n,l\rangle$	$\varepsilon_{n,l}$
$\lvert 1,0\rangle$	9.590 663 46
$\lvert 2,0\rangle$	11.063 676 72
$\lvert 2,1\rangle$	11.133 578 09
$\lvert 3,0\rangle$	12.401 451 03
$\lvert 3,1\rangle$	12.462 767 26
$\lvert 3,2\rangle$	12.637 819 53

5 Klein–Gordon Equation in the Presence of the Psudoharmonic Potential in Uniform Magnetic Field in Noncommutative Space

The time-independent Klein–Gordon equation in noncommutative space can be written in the form

$$\Big(\Big(\boldsymbol{p}-e\frac{\boldsymbol{B}\times\boldsymbol{r}}{2c}\Big)-\mathrm{i}m\omega\boldsymbol{r}\Big)\cdot\Big(\Big(\boldsymbol{p}-e\frac{\boldsymbol{B}\times\boldsymbol{r}}{2c}\Big)+\mathrm{i}m\omega\boldsymbol{r}\Big)*\psi_{n,l}(r)=\{\varepsilon_{n,l}^2-m^2-2V(\varepsilon_{n,l}+m)\}*\psi_{n,l}(r)\,, \tag{29}$$

So by using the Moyal product, we can map the noncommutative space onto the commutative space, so the coordinates change as follows

$$\boldsymbol{r}\to\boldsymbol{r}+\frac{1}{2\hbar}\boldsymbol{\theta}\times\boldsymbol{p},\quad\text{or}\quad x\to x-\frac{1}{2\hbar}\theta p_y,\quad y\to y+\frac{1}{2\hbar}\theta p_x\,. \tag{30}$$

Thus, we can rewrite Eq. (29),

$$\Big(\boldsymbol{p}-\frac{e}{2c}(\boldsymbol{B}\times\boldsymbol{r})-\frac{e}{4c\hbar}(\boldsymbol{B}\times(\boldsymbol{\theta}\times\boldsymbol{p}))+\mathrm{i}m\omega\boldsymbol{r}+\frac{\mathrm{i}m\omega}{2\hbar}(\boldsymbol{\theta}\times\boldsymbol{p})\Big)\cdot\Big(\boldsymbol{p}-\frac{e}{2c}(\boldsymbol{B}\times\boldsymbol{r})-\frac{e}{4c\hbar}(\boldsymbol{B}\times(\boldsymbol{\theta}\times\boldsymbol{p}))-\mathrm{i}m\omega\boldsymbol{r}$$
$$-\frac{\mathrm{i}m\omega}{2\hbar}(\boldsymbol{\theta}\times\boldsymbol{p})\Big)\psi_{n,l}(r)=\{\varepsilon_{n,l}^2-m^2-2V(\varepsilon_{n,l}+m)\}\psi_{n,l}(r)\,. \tag{31}$$

We have

$$\Big\{\Big(1+\frac{eB\theta}{2c\hbar}+\Big(\frac{eB\theta}{4c\hbar}\Big)^2+\Big(\frac{m\omega\theta}{2\hbar}\Big)^2\Big)p^2+\Big(m^2\omega^2+\Big(\frac{eB}{2c}\Big)^2+2\alpha\varepsilon_{n,l}+2\alpha m\Big)r^2+\frac{1}{r^2}(2\beta\varepsilon_{n,l}+2\beta m)$$
$$-\Big(\frac{eB}{c}+\frac{\mathrm{m}^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4c^2\hbar}\Big)\mathrm{L}_z-2m\omega\hbar-\frac{m\omega eB\theta}{2c}-\varepsilon_{n,l}^2+m^2+2\gamma\varepsilon_{n,l}+2\gamma m\Big\}\psi_{n,l}(r)=0\,. \tag{32}$$

So by choosing the coefficient of the L_z being zero we have,

$$\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4c^2\hbar}=0\,, \tag{33}$$

Non-commutative parameter

$$\theta=-\frac{eB/c}{m^2\omega^2/\hbar+e^2B^2/4c^2\hbar}=-\frac{4eBc\hbar}{4m^2\omega^2c^2+e^2B^2}\,. \tag{34}$$

So the magnetic field can be obtained,

$$B=\Big(-2\hbar\pm\frac{2\sqrt{\hbar^2-\theta^2m^2\omega^2}}{e\theta}\Big)c\,. \tag{35}$$

We determine the values of magnetic field in Table 3.

Table 3 Magnetic fields in noncommutative space.

State $\|n,l\rangle$	$\varepsilon_{n,l}$	State $\|n,l\rangle$	$\varepsilon_{n,l}(\theta=\bar{\theta}=0,\ \lambda=1)$
$\|1,0\rangle$	9.952 161 161	$\|1,0\rangle$	9.590 122 686
$\|2,0\rangle$	11.059 780 52	$\|2,0\rangle$	11.057 776 96
$\|2,1\rangle$	11.129 746 96	$\|2,1\rangle$	11.127 736 47
$\|3,0\rangle$	12.398 277 34	$\|3,0\rangle$	12.197 095 85
$\|3,1\rangle$	12.459 641 44	$\|3,1\rangle$	12.458 469 28
$\|3,2\rangle$	12.634 823 89	$\|3,2\rangle$	12.633 677 12

With the change of variables, we have

$$\varsigma = 1+\frac{eB\theta}{2c\hbar}+\Big(\frac{eB\theta}{4c\hbar}\Big)^2+\Big(\frac{m\omega\theta}{2\hbar}\Big)^2,\quad \xi = m^2\omega^2+\Big(\frac{eB}{2c}\Big)^2+2\alpha\varepsilon+2\alpha m,\quad \zeta = 2\beta\varepsilon_{n,l}+2\beta m\,,$$
$$\wp = -\Big(\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4c^2\hbar}\Big)L_z-2m\omega\hbar-\frac{m\omega eB\theta}{2c}-\varepsilon^2+m^2+2\gamma\varepsilon+2\gamma m\,, \tag{36}$$

we arrive at

$$U''(r)+\frac{(1/2)}{s}U'(r)+\Big(\frac{1}{s^2}\Big)[-Cs^2+As-B]U(r)=0\,, \tag{37}$$

where

$$A=-\frac{\wp}{4\varsigma},\quad B=-\Big(\frac{1}{16}-\frac{l^2}{4}-\frac{\zeta}{4\varsigma}\Big),\quad C=\frac{\xi}{4\varsigma}\,. \tag{38}$$

Then the space noncommutative Klein–Gordon equation oscillator energy levels are given by

$$\begin{aligned}&(2n+1)\sqrt{\Big(m^2\omega^2+\Big(\frac{eB}{2c}\Big)^2+2\alpha\varepsilon_{n,l}+2\alpha m\Big)\Big(1+\frac{eB\theta}{2c\hbar}+\Big(\frac{eB\theta}{4c\hbar}\Big)^2+\Big(\frac{m\omega\theta}{2\hbar}\Big)^2\Big)}\\&\quad-\Big(\frac{eB}{2c}+\frac{\mathrm{m}^2\omega^2\theta}{2\hbar}+\frac{e^2B^2\theta}{8c^2\hbar}\Big)L_z-m\omega\hbar-\frac{m\omega eB\theta}{4c}-\frac{\varepsilon_{n,l}^2}{2}+\frac{m^2}{2}+\gamma\varepsilon_{n,l}+\gamma m\\&\quad+\sqrt{\Big(m^2\omega^2+\Big(\frac{eB}{2c}\Big)^2+2\alpha\varepsilon_{n,l}+2\alpha m\Big)\Big(l^2\Big(1+\frac{eB\theta}{2c\hbar}+\Big(\frac{eB\theta}{4c\hbar}\Big)^2+\Big(\frac{m\omega\theta}{2\hbar}\Big)^2\Big)+2\beta\varepsilon_{n,l}+2\beta m\Big)}=0\,.\end{aligned} \tag{39}$$

We determine the energy Eigen values relation as Eq. (39) then we show the energy levels vs. n,l in Table 4.

Table 4 Energy eigenvalues in noncommutative space with uniform magnetic field. ($e=L=m=\alpha=\beta=\gamma=1$, $B=0.079\ 884\ 046\ 5$, $\theta=0.05$).

State $\|n,l\rangle$	$\varepsilon_{n,l}(\theta=0.05)$
$\|1,0\rangle$	9.585 880 324
$\|2,0\rangle$	11.055 770 44
$\|2,1\rangle$	11.125 900 56
$\|3,0\rangle$	12.395 986 40
$\|3,1\rangle$	12.457 477 46
$\|3,2\rangle$	12.633 005 77

Table 5 Magnetic fields in noncommutative phase space.

ω	θ	$\bar{\theta}$	B	λ
0.1	0	0	0	±1
0.1	0.08	0.09	$-0.090\ 638\ 593\ 32,\ -50.089\ 038\ 61$	$\pm1.000\ 897\ 983$
0.1	0.09	0.05	$-0.050\ 483\ 894\ 40,\ -44.493\ 488\ 33$	$\pm1.000\ 561\ 711$
0.1	0.05	0.01	$-0.010\ 498\ 753\ 45,\ -80.009\ 469\ 86$	$\pm1.000\ 062\ 490$
0.1	0.01	0.001	$-0.001\ 099\ 997\ 995\ 525, -400.000\ 900$	$\pm1.000\ 001\ 250$

Thus, the wavefunctions can be obtained as

$$\psi_n(s)=s^{\alpha_{12}}\,\mathrm{e}^{\alpha_{13}s}\mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}s)=s^{\frac{1}{4}+\sqrt{\frac{\zeta}{4\varsigma}}}\,\mathrm{e}^{-(\frac{\xi}{4\varsigma})s}\mathrm{L}_n^{2\sqrt{\frac{\zeta}{4\varsigma}}}\left(2\sqrt{\Big(\frac{\xi}{4\varsigma}\Big)}s\right)$$

$$= s^{\frac{1}{4}+\sqrt{\frac{l^2}{4}+\frac{2\beta\varepsilon_{n,l}+2\beta m}{4(1+eB\theta/2c\hbar+(eB\theta/4c\hbar)^2+(m\omega\theta/2\hbar)^2)}}}\,\mathrm{e}^{-\left(\frac{m^2\omega^2+(eB/2c)^2+2\alpha\varepsilon_{n,l}+2\alpha m}{4(1+eB\theta/2c\hbar+(eB\theta/4c\hbar)^2+(m\omega\theta/2\hbar)^2)}\right)^{1/2}s}$$
$$\times \mathrm{L}_n^{2\sqrt{\frac{l^2}{4}+\frac{2\beta\varepsilon_{n,l}+2\beta m}{4(1+eB\theta/2c\hbar+(eB\theta/4c\hbar)^2+(m\omega\theta/2\hbar)^2)}}}\left(2\left(\frac{(m^2\omega^2+(\frac{eB}{2c})^2+2\alpha\varepsilon+2\alpha m)}{4(1+\frac{eB\theta}{2c\hbar}+(\frac{eB\theta}{4c\hbar})^2+(\frac{m\omega\theta}{2\hbar})^2)}\right)^{1/2}s\right), \tag{40}$$

for $\alpha=\beta=\gamma=m=l=e=1$, we have,

$$\psi_n(s) = s^{\frac{1}{4}+\sqrt{\frac{\frac{3}{4}+\frac{\varepsilon}{2}}{(1+\frac{B\theta}{2}+(\frac{B\theta}{4})^2+(\frac{\omega\theta}{2})^2}}}\,\mathrm{e}^{-\left(\frac{\frac{3}{4}+\frac{\varepsilon}{2}}{(1+\frac{B\theta}{2}+(\frac{B\theta}{4})^2+(\frac{\omega\theta}{2})^2)}\right)s}$$
$$\times \mathrm{L}_n^{2\sqrt{\frac{\frac{3}{4}+\frac{\varepsilon}{2}}{(1+\frac{B\theta}{2}+(\frac{B\theta}{4})^2+(\frac{\omega\theta}{2})^2)}}}\left(2\left(\frac{\frac{\omega^2}{4}+\frac{(B/2)^2}{4}+\frac{\varepsilon}{2}+\frac{1}{2}}{(1+\frac{B\theta}{2}+(\frac{B\theta}{4})^2+(\frac{\omega\theta}{2})^2)}\right)s\right). \tag{41}$$

if we do not consider the effect of the magnetic field,

$$\psi_n(s) = s^{\frac{1}{4}+\sqrt{(\frac{3}{4}+\frac{\varepsilon}{2})}}\,\mathrm{e}^{-(\frac{\omega^2}{4}+\frac{\varepsilon}{2}+\frac{1}{2})s}\mathrm{L}_n^{2\sqrt{(\frac{3}{4}+\frac{\varepsilon}{2})}}\Big(2\Big(\frac{\omega^2}{4}+\frac{\varepsilon}{2}+\frac{1}{2}\Big)s\Big). \tag{42}$$

It is shown that the Klein–Gordon oscillator in a noncommutative space has a similar behavior with commutative space. By solving Eq. (32) in the absence of magnetic field. We show that the energy eigenvalues vs. n, l in Table 5, so we have the same relations with Eqs. (20) and (21).

6 Klein–Gordon Equation in the Noncommutative Phase Space in the Presence of Psudoharmonic Potential with Magnetic Field

In this section, we analyze the Klein–Gordon equation in non-commutative phase space, to map noncommutative phase space into commutative space, we use the following expressions,

$$x \to x\lambda - \frac{1}{2\lambda}\theta p_y, \quad p_x \to p_x\lambda + \frac{1}{2\lambda}\bar{\theta}y, \quad y \to y\lambda + \frac{1}{2\lambda}\theta p_x, \quad p_y \to p_y\lambda - \frac{1}{2\lambda}\bar{\theta}x\,, \tag{43}$$

where the scale factor λ is an arbitrary constant parameter. So Eq. (29) becomes

$$\Big(\Big(\boldsymbol{p}\lambda - \frac{\bar{\boldsymbol{\theta}}\times\boldsymbol{r}}{2\lambda} - \frac{e\boldsymbol{B}}{2c}\times\Big(\boldsymbol{r}\lambda+\frac{\boldsymbol{\theta}\times\boldsymbol{p}}{2\lambda}\Big) - \mathrm{i}m\omega\Big(\boldsymbol{r}\lambda+\frac{\boldsymbol{\theta}\times\boldsymbol{p}}{2\lambda}\Big)\Big)\cdot\Big(\boldsymbol{p}\lambda - \frac{\bar{\boldsymbol{\theta}}\times\boldsymbol{r}}{2\lambda} - \frac{e\boldsymbol{B}}{2c}\times\Big(\boldsymbol{r}\lambda+\frac{\boldsymbol{\theta}\times\boldsymbol{p}}{2\lambda}\Big)$$
$$+\mathrm{i}m\omega\Big(\boldsymbol{r}\lambda+\frac{\boldsymbol{\theta}\times\boldsymbol{p}}{2\lambda}\Big)\Big)\Big)\psi_{n,l}(r) = \{\varepsilon_{n,l}^2 - m^2 - 2V(\varepsilon_{n,l}+m)\}\psi_{n,l}(r)\,, \tag{44}$$

with the Physical and mathematical calculations

$$\Big\{\Big\{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}\Big\}p^2+\Big\{\frac{\bar{\theta}^2}{4\lambda^2}+\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2+\frac{eB\bar{\theta}}{2c}+2\alpha m+2\alpha\varepsilon_{n,l}\Big\}r^2$$
$$+\frac{1}{r^2}(2\beta m+2\beta\varepsilon_{n,l}) - \Big\{+\bar{\boldsymbol{\theta}}+\frac{e\lambda^2\boldsymbol{B}}{c}+m^2\omega^2\boldsymbol{\theta}+\frac{e\theta\bar{\theta}B}{4c\lambda^2}+\frac{e^2B^2\theta}{4c^2}\Big\}L_z+2m\omega\hbar\lambda^2$$
$$+\frac{m\omega e\hbar B\theta}{c}-\frac{m\omega\theta\bar{\theta}\hbar}{2\lambda^2}-\varepsilon_{n,l}^2+m^2+2\gamma\varepsilon_{n,l}+2\gamma m\Big\}\psi_{n,l}(r)=0\,, \tag{45}$$

If the coefficient of angular momentum is zero, we can obtain the noncommutative parameters.

$$-\bar{\theta}+\frac{e\lambda^2B}{c}+m^2\omega^2\theta+\frac{e\theta\bar{\theta}B}{4c\lambda^2}+\frac{e^2B^2\theta}{4c^2}=0\,, \tag{46}$$

Noncommutative parameter,

$$\theta = -\frac{4(-\bar{\theta}c+e\lambda^2B)c\lambda^2}{4m^2\omega^2c^2\lambda^2+e\bar{\theta}Bc+e^2B^2\lambda^2}\,, \tag{47}$$

And the magnetic field,

$$B = \pm\frac{1}{2}\frac{-4\lambda^4\mp\theta\bar{\theta}+\sqrt{16\lambda^8-8\lambda^4\theta\bar{\theta}+\theta^2\bar{\theta}^2-16\theta^2m^2\omega^2\lambda^4}}{\theta\lambda^2e}c\,. \tag{48}$$

By Eq. (48) we can show that the values of the magnetic field in Table 5.

$$\eta = \lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}, \quad \nu = \frac{\bar{\theta}^2}{4\lambda^2}+\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2+\frac{eB\bar{\theta}}{2c}+2\alpha m+2\alpha\varepsilon, \quad \kappa = 2\beta m+2\beta\varepsilon\,,$$
$$\mu = -\Big(-\bar{\boldsymbol{\theta}}+\frac{e\lambda^2\boldsymbol{B}}{c}+m^2\omega^2\boldsymbol{\theta}+\frac{e\theta\bar{\theta}B}{4c\lambda^2}+\frac{e^2B^2\theta}{4c^2}\Big)L_z+2m\omega\hbar\lambda^2+\frac{m\omega e\hbar B\theta}{c}-\frac{m\omega\theta\bar{\theta}\hbar}{2\lambda^2}-\varepsilon^2+m^2+2\gamma\varepsilon+2\gamma m\,. \tag{49}$$

We can rewrite Eq. (45) in the form,

$$U''_{n,l}+\frac{(1/2)}{s}U'_{n,l}+\Big(\frac{1}{s^2}\Big)[-Cs^2+As-B]U_{n,l}=0\,, \tag{50}$$

where

$$A=-\frac{\mu}{4\eta},\quad B=-\Big(\frac{1}{16}-\frac{l^2}{4}-\frac{\kappa}{4\eta}\Big),\quad C=\frac{\nu}{4\eta}\,. \tag{51}$$

We obtain,

$$(2n+1)\sqrt{\nu\eta}+\frac{\mu}{2}+\sqrt{l^2\eta+\nu\kappa}=0\,, \tag{52}$$

we find,

$$\begin{aligned}&(2n+1)\sqrt{\Big(\frac{\bar\theta^2}{4\lambda^2}-\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2+\frac{eB\bar\theta}{2c}+2\alpha m+2\alpha\varepsilon\Big)\Big(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}\Big)}\\&-\Big(+\frac{\bar\theta}{2}+\frac{e\lambda^2\boldsymbol{B}}{2c}+\frac{m^2\omega^2\boldsymbol{\theta}}{2}+\frac{e\theta\bar\theta B}{8c\lambda^2}+\frac{e^2B^2\theta}{8c^2}\Big)\mathrm{L}_z+m\omega\hbar\lambda^2+\frac{m\omega e\hbar B\theta}{2c}-\frac{m\omega\theta\bar\theta\hbar}{4\lambda^2}+\frac{m^2}{2}-\frac{\varepsilon^2}{2}+\gamma\varepsilon+\gamma m\\&+\sqrt{\Big(\frac{\bar\theta^2}{4\lambda^2}+\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2+\frac{eB\bar\theta}{2c}+2\alpha m+2\alpha\varepsilon\Big)\Big(l^2\Big(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}\Big)+2\beta m+2\beta\varepsilon\Big)}=0\,.\end{aligned} \tag{53}$$

We determine the energy eigenvalues relation as Eq. (53) then we show the energy levels vs. n,l in Table 6.

Table 6 Energy eigenvalues in noncommutative phase space. ($\theta=0.05$, $\bar\theta=0.01$, $\lambda=1.000\ 062\ 490$, $B=0.079\ 884\ 046\ 5$). $B=0$.

State $\lvert n,l\rangle$	$\varepsilon_{n,l}$	State $\lvert n,l\rangle$	$\varepsilon_{n,l}(\theta=\bar\theta=0,\lambda=1)$
$\lvert 1,0\rangle$	9.590 548 501	$\lvert 1,0\rangle$	9.590 122 686
$\lvert 2,0\rangle$	11.058 417 66	$\lvert 2,0\rangle$	11.057 776 96
$\lvert 2,1\rangle$	11.128 397 88	$\lvert 2,1\rangle$	11.127 736 47
$\lvert 3,0\rangle$	12.397 082 65	$\lvert 3,0\rangle$	12.397 095 85
$\lvert 3,1\rangle$	12.458 456 10	$\lvert 3,1\rangle$	12.458 469 28
$\lvert 3,2\rangle$	12.633 663 98	$\lvert 3,2\rangle$	12.633 677 12

And,

$$\psi_n(s)=s^{\frac14+\sqrt{\frac{1}{16}-B}}\,\mathrm{e}^{-\sqrt{C}s}\mathrm{L}_n^{2\sqrt{\frac{1}{16}-B}}(2\sqrt{C}s)\,, \tag{54}$$

Thus,

$$\psi_n(s)=s^{\frac14+\sqrt{\frac{l^2}{4}+\frac{\kappa}{4\eta}}}\,\mathrm{e}^{-(\nu/4\eta)s}\mathrm{L}_n^{2\sqrt{\frac{l^2}{4}+\frac{\kappa}{4\eta}}}(2\sqrt{(\nu/4\eta)}s)\,. \tag{55}$$

We can rewrite the Eq. (55) as

$$\begin{aligned}\psi_n^{(NCp)}(s)=&\,s^{\frac14+\sqrt{\frac{l^2}{4}+\frac{2\beta\varepsilon_{n,l}+2\beta m}{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}}}}\exp\left\{-\Big[\frac{\frac{\bar\theta^2}{4\lambda^2}+\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2+\frac{eB\bar\theta}{2c}+2\alpha m+2\alpha\varepsilon_{n,l}}{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c}}\Big]^{1/2}s\right\}\\&\times\mathrm{L}_n^{2\sqrt{\frac{l^2}{4}+4\frac{2\beta\varepsilon_{n,l}+2\beta m}{(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c})}}}\left(2\sqrt{\frac{\frac{\bar\theta^2}{4\lambda^2}+\frac{e^2\lambda^2B^2}{4c^2}+m^2\omega^2\lambda^2-\frac{eB\bar\theta}{2c}+2\alpha m+2\alpha\varepsilon_{n,l}}{(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}+\frac{e^2B^2\theta^2}{16c^2\lambda^2}+\frac{eB\theta}{2c})}}s\right)\,.\end{aligned} \tag{56}$$

For $\alpha=\beta=\gamma=m=l=e=\lambda=1$ and $\bar\theta=0$, we have,

$$\begin{aligned}\psi_n(s)=&\,s^{\frac14+\sqrt{\frac{\frac34+\frac{\varepsilon}{2}}{1+\frac{\omega^2\theta^2}{4}+\frac{B^2\theta^2}{16}+\frac{B\theta}{2}}}}\exp\left\{-\Big[\frac{(\frac{B^2}{4}+\omega^2+2+2\varepsilon)}{4(1+\frac{\omega^2\theta^2}{4}+\frac{B^2\theta^2}{16}+\frac{B\theta}{2})}\Big]^{1/2}s\right\}\\&\times\mathrm{L}_n^{2\sqrt{\frac{\frac34+\frac{\varepsilon}{2}}{1+\frac{\omega^2\theta^2}{4}+\frac{B^2\theta^2}{16}+\frac{B\theta}{2}}}}\left(2\sqrt{\Big(\frac{(\frac{B^2}{4}+\omega^2+2+2\varepsilon)}{4(1+\frac{\omega^2\theta^2}{4}+\frac{B^2\theta^2}{16}+\frac{B\theta}{2})}\Big)s}\right)\,.\end{aligned} \tag{57}$$

Equations in the absence of magnetic field in noncommutative phase space,

$$\Big[\Big(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}\Big)p^2+(\bar\theta^2/4\lambda^2+m^2\omega^2\lambda^2+2\alpha m+2\alpha\varepsilon)r^2+\frac{1}{r^2}(2\beta m+2\beta\varepsilon)-(\vec{\bar\theta}+m^2\omega^2\vec\theta)\mathrm{L}_z$$

$$+ 2m\omega\hbar\lambda^2 - \frac{m\omega\theta\bar{\theta}\hbar}{2\lambda^2} - \varepsilon^2 + m^2 + 2\gamma\varepsilon + 2\gamma m\Big]\psi = 0\,. \tag{58}$$

Using the change of variables for Eq. (58)

$$\eta = \lambda^2 + \frac{m^2\omega^2\theta^2}{4\lambda^2}, \quad \nu = \frac{\bar{\theta}^2}{4\lambda^2} + m^2\omega^2\lambda^2 + 2\alpha m + 2\alpha\varepsilon\,,$$

$$\kappa = 2\beta m + 2\beta\varepsilon, \quad \mu = -\bar{\theta}L_z - m^2\omega^2\theta L_z - 2m\omega\hbar\lambda^2 - \frac{m\omega\theta\bar{\theta}\hbar}{2\lambda^2} + \varepsilon_{n,l}^2 - m^2 - 2\gamma\varepsilon_{n,l} - 2\gamma m\,, \tag{59}$$

We can rewrite the Eq. (58) as

$$U'' + \frac{(1/2)}{s}U' + \Big(\frac{1}{s^2}\Big)[-Cs^2 + As + B]U = 0\,, \tag{60}$$

where

$$A = \frac{\mu}{4\eta}, \quad B = \frac{1}{16} - \frac{l^2}{4} - \frac{\kappa}{4\eta}, \quad C = \frac{\nu}{4\eta}\,. \tag{61}$$

We obtain

$$(2n+1)\sqrt{\nu\eta} + \mu/2 + \sqrt{l^2\eta + \nu\kappa} = 0\,, \tag{62}$$

$$(2n+1)\sqrt{\Big(\frac{\bar{\theta}^2}{4\lambda^2} + m^2\omega^2\lambda^2 + 2\alpha m + 2\alpha\varepsilon\Big)\Big(\lambda^2 + \frac{m^2\omega^2\theta^2}{4\lambda^2}\Big)} + \frac{\bar{\theta}L_z}{2} - \frac{m^2\omega^2\theta L_z}{2} - m\omega\hbar\lambda^2 - \frac{m\mu\theta\bar{\theta}\hbar}{4\lambda^2} - \frac{\varepsilon_{n,l}^2}{2} + \frac{m^2}{2}$$
$$+ \gamma\varepsilon_{n,l} + \gamma m + \sqrt{\Big(\frac{\bar{\theta}^2}{4\lambda^2} + m^2\omega^2\lambda^2 + 2\alpha m + 2\alpha\varepsilon\Big)\Big(l^2\Big(\lambda^2 + \frac{m^2\omega^2\theta^2}{4\lambda^2}\Big) + 2\beta m + 2\beta\varepsilon\Big)} = 0\,. \tag{63}$$

We determine the energy eigenvalues relation as Eq. (63) then we show the energy levels vs. n, l in Table 7.

Table 7 ($\bar{\theta} = 0.01$, $\lambda = 1.000\ 062\ 490$, $B = 0$, $\theta = 0.05$, $\omega = 0.1$).

State $\lvert n, l\rangle$	$\varepsilon_{n,l}$
$\lvert 1, 0\rangle$	9.591 077 675
$\lvert 2, 0\rangle$	11.058 683 42
$\lvert 2, 1\rangle$	11.128 643 41
$\lvert 3, 0\rangle$	12.397 143 36
$\lvert 3, 1\rangle$	12.458 500 76
$\lvert 3, 2\rangle$	12.633 664 91

So the wavefunctions can be derived as,

$$\psi_n(s) = s^{\frac{1}{4}+\sqrt{\frac{1}{16}-B}}\,\mathrm{e}^{-cs}\mathrm{L}_n^{2\sqrt{\frac{1}{16}-B}}(2\sqrt{C}s)\,, \tag{64}$$

Other hand,

$$\psi_n(s) = \mathrm{s}^{\frac{1}{4}+\sqrt{\frac{l^2}{4}+\frac{\kappa}{4\eta}}}\,\mathrm{e}^{-(\frac{\nu}{4\eta})s}\mathrm{L}_n^{2\sqrt{\frac{l^2}{4}+\frac{\kappa}{4\eta}}}\Big(2\sqrt{\Big(\frac{\nu}{4\eta}\Big)}s\Big)\,. \tag{65}$$

We can rewrite the Eq. (65) as,

$$\psi_{n,l}(s) = s^{\frac{1}{4}+\sqrt{\frac{l^2}{4}+\frac{2\beta\varepsilon_{n,l}+2\beta m}{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}}}}\,\mathrm{e}^{-\left(\frac{\frac{\bar{\theta}^2}{4\lambda^2}+m^2\omega^2\lambda^2+2\alpha m+2\alpha\varepsilon_{n,l}}{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}}\right)^{1/2}s}$$
$$\times\,\mathrm{L}_n^{2\sqrt{\frac{l^2}{4}+4\frac{2\beta\varepsilon_{n,l}+2\beta m}{(\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2})}}}\left(2\sqrt{\frac{\frac{\bar{\theta}^2}{4\lambda^2}+m^2\omega^2\lambda^2+2\alpha m+2\alpha\varepsilon_{n,l}}{\lambda^2+\frac{m^2\omega^2\theta^2}{4\lambda^2}}}s\right)\,, \tag{66}$$

for $\alpha = \beta = \gamma = m = l = \lambda = 1$ and $\bar{\theta} = 0$, we have,

$$\psi_n(s) = s^{1/4+\sqrt{(3/4+\varepsilon/2)}}\,\mathrm{e}^{-(\omega^2/4+1/2+\varepsilon/4)s}\mathrm{L}_n^{2\sqrt{(3/4+\varepsilon/2)}}\Big(2\sqrt{(\omega^2/4+1/2+\varepsilon/4)}\,s\Big)\,. \tag{67}$$

7 Conclusion

In this work we study commutative space, noncommutative space and noncommutative phase space and obtain the energy eigenvalues and eigenfunctions for Klein–Gordon equation in the presence of Pseudoharmonic potential in commutative space. Our results are shown in Table 1. Therefore we solve the recent problem under uniform magnetic

field in commutative space. Also we can see the energy eigenvalues for different levels in Table 2. In addition we found the energy spectrum corresponding the eigenfunctions for Klein–Gordon equation under Pseudoharmonic potential in uniform magnetic field in noncommutative space and noncommutative phase space. Due to the noncommutative effect, the result of these sections will be reduced to the results in commutative space. Let us note finally, the exact wave functions and corresponding energy eigenvalues are obtained by NU method contrary to some other work which uses perturbations theory, we used an exact treatment.

Appendix: Nikiforov–Uvarov Method

This analytical technique of mathematical physics solves second order differential equations. Let us consider the following differential equation[30]

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{(\alpha_1-\alpha_2 s)}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{\{-\xi_1 s^2+\xi_2 s-\xi_3\}}{[s(1-\alpha_3 s)]^2}\Big\}\psi=0\,, \tag{A1}$$

According to the NU method, the eigen-functions and eigen-energies respectively are[31]

$$\psi_n(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}\mathrm{P}_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s)\,, \tag{A2}$$

where the Jacobi polynomial is,

$$\mathrm{P}_n^{(c,d)}(z)=2^{-n}\sum_{p=0}^{n}\binom{n+c}{P}\binom{n+d}{n-p}(1-z)^{n-p}(1+z)^p, \tag{A3}$$

$$\mathrm{P}_n^{(c,d)}(z)=\frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\Big(\frac{z-1}{2}\Big)^r, \tag{A4}$$

and

$$\binom{n}{r}=\frac{n!}{r!\,(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}\,. \tag{A5}$$

And

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)\big(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\big)+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0\,, \tag{A6}$$

where

$$\begin{aligned}&\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\quad \alpha_6=\alpha_5^2+\xi_1,\quad \alpha_7=2\alpha_4\alpha_5-\xi_2,\quad \alpha_8=\alpha_4^2+\xi_3\,,\\ &\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\quad \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\quad \alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\quad \alpha_{12}=\alpha_4+\sqrt{\alpha_8}\,,\\ &\alpha_{13}=\alpha_5-\big(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\big)\,.\end{aligned} \tag{A7}$$

In the special case of $\alpha_3=0$,

$$\lim_{\alpha_3\to 0}\mathrm{P}_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-\alpha_3 s)=\mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}s),\quad \lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=\mathrm{e}^{\alpha_{13}s} \tag{A8}$$

and

$$\psi_n(s)=s^{\alpha_{12}}\,\mathrm{e}^{\alpha_{13}s}\mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}s)\,. \tag{A9}$$

References

[1] C.R. Hagen, Phys. Rev. D **48** (1993) 5935.

[2] A. Arda and R. Sever, Phys. Scr. **82** (2010) 065007.

[3] H. Hassanabadi, S. Zarrinkamar, and H. Rahimov, Commun. Theor. Phys. **56** (2011) 423.

[4] A. Arda and R. Sever, J. Math. Phys. **52** (2011) 092101.

[5] N. Seiberg and E. Witten, J. High Energy Phys. **09** (1999) 032.

[6] N. Seiberg, L. Susskind, and N. Toumbas, J. High Energy Phys. **06** (2000) 044.

[7] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977.

[8] A. Connes, M.R. Douglas, and A. Schwarz, J. High Energy Phys. **02** (1998) 003.

[9] L. Susskind, [arXiv:hep-th/0101029].

[10] J.W. Moffat, Phys. Lett. B **493** (2000) 142.

[11] C. Duval and P.A. Horvathy, Phys. Lett. B **479** (2000) 284.

[12] M. Chaichian, M.M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86** (2001) 2716.

[13] B. Mirza and M. Zarei, Eur. Phys. J. C **32** (2004) 0.

[14] P.A. Horvathy, Ann. Phys. (N.Y.) **299** 128 (2002).

[15] I.I. Gol'dman, V.D. Krivchenkov, V.I. Kogan, and V.M. Galitskii, *Problems in Quantum Mechanics*, Academic Press, New York (1960) p. 308.

[16] S.H. Dong, J. Phys. A: Math. Gen. **36** (2003) 4977.
[17] S.H. Dong, D. Morales, and J. García-Ravelo, Int. J. Mod. Phys. E **16** (2007) 189.
[18] H. Hassanabadi, Z. Molaee, and A. Boumali, Found. Phys. **43** (2013) 225.
[19] S.H. Dong and M. Lozada-Cassou, Int. J. Mod. Phys. B **19** (2005) 4219.
[20] M. Chaichian, M.M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86** (2001) 2716.
[21] K. Li and J. Wang, Eur. Phys. J. C **50** (2007) 1007.
[22] X. Calmet and M. Selvaggi, Phys. Rev. D **74** (2006) 037901.
[23] H. Hassanabadi, Z. Molaee, and S. Zarrinkamar, Eur. Phys. J. C **72** (2012) 2217.
[24] K. Li, J. Wang, and C. Chen, Mod. Phys. Lett. A **20** (2005) 2165.
[25] S. Dulat and K. Li, Chin. Phys. C **32** (2008) 92.
[26] C. Acatrinei, J. High Energy Phys. **0109** (2001) 007.
[27] J. Gamboa, *et al.*, Int. J. Mod. Phys. A **17** (2002) 2555.
[28] J. Gamboa, *et al.*, Mod. Phys. Lett. A **16** (2001).
[29] B. Muthukumar and P. Mitra, Phys. Rev. D **66** (2002).
[30] A.F. Nikiforov and V.B. Uvarov, *Special Function of Mathematical Physics*, Birkhauser, Basel (1988).
[31] Z. Molaee, M. Ghominejad, H. Hassanabadi, and S. Zarrinkamar, Eur. Phys. J. Plus **127** (2012) 116.