LETTERS AND COMMENTS

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LETTERS AND COMMENTS

A little help for a better understanding and application of Faraday’s law

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Abstract

In this letter, we examine Faraday’s law of induction, analysing the electromotive force generated by a Lorentz force and the one generated by an electric field due to a changing magnetic field. We obtain the result in a didactically simple and appealing way. The final formula is derived considering explicitly the dependence of the magnetic field on the space coordinates, which is often neglected in standard textbooks.

(Some figures may appear in colour only in the online journal)

1. Introduction

Let us remember the following relation that describes well-known electromagnetic induction:

\[
\text{EMF} = -\frac{d\Phi(t)}{dt} = -\frac{d}{dt} \int_{S(t)} \vec{B}(t, \vec{r}(t)) \cdot \hat{n} \, dS,
\]

(1)

where EMF is the electromotive force, \(S(t)\) is a surface that has a circuit, which we will denote with \(I\), as its boundary, and the integral is the magnetic flux. First of all, we stress that Faraday’s law describes two different phenomena: the EMF generated by a Lorentz force on a moving wire and the one generated by an electric field due to a time-varying magnetic field. This is well explained in many standard textbooks (see, e.g. [1, 2]). While there are no doubts about the way to calculate the derivative of the flux (1) in the two extremal cases, many interesting new books (see, e.g. [3, 4]) and papers [5–16] focussed on the study of the right approach to the more general case, the right way to define the EMF as well as the relativistic point of view. The debate is still alive after so many years. In this letter, we want to suggest an appealing way to demonstrate Faraday’s law in the general case of surface and field both simultaneously changing, considering explicitly the dependence of the magnetic field on the space coordinates. This dependence is often neglected.
in most standard textbooks that usually put in evidence only the time-dependent part of the field.

So we have to consider a closed circuit embedded in a non-constant magnetic field, with a wire moving with velocity \( \vec{v} \). This motion of the circuit and the changing field induce along the wire a displacement of charges with velocity \( \vec{u} = \vec{v} + \vec{u} \) (where \( \vec{u} \) is an infinitesimal part of the circuit) so that the total velocity of the charge is \( \vec{V}_{\text{charge}} = \vec{v} + \vec{u} \). A point of the circuit, which has a position \( \vec{r}(t) \) at the initial time, will be found at a position \( \vec{r}(t + \Delta t) \) after a time interval \( \Delta t \). Now our aim is to perform the time differentiation of the flux integral (1) in order to obtain explicitly a didactically useful formula to understand better the physics of electromagnetic induction. We also want to advise that other different forms of Faraday’s law, often considered equivalent, may create confusion because they give the right results only if some very special hypotheses are satisfied.

This letter is organized as follows. First, we write the demonstration of Faraday’s law in a general form in section 2, and then, in section 3, conclusions are drawn.

### 2. Faraday’s law

We consider the following integral function:

\[
\Phi(t) = \int_{S(t)} \vec{B} \cdot (\vec{\nabla} \times \vec{B}) \cdot \hat{n} dS
\]

and its derivative

\[
\frac{d\Phi(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{S(t+\Delta t)} \vec{B} \cdot (\vec{\nabla} \times \vec{B}) \cdot \hat{n} dS - \int_{S(t)} \vec{B} \cdot (\vec{\nabla} \times \vec{B}) \cdot \hat{n} dS}{\Delta t}.
\]

We start from the second Maxwell equation \( \vec{\nabla} \cdot \vec{E} = 0 \) and its integral form \( \Phi_\sigma = \int_\sigma \vec{B} \cdot \hat{n} dS = 0 \), where \( \sigma \) is a closed surface. From figure 1, we have

\[
\hat{n}_3 \, d\Sigma = -d\vec{\nabla} \cdot d\vec{T},
\]

\[
\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \Delta \vec{B} = \int_0^{\Delta \vec{B}} d\vec{\nabla} = \vec{\nabla} \Delta \vec{r},
\]

\[
\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \Delta \vec{B} = \int_0^{\Delta \vec{B}} d\vec{\nabla} = \vec{\nabla} \Delta \vec{r},
\]

where \( \vec{\nabla} = \frac{d\vec{\nabla}}{dt} = \frac{d\vec{r}}{dt} \) is constant during the small interval of time \( \Delta t \) and \( 0 < \Delta t < \Delta \vec{r} \). So we can compute the area of the surface \( \Sigma \) that, for small \( \Delta t \), is

\[
A_\Sigma = \frac{1}{2} \left[ \oint_{\Gamma(t)} |\Delta \vec{\nabla} \cdot d\vec{T} | + \oint_{\Gamma(t+\Delta t)} |\Delta \vec{\nabla} \cdot d\vec{T} | \right] \simeq \oint_{\Gamma(t)} |\vec{\nabla} \Delta t \cdot d\vec{T} |.
\]

and considering the closed surface \( \sigma \) formed by \( S(t), S(t + \Delta t) \) and the lateral surface \( \Sigma \) at the fixed instant \( t + \Delta t \), we can write

\[
\Phi_\sigma = \int_{S(t+\Delta t)} \vec{B} \cdot (\vec{\nabla} \times \vec{B} + \Delta \vec{B}) \cdot \hat{n}_1 dS + \int_{S(t)} \vec{B} \cdot (\vec{\nabla} \times \vec{B} + \Delta \vec{B}) \cdot \hat{n}_2 dS
\]

\[
+ \int_{\Sigma} \vec{B} \cdot (\vec{\nabla} \times \vec{B} + \Delta \vec{B}) \cdot \hat{n}_3 d\Sigma = 0.
\]
Figure 1. The evolution of the circuit, embedded in a magnetic field, from the time $t$ to the time $t + \Delta t$. (The magnetic field is not explicitly drawn because the demonstration of Faraday’s law in the text is general and does not depend on the particular form of the vector field $\mathbf{B}$.)

Then, using equation (4) and the convention for the normal $\hat{n}$ of a surface enclosed in a circuit, which must have the current flow counterclockwise in the loop $\hat{n} = \hat{n}_1 = -\hat{n}_2$, we obtain

$$\int_{S(t + \Delta t)} \mathbf{B}(t + \Delta t, \mathbf{r} + \Delta \mathbf{r}) \cdot \hat{n} dS - \int_{S(t)} \mathbf{B}(t + \Delta t, \mathbf{r}) \cdot \hat{n} dS$$

$$- \int_{\Sigma} \mathbf{B}(t + \Delta t, \mathbf{r} + \Delta \mathbf{r}) \cdot (d\mathbf{r} \wedge d\mathbf{T}) = 0.$$  

(9)

We have, using Taylor’s theorem,

$$\mathbf{B}(t + \Delta t, \mathbf{r}) = \mathbf{B}(t, \mathbf{r}) + \frac{\partial \mathbf{B}}{\partial t} \bigg|_{\Delta t = 0} \Delta t,$$

(10)

$$\mathbf{B}(t + \Delta t, \mathbf{r} + \Delta \mathbf{r}) = \mathbf{B}(t + \Delta t, \mathbf{r}) + (\mathbf{\nabla} \cdot \Delta \mathbf{r}) \mathbf{B} \bigg|_{\Delta t = 0} \Delta t.$$  

(11)
Substituting these equations into (9), we can write
\[
\int_{S(t+\Delta t)} \mathbf{B} (t + \Delta t, \mathbf{r} + \Delta \mathbf{r}) \cdot \mathbf{n} dS - \int_{S(t)} \mathbf{B} (t, \mathbf{r}) \cdot \mathbf{n} dS
\]
\[
= \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \bigg|_{\Delta t=0} \Delta t \cdot \mathbf{n} dS + \int_{\Sigma} (\mathbf{\nabla} \cdot \mathbf{\nabla}) \mathbf{B} (t + \Delta t, \mathbf{r}) \cdot \mathbf{d} \mathbf{r} \wedge d \mathbf{T}
\]
\[
+ \int_{S(t)} \mathbf{n} \cdot \Delta \mathbf{r} \mathbf{d}(\Delta r) \cdot (\mathbf{\nabla} \cdot \mathbf{\nabla}) \mathbf{B} \bigg|_{\Delta \mathbf{r}=0} \cdot (\mathbf{\nabla} \wedge d \mathbf{T})
\]
\[
= \Delta t \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \bigg|_{\Delta t=0} \cdot \mathbf{n} dS + \Delta t \oint_{l} \mathbf{B} \cdot \mathbf{\nabla} \cdot d \mathbf{T}
\]
\[
+ \frac{\Delta t^2}{2} \oint_{l} (\mathbf{\nabla} \cdot \mathbf{\nabla}) \mathbf{B} \bigg|_{\Delta \mathbf{r}=0} \cdot \mathbf{\nabla} \wedge d \mathbf{T}. \quad (12)
\]

Then, we can compute the derivative of the integral function using the definition of equation (3):
\[
\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{S(t+\Delta t)} \mathbf{B} (t + \Delta t, \mathbf{r} + \Delta \mathbf{r}) \cdot \mathbf{n} dS - \int_{S(t)} \mathbf{B} (t, \mathbf{r}) \cdot \mathbf{n} dS \right]
\]
\[
= \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS - \oint_{l} (\mathbf{\nabla} \cdot \mathbf{\nabla}) \mathbf{B} \cdot d \mathbf{T}. \quad (13)
\]

Finally, we obtain
\[
\text{EMF} = - \frac{d\Phi(t)}{dt} = - \int_{S(t)} \left( \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} dS + \oint_{l} (\mathbf{\nabla} \cdot \mathbf{\nabla}) \mathbf{B} \cdot d \mathbf{T}. \quad (14)
\]

### 3. Conclusions

In our opinion, equation (14) is the only correct way to write down Faraday’s law in the general case, and comparing it with the correct definition of the EMF in terms of the Lorentz force divided by the charge \( q \)
\[
\text{EMF} = \oint_{l} (\mathbf{E} + \mathbf{\nabla} \times \mathbf{\text{charge}}) \cdot d \mathbf{T} = \oint_{l} (\mathbf{E} + \mathbf{\nabla} \times \mathbf{B}) \cdot d \mathbf{T}, \quad (15)
\]
the Maxwell equation
\[
\mathbf{\nabla} \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (16)
\]
is recovered.

But some other versions of Faraday’s law often proposed in several contexts do not always give the right result. For example, we examine
\[
\text{EMF} = - \frac{d\Phi(t)}{dt} = - \int_{S(t)} \left[ \frac{\partial \mathbf{B}}{\partial t} - \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{B}) \right] \cdot \mathbf{n} dS. \quad (17)
\]
obtained from (14) using Stokes’ theorem and
\[ \text{EMF} = -\frac{d\Phi(t)}{dt} = -\int_{S(t)} \left[ \frac{\partial \vec{B}}{\partial t} + (\vec{\nabla} \cdot \vec{B}) \right] \cdot \hat{n} d\Sigma \]  (18)
derived from (17) using the well-known vectorial identity
\[ \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{B}) = (\vec{B} \cdot \vec{\nabla}) \vec{\nabla} - (\vec{\nabla} \cdot \vec{B}) \vec{B} + \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{\nabla}) \]  (19)
and supposing a constant velocity field. When the field \( \vec{B} \) is constant everywhere and the
velocity has a constant value just on a moving line (a part of the circuit) and is zero elsewhere
(i.e. \( \vec{\nabla} \) is not a continuous field with continuous space derivatives), equations (17) and (18)
erroneously predict a vanishing EMF.

We remember that Stokes’ theorem can be applied to equation (14) only if the vector
field \( \vec{\nabla} \wedge \vec{B} \) is a continuously differentiable vector function. Even if we generalize our
framework taking into account also discontinuous functions, the result does not change.
If we consider the famous example of a rectangular circuit on the xy-plane with only one side
moving along the x-axis with constant velocity and a magnetic field directed along the z-axis,
we can approximate the velocity field with a constant in an infinitesimal part \( \epsilon \) (the thickness
of the moving wire) of the domain. In this way, the velocity field can be described by a
rectangular function
\[ \vec{v}(x, y) = \vec{v}_0 \text{rect}\left[ \frac{x - x_0(t)}{\epsilon} \right] = u(x - x_0(t) + \epsilon/2) - u(x - x_0(t) - \epsilon/2) \]
where \( x_0(t) = x_0 + v_0 t \) and \( u(x) \) is the step function. Computing the divergence of velocity in
the last term of equation (19), we obtain a difference of two delta functions that, after being
integrated, give the difference between two values of the magnetic field at a distance \( \epsilon \) from
each other. Hence, if \( \vec{B} \) is constant, the surface integral in (17) is vanishing.

So, from the didactic point of view, it is better to show Faraday’s law in a way that obtains
as its final result formula (14), without introducing other almost equivalent equations that
can generate confusion when applied by the students in solving exercises. In conclusion, we
have derived the right formula (14) in the general case of a field dependent on time and space
coordinates and we have also clarified that some other ways to express Faraday’s law cannot
be considered fully equivalent and could lead to incorrect results.

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