European Journal of Physics

The solar neutrino problem

To cite this article: J Konijn 1999 *Eur. J. Phys.* **20** 399

Eur. J. Phys. **20** (1999) 399–407. Printed in the UK PII: S0143-0807(99)95377-1

The solar neutrino problem

J Konijn
NIKHEF, PO Box 41882, 1009 DB Amsterdam, The Netherlands

Received 30 June 1999, in final form 5 July 1999

Abstract. One of the most intriguing problems in science is and has always been the understanding of our Sun. Despite their elusiveness, millions upon millions of neutrinos hit each human being every second. Unless a neutrino scores a direct hit on an atomic nucleus (which only rarely occurs), it passes through without leaving a hint of its passage. Yet neutrinos are amongst the most important particles in the Universe. Because they emerge unscathed from the core of the Sun, they carry information about solar processes that we cannot observe otherwise. Approximately 97% of the energy released by the Sun is produced in the form of charged particles or photons. The remaining $\sim$3% appears in the form of the kinetic energy of neutrinos. Ever since Davis and his collaborators started to observe the solar neutrino flux in the Homestake mine, we have for almost 30 years been faced with the problem *of observing only about 30% of the expected neutrino flux*. The solar neutrino problem originates from the discrepancy between the expected solar neutrino flux, as calculated by the standard solar model (SSM), and experimental results.

(Some figures in this article appear in black and white in the printed version.)

1. Introductory remarks

Neutrinos are elusive but essential ingredients in the electro-weak standard model. They have neither electric charge nor do they feel the strong interaction; consequently, they can only be detected through their weak interactions.

Apart from their role in electro-weak interactions, they play an important role in the Universe and its history (cosmology). Neutrinos are all around us: they were produced by the early Big Bang nucleosynthesis and at present in the Sun and in supernovas. We know a great deal about the interaction of neutrinos with matter and about the sources of neutrinos, but very little about the fundamental properties of neutrinos, such as their mass, magnetic moment, lifetime and number of lepton families. As there are $\sim 110\ \nu/\mathrm{cm}^3$ of each type in our Universe, a massive neutrino is a good candidate for explaining part of the cosmological dark-matter puzzle. Of all the dark matter candidates, the light neutrino is the only one that is actually known to exist in nature; its mass should then be in the range of a few eV. The search for a non-zero neutrino mass involves the interplay between different subfields of physics: astrophysics, particle physics, nuclear physics, cosmology, and to some extent geophysics. Until now, the masses of the neutrinos have been assumed to be zero, but experimentally only upper limits have been set.

The present upper limits are 18 MeV for ν_τ, 160 keV for ν_μ to $\sim$ 3.5 eV for ν_e. In contrast, the photon mass is known to be less than 10^{-21} eV owing to the long range of the electromagnetic interaction.

The original motivation for carrying out a solar neutrino experiment 30 years ago was to use the neutrinos produced by nuclear reactions between light elements in the Sun's core to test stellar evolution. Traditionally, the solar neutrino calculations assume that neutrinos do not engage in any exotic mixing once they have been created. Conventional neutrino scattering in the outer reaches of the Sun and the Earth's atmosphere is negligible. If neutrinos experience no such exotic metamorphosis between their creation in the Sun and the detectors here on Earth, we should simply observe the calculated spectra (see figure 2 later) and measure the fluxes predicted by our standard solar models.

Puzzles like the solar neutrino deficiency, the atmospheric neutrino flux ratios and the observation of neutrinos from the supernova SN 1987A have raised a number of very important questions that need to be answered in the near future.

If the neutrino has mass, then it can be a superposition (just as is the case for the neutral kaons: $K^{\circ}_{\mathrm{Short}}$ and $K^{\circ}_{\mathrm{Long}}$) of other neutrino eigenstates of the mass matrix as suggested in theoretical models of grand unification, in which case neutrino oscillations may take place.

2. The solar model

Natural neutrino sources in the cosmos range in energy from 10^{-4} eV (the 3 K neutrino background) to 10^{19} eV (from diffuse Galactic and extra-Galactic sources). The cross section for neutrino interactions varies with E_ν and ranges from 10^{-66} cm^2 to $\sim 10^{-32}$ cm^2 over the same energy range. This makes neutrino detection extremely difficult at very low energy, where the neutrino flux is large. In contrast, at very high neutrino energies when the cross section is large, the cosmic neutrino flux is small.

The Sun is a source for neutrinos, as various nuclear reactions take place in the centre of the Sun. The net result is that four protons convert into an alpha particle:

$$2e^- + 4p \rightarrow {}^4\mathrm{He} + 2\nu_e + 26.7\ \mathrm{MeV}$$

where about 3% of the energy is carried away by neutrinos (each having about 13 MeV). The thermal energy thus gives rise to an electron neutrino, ν_e, flux on Earth of $\Phi_\nu = 6 \times 10^{10} \nu_e/\mathrm{cm}^2$ s.

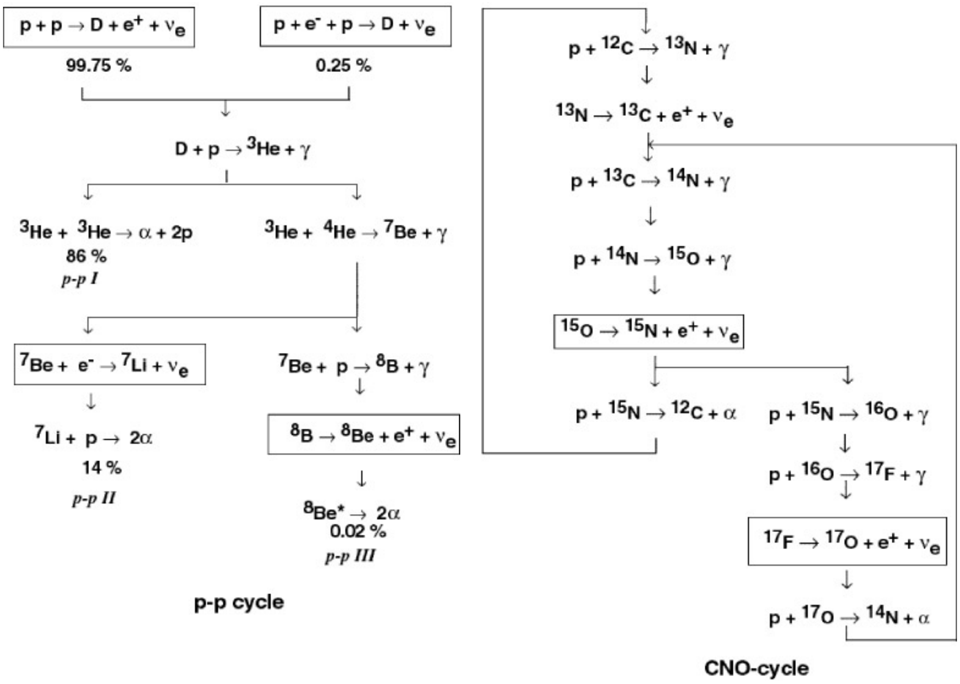

Figure 1. Diagram showing the two most important reaction cycles in the standard solar model.

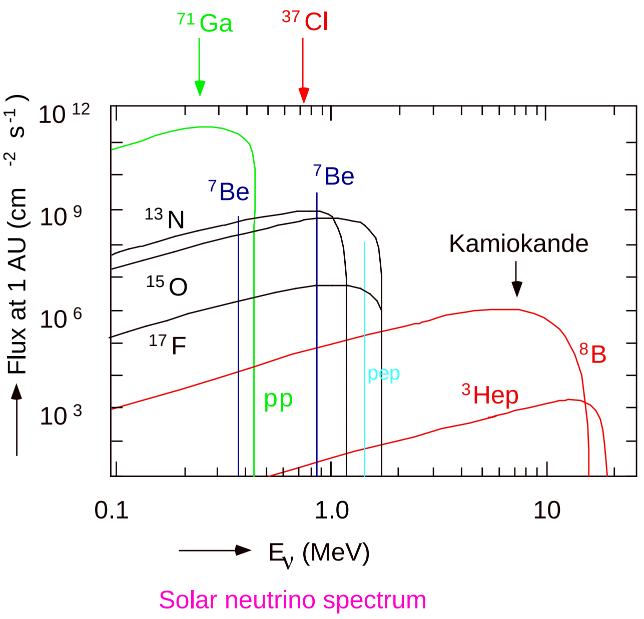

Figure 2. Theoretical calculated neutrino fluxes of the most important nuclear reactions in the Sun. The horizontal energy and vertical intensity axes are on different logarithmic scales.

Various authors [1–5] have calculated the nuclear reactions taking place in the interior of the Sun. The predicted fluxes are commonly called the 'standard solar model' (SSM).

The two most important reaction cycles (p–p and CNO) are shown in figure 1. The most energetic neutrinos are produced in the reaction

$$^{3}\mathrm{He} + \mathrm{p} \rightarrow {}^{4}\mathrm{He} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \qquad (E_{\nu} < 18.8\ \mathrm{MeV}).$$

By analysing the energy balance of each reaction, it can be concluded that the solar neutrinos exhibit an energy spectrum with a maximum at about 18 MeV.

In figure 2 the neutrino energy spectra originating from the most important nuclear reactions in the Sun are shown on a log–log scale. Not only are the solar reactions producing neutrinos shown, but also the energy thresholds of the various radiochemical detection reactions, used to identify the solar neutrino fluxes, are indicated along the energy axis.

3. The solar neutrino observations

Neutrinos only feel the weak interaction and belong to the particle group of leptons, together with electrons, muons, and tau leptons. The detection of solar neutrinos can, therefore, only take place through the charged current weak interaction:

$$\nu_{\mathrm{e}} + \mathrm{A}_{N}^{Z} \rightarrow \mathrm{e}^{-} + \mathrm{A}_{N-1}^{Z+1}.$$

If ν_{e} were to oscillate into ν_{μ} these muons would, however, be 'sterile'. This is due to the fact that the neutrinos created in the Sun have a maximum neutrino energy $E_{\nu} < 18$ MeV. The reaction, induced by a ν_{μ},

$$\nu_{\mu} + \mathrm{A}_{N}^{Z} \rightarrow \mu^{-} + \mathrm{A}_{N-1}^{Z+1}$$

then *cannot take place, as the energy is too small* to produce the rest mass of the muon, $M_{\mu} = 105$ MeV.

3.1. The Homestake Gold Mine chlorine experiment

The first, pioneering solar neutrino experiment was started in 1967 by Davis *et al* in the Homestake Gold Mine in Lead, South Dakota [6]. They extracted ^{37}Ar atoms that were created in the reaction

$$\nu_e + {}^{37}\mathrm{Cl} \rightarrow {}^{37}\mathrm{Ar} + e^-$$

with a reaction threshold of 814 keV and a half-life of 35 days. The radioactive ^{37}Ar decays by electron capture:

$$^{37}\mathrm{Ar} + e^- \rightarrow {}^{37}\mathrm{Cl} + \nu_e + \text{x-rays}.$$

The neutrino target here is a tank containing 615 metric tons of tetrachlorethylene C_2Cl_4. Because (p, n) reactions will also give rise to ^{37}Ar production, the tank must be very thoroughly shielded against cosmic rays. Hence, the tank is located deep underground at a depth corresponding to 4200 m of water, or about 1400 m of rock.

In the experiment, the tank is exposed for between 35 to 50 days, after which 0.2 cm^3 of stable ^{36}Ar or ^{38}Ar is added. After the exposure Ar is removed by circulating 6×10^5 l of He gas. Ar is separated from He by a cooled charcoal trap at liquid-N_2 temperature. Ar is transferred from the heated charcoal trap to a line where it is purified and the volume is measured. The gas is then loaded into the proportional counter that is filled with tritium-free methane as a counting gas. The x-rays and Auger electrons are measured in a proportional counter. In a window where the rise time versus energy is plotted the fast events will show up with a well-defined energy. The background due to radioactivity occurs at random times and thus shows a flatter rise time and energy distributions. Each sample is counted for $\sim$8 months. The carrier yield is verified by mass spectroscopy.

The result of the chlorine experiment over the 30 years that the experiment has been running so far is (2.56 $\pm$ 0.16) SNU (solar neutrino units). Compared with the predicted value of (7.7 $\pm$ 1.1) SNU [5], this experimental result yields a ratio (experiment/theory) of (33 $\pm$ 3)% [5]. *This experiment is the strongest evidence of the solar neutrino problem.* Moreover, statistical analysis shows that the experiment has the expected Poisson statistics and no individual run can be discarded. The chlorine experiment has a stable behaviour as a function of time at the 2σ level.

3.2. The Kamiokande experiment

A large water-based Cherenkov detector in Japan, originally constructed to search for proton decay, was converted into a low-energy neutrino detector. This detector was operational for 9 years and data collection came to an end in early 1995. The Kamiokande detector was located 1 km (2700 m water-equivalent) underground in the Mozumi zinc mine, in the town of Kamioka. The total volume of Kamiokande contained 4.5 kton, with an inner volume of 2.14 kton of water, viewed by 948 photo-multipliers. The outer part of the detector contained 1500 ton of water with 123 photo-multipliers tagging cosmic ray muons and absorbing neutrons and γ-rays. For solar neutrino observations a fiducial volume containing 680 ton of water is used. The scattering cross section for a 10 MeV ν_e to produce a recoil electron with an energy of at least 5 MeV is very small, only 4×10^{-44} cm^2. For the other neutrino flavours ν_μ and ν_τ, it is even smaller. The kinematics dictate that the electrons are scattered mainly within an angle of 15° of the incident neutrino direction. The relativistic charged particles traversing faster than the velocity of light in water emit Cherenkov radiation in a cone with an opening angle θ_c:

$$\cos\theta_c = 1/\beta n$$

where $\theta_c = 42^\circ$ for particles with $\beta = 1$ and n is the refractive index of water. The threshold energy of the Kamiokande detector was 6.5 MeV with an event rate of a few neutrinos per week. This high-energy threshold implies that the detector is sensitive only to the ^{8}B neutrino

Table 1. Ratio of solar neutrino flux and theoretical calculated flux.

Experiment	E_ν (MeV)	$\frac{\text{Experiment}}{\text{Theory}}$	Source
SAGE and GALLEX ^{71}Ga	> 0.236	0.57 ± 0.06	[5] (Mean of SAGE and GALLEX)
Homestake ^{37}Cl	> 0.814	0.33 ± 0.03	[5]
Kamiokande (water)	> 7.5	0.54 ± 0.07	[5] (2079 days)
Super-Kamiokande (water)	> 6.5	0.471 ± 0.019	[5] (January 1999 analysis of 504 live time days)

flux. Kamiokande's total recorded flux was just about half ((54 $\pm$ 7)%) the rate of what the standard model [5] predicted (see also table 1).

3.3. Super-Kamiokande

The ring imaging water Cherenkov detector Super-Kamiokande (SK) was built to replace the old Kamiokande detector. This detector, at a depth of 1 km, is located in a large hall, housing a detector consisting of the main cylindrical tank ($\varnothing = 39.3$ m and height 42 m) filled with 50 kton of pure water. The main tank is optically separated from a 32.5 kton inner detector ($\varnothing = 34$ m and height 36 m), viewed by 11 146 photo-multipliers ($\varnothing = 50$ cm) from a 3 m thick outer detector watched by 1885 photo-multipliers ($\varnothing = 20$ cm). The outer detector vetoes cosmic rays and radioactivity from the surroundings. Data collection started in April 1996. The actual observation of solar neutrinos began in May 1996 with a threshold energy of 5.6 MeV. The detector provides ~ 14 solar neutrino events per day. Because of the high signal rate, it is expected to examine day/night variation, seasonal variation and spectrum distortion with sufficient significance. At present the detection energy threshold is 6.5 MeV and therefore SK can only detect the ^{8}B neutrino flux. The main interaction in this energy region is neutrino–electron elastic scattering, since neutrino–nucleus scattering is suppressed. The cross section of the ν_e + e interaction can be well calculated using the standard theory of the electro-weak interaction:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_e} = \frac{2G_F^2 m_e}{\pi}\left[c_L^2 + c_R^2\left(1 - \frac{E_e}{E_\nu}\right)^2 + c_L c_R \frac{m_e}{E_\nu}\right]$$

where E_e is the kinetic energy of the scattered electrons, $c_L = \frac{1}{2} + \sin^2\theta_W$ and $c_R = \sin^2\theta_W$ for ν_e + e scattering, and $c_L = -\frac{1}{2} + \sin^2\theta_W$ and $c_R = \sin^2\theta_W$ for ν_μ + e scattering. With $\sin^2\theta_W = 0.255$, the total cross sections are

$$\sigma(\nu_e + e \to \nu_e + e) = 0.920 \times 10^{-43}\ (E_\nu/10\ \text{MeV})\ \text{cm}^2$$

$$\sigma(\nu_\mu + e \to \nu_\mu + e) = 0.157 \times 10^{-43}\ (E_\nu/10\ \text{MeV})\ \text{cm}^2.$$

Although the cross sections are small, the $\nu + e^-$ interactions have the advantage that the direction of the recoil electron is constrained to the forward direction by $E_e\theta^2 \leqslant 2m_e - 18^\circ$ for 10 MeV recoil electrons. Hence, the direction and energy of the recoil electron reflect the direction and energy of the incident neutrino. This makes it possible to study the neutrino energy spectrum by the recoil electron energy. The result of the SK (based on 86 637 events) for the ^{8}B neutrino flux was $(2.43 \pm 0.05(\text{stat.}) \pm 0.08(\text{syst.}))$ SNU [9]. This was obtained from data collected in 504 days (live time of solar neutrino observations). The predicted absolute flux is (5.15 $\pm$ 0.85) SNU [5]. The result is presented in table 1, together with those of the other experiments. The data are consistent with the old Kamiokande data.

3.4. The gallium experiments

The gallium experiments are able to detect much lower energy neutrinos. The p–p neutrinos have an energy end-point of 420 keV and can be observed by ^{71}Ga targets. There are two experimental sites.

SAGE, started in 1990, is a Russian–American experiment located in the Baksan Valley in the Caucasus at a depth of 4.7 km water-equivalent with 60 metric tons of gallium metal as target.

GALLEX, started in 1991, is a collaboration between Heidelberg, Karlsruhe, Gran Sasso, Milan, Rome, Nice, Saclay, Rehovoth and Brookhaven located in the Gran Sasso Underground Laboratory, with 30 tons of gallium in a solution of $GaCl_3$. Both experiments are performed by extracting ^{71}Ge atoms created in the reaction $\nu_e + {}^{71}\mathrm{Ga} \rightarrow {}^{71}\mathrm{Ge} + e^-$ *with a reaction threshold of 236 keV*. The radioactive ^{71}Ge decays with a half-life of 11.4 days by electron capture:

$$^{71}\mathrm{Ge} + e^- \rightarrow {}^{71}\mathrm{Ga} + \nu_e + \text{x-rays}.$$

The GALLEX experiment has been running since May 1991. This experiment has been calibrated twice, with two intense ^{51}Cr (half-life 27.7 days) neutrino sources (> 60 PBq) produced in the Siloé nuclear reactor [7].

The observed rate for each experiment yields ratios (experiment/theory) of $(60 \pm 6)\%$ [5] and $(52 \pm 6)\%$, for GALLEX and SAGE, respectively. The mean value of the SAGE and GALLEX experiments is also given in table 1. It shows that a 40% deficit in the solar neutrino flux has now been observed by four independent experiments.

4. Discussion

All solar neutrino experiments have detected significant quantitative differences of 30–50% with respect to the rates predicted by the standard solar models. The predicted rates depend on the type of detector, because each solar process has its own energy spectrum and each detector has its own lowest-energy threshold (indicated in figure 2). Only the gallium detectors are sensitive to neutrinos from the proton–proton fusion reaction, which is assumed to be the principal power source of the Sun. If one believes that the neutrinos from the Sun are reaching the earth unscathed after their direct creation in the various nuclear processes, we have another problem to explain: the imbalance between the neutrino fluxes arising from the ^{8}B and ^{7}Be reactions. (If we subtract the results from Kamiokande, ^{8}B neutrinos, from the Homestake ^{37}Cl results to obtain the ^{7}Be flux, we arrive at a flux that is essentially zero). The ^{7}Be reactions are the only conceivable source of ^{8}B in the Sun, while both the p–p and ^{7}Be predictions are very solid. Given the observed parameters of the Sun these predictions cannot be changed very much.

There is, however, a way out of this problem. Neutrinos come in three flavours (ν_e, ν_μ and ν_τ), associated with the three charged leptons, respectively: the electron, the muon and the tau lepton. The only neutrinos in the Sun's energy cycle to which our detectors are sensitive are the electron neutrinos, ν_e. In 1967 Pontecorvo suggested the possibility that neutrino flavours could mix if they have a mass. The effect is called *neutrino oscillation* as the probability of metamorphosis between two flavours has a sinusoidal dependence on path length. These oscillations can only happen if there is a tiny but finite mass difference between the two neutrino varieties assumed (analogous to the mass difference of the two neutral kaons: K°_{Short} and K°_{Long}). In 1985 an elegant theory was developed by Mikheyev and Smirnov based on earlier work by Wolfenstein. They remarked that the ν_e cross section with matter was different from that for the other two flavours ν_μ and ν_τ. On the way out from the centre of the Sun the ν_e can interact with an electron through a charge current (CC) interaction, i.e. the electron becomes a ν_e and the incident ν_e an electron. The other two flavours cannot have such CC interactions with electrons. As a consequence the total ν_e cross section in the Sun is electron density dependent. If we write the general neutrino mass Lagrangian in a matrix form, such a behaviour can be translated into an extra neutrino mass term in the mass matrix of the electron neutrino. The basic assumption is then that the physical neutrinos emerging from weak interactions (ν_e, ν_μ and ν_τ) do not have definite masses. Instead they are superpositions (see also the appendix) $\nu_\ell = \sum U_{\ell i}\nu_i$ of mass-eigenstate neutrinos: ν_1, ν_2 and ν_3, where ℓ stands for e, μ or τ. It is

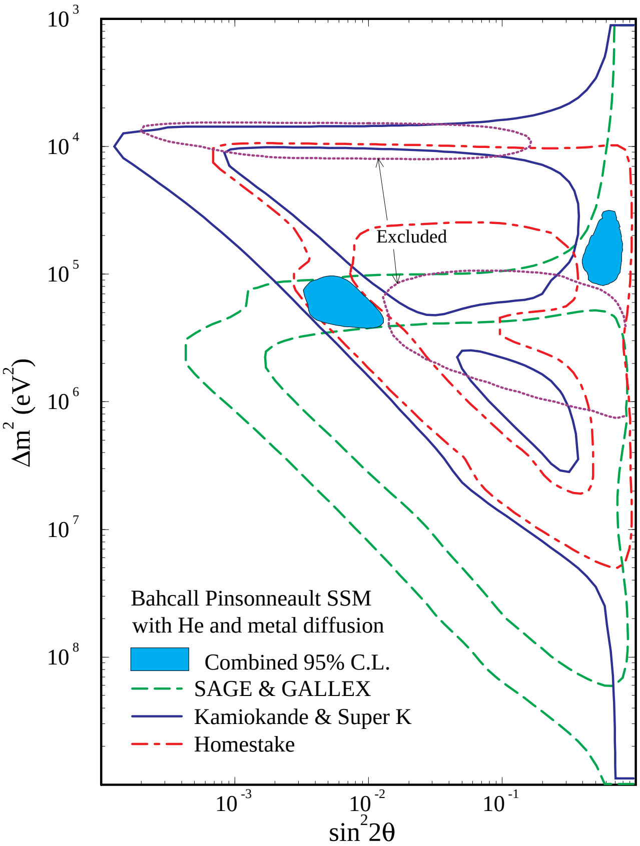

Figure 3. The results of a calculation by Hata and Langacker [8] of the MSW parameter space (shaded region) allowed by the combined observations at 95% C.L. within the SSM [7]. The constraints from Homestake, combined Kamiokande and Super-Kamiokande and combined SAGE and GALLEX are shown by the dot-dashed, solid, and dashed lines, respectively. Also shown are the regions excluded by the day–night data (dotted lines).

then plausible that the ν_e-mass-eigenstate at a certain electron density in the Sun comes close to the ν_μ-mass-eigenstate and that considerable mixing between the two mass-eigenstates occurs. From MSW oscillation calculations by Hata and Langacker [8], considering only two-flavour oscillations (see figure 3), small and large mixing-angle solutions are viable: $\sin^2 2\theta \sim 0.008$ and $\Delta m^2 \sim 5 \times 10^{-6}$ eV^2, and $\sin^2 2\theta \sim 0.6$ and $\Delta m^2 \sim 1.6 \times 10^{-5}$ eV^2, in terms of the standard solar model (SSM) of Bahcall and Pinsonneault [4]. This implies a mass difference of 2 to 4 meV between ν_e and ν_μ. Oscillations into sterile neutrinos are only possible for small angles. According to their calculations Hata and Langacker exclude as a consequence of the Kamiokande day–night data a large parameter space for MSW, independent of SSM predictions. If the Mikheyev–Smirnov–Wolfenstein (MSW) resonance is the reason for the neutrino metamorphosis in the Sun, the shape of the neutrino energy spectra is heavily distorted and we should try to measure this.

It is not known whether the neutrinos have any mass, but if they possess mass it could be the answer to many of our questions. Clear evidence indicative of non-vanishing neutrino masses is given by the measurement of atmospheric neutrinos, reported last year by the Super-Kamiokande group [10,11]. Primary cosmic rays, like protons and nuclei, stream continuously into the atmosphere of the Earth. As they propagate in the atmosphere, they interact with air and create secondary particles (mostly pions and kaons). Atmospheric neutrinos are produced by the decay of these particles: $\pi^{\pm} \rightarrow \mu^{\pm} + \nu_{\mu}$ and $\mu^{\pm} \rightarrow e^{\pm} + \nu_{e} + \nu_{\mu}$. In this way about twice as many ν_{μ} are produced as ν_{e}. In June last year the Super-Kamiokande group reported that measured atmospheric neutrino flux ratio ν_{μ}/ν_{e} and its zenith angular dependence were compared with detailed Monte Carlo simulations.

The ratio $(\nu_{\mu}/\nu_{e})_{\mathrm{Data}}/(\nu_{\mu}/\nu_{e})_{\mathrm{MC}}$ was measured to be 0.63 ± 0.03(stat.) $\pm$ 0.04(sys.) for the sub-GeV data (fully contained in their detector) and 0.65 ± 0.05(stat.) $\pm$ 0.08(sys.) for the multi-GeV data, collected from 535 live-time days (taken from May 1996 to May 1998). In addition a strong distortion in the shape of the event zenith angle distribution was observed. The ratio of the number of upward to downward μ-like events was found to be $N_{\mathrm{up}}/N_{\mathrm{down}} = 0.78 \pm 0.05(\mathrm{stat.}) \pm 0.01(\mathrm{sys.})$ for the sub-GeV data and $N_{\mathrm{up}}/N_{\mathrm{down}} = 0.53 \pm 0.06(\mathrm{stat.}) \pm 0.01(\mathrm{sys.})$ for the multi-GeV data, with an expected value of 0.99 ± 0.03(stat.) $\pm$ 0.02(sys.). The same ratio for the observed e-like events was consistent with unity. The deficit of the upward-going muons could very well be explained in terms of neutrino oscillations. This could be a crucial pointer to a more comprehensive theory beyond the standard model of the fundamental particles.

In the next few years three large new solar neutrino detectors will generate new and precise data that should clarify our view and understanding of how the Sun shines and how neutrinos behave. These experiments include Super-Kamiokande, the Sudbury Neutrino Observatory (SNO) in a mine in northern Ontario, and Borexino in the Apennines, east of Rome. The SK experiment has already been reporting some very interesting data and has now been operating for about two years. Each of these detectors will register more detailed neutrino interactions than all of the previous solar neutrino experiments put together.

Appendix

Most methods to determine the neutrino mass are limited in sensitivity to neutrino masses $m_{\nu} \geqslant 10$ eV. It appears that the only practical way to study smaller masses is through the dynamic effect of neutrino oscillations. The basic assumption is that the 'physical' neutrinos ν_{ℓ}, i.e. the particles emerging from weak decays and each associated with its charged lepton partner, do not have a definite value of mass. Instead, they are superpositions of 'mass-eigenstate' neutrinos ν_i:

$$\nu_{\ell} = \sum_{i} U_{\ell i} \nu_i \tag{A1}$$

where the summation is over all mass eigenstates ν_1, ν_2 and ν_3. It is assumed that the unitary matrix **U** has at least one non-vanishing non-diagonal matrix element, giving rise to mixing between flavours.

For simplicity let us consider the case of only two neutrino flavours with non-zero rest mass, which we call ν_{e} and ν_{μ}. This mass term can be diagonalized by substituting

$$\nu_{e} = \nu_1 \cos\theta + \nu_2 \sin\theta \tag{A2a}$$

$$\nu_{\mu} = -\nu_1 \sin\theta + \nu_2 \cos\theta \tag{A2b}$$

with θ as the 'mixing angle'. From this we can solve

$$\nu_1 = \nu_{e} \cos\theta - \nu_{\mu} \sin\theta \tag{A3a}$$

$$\nu_2 = \nu_{e} \sin\theta + \nu_{\mu} \cos\theta. \tag{A3b}$$

The mass eigenstates ν_1 and ν_2, i.e. the physical particles, have masses m_{ν_1} and m_{ν_2}. The time evolution of these particles is given by

$$|\nu_i(t)\rangle = e^{-iE_i t}|\nu_i(0)\rangle \tag{A4}$$

where $E_i = (p_i^2 + m_i^2)^{1/2}$ is the neutrino energy. From this we find $E_i/p_i \approx 1 + m_i^2/(2p_i)$. Substituting (A2) into this equation, one can work out the evolution of the mass eigenstates ν_i in terms of the flavour eigenstates.

We then find

$$\begin{aligned}\nu_e &= \nu_1(0)\cos\theta\,\mathrm{e}^{-\mathrm{i}E_1t} + \nu_2(0)\sin\theta\,\mathrm{e}^{-\mathrm{i}E_2t}\\ &= (\nu_e(0)\cos\theta - \nu_\mu(0)\sin\theta)\cos\theta\,\mathrm{e}^{-\mathrm{i}E_1t} + (\nu_e(0)\sin\theta + \nu_\mu(0)\cos\theta)\sin\theta\,\mathrm{e}^{-\mathrm{i}E_2t}\\ &= \nu_e(0)(\cos^2\theta\,\mathrm{e}^{-\mathrm{i}E_1t} + \sin^2\theta\,\mathrm{e}^{-\mathrm{i}E_2t}) + \nu_\mu(0)\sin\theta\cos\theta\big[\mathrm{e}^{-\mathrm{i}E_1t} - \mathrm{e}^{-\mathrm{i}E_2t}\big]. \qquad \text{(A5)}\end{aligned}$$

After a time t, the probability P of finding a ν_μ in a neutrino created as ν_e is then *the square of the amplitude* of finding the ν_μ neutrino:

$$P(\nu_e \to \nu_\mu) = \sin^2\theta\cos^2\theta\big[\mathrm{e}^{-\mathrm{i}E_1t} - \mathrm{e}^{-\mathrm{i}E_2t}\big]^2$$

which can be written (recall $p \approx p_1 \approx p_2 \gg m_{1,2}$) as

$$P(\nu_e \to \nu_\mu) = \frac{1}{2}\sin^2 2\theta\left\{1 - \cos\left[\left(\frac{m_2^2 - m_1^2}{2p}\right)t\right]\right\}$$

or rewritten as

$$P(\nu_e \to \nu_\mu) = \frac{1}{2}\sin^2 2\theta\sin^2\left[\left(\frac{m_2^2 - m_1^2}{4p}\right)t\right] \qquad \text{(A6)}$$

where $\sin^2 2\theta$ is the oscillation amplitude.

The mechanism for this 'flavour oscillation' is driven by the phase factor e^{-E_it} in (A5). For a given momentum p, the energy E_i differs when the masses of ν_1 and ν_2 differ and, therefore, the periods of the phases are unequal. Hence the coefficients of the superposition of ν_1 and ν_2 change, inducing an appearance of the other neutrino flavour in the wavefunction.

With the definition $\Delta m^2 = |m_2^2 - m_1^2|$ and $p \sim E$, one finds the common expression for the oscillation length

$$L_{\mathrm{osc}} = \frac{4\pi E}{\Delta m^2}$$

or when expressed in practical units

$$L_{\mathrm{osc}}\,[\mathrm{m}] = \frac{2.48\,E\,[\mathrm{MeV}]}{\Delta m^2\,[\mathrm{eV}^2]}. \qquad \text{(A7)}$$

In order to generate neutrino oscillations, one needs not only non-vanishing masses, but also a mass difference and a non-diagonal mass matrix. In analogy with the quark system, however, one would be surprised if non-zero neutrino masses were to materialize with an exactly diagonal mass matrix. Equation (A6) can be rewritten in terms of a choice of practical units:

$$P(\nu_e \to \nu_\mu) = \sin^2 2\theta\sin^2\left[\frac{1.27\Delta m^2\,[\mathrm{eV}^2]\,L_{\mathrm{osc}}\,[\mathrm{m}]}{2E\,[\mathrm{MeV}]}\right].$$

References

[1] Bahcall J N *et al* 1982 *Rev. Mod. Phys.* **54** 767
[2] Turck-Chièze S *et al* 1988 *Astrophys. J.* **335** 415
[3] Turck-Chièze S and Lopes I 1993 *Astrophys. J.* **408** 347
[4] Bahcall J N and Pinsonneault M H 1995 *Rev. Mod. Phys.* **67** 781
[5] Bahcall J N, Basu S and Pinsonneault M H 1998 *Phys. Lett.* B **433** 1
[6] Davis R *et al* 1968 *Phys. Rev. Lett.* **20** 1205
Cleveland B T *et al* 1998 *Astrophys. J.* **496** 505
[7] Hampel W *et al* 1988 *Phys. Lett.* B **420** 114
[8] Hata N and Langacker P 1997 *Preprint* IASSNS-AST 97/29, UPR-751T, May
[9] Ishino H 1999 *Thesis* University of Tokyo, January
[10] Okumura K 1999 *Thesis* University of Tokyo, February
[11] Fukuda Y *et al* 1998 *Phys. Lett.* B **433** 9