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Interaction of a moving magnetic dipole with a static electric field

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Abstract. We examine the dynamics of a magnetic dipole in a static electric field. Applications of interest are the Aharonov–Casher effect and spin–orbit coupling. The ideal Aharonov–Casher effect is shown to involve a correlation between the moving particle and the internal coordinates of a voltage source. A similar relation has been demonstrated in the Aharonov–Bohm experiment. We also give a simple derivation of spin–orbit coupling based purely on non-relativistic physics.

Zusammenfassung. Wir untersuchen die Dynamik eines magnetischen Dipols in einem statischen elektrischen Feld. Anwendungen von Interesse sind das Aharonov–Casher Effekt und die Spin–Bahn Kopplung. Beim idealen Aharonov–Casher Effekt, handelt es sich um eine Korrelation zwischen dem bewegenden Teilchen und den inneren Koordinaten einer Spannugsquelle. Eine ähnliche Beziehung ist bereits im Falle des Aharonov–Bohm Effekts nachgewiesen worden. Eine einfache Herleitung der Spin–Bahn Kopplung die ausschliesslich auf nichtrelativistische Physik beruht wird angegeben.

1. Introduction

Recent interest in the Aharonov–Casher effect [1] is the stimulus for a fresh look at the general problem of moving magnetic moments in static electric fields. We restrict our attention to two specific examples: (1) the Aharonov–Casher effect, and (2) spin–orbit coupling.

We will show that both problems can be treated in a direct way, strictly within the framework of non-relativistic theory and without reference to hidden momenta.

2. The Aharonov–Casher effect

A few years ago, Aharonov and Casher [1] showed that a magnetic dipole (a neutron) aligned parallel to a static line charge (a charged wire) will experience no force, but paths in the plane perpendicular to the line charge passing it on opposite sides lead to a relative quantum phase shift. This effect has been verified experimentally [2, 3]. The derivation has not been straightforward [4–6].

The Aharonov–Casher (AC) effect can be obtained most directly by observing that the moving magnetic moment gives rise to an electric dipole moment

$$\boldsymbol{P} = \frac{1}{c}\boldsymbol{v} \times \boldsymbol{\mu} \tag{1}$$

in the rest system of the line charge [7]. The interaction energy is then given by

$$U = -\boldsymbol{P} \cdot \boldsymbol{E} = \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{E} \times \boldsymbol{\mu}. \tag{2}$$

The force law is obtained from this velocity-dependent potential from the relation

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla_v U) - \nabla U \tag{3}$$

where

$$\nabla_v = \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z}.$$

It is not difficult to show that the momentum of the electromagnetic field in the rest frame of the wire is [8]

$$\boldsymbol{P}_{\text{field}}(r) = \frac{1}{c}\boldsymbol{E} \times \boldsymbol{\mu} \tag{4}$$

so that, in terms of $\boldsymbol{P}_{\text{field}}$, the interaction becomes

$$U = \boldsymbol{v} \cdot \boldsymbol{P}_{\text{field}} \tag{5}$$

just as in the Aharonov–Bohm effect [9]. Straightforward computation then leads to the force

$$\boldsymbol{F} = -\boldsymbol{\mu} \cdot \nabla\left(\frac{1}{c}\boldsymbol{v} \times \boldsymbol{E}\right) + \frac{1}{c}\boldsymbol{E} \times \frac{\partial \boldsymbol{\mu}}{\partial t} \tag{6}$$

in the rest frame of the line charge. The force of equation (6) is that found by AC [1] and by Goldhaber [8].

The point we wish to make here is that the interaction of equation (2) also may be written as

$$U = \int_{-\infty}^{\infty} \lambda\, \Phi(z)\, \mathrm{d}z \tag{7}$$

where $\Phi(z)$ is the electrostatic potential due to the dipole $\boldsymbol{P}$ at points along the z axis. If the line charge λ is not to be redistributed along the conducting wire (the theory assumes that λ remains constant!), the power source (or better, sources distributed along the wire) must supply an additional electric field that cancels the z component of the electric field of the neutron. The sources must therefore provide an energy given by the negative of equation (7).

In quantum theory, this is a troublesome situation. Equation (2) shows that U has a different sign on different sides of the wire. This indicates a correlation between the position of the neutron and the state (higher or lower voltage output) of the power supply. Quantum theory dictates that the neutron may not be treated as an isolated particle, since it is correlated with the state or condition of its surroundings. A similar situation arises in the Aharonov–Bohm effect [9]. The correlation between the particle's path and state of the voltage (in AC) or current (in AB) source provides physical insight into the discrepancy between results of AB scattering calculations [10] and the recently reported results of Shapiro and Henneberger [11] based on Feynman path integrals. The path integral method is based upon the Lagrangian of the particle. The method is applicable to the AC and AB problems, while the scattering computation is not, for the reason given above.

In the AC case, the Lagrangian is given by

$$L = T - U = \tfrac{1}{2} m v^2 - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{E} \times \boldsymbol{\mu}. \tag{8}$$

The canonical momenta are

$$P_k = \frac{\partial}{\partial v_k} L = m v_k - \frac{1}{c}(\boldsymbol{E} \times \boldsymbol{\mu})_k. \tag{9}$$

In analogy with the AB case, one can now simply write down the topological phase factor (which appears in path integral calculations) for the AC case:

$$\exp\left(\frac{\mathrm{i}}{c\hbar}\oint \boldsymbol{\mu} \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}\right).$$

This is the AC result.

3. Spin–orbit coupling as a non-relativistic effect

The interaction energy of equation (2) has a long history in physics. It can be applied to the problem of spin–orbit coupling, if $\boldsymbol{\mu}$ is the electron magnetic moment. The interaction is the same in the rest frame of the moving electron as it is in the rest frame of the nucleus. It was used by Frenkel [12] in 1926 in a derivation of Thomas precession [13]. (This is the precession of the electron spin in the instantaneous rest system of the centre of mass of the moving electron). Our purpose here is to demonstrate that spin–orbit coupling can be derived entirely in the framework of non-relativistic theory. One need not make use of either the Dirac equation or of Lorentz transformations.

To first order in v/c, we may treat U as being time independent. The magnetic moment $\boldsymbol{\mu}$ is not a fixed vector, but $\mathrm{d}\boldsymbol{\mu}/\mathrm{d}t$ depends again upon the interaction U and hence corrections to U due to the motion of $\boldsymbol{\mu}$ are of order v^2/c^2. This makes possible a comparison of equation (2) with the electromagnetic interaction of a charged particle,

$$U = -\frac{e}{c}\boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}). \tag{10}$$

Comparison indicates that we may formally consider the problem as an electromagnetic one in which there is an effective vector potential

$$\boldsymbol{A}_{\text{eff}} = -\frac{1}{e}(\boldsymbol{E} \times \boldsymbol{\mu}) \tag{11}$$

which gives an effective magnetic field

$$\boldsymbol{B}_{\text{eff}} = \nabla \times \boldsymbol{A}_{\text{eff}} = -\frac{1}{e}\nabla \times (\boldsymbol{E} \times \boldsymbol{\mu}) = -\frac{1}{e}(\boldsymbol{\mu} \cdot \nabla)\boldsymbol{E}. \tag{12}$$

In the last step of equation (12), we have omitted a term $\boldsymbol{B}'_{\text{eff}} = (\boldsymbol{\mu}/e)(\nabla \cdot \boldsymbol{E})$. In the absence of screening, such a term can affect only S states, and hence is of no interest for spin–orbit coupling. It is intriguing to note that writing $-\boldsymbol{\mu} \cdot \boldsymbol{B}'_{\text{eff}}$ gives the Darwin term [14] up to a factor of the order of unity. We leave it to the reader to investigate whether this is more than a mere coincidence.

In the screened Coulomb case, let $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{r} f(r)$. Then

$$(\boldsymbol{\mu} \cdot \nabla)\boldsymbol{E} = \boldsymbol{\mu} f(r) + \hat{\boldsymbol{r}}(\boldsymbol{\mu} \cdot \boldsymbol{r})\frac{\mathrm{d}f(r)}{\mathrm{d}r}. \tag{13}$$

If there is no screening, then $f(r) = q/r^3$, and

$$(\boldsymbol{\mu} \cdot \nabla)\boldsymbol{E} = q(\boldsymbol{\mu} \cdot \nabla)\frac{\boldsymbol{r}}{r^3} = q(\boldsymbol{\mu} r^2 - 3\boldsymbol{r}(\boldsymbol{\mu} \cdot \boldsymbol{r}))\frac{1}{r^5}. \tag{14}$$

The last expression is just q multiplied by the magnetic field at the origin due to a magnetic moment $\boldsymbol{\mu}$ at the point $\boldsymbol{r}$.

The average value of $\boldsymbol{B}_{\text{eff}}$ experienced by the electron in its orbit is thus

$$\langle \boldsymbol{B}_{\text{eff}} \rangle = -\frac{1}{e}\langle \boldsymbol{\mu} f(r) \rangle, \tag{15}$$

since a term proportional to $\hat{\boldsymbol{r}}$ cannot contribute to the precession (see equation (17)).

We may now simply adopt the theory of Larmor precession [15] and observe that the orbit will precess with the angular frequency

$$\omega_L = -\frac{e}{2mc}\boldsymbol{B}_{\text{eff}} = \frac{1}{2mc}\boldsymbol{\mu} f(r). \tag{16}$$

The kinetic energy in the rest frame of the nucleus (laboratory inertial frame) is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\boldsymbol{v}' + \omega_L \times \boldsymbol{r})^2 \cong \tfrac{1}{2}mv'^2 + m\boldsymbol{v}' \cdot (\omega_L \times \boldsymbol{r}) \tag{17}$$

to first order in ω_L. In the above, $\boldsymbol{v}'$ is the velocity of the electron in the rotating (and thus non-inertial) frame.

The spin–orbit coupling now emerges as a correction to the kinetic energy due to the Larmor precession of the electron orbit. The essential point here is that the kinetic energy that is relevant to the internal dynamics of the atom is the kinetic energy in the rotating system. In the rotating frame,

$$\begin{aligned} T &= \tfrac{1}{2}mv^2 - m\boldsymbol{v}' \cdot (\omega_L \times \boldsymbol{r}) \\ &\cong \tfrac{1}{2}mv^2 - m\boldsymbol{v} \cdot (\omega_L \times \boldsymbol{r}). \end{aligned} \tag{18}$$

Equation (16) then yields

$$T = \tfrac{1}{2}mv^2 - \frac{\boldsymbol{v}}{2c} \cdot (\boldsymbol{\mu} \times \boldsymbol{E}). \tag{19}$$

The effect of the Larmor precession is to eliminate the effect of the interaction to lowest order in ω_L in the rotating frame. In the inertial frame it gives rise to a kinetic energy correction

$$U_{\text{eff}} = \frac{\boldsymbol{v}}{2c} \cdot (\boldsymbol{E} \times \boldsymbol{\mu}). \tag{20}$$

With $\boldsymbol{E}(\boldsymbol{r}) = -\hat{\boldsymbol{r}}\,\mathrm{d}V/\mathrm{d}r$, this can be written

$$\begin{aligned} U_{\text{eff}} &= -\frac{\boldsymbol{\mu}}{2c} \cdot (\boldsymbol{E} \times \boldsymbol{v}) = \frac{\boldsymbol{\mu}}{2c} \cdot \frac{\mathrm{d}V}{\mathrm{d}r}(\hat{\boldsymbol{r}} \times \boldsymbol{v}) \\ &= \frac{1}{2mrc}\frac{\mathrm{d}V}{\mathrm{d}r}(\boldsymbol{\mu} \cdot \boldsymbol{L}). \end{aligned} \tag{21}$$

This is the result obtained from the Dirac equation [14].

It is interesting to compare the Larmor frequency with the frequency of Thomas precession. The relation $\boldsymbol{\mu} = (e/mc)\boldsymbol{S}$ together with equation (16) yields

$$\omega_L = -\frac{e}{2m^2c^2}\boldsymbol{S}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{22}$$

The Thomas precession is given by [16]

$$\omega_T = -\frac{e}{2m^2c^2}\boldsymbol{L}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r} \tag{23}$$

so that ω_L and ω_T are obtained from each other by the interchange $\boldsymbol{L} \leftrightarrow \boldsymbol{S}$. The preceding treatment shows that spin–orbit coupling may be visualized as a normal Zeeman effect on the electron's orbital motion due to the average magnetic field produced by its magnetic moment.

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