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Sparsity regularization in dynamic elastography

M Honarvar\(^1\), R S Sahebjavaher\(^2\), S E Salcudean\(^2\) and R Rohling\(^{1,2}\)

\(^1\) Department of Mechanical Engineering, University of British Columbia, Vancouver, BC, Canada
\(^2\) Department of Electrical Engineering, University of British Columbia, Vancouver, BC, Canada

E-mail: honarvar@interchange.ubc.ca, ramins@ece.ubc.ca, tims@ece.ubc.ca and rohling@ece.ubc.ca

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Abstract

We consider the inverse problem of continuum mechanics with the tissue deformation described by a mixed displacement–pressure finite element formulation. The mixed formulation is used to model nearly incompressible materials by simultaneously solving for both elasticity and pressure distributions. To improve numerical conditioning, a common solution to this problem is to use regularization to constrain the solutions of the inverse problem. We present a sparsity regularization technique that uses the discrete cosine transform to transform the elasticity and pressure fields to a sparse domain in which a smaller number of unknowns is required to represent the original field. We evaluate the approach by solving the dynamic elastography problem for synthetic data using such a mixed finite element technique, assuming time harmonic motion, and linear, isotropic and elastic behavior for the tissue. We compare our simulation results to those obtained using the more common Tikhonov regularization. We show that the sparsity regularization is less dependent on boundary conditions, less influenced by noise, requires no parameter tuning and is computationally faster. The algorithm has been tested on magnetic resonance elastography data captured from a CIRS elastography phantom with similar results as the simulation.

(Some figures may appear in colour only in the online journal)

1. Introduction

Elasticity imaging, or elastography, is an emerging technique in medical imaging (Ophir \textit{et al} 1991). The goal is to accurately depict the differences in the elastic modulus of tissues in order to distinguish them based on tissue composition and pathology. Elastography involves exciting the tissue by either an external or an internal excitation, measuring the displacement response using ultrasound (Bercoff \textit{et al} 2004a, Turgay \textit{et al} 2006, Sinkus \textit{et al} 2006, Basarab \textit{et al} 2007) or magnetic resonance (MR) imaging techniques (Manduca \textit{et al} 1996, Sinkus
et al 2005a, Kruse et al 2008) and then calculating the tissue elasticity distribution from the measured displacements by solving a parameter identification problem. This problem can be considered an inverse problem. A range of different approaches are used in elastography with different types of excitation and displacement measurements and different techniques for solving the inverse problem. The excitation can be quasi-static (Ophir et al 1991, 1999, Yanning et al 2003) or dynamic (Bercoff et al 2004a, Park and Maniatty 2006, Eskandari et al 2008). In quasi-static elastography, the images before and after compression are compared to measure the displacements, then the displacements are used to calculate the strain distribution, which is either imaged directly or converted to a relative elasticity distribution. In these quasi-static techniques, since the force boundary conditions are not readily available, the absolute values of the elastic modulus cannot be obtained. Conversely, in dynamic elastography, having inertial forces inside the tissue enables the calculation of the absolute values of elasticity. The dynamic excitation can be transient or harmonic. With transient excitation, an ultra-fast ultrasound system can be used to measure the displacements in time and space to capture the traveling shear wave inside the tissue and infer elasticity from sound speed (Sarvazyan et al 1998, Bercoff et al 2004b). Using harmonic low-frequency excitation, the displacement distribution can be measured by a conventional ultrasound or MR imaging system and the elasticity can be inferred from the shear wave speed (Baghani et al 2010, 2011, Muthupillai et al 1995, Oliphant et al 2001, Sinkus et al 2005b).

In all of these approaches, an elasto-dynamic wave equation is used to solve the inverse problem. To obtain the full inversion of the wave equation, one needs all components of the displacement field in a 3D volume, which are not readily available in all cases such as ultrasound elastography. Therefore, the problem is simplified in some approaches such as direct inversion (DI) (Oliphant 2001) and local frequency estimation (LFE) (Manduca et al 1996), where it is assumed that the gradients of the pressure and shear modulus are negligible, and which results in independent Helmholtz equations for each component of the displacement. In this way, measuring just one displacement component is sufficient to solve for the shear modulus. Park and Maniatty (2006) showed that neglecting the pressure gradient term in heterogeneous media unfortunately introduces inaccuracy in the modulus estimates.

The other most commonly used method for solving the inverse problem is the finite element method (FEM) which can be solved directly (Yanning et al 2003, Park and Maniatty 2006) or iteratively (Eskandari et al 2008, Oberai et al 2003) in both dynamic and quasi-static elastography to obtain the elasticity distribution. With the FEM, the equation of motion can be used without any simplification, which means that the gradients of the pressure or shear modulus are not assumed to be negligible. However, for nearly incompressible materials with Poisson’s ratio of close to 0.5, using a low-order FEM with a standard displacement-based formulation is prone to the so-called locking problem which results in poor convergence and instability (Babuska and Suri 1992). The typical solution to this problem is to use a mixed formulation of the FEM. In Park and Maniatty (2006), a mixed finite element based DI method has been used to solve the inverse problem for linear, isotropic, elastic materials. This method is fast since it solves for the shear modulus directly, without any iterations. Furthermore, with this method there is no need to know the boundary conditions. The only disadvantage of this method when compared to iterative techniques is that the displacement data needs to be differentiated once in space, making this method more sensitive to noise (Park and Maniatty 2006). Therefore, the displacement data needs to be filtered before being used in the reconstruction which reduces the spatial resolution. Moreover, a mixed FEM inversion technique solves for both pressure and shear modulus distributions so it has more unknowns than a standard formulation. This makes the inverse problem more ill-conditioned, and therefore it is necessary to utilize regularization techniques when solving for the unknowns.
The most common regularization method is the Tikhonov regularization (Tikhonov et al. 1977). In this method, the low-order spatial derivatives of the solution are penalized which results in smooth solutions. However, as mentioned above, this reduces the resolution and structural detail of the reconstruction. Tikhonov regularization confines the variation of the reconstructed values, and this usually results in the underestimation of the parameters. In geophysics, an alternative approach to the regularization was developed that uses a sparse representation of the original variables (Jafarpour et al. 2009). In this method, the original variables are transformed into another set of variables using an appropriate transformation which results in the sparse representation of the original variables. The goal is to use a small subset of new variables to provide a good approximation of the original field. This technique is generally used in image compression (Jain 1989). Applying this transformation on the problem’s variables and then using truncating approximation on the new variables results in a new inverse problem with fewer unknowns and improved conditioning.

This technique could be helpful in cases where the number of measurement points is limited, such as the problem of assessing subsurface structures from limited surface measurements. Jafarpour and colleagues (Jafarpour et al. 2009) used the sparsity regularization in solving the subsurface inverse problem. They used a discrete cosine transform (DCT) as the sparsifying transform and solved the inverse problem using an iterative method to minimize the sum of an $l_2$-norm measurement misfit term and an $l_1$-norm regularization term.

This paper is the first to use the direct mixed FEM technique with the DCT-based sparsity regularization to solve the elasticity inverse problem. The results are compared with the traditional Tikhonov regularization. The rest of this paper is organized as follows. In section 2, the theory of mixed finite elements and inverse problems are described. Then the sparsity regularization is introduced and discussed. Simulation results are presented in section 3 and section 4 concludes the paper.

2. Theory

2.1. Finite element formulation

The governing equation of motion for linear, isotropic and elastic materials with a time harmonic excitation is as follows:

\[
\nabla \cdot [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}] = -\rho \omega^2 \mathbf{u} \quad \text{in} \; \Omega \\
\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \; \Gamma_1 \\
[\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}] \cdot \mathbf{n} = \hat{T} \quad \text{on} \; \Gamma_2,
\]

where $\Omega$ is the region of interest in which the boundary is $\Gamma = \Gamma_1 \cup \Gamma_2$ and the displacement field $\mathbf{u}(x)$ is measured as a function of the position ($x \in \Omega$), $\mathbf{n}$ is the unit normal vector on $\Gamma$. $\mu$ and $\lambda$ are Lamé parameters and the vector $\hat{T}$ is a traction vector on a part of the boundary denoted by $\Gamma_2$, and $\hat{\mathbf{u}}$ is the prescribed displacement vector on a part of the boundary denoted by $\Gamma_1$. In other words, $\Gamma_1$ is related to the displacement boundary condition and $\Gamma_2$ is related to the force boundary condition.

Lamé parameters $\mu$ and $\lambda$ are related to Young modulus ($E$) and Poisson ratio ($\nu$) via

\[
\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}.
\]

For nearly incompressible materials, the limit $\lambda \to \infty, \nu \to 0.5$ implies that $\nabla \cdot \mathbf{u} \to 0$. This effect causes the locking problem in the FEM forward problem. This also causes a similar problem in solving the inverse problem. Since the divergence of the displacement is close
to zero, this term cannot be calculated directly from the noisy displacement data. In order to overcome this problem, a mixed formulation of the FEM is used in which pressure is introduced as a new variable \( p \):

\[
p = \lambda (\nabla \cdot \mathbf{u}) .
\]  

(5)

This variable is determined independently of the displacement. Substituting (5) into both (1) and (3) gives

\[
\nabla \cdot [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + p \mathbf{I}] = -\rho \omega^2 \mathbf{u} \quad \text{in} \quad \Omega \tag{6}
\]

\[
\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \Gamma_1 \tag{7}
\]

\[
[\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + p \mathbf{I}] \cdot \mathbf{n} = \hat{T} \quad \text{on} \quad \Gamma_2 . \tag{8}
\]

The following formulation follows the DI approach suggested by Park and colleagues (Park and Maniatty 2006). To solve (6) numerically, it should be discretized, which can be done using the FEM. According to Galerkin’s method, equations (6)–(8) are written in a weak formulation or integral form, which involves multiplying (6) with a test function \( w \in \mathbb{R}^3 \) and integrating over the entire domain. If \( \{\mathbf{u}(x), p(x)\} \) is a solution of the boundary value problem (6)–(8), then for any smooth function \( w \), that is, zero on \( \Gamma_1 \) we have

\[
\int_{\Omega} \nabla \cdot [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] \cdot \nabla w \, d\Omega + \int_{\Omega} p \nabla \cdot w \, d\Omega = -\rho \omega^2 \int_{\Omega} \mathbf{u} \cdot w \, d\Omega . \tag{9}
\]

Conversely, if \( \{\mathbf{u}(x), p(x)\} \) satisfies (9) for each smooth function \( w \), then one may show that this solution will solve (6). After performing integration by parts, which reduces one order of the derivatives of the displacement, we obtain

\[
-\int_{\Omega} [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] \cdot \nabla w \, d\Omega - \int_{\Omega} p \nabla \cdot w \, d\Omega + \int_{\Gamma_2} [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + p \mathbf{I}] \cdot \mathbf{n} \cdot w \, d\Gamma = -\rho \omega^2 \int_{\Omega} \mathbf{u} \cdot w \, d\Omega . \tag{10}
\]

By substituting the traction boundary condition (8) into (10), we have

\[
\int_{\Omega} [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] \cdot \nabla w \, d\Omega + \int_{\Gamma_2} p \nabla \cdot w \, d\Omega + \int_{\Gamma_2} \hat{T} \cdot w \, d\Gamma = -\rho \omega^2 \int_{\Omega} \mathbf{u} \cdot w \, d\Omega . \tag{11}
\]

Then the infinite dimensional problem (11) is replaced with a finite dimensional version:

\[
\int_{\Omega} [\mu^h (\nabla \mathbf{u}^h + (\nabla \mathbf{u}^h)^T)] \cdot \nabla \mathbf{w}^h \, d\Omega + \int_{\Omega} p^h \nabla \cdot \mathbf{w}^h \, d\Omega = -\rho \omega^2 \int_{\Omega} \mathbf{u}^h \cdot \mathbf{w}^h \, d\Omega + \int_{\Gamma_2} \hat{T} \cdot \mathbf{w}^h \, d\Gamma , \tag{12}
\]

where \( \mu^h, p^h, \nabla \mathbf{u}^h, \mathbf{u}^h \) and \( \mathbf{w}^h \) are finite dimensional approximations of the actual values. In the FEM, a finite number of nodes are considered on the region for each of the parameters and these values are approximated over the region using finite element shape functions:

\[
\mu^h(x) = \sum_{a=1}^{N_\mu} \tilde{\mu}_a \psi^h_{\mu a}(x) , \tag{13}
\]

\[
p^h(x) = \sum_{b=1}^{N_p} \tilde{p}_b \psi^h_{pb}(x) , \tag{14}
\]

\[
\mathbf{u}^h(x) = \sum_{c=1}^{N_u} \tilde{\mathbf{u}}_c \psi^h_{uc}(x) . \tag{15}
\]
\[
\n\nabla \mathbf{u}^h(x) = \sum_{\gamma=1}^{N_u} \nabla \mathbf{u}_\gamma \psi_\gamma^h(x),
\]

(16)

where \(\psi_\gamma^\mu\), \(\psi_\gamma^p\) and \(\psi_\gamma^u\) are shape functions for the shear modulus, pressure and displacement respectively and \(\bar{\mu}_\gamma\), \(\bar{p}_\gamma\) and \(\nabla \mathbf{u}_\gamma\) are nodal values of the parameters. \(N_\mu\), \(N_p\) and \(N_u\) are the numbers of shear modulus, pressure and displacement nodes. The shape functions have a value of 1 at their corresponding node and a value of 0 at all other nodes. Normally in a finite element formulation, when solving the forward problem, the gradient of displacement \(\nabla \mathbf{u}^h\) is approximated by differentiating the displacement shape functions in (15), but in the case of an inverse problem where we have noisy displacement measurements, it is better to take the derivatives of the displacements separately using a more robust method such as the least-squares method and then use them in the formulation.

In Galerkin’s method, the test functions \(\mathbf{w}^h\) are also approximated using the displacement shape functions as

\[
\mathbf{w}^h(x) = \sum_{\alpha=1}^{N_u} \tilde{\mathbf{w}}_\alpha \psi_\alpha^h(x).
\]

(17)

Since equation (12) must be satisfied for an arbitrary \(\tilde{\mathbf{w}}_\alpha\), then \(\tilde{\mathbf{w}}_\alpha\) can be factored out and eliminated, resulting in

\[
\int_{\Omega} \left[ \mu^h (\nabla \mathbf{u}^h + (\nabla \mathbf{u}^h)^T) \right] \cdot \nabla \psi \, d\Omega + \int_{\Omega} \rho \nabla \psi \cdot \mathbf{d} \Omega = \rho \omega^2 \int_{\Omega} \mathbf{u}^h \psi \, d\Omega + \int_{\Gamma_t} \mathbf{T} \psi \, d\Gamma,
\]

(18)

where \(\psi\) is a row vector of all the displacement shape functions:

\[
\psi = \begin{bmatrix} \psi_1^u, \psi_2^u, \ldots, \psi_{N_u}^u \end{bmatrix}
\]

(19)

\[
\nabla \psi = \begin{bmatrix} \psi_1^u, \psi_2^u, \ldots, \psi_{N_u}^u \\ \psi_1^p, \psi_2^p, \ldots, \psi_{N_p}^p \\ \psi_1^\mu, \psi_2^\mu, \ldots, \psi_{N_\mu}^\mu \end{bmatrix}.
\]

(20)

There is one point about the test function which should be mentioned here. The second term on the RHS of the equation (12) contains the traction vector on the boundary which we are not able to measure. By setting the interpolation functions for the test functions \(\mathbf{w}^h\) to be zero at the nodes on the boundary, the equations associated with the boundary nodes are eliminated.

It should be mentioned that the order of shear modulus and pressure shape functions should be chosen to be equal or less than the order of the displacement shape functions in order to have enough equations to solve for the unknowns (Park and Maniatty 2006). In this paper, we use constant shape functions for the shear modulus and pressure, and linear shape functions for the displacement.

In elastography, the displacement distribution is known from measurements. The unknown is the elasticity distribution. Therefore, equation (12) should be rearranged with respect to shear modulus and pressure unknowns and written in the matrix form as follows:

\[
\begin{bmatrix} \mathbf{A} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \bar{\mu} \\ \bar{p} \end{bmatrix} = \{f\},
\]

(21)

in which vectors \(\bar{\mu}\) and \(\bar{p}\) are the global shear modulus and pressure vectors, and the \(i\)th column of the matrices \(\mathbf{A}\) and \(\mathbf{C}\) and the vector \(f\) are defined as follows:

\[
\mathbf{A}_i = \text{reshape} \left( \int_{\Omega} \left[ \psi_i^\mu (\nabla \mathbf{u}^h + (\nabla \mathbf{u}^h)^T) \right] \cdot \nabla \psi \, d\Omega \right),
\]

(22)

\[
\mathbf{C}_i = \text{reshape} \left( \int_{\Omega} \psi_i^p \nabla \psi \, d\Omega \right),
\]

(23)
\[ f = \text{reshape} \left( \rho \omega^2 \int_\Omega \mathbf{u}^T \psi \, d\Omega \right). \]  

(24)

where \text{reshape} (\cdot) of a matrix means all the elements of that matrix are regarded as a single column.

After defining the system of equations (21), the question that arises is whether this system of equations is solvable. First of all, as explained before, since we are eliminating the boundary force term in equation (12) by using the test functions which have zero value on the boundary, we are not able to find the absolute value of the pressure. The reason is that eliminating the boundary forces means we are not considering equation (8). Equation (6) alone has only information about the gradient of the pressure. Therefore, only the relative values of the pressure can be determined by that equation and, in order to solve for pressure, we need to determine the value of the pressure in at least one element. Moreover, depending on the order of the shape functions and the number of nodes used for unknown parameters and the displacements, the number of equations might be less or more than the number of the unknown parameters. Therefore, the system might be under or over determined. For example, in the 2D case with a mesh size of \((n + 1) \times (n + 1)\), if we use rectangular four-node elements with constant shape functions for the shear modulus and pressure and a linear shape function for displacements, we will have \(n \times n\) elements, therefore \(2n^2\) unknowns and \(2(n - 1)^2\) equations (two equations for each interior node). In this case the system is underdetermined. But for the 3D case, using the same type of element and shape functions, the number of unknowns would be \(2n^3\) and the number of equations would be \(3(n - 1)^3\). In this case for \(n > 7\), we have more equations than unknowns. Yet even in this case, the system is rank deficient due to the pressure term and a constraint needs to be added to the pressure parameters to make the system determined.

Removing the boundary equations not only decreases the rank of the matrix \(C\), but also degrades the condition number of the matrices \(A\) and \(C\). Furthermore, the existence of noise and points of small displacements or small derivatives of displacements degrades the conditioning of the inverse problem. Therefore, a degree of regularization is always needed to improve the condition of the inverse problem and make the problem solvable.

In the following section, different ways of adding constraints to the problem parameters are discussed.

2.2. Sparsity regularization

As mentioned before, regularization is necessary to add constraints on the system parameters to make the system determined and well conditioned. A common approach is the Tikhonov regularization in which the objective function of the inverse problem is changed to a sum of two terms: a data misfit term and a regularization term (Park and Maniatty 2006). The regularization consists of the norm of the parameters plus the norm of the low-order spatial derivatives of the parameters. So, in that case, the new inverse problem becomes

\[
\min_{\bar{\mu}, \bar{p}} \left[ (A\bar{\mu} + C\bar{p} - f)^T (A\bar{\mu} + C\bar{p} - f) + \alpha (G\bar{\mu})^T (G\bar{\mu}) + \beta_1 \bar{p}^T \bar{p} + \beta_2 (G\bar{p})^T (G\bar{p}) \right],
\]

(25)

where \(\alpha, \beta_1\) and \(\beta_2\) are regularization parameter weights and \(G\) is the differential operator. The reason the norm of the pressure parameter and its gradient are penalized, while only the norm of the spatial derivatives of the shear modulus is penalized, is the rank deficiency of the matrix \(C\). This deficiency, inhibiting our ability to determine the absolute value of the pressure, was created after the removal of the boundary equations. However, the absolute values can be still determined for the shear modulus, and there is no need to add an absolute constraint for
the shear modulus. Differentiating equation (25) with respect to the unknown parameters and setting it equal to zero, we obtain the new system of equations as follows:

\[
\begin{bmatrix}
  A^T A + \alpha G^T G \\
  C^T A \\
  C^T C + \beta_1 I + \beta_2 G^T G
\end{bmatrix}
\begin{bmatrix}
  \bar{\mu} \\
  \bar{\phi}
\end{bmatrix}
= \begin{bmatrix}
  A^T f \\
  C^T f
\end{bmatrix}.
\] (26)

The proposed alternative regularization is to use a sparse transformation. Depending on the structure of the underlying field of parameters, a suitable sparsifying transform should be chosen so that only a small subset of the new parameters in the transformed field can provide a good approximation of the original field. The DCT is often used for audio (MP3) and image (JPEG) compression because it has a strong energy compaction property. The 1D DCT of a signal \(x(n)\) of length \(N\) is as follows:

\[
X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \cos \left( \frac{\pi}{N} \left(n + \frac{1}{2}\right) k \right) \quad k = 0, \ldots, N-1.
\] (27)

To make this transform orthonormal, the first term \(X(0)\) should be multiplied by \(\frac{1}{\sqrt{N}}\) and the other terms should be multiplied by \(\sqrt{\frac{2}{N}}\). For the 2D case, it is simply a separable product of DCTs along each dimension:

\[
X(k_1, k_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \cos \left( \frac{\pi}{N_1} \left(n_1 + \frac{1}{2}\right) k_1 \right) \cos \left( \frac{\pi}{N_2} \left(n_2 + \frac{1}{2}\right) k_2 \right).
\] (28)

For illustration, figures 1(a) and (d) show two different sample elasticity maps. One map has an inclusion with a Gaussian profile and the other a circular inclusion with a step profile. We can see from figures 1(b) and (e), which show the logarithm of the absolute values of the DCT coefficients for (b) Gaussian and (e) step distributions. Reconstructed (c) Gaussian, (f) step distributions maps from truncated DCT coefficients, i.e. the coefficients inside the square regions in (b) and (e).
coefficients that elasticity maps with sharp edges are less sparse after transformation than maps which are smooth. Figures 1(c) and (f) show the inverse DCT of the truncated coefficients located inside the square region shown in figures 1(b) and (e). As we can see, by using only $\frac{1}{16}$ of the low-frequency DCT coefficients, we can reconstruct the original map with a very good approximation. In figure 2, the upper row shows cross-section profiles of the reconstructed maps obtained from truncated DCT coefficients using different cut-off ratios $r = \frac{N_s}{N_t}$, which is the ratio of the size of the selected region to the original region. The second row in figure 2 shows the reconstructed map after penalizing the spatial derivatives of the map similar to what happens in the Tikhonov regularization. This is done by minimizing the functional below:

$$\min_\mathbf{E} [\mathbf{E} - \mathbf{E}^*]^T (\mathbf{E} - \mathbf{E}^*) + \alpha (\mathbf{GE}^*)^T (\mathbf{GE}^*)]$$

(29)

in which $\mathbf{E}^*$ is the original elasticity map in a vector and $\mathbf{G}$ is the differential operator and $\alpha$ is the regularization parameter. After minimization we have the system of equations as follows:

$$[\mathbf{I} + \alpha \mathbf{G}^T \mathbf{G}] [\mathbf{E}] = [\mathbf{E}^*].$$

(30)

As shown in figures 2(a) and (c), the reproduced image of Gaussian profile using truncated DCT coefficients is almost exact but the step profile has some ringing artifacts. Although there are some oscillations around the exact values, the contrast between the inclusion and background is almost exact on average and we do not see any reduction in contrast. But if we use the Tikhnonov regularization to limit the spatial gradient of the variables, this will cause a reduction in contrast as shown in figures 2(b) and (d).

**Figure 2.** Profile of the reconstructed maps using truncated DCT coefficients of the (a) Gaussian map and (c) circular step map. The resulting profiles using the Tikhonov regularization which penalizes the gradient of the original parameters for maps with (b) Gaussian and (d) step inclusions.
It is anticipated that for smooth fields, the DCT is a suitable sparsifying transform which can be used to approximate the field with significantly fewer variables. Therefore, using the sparsity regularization with the DCT as a sparsifying transform can be helpful by decreasing the number of variables and consequently improving the condition of the inverse problem for such fields. Moreover, the other advantage of this method is its higher speed. Since the number of the unknowns is reduced significantly, the system of equations can be solved significantly faster than the original system of equations.

Since the DCT is a linear transform, it can be written in the matrix form

\[ X = T x \]  

in which \( T \) is the DCT transform matrix, \( x \) is a vector containing original field variables and \( X \) is the transform-domain vector. We can reconstruct \( x \) from \( X \) using the inverse of \( T \):

\[ x = T^{-1} X. \]  

Since \( X \) is sparse, we can choose a small number of dominant coefficients with large values and assume the others to be zero. This means we can choose the appropriate columns of \( T^{-1} \) corresponding to those selected dominant coefficients and call the new matrix \( R \equiv (T^{-1})^* \), and let \( X^* \) be the truncated transform-domain vector. In this way:

\[ x \approx (T^{-1})^* X^* = RX^*. \]  

In order to apply the sparsity regularization on the direct FEM inverse problem explained in the previous section, we need to replace the vectors \( \bar{\mu} \) and \( \bar{p} \) in equation (21) with approximated vectors obtained by using equation (33). So the new regulated system of equations becomes

\[
\begin{bmatrix}
(AR_1)^T (AR_1) & (AR_1)^T (CR_2) \\
(CR_2)^T (AR_1) & (CR_2)^T (CR_2)
\end{bmatrix}
\begin{bmatrix}
\bar{\mu}^* \\
\bar{p}^*
\end{bmatrix} =
\begin{bmatrix}
(AR_1)^T f \\
(CR_2)^T f
\end{bmatrix},
\]  

in which \( R_1 \) and \( R_2 \) are appropriate selected transforms for shear modulus and pressure variables, respectively. The challenging part of the sparsity regularization is determining how to choose the dominant transformed variables. Depending on the structure of the field, the maximum coefficients of the transformed field are placed in different positions. If prior knowledge of the structure of unknown variables is available, it can be used to find the pattern of DCT coefficients distribution. For example, assuming a smooth distribution of variables, we can conclude that most of the larger coefficients in the DCT domain are located in the lower frequency part of the domain. So in this paper, we roughly select the coefficients located in a rectangular window at the lower frequency corner of the domain as shown in figure 2(a).

### 3. Numerical simulations

A 2D 20 mm \( \times \) 20 mm region under plane strain is used to create the synthetic data. The region is bounded from the top and left as shown in figure 3 and excited from the bottom with a harmonic displacement excitation. There are two Gaussian hard inclusions with the distribution:

\[ E(x, y) = 10 + 15 \times e^{(-\frac{(x-10)^2 + (y-5)^2}{36})} + 30 \times e^{(-\frac{(x-10)^2 + (y-15)^2}{36})}, \]  

where \( E \) is the elasticity value in kPa. A constant density of 1000 kg m\(^{-3}\) is assumed for the medium and the Poisson ratio is assumed to be \( \nu = 0.4995 \). While typical values for \( \mu \) and \( \lambda \) for soft tissue are 10 kPa and 2 GPa, respectively, which lead to the Poisson ratio of \( \nu = 0.499997 \), we have used a smaller Poisson ratio of 0.4995 in all of our computations. To justify this use, we have solved the forward problem for different values of the Poisson ratios, and found that for Poisson ratios greater than 0.4995, changes in the results are negligible. The
Figure 3. Schematic showing geometry and boundary conditions of the model used. The region is subjected to a harmonic excitation from the bottom and a sliding constraint at the top and left edges.

Figure 4. (a) The axial displacement (mm) and (b) pressure distributions (kPa) using an excitation frequency of 100 Hz.

The forward problem is solved using four-node square elements with a mesh size of $L = 0.25$ mm ($80 \times 80$) to obtain synthetic data. The element size used in solving the inverse problem should be larger than the element size used in the forward problem so that the discretization error of the forward problem is small compared to the discretization error of the inverse problem. Therefore, the generated data are first down-sampled to a coarser mesh size of $30 \times 30$ and then used in the inverse problem. The sample data generated with an excitation frequency of 100 Hz are shown in figure 4.

The elasticity reconstruction is performed using the direct mixed FEM technique with both the Tikhonov regularization and the sparsity regularization. In order to complete the rank of the matrix $C$, in sparsity regularization, $p^*(0, 0)$, which is the coefficient of the constant base function in the DCT domain, is assumed to be zero and is removed from the unknowns. In the Tikhonov regularization, the term which minimizes the norm of the pressure deals with this problem. The reconstructed elasticity and pressure for the excitation frequency of 100 Hz, using the sparsity regularization with cut-off ratios of $r_{\mu} = 0.4$ and $r_p = 0.9$ for the shear modulus and pressure respectively, are shown in figure 5. Figure 6 shows the reconstructed elasticity using both sparsity and Tikhonov regularizations for two different excitation frequencies. As
Figure 5. (a) and (b) Reconstructed elasticity (kPa) and (c) pressure (kPa + constant) using our sparsity regularization method with cut-off ratios of $r_\mu = 0.4$, $r_p = 0.9$ and excitation frequency of 100 Hz.

Figure 6. Reconstructed elasticity (kPa) using the sparsity (top row) and Tikhonov (bottom row) regularization methods for 150 and 200 Hz excitation frequencies.

one can see, parameter settings for the Tikhonov regularization in one case may not work for a different case, for example, when the frequency of excitation or boundary conditions are changed. In this example, the parameters of the Tikhonov regularization were set for the excitation frequency of 150 Hz ($\alpha = 10^{-17}$, $\beta_1 = 10^{-14}$ and $\beta_2 = 10^{-21}$) and then the same parameters were used for the excitation frequency of 200 Hz. The parameters are optimized by performing a full search on a discretized domain to minimize the RMS error for elasticity. As the results shown in figure 6(b), the Tikhonov regularization is not working at 200 Hz as well, and the regularization parameters need to be readjusted, while the sparsity regularization does not need any changes. The elasticity reconstruction was performed for different excitation frequencies and different boundary conditions using both regularization methods, and the RMS error was calculated for each case using equation (36):

$$\text{RMS} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \frac{(E_i - E_0^i)^2}{(E_0^i)^2}}.$$  

(36)

The mean of RMS errors for the sparsity regularization and that of the Tikhonov method are 0.064 and 0.148, respectively, with the standard deviations of 0.027 and 0.0363. The mean
Figure 7. The reconstructed elasticity of the region with circular step inclusion using the Tikhonov regularization (upper row (a)–(c)) and the sparsity regularization (lower row (d)–(f)). For the middle column, the optimized parameters are used and the left and right columns are the results of under-regularized and over-regularized conditions, respectively. (g) and (h) are the RMS error versus regularization parameters for Tikhonov and sparsity regularization methods.

and standard deviation for the sparsity regularization are smaller than those for the Tikhonov method, which shows that the Tikhonov method is more dependent on the frequency and boundary condition.

As discussed in the previous section and shown in figure 1(e), in the case of elasticity patterns with sharp edges, the transformed variables in the DCT domain are not as sparse as when the elasticity pattern is smooth and without discontinuity. This can affect the performance of the sparsity regularization and cause artifacts in the reconstruction results. In order to investigate the effect of discontinuities in elasticity distribution, a region with a circular step inclusion has been used to create synthetic data with the same boundary conditions as in figure 3, with an excitation frequency of 200 Hz. White Gaussian noise was added to the data to achieve an SNR of 35 dB, and the inverse problem was solved using both regularization methods. The regularization parameters were optimized for both methods. The parameters obtained for the sparsity regularization are \( \alpha = 0.6 \) and \( \epsilon_p = 0.9 \), and for the Tikhonov regularization are \( \alpha = 10^{-17}, \beta_1 = 10^{-13} \) and \( \beta_2 = 10^{-25} \). Figure 7 shows the reconstructed results. To show the effect of the regularization parameters, these parameters were changed around the optimized value and the inverse problem was solved. For this test, only the parameters related to the shear modulus were changed and the others were kept constant. The figures in the central column show the results when the optimized parameters are used, and the left and right columns show the results of under- and over-regularized conditions, respectively. The two plots in figures 7(g) and (h) show the RMS errors as the
regularization parameters change. As can be seen, for under-regularized conditions, errors appear and some artifacts related to discretization error and noise appear in the results with both methods. For over-regularized conditions, the results are smooth and, as expected, the Tikhonov regularization reduces the contrast, but the sparsity regularization only causes some ringing artifacts but does not reduce the contrast.

In another test, white Gaussian noise was added to the displacement data, obtained from both models with Gaussian and step inclusion, to achieve an SNR of 25 dB and the inverse problem was solved for different values of excitation frequencies using both regularization methods. For each case, the model was simulated 30 times and the means and standard deviations of the RMS error for elasticity are shown in figure 8. The regularization parameters which were used are the ones optimized for the excitation frequency of 200 Hz. As can be seen from this figure, the mean RMS values for the Tikhonov method are higher than that for the sparsity regularization and also are much more dependent on the excitation frequency. The standard deviation of RMS errors for the sparsity regularization is smaller than for the Tikhonov method which shows the higher stability of this method compared to that of the Tikhonov method in the presence of noise. As expected, it can also be seen that the sparsity regularization works better for the region with Gaussian inclusion due to its higher level of sparsity.

4. MRE experiment data

The direct mixed FEM inversion technique with the sparsity regularization was applied to the MRE experiment data captured from a CIRS elastography phantom model 049 (CIRS Inc., Norfolk, VA, USA). The phantom has eight spherical inclusions with different values of elasticity and sizes as shown in figure 9. The assumed values of elasticity for background and inclusions are reported by the manufacturer and are shown in table 1. The sizes of the inclusions are 10 and 20 mm in diameter. The phantom was excited with a frequency of 200 Hz in the Z-direction using a MRI compatible actuator developed in our lab (Sahebjavaher et al 2012).
Figure 9. (a) CIRS elasticity phantom. (b) Schematic showing the region of interest and the exciter position.

Figure 10. All three components of displacement (mm) data in a plane normal to the Y-direction. (a), (b) and (c) show X, Y and Z directional displacements, respectively.

Table 1. Elasticity values of inclusions and background for CIRS phantom reported by the manufacturer.

<table>
<thead>
<tr>
<th>Region</th>
<th>Elasticity (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>6</td>
</tr>
<tr>
<td>Type II</td>
<td>17</td>
</tr>
<tr>
<td>Type III</td>
<td>54</td>
</tr>
<tr>
<td>Type IV</td>
<td>62</td>
</tr>
<tr>
<td>Background</td>
<td>29</td>
</tr>
</tbody>
</table>

The MRE images were acquired using a 2D multi-slice multi-shot SE-EPI method with turbo factor 11 (Herzka et al 2009). A single sinusoidal cycle was used as the motion encoding gradient with a strength of 60 mT m$^{-1}$, which captured eight states of the mechanical motion. All three components of displacement were acquired on a 224 $\times$ 112 $\times$ 48 matrix with 1.5 mm isotropic voxel size. Total image acquisition time was about 5 min. The acquisition matrix was cropped to 96 $\times$ 48 $\times$ 32, which includes the four larger inclusions as shown in figure 9(b). Figure 10 shows all the three components of the displacement in a plane normal to the Y-direction. From this figure, especially the displacement in the Z-direction, the reader can recognize the position of the actuator where the displacement is maximum. The displacements are attenuated further away from the excitation source because of the damping effect in the material.

The region of interest is divided into overlapping sub-domains of size 15 $\times$ 15 $\times$ 15 to decrease the processing time. The sub-domains are overlapped by three pixels and the values in the overlapping region are averaged to retain the uniformity of the reconstructed
shear modulus. The derivatives of the displacements used in the reconstruction are calculated using a planar fit on overlapping windows of size $4 \times 4 \times 4$. A constant density of 1000 kg m$^{-3}$ is assumed. The sparsity regularization is used for both the shear modulus and pressure. The sparsity pattern or the selected new coefficients in the transformed domain are the lower frequency coefficients located in a rectangular window of size $\frac{3}{2}$ of the entire domain which in this case is $10 \times 10 \times 10$.

Figure 11 shows the reconstructed elasticity on different cross-sections. All the inclusions are visible with similar shapes, sizes and elasticities as the manufacturer’s values. The regions are segmented by assuming spheres of radius 10 mm, locating them manually in each region and moving them around the assigned positions to minimize the elasticity error. The mean and standard deviation of the elasticity of each region are calculated and compared with the values reported by the manufacturer. The quantitative results are shown in figure 12. In order to estimate the size of the inclusions in the reconstructed elasticity map, the average of the elasticity of the elements located at a distance $r$ from the estimated centers are plotted in figure 13. Then, the radius of the inclusion is defined at the midpoint between the points of maximum curvature. The estimated radii of the inclusions 1, 2, 3 and 4 are 10.6, 11.4, 9.5 and 9.9 mm, respectively, which are close to the manufacturer’s nominal value of 10 mm.

The processing time for this case using the sparsity regularization was approximately 20 min, while the Tikhonov regularization took approximately 5 h to process. Since similar quality results can be obtained using the Tikhonov regularization by adjusting the regularization parameters properly, only the results from the sparsity regularization are shown here. The estimated elasticity values are slightly different from the values reported by the manufacturer. For hard inclusions, we have an underestimation of the true parameters and for soft inclusions we have an overestimation of the true parameters. The same discrepancy has also been reported by Baghani et al (2011). The reasons for this discrepancy can be the difference in temperature, excitation frequency or the changes of the material properties due to ageing.

As mentioned earlier, only the relative values of the pressure distribution can be evaluated. Since we are using small overlapping windows and the pressure values obtained are not absolute values, the pressure distribution is not continuous at the boundaries of the overlapping windows as shown in figure 14(a). In order to be able to show a continuous pressure distribution for the entire domain, we down-sampled the data to $\frac{3}{2}$ of the original mesh size and solved the inverse problem for the entire domain at once without dividing it into smaller windows. Figure 14(b) shows the reconstructed pressure distribution for this case. Unlike the shear modulus, the unit of the pressure depends on the unit of the displacement data, but since only the shear modulus value is of our interest in this study, we are not concerned about the unit of the pressure obtained. However, from this figure we can see that the gradient of the pressure is not negligible in some areas especially near the boundaries and this term should be considered in the reconstruction formulation as reported by Park and Maniatty (2006). Also, from the pressure distribution, the position of the actuator is easily detectable in the middle at top where the pressure concentration is higher. This shows the consistency of the pressure solution.

In order to show the effect of windowing on the reconstruction, the down-sampled data which were used to solve the inverse problem without windowing were also used in the window-based reconstruction method using overlapping windows of size $15 \times 15 \times 15$ and the results were compared. The mean and standard deviation of the RMS error between these two results are 0.0179 and 0.0142, respectively. This shows that the results obtained from the
Figure 11. The reconstructed elasticity (kPa) using the sparsity regularization. The reconstructed elasticity distribution on different cross-sections normal to the Z-direction (left column) and the Y-direction (right column) are shown.
Figure 12. (a) The segmented regions with different elasticity. (b) The mean and standard deviation of the elasticity over each region compared to the values reported by the manufacturer.

Figure 13. Profiles of the average elasticity versus distance from the center of the inclusions (a) type I, (b) type II, (c) type III and (d) type IV.

windowing method are very close to the results obtained when the whole region is processed at once.
5. Conclusion

This paper presented a comparison of sparsity and Tikhonov regularizations in solving the inverse problem of tissue elasticity using a direct finite element based algorithm with mixed $u$–$p$ formulation. Synthetic data were generated using different boundary conditions and excitation frequencies and the reconstructed results from sparsity and Tikhonov regularizations were compared. The results show the higher dependence of the Tikhonov regularization on the excitation condition and the higher dependence of its results on the regularization parameters. Sparsity regularization is independent of the boundary conditions and excitation and no readjustment is needed in this method. The noise analysis shows the higher robustness of the sparsity regularization compared to that of the Tikhonov regularization method. The results also show the higher speed of the proposed method. Results from MRE experiment data were also presented. The data were captured from a CIRS elastography phantom. All the inclusions were discernible as spherical inclusions with identifiable edge locations within 1.4 mm of the expected edge, and absolute elasticity values for each inclusion matching the order from smallest to highest provided by the manufacturer. Some discrepancies in absolute elasticity remain, likely due to differences in test conditions or ageing of the phantom material, as reported previously.

For future work, the sparsity pattern used in the sparsity regularization can be improved. In the sparsity regularization, one challenge is to decide which transformed variables should be retained and which should be neglected. The pattern we used in this paper was simply a rectangular window of the new coefficients at the lower frequency part of the frequency domain. An alternative way is to choose an optimum sparsity pattern based on the prior knowledge of the unknown parameters.

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