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Uncertainty of nuclear counting

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Abstract
Nuclear counting is affected by pulse pileup and system dead time, which induce rate-related count loss and alter the statistical properties of the counting process. Fundamental equations are presented to predict deviations from Poisson statistics due to non-random count loss in nuclear counters and spectrometers. Throughput and dispersion of counts are studied for systems with pileup, extending and non-extending dead time, before and also after compensation for count loss. Equations are provided for random fractions of the output events, applicable to spectrometry applications. Methods for loss compensation are discussed, including inversion of the throughput equation, live-time counting and loss-free counting. Secondary effects in live-time counting are addressed: residual interference from pileup in systems with imposed dead times and errors due to varying count rate when measuring short-lived radionuclides.

Keywords: nuclear counting, dead time, pulse pileup, radionuclide metrology, counting statistics

(Some figures may appear in colour only in the online journal)
time intervals when the measuring system is free to accept or record events.

In the field of primary standardization of activity [35], counting is usually performed with a single-channel system and an artificial dead time of selectable length and type (extending or non-extending) imposed on every detected event. This imposed dead time dominates the effects of instrumental dead time, yielding greater control over the system live time. Even though the major contribution to count loss may be compensated for, additional extension of the artificial dead time may be required to cover long saturation pulses, delayed signals from metastable states and afterpulses (e.g. due to delayed fluorescence generated in liquid scintillation counting systems), pickup of electrical oscillations, etc.

Ultimate precision requires additional consideration of secondary effects. For example, in systems with imposed dead time there is a residual count loss effect due to pulse pileup which is mathematically equivalent to a series arrangement of dead times [36–45]. Another deviation to normal counting statistics occurs if the Poisson process is inhomogeneous, e.g. in the case of short-lived radionuclides for which the activity varies significantly during a measurement [7, 46–50]. Adaptive ‘loss-free’ counting systems can deal with the simultaneous variation of the count rate and the live-time fraction [29–34], but classical live-time counting applying an average dead-time correction is in error [44, 45].

In this work, three basic types of count loss mechanisms, their effects on a stationary Poisson process in the time domain, and statistical uncertainty formulas are discussed for both integral counting and for spectrometric analysis of a fraction of the event spectrum. Equations are presented to gain statistical control over nuclear counting, applicable from routine spectrometry to high-level standardization work.

2. Basics of nuclear counting

2.1. Pileup and dead time

In the literature it is often stated that there are two basic types of dead time, ‘extending dead time’ (EDT) and ‘non-extending dead time’ (NEDT), and that all nuclear counters can be described by one of these two types, or by a combination of them (see e.g. [51, 52]). However, this does not account for the statistical effects caused by pulse pileup (PU), which is the main loss mechanism in contemporary spectrometry chains [11, 13, 20].

‘Non-extending’ means that the dead-time period, $\tau_n$, is not prolonged by the arrival of a new pulse during the dead-time period. The dead time created by an ADC is a typical example. For the so-called ‘paralysable’ systems, with dead time of the ‘extending’ type, each incoming event prolongs the system dead time by an amount $\tau_e$. A clear distinction should be made between count loss by ‘pileup (rejection)’ and ‘extending dead time’. Pulse pileup occurs when a new pulse from the preamplifier is fed through the shaping amplifier before the ADC has had the chance to complete processing the previous pulse. In such cases, an amplifier-ADC combination with pileup rejection has the ability to inhibit the ADC from processing the composite pulses. Contrary to the case of ordinary dead-time effects, the ‘first’ incoming pulse is lost for spectrometric analysis, either by a built-in electronic pileup rejection system or by the pulse count being assigned to another spectrum channel [13]. Therefore we are facing at least three different types of pulse loss distortions of a Poisson process, each requiring a particular statistical treatment.

In figure 1 the pulse shapes of a semi-Gaussian (left) and a gated integrator (right) amplifier are shown, as well as a
schematic pulse representation (below). Pileup, with loss of at least two signals, occurs when the arrival of a pulse falls within the ‘leading edge’ of the previous pulse. This corresponds to the time zone ‘A’ in the schematic graph and has a characteristic duration $T_P$, i.e. from when the pulse rises above the noise level up to the peak height detection by the ADC. If the pulse falls within the ‘trailing edge’ portion, time zone ‘B’, the first pulse is successfully analysed and only the second pulse is lost. Therefore, pileup in the trailing edge is fully equivalent with EDT of duration $T_W - T_P$, where $T_W$ is the pulse width.

Double count loss by pileup can also occur in the absence of a pileup rejector. In spectrometry, small fractions of the pulse height spectrum are usually considered. The composite pulse of coincident events will (most often) fall outside the region of interest (ROI) and hence be lost in a comparable manner as with pileup rejection. However, when performing ‘integral counting’ of all pulses above a threshold level, the summed pulse is valid as one count and pileup is equivalent to EDT. This applies to most of the primary standardization methods [35] and Geiger–Müller counters.

2.2. Time interval distributions

To understand counting statistics, i.e. the probability for observing a given number of events, one should look upon the counting process as a sequence of events developing in time, where loss mechanisms directly affect the statistical distributions of the time intervals between successive events. These time intervals are independent, and form a renewal process [1, 5]. Multiple intervals, i.e. the waiting time for $N$ subsequent events, are obtained from a $N$-fold convolution of the single time-interval density distributions. The transition from interval densities to counting probabilities can be done via differentiation of the corresponding Laplace transform (see e.g. [5, 13, 16]). Experimental as well as theoretical work for EDT or NEDT and their combinations can be found in [1, 5, 8, 52]. More recent work [11, 13, 16] includes the effects of pileup and limited fractions of spectra.

In figure 3, the time distortion of a Poisson process is shown for NEDT, EDT and ‘PU’ at an incoming count rate of $\rho = 1/\tau$. NEDT preserves the exponential shape of the interval distribution, except that events falling within the dead time $\tau$ are eliminated. EDT also perturbs the slope of the interval distribution after the dead-time period $\tau$, which is most clearly visible in the time interval $[\tau, 2\tau]$ and piecewise fades out at higher time intervals. PU has yet another effect on the time-interval distribution and does not preserve the slope of the original Poisson process.

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2.3. Detector throughput

The time-interval distribution between successive counts determines the counting statistics. Expressions for the expectation value of the number of counts obtained during a fixed measuring time can be derived from the first moment of the \( N \)-fold time-interval distribution. For input pulses forming a Poisson process with rate \( \rho \), the expected output rate \( R \) for a counter with non-extendable dead time \( \tau_n \) is \([1, 5]\)

\[
R = \frac{\rho}{1 + \rho\tau_n},
\]

(1)

whereas with extendable dead time \( \tau_e \) it is \([1, 5]\)

\[
R = \rho e^{-\rho\tau_e},
\]

(2)

and in the case of pileup it is also exponential \([13]\):

\[
R = \rho e^{-\rho\tau_p},
\]

(3)

in which \( \tau_p = T_p + T_W = 2T_p + (T_W - T_p) \). As far as the average throughput is concerned, pileup is equivalent to EDT in which the resolving time between two pulses is counted twice. The pulse width may vary as a function of pulse height \([13]\), but in general the statistical equations in this paper apply to the mean characteristic width.

Correcting for count loss can be done by measuring in ‘live-time mode’, implicitly relying on the obtained real-time to live-time ratio correction factor, \( T_R/T_L \):

\[
\tilde{\rho} = \frac{R T_R}{T_L},
\]

(4)

or in the old-fashioned ‘real-time mode’ by explicitly applying the inverse throughput formula \( X^{-1}(R) \) (equations (1)–(3)), assuming that the characteristic dead time is known. The expression for NEDT is straightforward:

\[
\tilde{\rho} = \frac{R}{1 - R\tau_n}.
\]

(5)

For PU and EDT, the inversion of the throughput formula (equations (2) and (3)) is solved as a special case of Takacs’ formula \([8]\):

\[
\tilde{\rho} = R\sum_{k=0}^{\infty} \frac{(k + 1)^{k-1}}{k!} (R\tau_p)^k.
\]

(6)

The dead-time correction method is sensitive to uncertainty on the characteristic dead times, which propagate to the corrected rate via

\[
\frac{\sigma(\tilde{\rho})}{\tilde{\rho}} \approx \tilde{\rho} \tau_n \frac{\sigma(\tau_n)}{\tau_n}
\]

(7)

for NEDT (derivation in section 2.5), and

\[
\frac{\sigma(\tilde{\rho})}{\tilde{\rho}} \approx \tilde{\rho} \tau_p \frac{\sigma(\tau_p)}{\tau_p}
\]

(8)

for PU and EDT (derivation in section 2.6). A relative error of 10% on the characteristic dead times at a count rate of \( \rho\tau = 0.3 \) leads to errors of 3% (NEDT) and 4.3% (PU, EDT) on the count rate estimate.

2.4. Dispersion

At high event rates, the scatter on the number of counted events \( N \) is relatively lower than for a pure Poisson process. In figure 4 the standard deviation is plotted as a function of the input rate for the three basic types of count loss. The functions were derived theoretically from the first two moments of the Laplace transform of the \( N \)-fold time interval distribution \([5, 13]\). The asymptotic counting uncertainty in a counter with NEDT is \([1, 5]\)

\[
\frac{\sigma(N)}{\sqrt{N}} = \frac{1}{1 + \rho\tau_n},
\]

(9)

and in the case of EDT it is \([1, 5]\)

\[
\frac{\sigma(N)}{\sqrt{N}} = \sqrt{1 - 2\rho\tau_e e^{-\rho\tau_e}}
\]

(10)

on the condition that the measurement time \( t \) is much longer than the dead time \( \tau \) (\( t \gg \tau \)).

Whereas the effect of dead time on the count rate dispersion has long been known, the mathematical solution for counting with pileup (rejection) was derived only fairly recently \([13]\):
\[
\sigma(N) \over \sqrt{N} = \sqrt{1 + 2e^{-\rho \tau_p} \left[ 1 - (1 + \rho(T_p + T_W) e^{-\rho \tau_p}) \right]}.
\]  

(11)

The \( \sigma(N) / \sqrt{N} \) curve for pileup in figure 4 shows a resemblance with the one for EDT, but the deviation from \( \sqrt{N} \) is smaller and even becomes positive for \( \rho \tau_p > 2.5 \) (here \( \tau_p = 2T_p \)). Such effects cannot be reproduced by a generalized dead time as proposed by Albert and Nelson [1, 3, 8, 53].

As a rule, counting with significant non-random count loss (in the time domain) is underdispersed relative to Poisson statistics. However, after proper correction for the count loss, the result is always overdispersed. The true input rate is found only after correction for count loss. It is a mistake to assume that one can directly apply the relative uncertainties represented in equations (9)–(11), since the proper uncertainty propagation factor has to be taken into account. Consequently, the uncertainty on the loss-corrected number of counts, \( N_{\text{L}} \), always exceeds the value expected from Poisson statistics by at least the square root of the inverse throughput factor (equations (5) and (6)):

\[
\frac{\sigma(N)}{\sqrt{N}} \geq \sqrt{\omega_{R}}
\]  

(12)

for counters with PU and EDT, and

\[
\frac{\sigma(N)}{\sqrt{N}} \geq \sqrt{1 + \rho \tau}
\]  

(13)

for counters with NEDT. Explicit equations are derived in the next section.

2.5. Uncertainty after count-loss correction (NEDT)

Measurements performed with a real-time clock, for a fixed measurement time \( \ell_m = T_R \), have to be corrected for the non-linearity of the detector throughput to obtain the true count rate. The variance of the calculated counts \( N \) is obtained by uncertainty propagation on the variance of the measured counts \( N \). The loss correction can be done either by applying the ratio between the measurement real time and the observed system live time, \( \tilde{N} = NT_R / T_L \), or by relying on the inverse throughput formula (equations (5) and (6)), \( \tilde{N} = X^{-1}(N) \), if the characteristic dead time \( \tau \) of the counter is known.

Both methods yield equally precise results for NEDT. The uncertainty from the throughput of the formula via equation (5) is calculated from [23]:

\[
\sigma^2(N_{\text{L}}) \approx \left( \frac{\partial N_{\text{L}}}{\partial N} \right)^2 \sigma^2(N) = \left( \frac{\partial N}{\partial N_{\text{L}}} \right)^2 \sigma^2(N)
\]

\[
= (1 + \rho \tau_p)^4 \sigma^2(N),
\]  

(14)

which, in combination with equation (9) for the dispersion, leads to the statistical uncertainty on the loss-corrected counting result in the case of NEDT:

\[
\frac{\sigma(N_{\text{L}})}{\sqrt{N_{\text{L}}}} = \sqrt{1 + \rho \tau_p}.
\]  

(15)

A systematic component has to be included for the uncertainty on the characteristic dead time using the propagation formula in equation (7), which can be derived directly from

\[
\sigma^2(\rho) \approx (\partial \rho / \partial \tau_p)^2 \sigma^2(\tau_p) = \rho^4 \sigma^2(\tau_p).
\]

The alternative method to correct for count loss, applying the real-time to live-time ratio \( T_R / T_L \) via equation (4) leads to a similar result. One applies the uncertainty formula for the product of two interdependent variables [23]

\[
\frac{\sigma^2(N_{\text{L}})}{N_{\text{L}}^2} = \frac{\sigma^2(N)}{N^2} + \frac{\sigma^2(T_R / T_L)}{(T_R / T_L)^2} + 2 \text{cor} \frac{\sigma(N) \sigma(T_R / T_L)}{T_R / T_L},
\]  

(16)

in which \( \text{cor} \) represents the correlation factor between the number of counts \( N \) and the \( T_R / T_L \) ratio, which is equal to one since \( T_L = T_R - N \tau_p \) in the case of NEDT. Considering that

\[
\frac{\sigma^2(T_R / T_L)}{(T_R / T_L)^2} = \frac{\sigma^2(T_L)}{T_L^2} = \frac{\sigma^2(N)}{\sqrt{T_L^2}} = \frac{\sigma^2(N)}{T_L^2} (\rho \tau_p)^2,
\]  

(17)

one obtains from equation (16) the same asymptotic uncertainty as in equation (15). In practice this means that with NEDT the purely statistical dispersion (excluding systematic errors) on the estimated input rate \( \rho = N / T_R = N / T_L \) can be calculated directly from the number of counted events relative to the live-time:

\[
\sigma(\rho) = \sigma \left( \frac{N}{T_L} \right) = \sqrt{\frac{N}{T_L}}.
\]  

(18)

2.6. Uncertainty after count-loss correction (PU + EDT)

The throughput inversion method is not equivalent to live-time correction in the case of PU and EDT. Better statistical accuracy can be obtained by using the \( T_R / T_L \) ratio when available, since this contains additional information on the original count rate.

For a combination of PU and EDT (\( \tau_p = T_p + T_W \)), the uncertainty of the loss-corrected count integral, \( \tilde{N} = X^{-1}(N) = Ne^{\rho \tau_p} \) (equation (6)) is calculated from [23]

\[
\sigma^2(N_{\text{L}}) \approx \left( \frac{\partial N}{\partial N_{\text{L}}} \right)^2 \sigma^2(N)
\]

\[
= \left( \frac{1 - \rho \tau_p}{e^{\rho \tau_p}} \right)^2 \left\{ 1 + 2e^{-\rho \tau_p} \left[ 1 - (1 + \rho \tau_p)e^{-\rho \tau_p} \right] \right\} N,
\]  

(19)

which leads directly to [23]

\[
\frac{\sigma(N_{\text{L}})}{\sqrt{N_{\text{L}}}} \approx \sqrt{\frac{e^{\rho \tau_p} + 2 \left[ e^{\rho \tau_p} - (1 + \rho \tau_p) \right]}{(1 - \rho \tau_p)^2}}.
\]  

(20)

The specific case of EDT follows from equation (20), by setting \( T_p = 0 \):

\[
\frac{\sigma(N_{\text{L}})}{\sqrt{N_{\text{L}}}} \approx \sqrt{\frac{e^{\rho \tau_p} - 2 \rho \tau_p}{(1 - \rho \tau_p)^2}}.
\]  

(21)
formula in equation (8), which can be derived in a similar way as equation (19) via $\sigma^2(\rho) \approx (\partial R / \partial \rho)^{-2} (\partial R / \partial \rho_T)^2 \sigma^2(\rho_T) = ((1 - \rho_T e^{-\rho_T})^2 (-\rho^2 e^{-\rho_T})^2 \sigma^2(\rho_T))$

Figure 5 shows theoretical uncertainties from equations (20) ($T_P = T_W$) and (21) ($T_P = 0$), pertaining to the count integral as a function of the input rate. At low input rates, the relative uncertainty is proportional to the inverse square root of the number of counts. Clear deviations from this trend occur at high count rates exceeding $\rho_T = 0.1$. A singularity occurs at $\rho_T = 1$, which is merely a technical consequence of taking the inverse of the derivative at one point: the maximum of the throughput curve. In practice one gets a finite uncertainty, depending highly on $\sigma(N)$. The count rates corresponding to optimum counting uncertainty are situated rather close to $\rho_T = 2 \pm 3 \sqrt{3}$.

Equations (20) and (21) should be used with caution for low count numbers, corresponding to a high relative uncertainty $\sigma(N)$, because the derivative introduced via equation (19) can be assumed constant only for relatively small variations of $N$, i.e. for sufficiently large count numbers.

When applying the live-time to real-time ratio $T_R / T_L$, the uncertainty is again calculated from equation (16). With loss of the ‘extending’ type, there is no one-to-one relationship between the counted events $N$ and the system live time $T_L$. Hence the correlation factor generally does not equal 1. The uncertainties have been studied by computer simulation. For EDT ($T_P = 0$) one finds that the standard deviation of the loss-corrected number of counts in an arbitrary ROI equals the square root of the inverse throughput factor. However, the variance is higher with pileup rejection. Moreover, the relative deviation from Poisson statistics varies as a function of the considered spectrum fraction $f$ and the $T_P / T_W$ ratio. The following approximation can be applied for the statistical uncertainty on the loss-corrected counts for a spectrometer with PU and/or EDT:

$$\frac{\sigma(N)}{\sqrt{N}} \approx r \sqrt{\rho_T}, \quad (22)$$

in which $r$ is a correction factor that was derived from a detailed analysis of counting statistics in live-time mode [14]:

$$r = 1 + \frac{A}{2} \left( \frac{2 + A A}{1 + A} + f \right), \quad (23)$$

where $A = 1 - e^{-\rho_T}$, $A = 1 - e^{-\rho_T}$ and $f$ is the fraction of the spectrum represented by the considered ROI. Dependence of counting statistics on $f$ is discussed in section 3 (Spectrometry). The difference in statistical accuracy of both pulse loss correction methods is demonstrated in figure 6 for different combinations of PU and EDT (cf. $T_R / T_W$ ratio).

3. Spectrometry

3.1. Detector throughput

In routinely used spectrometers ($\gamma$ ray, x ray, alpha particle, etc.), the three types of count loss can be present at the same time. For a spectrometer counting electronic pulses with a finite width $T_W$ and leading edge $T_P$ followed by an ADC with a characteristic NEDT $T_M$, the throughput has been approximated by [52]:

$$R \approx \frac{\rho}{\exp(\rho(T_P + T_W)) + \max(0, \rho[T_M - (T_W - T_P)])}, \quad (24)$$

in which $\rho$ is the input rate and $R$ the average output rate. This formula works perfectly when only one type of count loss is involved, since it leads to the classical expressions in equations (1)–(3) in the case of extending ($R = \rho e^{-\rho T_W}$) and non-extending ($R = \rho / (1 + \rho T_M)$) dead time and also for combinations of PU with EDT ($R = \rho e^{-\rho (T_P + T_W)}$). It is not rigorous for a combination of PU and EDT with NEDT at extremely
high count rates. Such systems require specific calculations for dead times in series (see section 4).

3.2. Dispersion with NEDT

A spectrometer with narrow pulses and a slow ADC is considered ($T_M >> T_W$), for which NEDT is dominant. From figure 3 it could be concluded that time-interval distributions for a fraction $f$ of the energy spectrum is different from the full spectrum, hence also differences in statistical behaviour seen in figure 4 could be expected. Time interval distributions have been studied for random (in the time domain) fractions of the output events, with the aim of finding equations for the counting uncertainty in a ROI in the energy spectrum. This is of major concern for spectrometry measurements with a slow ADC.

The expression for the throughput for counting with NEDT in a ROI is

$$\frac{\sigma(N)}{\sqrt{N}} = \frac{\sqrt{f + (1-f)(1 + \rho <\tau_\text{a}>)^2}}{1 + \rho <\tau_\text{a}>}.$$ (25)

In the limit of extremely small ROIs, with $f = 0$, the uncertainty in equation (25) approaches Poisson behaviour, i.e. $\sigma(N) = \sqrt{N}$. For the full spectrum, $f = 1$, equation (25) reduces to the well-known expression in equation (9).

An even more general formula can be applied for a ROI taken with a Wilkinson type ADC, in which the characteristic dead time increases linearly with the spectral channel number [19]:

$$\frac{\sigma(N)}{\sqrt{N}} \approx \frac{\sqrt{f + (1-f)(1 + \rho <\tau_\text{a}>)^2}}{1 + \rho <\tau_\text{a}>},$$ (26)

in which $<\tau_\text{a}>_\text{a}$ is the average characteristic dead time for counts outside the ROI and $<\tau_\text{a}>$ is the average for the full spectrum. Equation (25) for a fixed dead time can be regarded as a special case of equation (26), in which $<\tau_\text{a}> = <\tau_\text{a}>_\text{a} = \tau_\text{a}$.

Computer simulation results reveal that the loss-corrected counting uncertainty with NEDT follows a surprisingly simple rule, which was confirmed with experimental tests using a Wilkinson type ADC [19]. Some evidence is shown in figure 7. The asymptotic statistical uncertainty on the loss-corrected number of counts in any ROI of a spectrum obtained by counting a Poisson process with a 'slow' Wilkinson ADC is given by

$$\frac{\sigma(N_i)}{\sqrt{N_i}} = \sqrt{1 + \rho <\tau_\text{a}>}.$$ (27)

For spectrum channels corresponding to a long characteristic dead time, it seems that the relatively low count scatter $\sigma(N)$ is compensated by the strong correlation with the system live time (cf. application of $T_R/T_L$) or the observed throughput (cf. application of inverse throughput). The opposite is true for channels associated with a short dead time. In practice, equation (27) infers that the dispersion of the loss-corrected count rate in any ROI is simply calculated from the counted events per live-time in the ROI, applying equation (18).

3.3. Dispersion with pileup and EDT

Pileup (rejection) is the dominant loss mechanism in a spectrometer with a fast ADC ($T_W > T_M$). The asymptotic counting uncertainty in a ROI representing a fraction $f$ of a spectrum obtained by a counter with pileup rejection (leading pulse edge $T_P$) and EDT (trailing pulse edge $T_W - T_P$) equals [13, 18]

$$\frac{\sigma(N)}{\sqrt{N}} = \sqrt{1 + 2\rho <\tau_\text{p} > \left(1 - (1 + \rho <\tau_\text{p} > e^{-\rho <\tau_\text{p} >})\right)},$$ (28)

in which $\tau_p = T_P + T_W$. The equation for EDT follows as a special case of equation (28) by introducing $T_P = 0$.
Equations (28) and (29) are generalizations of equations (11) and (10), respectively. Together with equation (25) they reproduce the statistics of a 30% spectral fraction seen in figure 4.

Contrary to the case of NEDT, the dispersion of the loss-corrected count rate depends on the fraction of the spectrum represented. The dispersion for counting with live-time correction has already been presented in equations (22) and (23). When opting for the inversion of the throughput formula, a linear relationship as a function of the considered fraction \(0 < f < 1\) can be used as an approximation [23]:

\[
\frac{\sigma(N)}{\sqrt{N}} = \sqrt{1 - 2f\rho\tau e^{-\rho\tau}}.
\]  

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\[
\frac{\sigma(N_i)}{\sqrt{N_i}} \approx \sqrt{\frac{e^{\rho\tau t} + 2[e^{\rho\tau t}-(1+\rho\tau t)]}{(1-\rho\tau)^2}} f + \sqrt{e^{\rho\tau t}} (1-f).
\]  

In figure 8 the statistical uncertainties obtained with both loss-correction methods, i.e. via \(N_i = X^{-1}(N)\) and \(N_i = N T_R/T_L\), are compared as a function of the size of the ROI. Live-time counting is more accurate (see also figure 6), but for small parts in a spectrum Poisson statistics is a good approximation and equation (18) can be applied.

3.4. Extended live-time clock

The Gedcke-Hale live-time clock method [52] is a modification of the simple live-time clock to include compensation for
the two-fold count loss due to leading edge pileup. Whereas a simple clock accounts for the loss of a second pulse by turning off the clock during the duration of the first pulse, the Gedcke-Hale clock applies backward live-time recording during the ‘leading edge’, up to the pulse height identification of the first pulse.

The accuracy of the Gedcke-Hale pulse-loss correction method was tested on a digital spectrometer using the double-source method [18]. In figure 9, the relative peak area for a live time of 200 s is shown as a function of the average pulse loss. Up to a count loss of 30%, the observed peak area remains within 1% of the reference value (no loss). At extreme count-loss regimes, one notices a linearly increasing undercompensation amounting to about 3% at 80% count loss. This was within the specifications of the manufacturer.

The statistical behaviour of a digital spectrometer is identical to comparable analog systems. It has been confirmed by experiment that the statistical formulas in equations (28) and (30) are rigorously applicable to the dispersion of counted and loss-corrected number of events in selected ROIs as well as the full spectrum obtained with these digital spectrometers [18].

The artificial extension of the live-time by counting backwards has consequences on the count distribution of accepted events per live-time period. Whereas dead time does not affect the Poisson distribution of the number of accepted counts, a statistical broadening occurs with pileup. An example is shown in figure 10: for the same average count rate, relatively more low count numbers arise from double event losses, which is compensated by higher count numbers made possible due to the live-time extension. Detailed theoretical studies of these
effects [14] are the basis of the correction factor \( r \) (equation (23)) appearing in live-time corrected counting statistics (cf. equations (22) and (31)).

3.5. Loss-free counting

‘Loss-free counting’ (LFC) [29–32, 34] and equivalently ‘zero dead-time’ (ZDT) counting [33, 34] is one of the most powerful tools for correcting rate-related losses in nuclear spectrometry. By performing ‘add-\( n \)’ operations to the spectrum instead of ‘add-1’, the LFC principle is able to restore the linearity of the spectrometer throughput. Since the integer correction factor \( n \) is determined from the real-time to live-time ratio over short time intervals, the LFC is able to deal with rapidly varying count rates. It is particularly well suited for application in neutron activation analysis, since the commonly enhanced loss of pulses from short-lived activation products is accounted for. Such is not the case with live-time correction techniques supplying one average correction factor for the entire measurement.

Pulse pileup can be compensated for provided that the width of the leading edge pulse \( T_P \) is added to the system busy time (as shown in figure 1) [14, 30–32, 52]. In an analog counting chain, \( T_P \) is introduced manually into the LFC module and fine-tuned for optimum loss compensation by means of the double source method. This is not required in...
the digital implementation of the ZDT method [33], which uses the Gedcke-Hale live time clock to calculate the required short-term factor $n$.

Double source tests have demonstrated that manually fine-tuned LFC can correct for count losses—due to pulse pileup rejection and dead time—approaching 100% (see figure 11) [12] and a digital spectrometer performed well up to 60% count loss, but exhibited problems at more than 90% loss [20]. Extra care is advised when using detectors with reset preamplifier, since an incorrect setting of the inhibit signal can make the loss correction energy-dependent [12].

The main trade-off of LFC is an increase of the count scatter as the clustering of true and artificial events introduce unavoidable deviations from Poisson conditions. As demonstrated in figure 12, the standard deviation of loss-corrected counts from a Poisson process can be calculated from [14]:

$$\sigma(n) \approx \sqrt{n} \left( n + 0.45 \alpha^{\beta+\Delta} \right), \quad (33)$$

in which $\Delta = 0.6(1 - e^{-\beta})$ and $\beta = -2.6$.

An interesting development for the direct assessment of uncertainty with ZDT, is the implementation of a so-called ‘variance’ ($\Sigma n^2$) spectrum next to the ZDT ($\Sigma n$) spectrum [33]. Whereas the $\Sigma n^2$ spectrum gives only an approximation of the true variance [21, 22], it is almost rigorously applicable to a small ROI ($r = 1$) when applying a relatively long inspection period ($\sigma(n)/n = 0$). Indeed, considering that the average correction factor $<n/\sigma>$ follows directly from the ratio of the number of counts in both spectra, $<n/\sigma> = \Sigma n^2/\Sigma n$, it is clear that the ‘variance’ spectrum contains the value $\Sigma n^2 = <n/\sigma> \Sigma n = <\sigma> N_{LFC}$, which is a first order approximation of the variance in equation (27). It is comparably larger than the square root of the uncorrected number of counts: $\sigma(N_{LFC})/N_{LFC} \approx \sqrt{<\sigma>}/N_{LFC} \geq \sqrt{1/N}$, since the average $<\sigma>$ is taken over the ZDT counts and not over the uncorrected counts [22].

4. Cascades of pileup and dead time

4.1. A series arrangement of pileup and dead time

In primary standardization of activity [35], it is common practice to count events triggered by detector pulses in a single-channel analyser and to impose a NEDT of fixed length that exceeds the pulse width of the detector signals. This principle can also be applied by software in the off-line analysis of list-mode files obtained with modern digital data acquisition methods. It is usually assumed that the recorded real-time to live-time ratio is an accurate correction factor for the incurred count loss. However, even if a very long dead time is selected, the effect of pileup is not completely compensated for, since any pulse occurring at the end of the imposed NEDT interval may block a partly overlapping...
event outside the dead time period that would normally be counted (see figure 14). This phenomenon is equivalent to a series arrangement of extending (EDT, i.e. pileup without rejection of the first pulse) and non-extending NEDT dead time [36, 40, 44]. At high count rates, if this effect is not properly taken into account, the live-time technique undercompensates for count loss.

Alternatively, one can impose an EDT instead. This dead time is in principle applied on every incoming event, whether it is falling within system live time and counted or within system dead time and excluded. In practice, most dead-time generators are triggered at the start of a pulse, which is recognized when the electronic detector signal rises above a threshold level, and are blind for a possible successive event until the trailing end of the pulse drops below the same threshold. Thus, only well-separated pulses prolong the system dead time, while piled-up pulses do not (see figure 14). Consequently, the system throughput will be slightly higher than what is expected from a perfect EDT counter. This situation is mathematically equivalent to a series arrangement of two EDTs [36–38, 42–44].

A convenient technical remediation of the problem is an EDT that is triggered not only at the start of the pulse, but also at the end [54]. By doing so, the dead time period is de facto extended by the full pulse width. Hence overlapping pulses also broaden the dead time period as if the counter had a perfect time resolution. Through this concept, the counter is insensitive to the pileup effect and acts as if it had a single EDT. One can also feed the live-time clock with an counter is insensitive to the pileup effect and acts as if it had a perfect time resolution. Through this concept, the pulses also broaden the dead time period as if the counter had an EDT extended by the full pulse width. Hence overlapping pulses also contribute to the pileup effect and act as if it had a perfect time resolution. Through this concept, the counter is insensitive to the pileup effect and acts as if it had a perfect time resolution.

4.2. Pileup and NEDT

For pulses with a width $T_W$ and imposed NEDT $\tau_n$ (equivalent to EDT+NEDT in series), the throughput equation is [40, 44]:

$$ R = \frac{\rho}{e^{\rho \tau_n} + \rho \max(0, \tau_n - T_W)} .$$  \hfill (34)

The relative error $\varepsilon(R)$ on the count rate by not taking into account the cascade effect of pileup with NEDT as a function of the expected count loss (when ignoring cascade effects) is calculated from the relative difference between the corrected (equation (34)) and uncorrected rate (equation (1)) [44]:

$$ \varepsilon(R) = \left[ \frac{1 + \rho \tau_n}{1 + \rho T_W - e^{\rho \tau_n}} - 1 \right]^{-1}. \hfill (35)$$

The corresponding relative error on the loss-corrected rate $\hat{R}$ is found by multiplying equation (35) with $T_K/T_L$ (cf. live-time mode) or the inverse throughput factor (cf. real-time mode), respectively [44]:

$$ \varepsilon(\hat{R}) = \varepsilon(R) \frac{T_K}{1 - \rho \tau_n} = \left[ e^{\rho \tau_n} - \rho T_W \right]^{-1} - 1 , \hfill (36)$$

whereas $\varepsilon(R)$ varies with $\tau_n$ as well as $T_W$, $\varepsilon(\hat{R})$ only varies with $T_W$. Consequently, the error cannot be reduced by imposing a longer NEDT. Figure 15 shows the error $\varepsilon(\hat{R})$ for $T_W = 2/3 \tau_n$. The validity of equation (36) has been verified experimentally [45].

This error on $\hat{R}$ (equation (36)) can be reduced by correcting the count rate $R$ mathematically via an iterative solution of the inverse of the mixed model ($\tau_n > T_W$) throughput formula in equation (34):

$$ \hat{R} = \frac{R}{1 - \rho \tau_n} \left[ e^{\rho \tau_n} - \rho T_W \right] . \hfill (37)$$

As initial value for $\hat{R}$, one can use $R/(1 - \rho \tau_n)$ in real-time mode or $R T_K/T_L$ in live-time mode. The uncertainty on the thus obtained loss-corrected count rate is [44]

$$ \sigma(N) = X^{-1/2} \frac{1 - 2 \rho T_W e^{\rho \tau_n}}{1 - \rho T_W} , \hfill (38)$$

in which $X$ is the throughput factor, hence $X^{-1/2} = \sqrt{\rho(\tau_n - T_W) + e^{\rho \tau_n}}$. Equation (38) is a generalized equation with respect to the simple cases of a counter with extending (equation (21)) or non-extending (equation (15)) dead time only. One has to include the propagation of the uncertainty on the pulse width $T_W$ and in real time mode also on $\tau_n$.

4.3. Pileup and EDT

The series arrangement of pileup and EDT, as described in section 4.1, is equivalent to an EDT–EDT cascade. Whereas the mathematical equation for the time interval distribution was found only recently [42, 43], the throughput factor could be derived earlier from probabilistic considerations [37, 38, 44], yielding:
Equation (39) turns out to be not smaller than the output rate and (Metrologia τ) dead time (τ) NEDT (cf. equation (36)). Indeed, a chain of piled-up events through, but create less dead time. When measuring in real-time (RT) or in live-time (LT) mode, one finds that both effects compensate each other.

Yet there is a residual error due to similar loss effects as with time mode, one should consider the relative error on the output count rate R due to the increased throughput for an EDT–EDT cascade (cf. equation (18)):

$$
\varepsilon(R) = \left[ \frac{e^{-\rho_{\text{loss}}}}{e^{-\rho_{\text{loss}}} (1 - \rho_{\text{loss}})} \right] - 1. \quad (42)
$$

Numerical examples [44] show that the results of equations (35) and (42) have opposite sign but similar magnitude for T_W ≤ τ_e/2 and combinations of τ_e and τ_s for which exp(ρτ_e) = (1 + ρτ_s). If one does not correct the output count rate R for the cascade effect, the error on the corresponding loss-corrected rate ρ is even enhanced via the inversion of the exponential throughput formula. Its propagation to the loss-corrected rate is well approximated by:

$$
\varepsilon(\hat{\rho}) \approx \left[ \frac{\partial R}{\partial \rho} \right]^{-1} \frac{R}{\rho} \varepsilon(R) \approx \frac{\varepsilon(R)}{1 - \rho \tau_c} \quad \text{for } \rho \tau_c << 1. \quad (43)
$$

The expected error for $T_W = 2/3 \tau_e$ is shown in figure 15. Also in the case of EDT ($\tau_e > T_W$), one can solve the inverse throughput relationship (equation (18)) by iteration:

$$
\hat{\rho} = \frac{R e^{\rho_{\text{loss}} T_W}}{1 - P_{\text{loss}}}, \quad (44)
$$

using $R T_W / T_L$ in live-time mode or e.g. $R/[1 - R \tau_c - 0.5(R \tau_c)^2]^{-1}$ in real-time mode as a first approximation for ρ and initial input for $P_{\text{loss}}$ in equation (40). Here, it is implicitly assumed that the count rate is inferior to the rate at which the throughput reaches its maximum.

5. Statistics of a decaying source

5.1. Deviation from Poisson statistics

The validity of Poisson statistics is based on the hypothesis of a constant source activity, even though this assumption is contradictory to the concept of decay and only applies by good approximation if the half-life of the source is long compared to the measurement. Many radionuclides are short-lived and the influence of decay can have a non-negligible effect on counting statistics. The modified distribution law of the expected number of counts in a measurement has been derived theoretically and compared with experimental data, allowing even the derivation of the half-life from the shape of the distribution [7,48].

5.2. Dead time of a decaying source

The throughput is biased by the increasing live-time fraction as the source decays. Live-time counting generates a mean output rate $R$ of a short-lived radionuclide with decay constant $\lambda$ over a measurement interval $T_R$. To establish the relationship between $R$ and the initial decay rate $\rho_0$, the throughput
formulas have to be integrated over \( T_R \), replacing \( \rho \) by the
time-dependent decay rate of the source and a constant back-
ground rate \( \rho_b \):

\[
\rho(t) = \rho_b e^{-\lambda t} + \rho_b. \tag{45}
\]

The integrals have been carried out \([7,46]\) and for NEDT, the throughput is

\[
R = \frac{\rho_b}{1 + \rho_b \tau_b} + \frac{1}{\lambda T_R \tau_b} \left[ \frac{1 + \rho_b \tau_a + \rho_0 \tau_a}{1 + \rho_b \tau_a + \rho_0 \tau_a e^{-\lambda T_R}} \right] \ln \left[ \frac{1 + \rho_b \tau_a + \rho_0 \tau_a}{1 + \rho_b \tau_a + \rho_0 \tau_a e^{-\lambda T_R}} \right]. \tag{46}
\]

Typically one has more control over the measurement duration than over the count rate, and thus can keep \( \lambda T_R \ll 1\). In that case, and if the background is negligible, then the correction to the measured (and already conventionally corrected for decay and dead time) average rate can be approximated by \([50]\)

\[
f \approx 1 + \frac{1}{12} \frac{\rho_0 \tau_a}{(1 + \rho_0 \tau_a)} (\lambda T_R)^2, \tag{47}
\]

which can be expressed in terms of measured throughput, \( R \):

\[
f \approx 1 + \frac{1}{12} R \tau_a (\lambda T_R)^2. \tag{48}
\]

For EDT, the mean count rate is \([7,46]\)

\[
R = \frac{e^{-\rho_0 \tau_a}}{\lambda \tau_a} \left[ e^{-\rho_0 \tau_a} - e^{-\rho_0 \tau_a} + \rho_1 \tau_e \left[ E_i (\rho_0 \tau_a) = E_i (\rho_0 \tau_e) \right] \right], \tag{49}
\]

where \( \Lambda = e^{-\tau_a} \) and the \( E_i \)'s are exponential integrals. The last term disappears for negligible background. From a serial expansion, a correction factor to the dead-time corrected rate was derived \([50]\)

\[
f \approx 1 + \frac{1}{12} \rho_0 \tau_a (\lambda T_R)^2. \tag{50}
\]

For example, for an \( ^{18}F \) source (half-life of 1.83 h) measured for \( T_R = 2 \times 10^4 \) s, with a true count rate of \( \rho_0 = 5 \times 10^4 \) s\(^{-1} \), on a system with dead time of \( \tau = 2 \times 10^4 \) s, we find correction factors for NEDT (equation (47)) and EDT (equation (50)) of \( f = 1.0018 \) and \( f = 1.0037 \), respectively.

These corrections would have uncertainties due to the expansion on the order of \( \rho T(\lambda T_R)^2 \), and may be most useful in designing experiments such that the correction is negligible, probably by limiting the counting duration, \( T_R \), of individual measurements, averaging the resulting live-time and decay-corrected rates from numerous measurements if necessary.

**6. Conclusions**

Pulse pileup and dead time unavoidably influence the throughput of a counter and alter the statistical properties of the number of decays counted. Rate-related count loss can be corrected for by measuring in system live time, but uncertainties on the dead-time compensation have to be propagated to the resulting loss-corrected count rate. Software can reproduce the role of live-time circuits in the analysis of list-mode data produced by digital spectrometers. Each pulse-loss correction method should be tested extensively at different count rates to assess rate-related errors and uncertainties. For high precision work, it is useful to utilize reduced count rates, e.g. via accurate dilution. The statistical dispersion depends on the type of count loss, i.e. pileup, extending or non-extending dead time, or combinations of the three, but also on the considered fraction of the registered events. While Poisson statistics do not rigorously apply in nuclear counting, the inverse square root of the uncorrected number of counted events is still a good estimator for the relative standard uncertainty of the loss-corrected number in various practical situations.

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