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Dimensionless units in the SI

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Abstract

The International System of Units (SI) is supposed to be coherent. That is, when a combination of units is replaced by an equivalent unit, there is no additional numerical factor. Here we consider dimensionless units as defined in the SI, e.g. angular units like radians or steradians and counting units like radioactive decays or molecules. We show that an incoherence may arise when different units of this type are replaced by a single dimensionless unit, the unit 'one', and suggest how to properly include such units into the SI in order to remove the incoherence. In particular, we argue that the radian is the appropriate coherent unit for angles and that hertz is not a coherent unit in the SI. We also discuss how including angular and counting units affects the fundamental constants.

Keywords: SI, international system of units, dimensionless units, radian, hertz

A parable of dimensionless units

Bert has a turntable operating at a rotation frequency of $33\frac{1}{3}$ revolutions per minute. His friend, Ernie, asks Bert, 'At what frequency is your turntable rotating?' If Bert were to answer '0.555 Hz' Ernie would know the rotation frequency. Similarly, if Bert were to answer '3.49 radians per second', Ernie would know the rotation frequency. And, if Bert were to say '200 degrees per second', Ernie would be well informed. On the other hand, if Bert were to respond 'the rotation frequency is 3.49', we would all agree that Ernie would not know the rotation frequency. Nor, would a response of '3.49 per second' be useful to Ernie, any more than a response of '200 per second'. In order to convey useful information, Bert must tell Ernie the units in which he is reporting the rotation frequency, including the so-called dimensionless units of cycles or radians or degrees. The current formulation of the SI specifically allows the units 'radian' or 'cycle' to be replaced by the dimensionless unit 'one', and it allows both radians per second and cycles per second (Hz) to be replaced with inverse seconds. Clearly, if Bert had followed this prescription, allowed by the current SI, he would have left Ernie in the dark about the rotation frequency.

If Bert had given an uninformative response about the rotation rate, Ernie might have asked 'what is the concentration of oxygen in the air you're breathing?' Bert could respond 'about 10^{25} atoms m^{-3}' or '5×10^{24} molecules m^{-3}'. These are clear answers. A factor-of-two ambiguity would arise if he had not specified the entity being counted; in fact, the current SI says that 'atoms' or 'molecules' are dimensionless units that should be set equal to 'one'. If Bert had said '5×10^{24} m^{-3}', Ernie might have interpreted that as being the atomic density and wonder if oxygen deprivation had compromised his friend's mental acuity.

Such situations allowed by ambiguous units are untenable, which is one of the main points of this paper. Fixing the problem is not going to be easy, as evidenced by the fact that it has persisted for so many years after the institution of the SI in 1960. This situation has led to such problematic pronouncements as 'the radian and steradian are special names for the number one …' [1]. One starting point is to recognize that replacing radians or cycles or similar dimensionless units by 'one' leads to trouble, and replacing molecules by 'one' in expressions for molecular concentration may also lead to trouble. These and similar arguments are made explicit below, as are our suggestions for a revision to the SI that goes a long way toward solving the problem[1].

[1] The hypothetical conversation in this section is not meant to suggest that there actually was such a conversation between Albert Einstein and Ernest Rutherford.

1. Introduction

The International System of Units (SI) defines units that are used to express the values of physical quantities [1]. In the foreseeable future, it is expected that there will be a redefinition of the SI based on specified values of certain fundamental constants [2]. This constitutes a dramatic change with one of the consequences being that there will no longer be a clear distinction between base units and derived units [3, 4]. In view of this change, it is timely to reexamine units in the SI and their definitions. One goal is to ensure that all such units are coherent, i.e. they comprise a coherent system of units.

In the current SI, various quantities are designated as being dimensionless. That is, they are deemed to have no unit or have what has been called the coherent derived unit 'one'. In some cases this designation leads to ambiguous results for these quantities. In this paper, we examine units in the SI that are considered dimensionless and other units not presently included in the SI that might be added to bring it into closer alignment with widespread scientific usage.

2. Units and dimensional analysis

In general, units are used to convey information about the results of measurements or theoretical calculations. To communicate a measurement of a length, for example, the result is expressed as a number and a unit, which in the SI is the meter. The number tells the length in meters of the result of the measurement.

For simple algebraic calculations involving units, one can write out the expression and separately collect the units, which may be replaced by an equivalent unit for convenience. For example, the kinetic energy E of a mass $m = 2$ kg moving at a velocity $v = 3\ \mathrm{m\ s^{-1}}$ is calculated as

$$E = \frac{1}{2}\mathrm{m}v^2 = \frac{1}{2}(2\,\mathrm{kg})\,(3\,\mathrm{m\,s^{-1}})^2 = \frac{2{\cdot}3^2}{2}\,\mathrm{kg\,m^2\,s^{-2}} = 9\,\mathrm{J}, \quad (1)$$

where J is the symbol for joule, the SI unit of energy. This calculation illustrates the important principle of the SI that the units are coherent. That is, when a combination of units is replaced by an equivalent unit, there is no additional numerical factor. For equation (1), this corresponds to the relation

$$\mathrm{kg\,m^2\,s^{-2}} = \mathrm{J}. \quad (2)$$

The notation $q = \{q\}[q]$ for a quantity with units distinguishes between the unit $[q]$ and the numerical value $\{q\}$ [5]. For example, for the speed of light, we have $c = 299\,792\,458\ \mathrm{m\ s^{-1}}$, where $\{c\} = 299\,792\,458$ and $[c] = \mathrm{m\ s^{-1}}$. Evidently both of these factors depend on the system of units, but the product $\{q\}[q]$ describes the same physical quantity. The factors $\{q\}$ and $[q]$ separately follow the algebraic rules of multiplication and division, which allows for a consistent dimensional analysis and conversion between different units.

In terms of this notation, the calculation in equation (1) can be written as

$$\{E\}\ [E] = \frac{1}{2}\{m\}\ [m]\,(\{v\}\ [v]\,)^2 \quad (3)$$

or

$$\{E\}\ \mathrm{J} = \frac{1}{2}\{m\}\ \{v\}^2\ \mathrm{kg\,m^2\,s^{-2}} = \frac{1}{2}\{m\}\ \{v\}^2\ \mathrm{J}\,. \quad (4)$$

In this way, calculations are separated into a purely numerical part and one involving units. For non-trivial equations, working separately with only the numerical values provides a practical way of carrying out the calculation. In particular, when mathematical functions such as exponential, trigonometric, or Bessel functions are involved, the arguments are necessarily numbers without units, and calculations are done with only the numerical values. This is further simplified if a coherent system of units is used, so there is no additional numerical factor.

Physical science is based on mathematical equations, which follow the rules of analysis spelled out in numerous mathematical reference works. Generally, in mathematical reference texts, distances, areas, and angles, for example, are all dimensionless. On the other hand, in physical science, one uses units.

One consequence of this difference is that mathematics provides no information on how to incorporate units into the analysis of physical phenomena. One role of the SI is to provide a systematic framework for including units in equations that describe physical phenomena.

3. Angles

Angles play an essential role in mathematics, physics, and engineering. They fall into the category of quantities with dimensionless units in the current SI, which leads to ambiguities in applications. This issue has been widely discussed in the literature, and arguments are given on both sides of the question of whether angles are quantities that should have units [6].

In part because units are rarely considered in mathematics, the unit of radian for angles is rarely mentioned in the mathematics reference literature, just as the meter is also rarely mentioned. Units are unnecessary in purely mathematical analysis. By the same token, caution is necessary in drawing conclusions about units based on purely mathematical considerations. For example, in the current SI, it is stated that angles are dimensionless based on the definition that an angle in radians is arc length divided by radius, so the unit is surmised to be a derived unit of one, or a dimensionless unit. However, this reasoning is not valid, as indicated by the following example. An angle can also be defined as 'twice the area of the sector which the angle cuts off from a unit circle whose centre is at the vertex of the angle' [7]. This gives the same result for the numerical value of the angle as the definition quoted in the SI brochure, however by following similar reasoning, it suggests that angles have the dimension of length squared rather than being dimensionless. This illustrates that conclusions about the dimensions of quantities based on such reasoning are clearly nonsense.

Regardless of whether we view angles as having dimension or not, they can be measured and the results can be expressed, for example, in units of degrees, radians, or revolutions. In elementary plane geometry or daily life, degrees are usually used, and it is intuitively familiar to think of a 45° or 90° angle and the fact that 360° is a complete revolution. In this case, the unit is degrees and [90°] = °.

In calculus and physics it is convenient to use radians or rad for angle units. The angle in radians between two lines that cross at a point is the length of circular arc s swept out between the lines by a radius vector of length r from the crossing point divided by the length of the radius vector. The angle θ is thus given by

$$\theta = \frac{s}{r}\,\text{rad}, \tag{5}$$

which corresponds to $\{\theta\} = s/r$ and $[\theta] = \text{rad}$.

The conversion between radians and degrees follows from the relation $360° = 2\pi$ rad, which gives, for example,

$$90° = \frac{2\pi\ \text{rad}}{360°}\,90° = \frac{\pi}{2}\,\text{rad}, \tag{6}$$

where the rules of algebra are applied to the units to cancel degrees from the equation.

In this context, units for angles obviously play a useful role. As with any measurable quantity, a given angle will have different numerical values depending on the units in which the angle is expressed. Units such as degrees or radians are converted to other units by algebraic calculations as in equation (6).

We consider the consequences of a consistent treatment of units for angles in the following. For an infinitesimal segment of a plane curve, the change in angle $\mathrm{d}\theta$ of the tangent to the curve is proportional to the infinitesimal change in position $\mathrm{d}s$ along the curve, where we define the constant of proportionality to be the angular curvature $\mathcal{C}$:

$$\mathrm{d}\theta = \mathcal{C}\,\mathrm{d}s, \tag{7}$$

where $\mathcal{C}$ has units of rad m^{-1}. Evidently, the angular curvature is a measure of the amount of bending of the segment of the curve. (This is different from curvature of a graph in elementary calculus or the curvature in differential geometry both of which are dimensionless and do not involve angles.) If an angular radius of curvature $\mathcal{R}$ is defined as

$$\mathcal{R} = \frac{1}{\mathcal{C}} \tag{8}$$

then

$$\mathrm{d}\theta = \frac{\mathrm{d}s}{\mathcal{R}}\,. \tag{9}$$

The quantity $\mathcal{R}$ with units m rad^{-1} should be distinguished from r in equation (5) which has units of m. If the curve is a portion of a circle, then we have

$$\theta = \frac{s}{\mathcal{R}}, \tag{10}$$

in analogy with equation (5) and $\{\mathcal{R}\} = \{r\}$. In fact, equation (5) is the same as equation (10) if the replacement rad → 1 is made.

Table 1. Quantities involving angles and their units.

Quantity	Expression	Units
Angle	θ	rad
Angular curvature	$\mathcal{C} = \dfrac{\mathrm{d}\theta}{\mathrm{d}s}$	rad m^{-1}
Angular radius of curvature	$\mathcal{R} = \dfrac{1}{\mathcal{C}} = \dfrac{\mathrm{d}s}{\mathrm{d}\theta}$	m rad^{-1}
Infinitesimal arc length	$\mathrm{d}s = \mathcal{R}\,\mathrm{d}\theta$	m
Infinitesimal angle	$\mathrm{d}\theta = \dfrac{\mathrm{d}s}{\mathcal{R}}$	rad
Solid angle	Ω	sr
Infinitesimal surface area	$\mathrm{d}a$	m^2
Infinitesimal solid angle	$\mathrm{d}\Omega = \dfrac{\mathrm{d}a}{\mathcal{R}^2}$	sr = rad^2

This extends naturally to steradians for solid angle, abbreviated sr, for which an infinitesimal solid angle subtended by the area $\mathrm{d}a$ on the surface of a sphere is given by

$$\mathrm{d}\Omega = \frac{\mathrm{d}a}{\mathcal{R}^2}, \tag{11}$$

which has units rad^2. Table 1 compiles a number of quantities involving angles and the associated units.

In applications, angles appear in the exponential and trigonometric functions, and these functions are defined for an argument that is a dimensionless number, i.e. the numerical value of the angle expressed in radians. The exponential function is given by its power series

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \dots, \tag{12}$$

and the relation

$$\mathrm{e}^{\mathrm{i}y} = \cos y + \mathrm{i}\sin y \tag{13}$$

follows from the series expansions of the cosine and sine functions. The unit 'radian' cannot be included as a factor in the arguments of these functions, because every term in the power series must have the same unit.

The connection of these functions to angles follows from the fact that equation (13) is a point in the complex plane on the unit circle at an angle $\theta = y$ rad in the counter-clockwise direction from the positive real axis. The periodicity of the function $\mathrm{e}^{\mathrm{i}y}$ fixes the unit of the angle to be $[\theta] = \text{rad}$, because both the angle $\theta = y$ rad and the function $\mathrm{e}^{\mathrm{i}y}$ go through one complete cycle as $y = \{\theta\}$ ranges from 0 to 2π. The choice of any other unit for $[\theta]$ would not align these two periods.

However, it is the general practice in physics to write the exponential function of an angle $\theta = y$ rad as $\mathrm{e}^{\mathrm{i}\theta}$ rather than $\mathrm{e}^{\mathrm{i}y}$ or $\mathrm{e}^{\mathrm{i}\{\theta\}}$. In fact, in carrying out calculations, scientists do not usually distinguish between θ and $\{\theta\}$, which amounts to treating rad as being 1.

This reveals a conflict between consistent application of dimensional analysis and common usage. A consistent application of dimensional analysis is needed in order for the SI to be used as the basis for any computer algebra program that takes units into account [8, 9]. This is likely to be an increasingly popular way of doing calculations, and having a consistent foundation is necessary to prevent errors. For such applications, it is important to use the numerical value of angles when expressed in radians, θ/rad, in exponential and trigonometric functions as well as more general functions of angles in mathematical physics to avoid errors of 2π which might otherwise occur. (For example, one could confuse Hz and rad s^{-1} as described in section 4.) On the other hand, for general use in printed equations following the common practice, the argument of the exponential and trigonometric functions is simply written as θ, which corresponds to replacing rad by 1. Of course, this replacement can only be done for the unit rad, and not revolutions (cycles) or degrees, replacements that would introduce numerical factors. In this sense, the unit rad is a coherent unit in the SI, whereas revolutions and degrees are not.

4. Periodic phenomena

Periodic phenomena in physics include rotations of an object, cycles or repetitions of a wave, or a series of any regular, repetitive events. Such periodic phenomena are characterized by a frequency whose units can be an angular factor or a cycle divided by time. In the SI, cycles/second = cyl/s is named hertz or Hz, and

$$1\,\mathrm{Hz} = 1\,\mathrm{cyl}\,\mathrm{s}^{-1} = 2\pi\,\mathrm{rad}\,\mathrm{s}^{-1}, \tag{14}$$

where the second equality follows from the fact that one cycle and 2π radians each correspond to the period of a periodic phenomenon. Hz may be viewed as being equivalent to rotations per second, but often, 'rotations' is used for mechanical motion and 'cycles' is used for waves.

We note that if cycle is not included in Hz, and radian is replaced by 1, both of which are indicated in the current SI brochure, then equation (14) would be nonsense.

The traditional symbol used for angular frequency is ω, which is understood to mean the frequency in units of rad s^{-1}, while the symbols ν or f are used to denote frequency expressed in hertz. The relation between the numerical value of a particular frequency expressed in Hz or rad s^{-1} is given by

$$\{\nu\}_{\mathrm{Hz}}\,[\mathrm{Hz}] = \{\omega\}_{\mathrm{rad\,s^{-1}}}\,[\mathrm{rad\,s^{-1}}] = \frac{\{\omega\}_{\mathrm{rad\,s^{-1}}}}{2\pi}\,[\mathrm{Hz}]\,, \tag{15}$$

or

$$\frac{\{\omega\}_{\mathrm{rad\,s^{-1}}}}{2\pi} = \{\nu\}_{\mathrm{Hz}}\,, \tag{16}$$

where the second equality in equation (15) follows from equation (14). As already noted, radians behave as coherent units for the SI, so we make the identification $\{\omega\}_{\mathrm{rad\,s^{-1}}} = \{\omega\}$, where the curly brackets with no subscript indicate that the numerical value corresponds to a coherent SI unit. However, a consequence of this convention is that the unit Hz is not a coherent SI unit as indicated by equation (16). This is in conflict with the current SI where Hz is treated as a coherent SI unit, only because cyl is replaced by 'one'. Since this leads to an inconsistency, we propose that the SI be modified in such a way that Hz is neither treated as a coherent SI unit nor replaced by s^{-1}.

We note that if both rad and cyl are replaced by 'one', as allowed in the current SI, then equation (16) takes the form of the (questionable) relation

$$\omega \overset{?}{=} 2\pi\,\nu; \tag{17}$$

we employ the symbol $\overset{?}{=}$ to emphasize that the equation is only true with those inappropriate replacements. In fact, the correct equation is given by equation (16) which only involves the numerical values. We recognize that when people write equation (17) as an equality, they mean what is stated in equation (16). In other words, when for a given frequency people mistakenly write ω is equal to $2\pi\nu$, they correctly mean that the numerical value in radians per second of ω is 2π times the numerical value in hertz of ν.

A basic equation for waves is the relation between the wavelength and the frequency. This is generally written

$$\lambda\nu = c, \tag{18}$$

where λ is the crest to crest wavelength and c is the wave velocity, which for electromagnetic radiation in free space is the speed of light. From the requirement that units on both sides of an equality must be the same, and the conventions that c has the unit m s^{-1} in the SI and ν has the unit Hz, equation (18) implies that the unit for λ is

$$[\lambda] = \frac{[c]}{[\nu]} = \mathrm{m\,s^{-1}\,Hz^{-1}} = \mathrm{m\,cyl^{-1}}, \tag{19}$$

which has a self-evident intuitive interpretation. Neither Hz nor m cyl^{-1} is a coherent unit. For a 'coherent' version of equation (18), that is, an equation in which c has the unit m s^{-1} and the frequency has the unit rad s^{-1}, we write

$$\lambda\!\!\!^{-}\omega = c \tag{20}$$

which implies that the reduced wavelength $\lambda\!\!\!^{-}$ has the units

$$[\lambda\!\!\!^{-}] = \frac{[c]}{[\omega]} = \frac{\mathrm{m\ s^{-1}}}{\mathrm{rad\ s^{-1}}} = \mathrm{m\,rad^{-1}} \tag{21}$$

and that

$$\{\lambda\!\!\!^{-}\} = \frac{\{\lambda\}_{\mathrm{m\ cyl^{-1}}}}{2\pi}, \tag{22}$$

where as before, the absence of a subscript on the curly brackets indicates that the numerical value refers to coherent SI units, which in this case are m rad^{-1}. Again, when the relation

$$\lambda\!\!\!^{-} \overset{?}{=} \frac{\lambda}{2\pi} \tag{23}$$

is treated as an equality, what is meant is equation (22).

Another quantity associated with waves is the wave vector

$$k = \frac{1}{\lambda\!\!\!^{-}}. \tag{24}$$

It has units of radians per meter or rad m^{-1}. (The magnitude of the wave vector is to be distinguished from the wavenumber λ^{-1} used in spectroscopy.) With these units for k, the covariant phase $kx - \omega t$ for a wave propagating in the x direction, where x is a coordinate and t is the time, is homogeneous in the unit

Table 2. Quantities involving rotational motion and their units.

Quantity	Equation	Units
Angular velocity	$\omega = \dfrac{d\theta}{dt}$	rad s^{-1}
Angular acceleration	$\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$	rad s^{-2}
Velocity	$v = \dfrac{ds}{dt} = \mathcal{R}\dfrac{d\theta}{dt} = \mathcal{R}\omega$	m s^{-1}
Moment of inertia	$I = m\mathcal{R}^2$	kg m^2 rad^{-2}
Angular momentum	$L = I\omega = m\mathcal{R}^2\omega$	J s rad^{-1}
Torque	$N = I\alpha$	J rad^{-1}
Energy	$E = I\dfrac{\omega^2}{2}$	J
Centrifugal force	$F_c = m\omega^2\mathcal{R} = \dfrac{mv^2}{\mathcal{R}}$	N rad

rad. This is consistent with the quantum mechanical expression for momentum.

In classical mechanics, rotational motion of a rigid body can be described by an angle θ about a fixed axis of rotation as a function of time t, an angular velocity ω

$$\omega = \frac{d\theta}{dt}, \tag{25}$$

and an angular acceleration α, given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{26}$$

Evidently, θ, ω, and α have units of rad, rad s^{-1}, and rad s^{-2}. Units for other quantities associated with rotational motion, such as the moment of inertia, may be deduced from the defining equations. As a rule of thumb, in order to obtain a coherent set of units it is necessary to take the radius r that appears in such expressions to be the angular radius of curvature $\mathcal{R}$ with units m rad^{-1} defined in equation (8). Table 2 lists various quantities associated with rotational motion of a point mass at a distance r from the axis of rotation, the relevant equations, and the corresponding units. A longer list is given by Eder [6].

In electromagnetism and quantum mechanics, the product ωt of angular frequency and time often appears in the exponential function. This is similar to the case for angles as discussed in section 3. In quantum mechanics for example, it is conventional to write

$$e^{-i\omega t}, \tag{27}$$

where actually what is meant is

$$e^{-i\{\omega t\}}. \tag{28}$$

Here, as for angles, it is common practice to treat rad as 'one', which does not lead to problems if rad is a coherent SI unit.

5. Counting quantities

Many scientific applications involve counting of events or entities. For example, in radioactive decay, events occur at random times, but still have a well-defined rate when averaged over a sufficiently long time with a large enough sample. The result of a measurement, where decays trigger counts in a detector, is counts/second or cnt/s. The SI unit for activity of a radiative sample is becquerel or Bq, meaning decays per second, which is related to counts per second through the overall detection efficiency. However, in the current SI, it is said that the becquerel has units of s^{-1}, which means that the decay or count in the numerator is dropped. Here we take issue with this prescription and argue that the unit 'decay' or 'count' should be retained, because it provides information about the number that precedes it in the expression for the quantity. In addition, since the current SI replaces both Hz and Bq by s^{-1}, the distinction between these units is lost and sometimes leads to the dangerous and sadly mistaken use of Hz, which refers to periodic cycles, for the rate of random events. (Non-radioactive decay e.g. decay of excited atomic states, is similarly a random process and is properly measured in decays per second, but not traditionally in Bq, and certainly not in Hz.)

This is a special case of counting in general. Things that can be counted include events, such as decays or clicks of a detector, and entities, such as atoms or molecules. For such countable things, it is useful to include a designation of what is being counted in the unit for the corresponding quantities. Quantities involving counting are not restricted to numbers and rates. For example, if in a certain time interval there are $\mathcal{D}$ = 200 decays = 200 dcy and the detector registers $\mathcal{N}$ = 20 counts = 20 cnt, then the efficiency η of the detection is

$$\eta = \frac{\mathcal{N}}{\mathcal{D}} = 0.1\ \frac{\text{cnt}}{\text{dcy}}. \tag{29}$$

Conversion between the count rate and the decay rate may be made using the detection efficiency as a conversion factor. For this detector, if a count rate of $\mathcal{Q}$ = 73 cnt s^{-1} is observed, it indicates a decay rate Γ given by

$$\Gamma = \frac{\mathcal{Q}}{\eta} = \frac{73\ \text{cnt s}^{-1}}{0.1\ \text{cnt/dcy}} = 730\ \frac{\text{dcy}}{\text{s}}. \tag{30}$$

In this case, the detector efficiency has units, unlike the recommendation of the current SI where it would be simply a number. The units provide useful information in a form that can be incorporated into calculations.

Counting also applies to entities such as atoms or molecules. The average number density n of molecules in a given volume is the number of molecules $\mathcal{M}$ divided by the volume V

$$n = \frac{\mathcal{M}}{V}, \tag{31}$$

which in the current SI has units of m^{-3}. However, this is another case where specification of what the density refers to is useful. This would make the number density consistent with other forms of density, such as mass density or charge density, which have units of kg m^{-3} and C m^{-3}, respectively. For number density, the unit should be mcl m^{-3}, which follows naturally when $\mathcal{M}$ has the unit mcl, where mcl is the

Table 3. Quantities involving counting and their unit symbols.

Quantity	Unit symbol
Events	evt
Number of counts	cnt
Number of decays	dcy
Entities	ent
Number of molecules	mcl
Number of atoms	atm
Number of particles	pcl

suggested unit for the number of molecules. For macroscopic numbers of molecules or atoms, it is convenient to use the unit mole or mol, where

$$1\ \mathrm{mol} = 6.02\ldots\times 10^{23}\ \mathrm{ent}, \tag{32}$$

where ent is the suggested symbol for entity. This expression makes it clear that the mole, which is the unit of amount of substance, is not just a number, but a number of entities. This relation can be used as a conversion factor between number density and molar density, which differ only in their units. As an example

$$n = 2.5\times 10^{25}\,\frac{\mathrm{mcl}}{\mathrm{m}^3} = \frac{1\ \mathrm{mol}}{6.02\ldots\times 10^{23}\ \mathrm{mcl}}2.5\times 10^{25}\,\frac{\mathrm{mcl}}{\mathrm{m}^3} = 42\,\frac{\mathrm{mol}}{\mathrm{m}^3}\,. \tag{33}$$

The presence of units makes the conversion more clear than it would be if the unit mcl were absent from equation (33) as the current SI prescribes, and for polyatomic molecules removes any ambiguity about whether atoms or molecules are being counted.

A list of suggested unit names for events and entities is given in table 3. Other items can be named as needed.

6. Fundamental constants

Fundamental constants are parameters in the equations that describe physical phenomena and have the units that are necessary for dimensional consistency. The CODATA recommended values and units for the constants [10] are based on the conventions of the current SI, and any modifications of those conventions will have consequences for the units.

For example, the equation

$$E = \hbar\omega \tag{34}$$

relates E, the energy of a photon, with its angular frequency ω. These quantities are related through the Planck constant $\hbar$, and for the equation to be dimensionally consistent, taking into account the modifications of the SI under consideration, the unit of $\hbar$ must be J s rad^{-1}, or more suggestively, J/(rad s^{-1}). This is in contrast with the CODATA tabulated value for $\hbar$ which has the unit J s. Similarly, the equation

$$E = h\nu, \tag{35}$$

where ν is the photon frequency in hertz, implies that the unit for h is J Hz^{-1}. Both J s rad^{-1} and J Hz^{-1} reduce to J s in the current SI, but they are distinct when units are treated consistently. The two expressions for the photon energy for a given frequency imply

$$\hbar\omega = h\nu, \tag{36}$$

and together with equation (16) lead to the conventional relation

$$\{\hbar\} = \frac{\{h\}_{\mathrm{J\ Hz^{-1}}}}{2\pi}, \tag{37}$$

between the numerical values of the Planck constant expressed in different units. One often sees

$$\hbar \overset{?}{=} \frac{h}{2\pi}, \tag{38}$$

but as before, when equation (38) is treated as an equality, what is meant is equation (37).

Another basic constant involving $\hbar$ is the reduced Compton wavelength of the electron $\lambda\!\!\!^{-}_{\mathrm{C}}$ given by

$$\lambda\!\!\!^{-}_{\mathrm{C}} = \frac{\hbar}{m_{\mathrm{e}}c} \tag{39}$$

which has the units

$$[\lambda\!\!\!^{-}_{\mathrm{C}}] = \frac{[\hbar]}{[m_{\mathrm{e}}]\ [c]} = \mathrm{m\,rad^{-1}} \tag{40}$$

consistent with equation (21). Similarly, the Bohr radius a_0 is related to the reduced Compton wavelength by

$$\lambda\!\!\!^{-}_{\mathrm{C}} = \alpha a_0, \tag{41}$$

where α is the dimensionless fine-structure constant, so that

$$[a_0] = \mathrm{m\,rad^{-1}}, \tag{42}$$

which is consistent with the use of the angular radius of curvature for mechanical rotational motion. For the Rydberg constant, the definition

$$R_\infty = \frac{\alpha}{4\pi a_0} \tag{43}$$

suggests the units

$$[R_\infty] = \mathrm{cyl\,m^{-1}} \tag{44}$$

in order to be consistent with the Rydberg formula

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right). \tag{45}$$

A corresponding angular version of the Rydberg constant is given by

$$R\!\!\!\!-_\infty = \frac{\alpha}{2a_0}, \tag{46}$$

with units rad m^{-1}, where

$$\{R\!\!\!\!-_\infty\} = 2\pi\{R_\infty\}\,. \tag{47}$$

The expression for the fine-structure constant α in the current SI is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \tag{48}$$

where e is the unit charge and ϵ_0 is the vacuum permittivity (electric constant). The $\hbar$ in that expression may be seen to arise from the form of electromagnetic interactions in the Schrödinger equation as follows. For the hydrogen atom

$$\left[\frac{\boldsymbol{p}^2}{2m_e} + V(\boldsymbol{x})\right]\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x})\,, \tag{49}$$

where $\boldsymbol{p} = -\mathrm{i}\hbar\,\nabla$, $V(\boldsymbol{x}) = -\,e^2/(4\pi\epsilon_0|\boldsymbol{x}|)$, $\psi(\boldsymbol{x})$ is the wave function, and E is the energy eigenvalue. If the coordinate is written as a dimensionless factor times the reduced Compton wavelength of the electron $\boldsymbol{x} = \tilde{\boldsymbol{x}}\hbar\,/\,(m_e c)$, then the equation is of the completely dimensionless form

$$\left[-\frac{\tilde{\nabla}^2}{2} - \frac{\alpha}{|\tilde{\boldsymbol{x}}|}\right]\tilde{\psi}(\tilde{\boldsymbol{x}}) = \tilde{E}\tilde{\psi}(\tilde{\boldsymbol{x}})\,, \tag{50}$$

where $\tilde{E} = E\,/\,m_e c^2$ and $\tilde{\psi}(\tilde{\boldsymbol{x}})\;\alpha\;\psi(\boldsymbol{x}m_e c\,/\,\hbar)$. However when an inverse radian is included in $\hbar$, equation (48) must be modified in order for α to be dimensionless. This can be done, although not uniquely, by using the freedom in the definition of electrical quantities as discussed by Jackson [11] in his appendix on units and dimensions. If replacements to the definitions of the unit factors given by $k_1 \to k_1$/rad and $k_2 \to k_2$/rad are made, then there is no change to the SI form of the Maxwell equations other than the modification of the units of ϵ_0 and μ_0 to be

$$[\epsilon_0] = \left[\frac{e^2}{\hbar c}\right] = \mathrm{C^2\,J^{-1}\,rad\,m^{-1}}\,. \tag{51}$$

and

$$[\mu_0] = \left[\frac{\hbar}{e^2 c}\right] = \mathrm{kg\,m\,rad^{-1}\,C^{-2}} \tag{52}$$

where μ_0 is the vacuum permeability (magnetic constant).

We now turn to constants related to counting. The Avogadro constant N_A is the number of entities in one mole which can be written as

$$N_A = 6.02\ldots\times 10^{23}\,\mathrm{ent\,mol^{-1}}, \tag{53}$$

in accord with equation (32). Evidently, this constant can be viewed as the conversion factor between entities and moles. It also provides the relation between the molar gas constant $R = 8.31\ldots\ \mathrm{J\ mol^{-1}\ K^{-1}}$ and the Boltzmann constant k, which is thus given by

$$k = \frac{R}{N_A} = 1.38\ldots\times 10^{-23}\,\mathrm{J\,K^{-1}ent^{-1}}. \tag{54}$$

Similarly, the Avogadro constant relates the Faraday constant $F = 9.64\ldots\times 10^4\ \mathrm{C\ mol^{-1}}$ to the unit charge, which can be written as

$$e = \frac{F}{N_A} = 1.60\ldots\times 10^{-19}\,\mathrm{C\,ent^{-1}}. \tag{55}$$

Evidently, this expression allows for an explicit conversion between number density and charge density, which takes the form $[\rho] = [\mathrm{C\ m^{-3}}] = [en]$, with n as defined in equation (31).

Table 4. Fundamental constants and their units.

Constants	Symbol	Units
Reduced Planck constant	$\hbar$	$\mathrm{J\ s\ rad^{-1}}$
Planck constant	h	$\mathrm{J\ Hz^{-1}}$
Electron reduced Compton wavelength	$\lambda\!\!\!^{-}_C$	$\mathrm{m\ rad^{-1}}$
Electron Bohr radius	a_0	$\mathrm{m\ rad^{-1}}$
Rydberg constant	$R\!\!\!\!-_\infty$	$\mathrm{rad\ m^{-1}}$
Vacuum permittivity (electric constant)	ϵ_0	$\mathrm{C^2\ J^{-1}\ rad\ m^{-1}}$
Vacuum permeability (magnetic constant)	μ_0	$\mathrm{kg\ m\ rad^{-1}}\ C^{-2}$
Avogadro constant	N_A	$\mathrm{ent\ mol^{-1}}$
Boltzmann constant	k	$\mathrm{J\ K^{-1}\ ent^{-1}}$
Elementary charge	e	$\mathrm{C\ ent^{-1}}$

Table 4 gives a partial list of fundamental constants that are affected by the explicit expression of radians or entities; the pattern for including such units in other constants where appropriate should be clear. On the other hand, the majority of constants remain unchanged.

7. Conclusion and recommendations

Modifications of the SI to eliminate the incoherence that results from dropping so-called dimensionless quantities have been identified and discussed. There is some latitude in how the modifications might be taken into account by users of the SI. However, one conclusion that is not optional is that the unit hertz cannot be regarded as a coherent unit of the SI, in contrast to its designation in the current form of the SI, where cycles are ignored and Hz may be replaced by $\mathrm{s^{-1}}$.

At the same time, we have shown that the unit radian can play a useful role in providing consistency of units and should be regarded as the coherent unit for angles in the SI. Therefore, we recommend that quantities involving rotation, angles, or angular frequencies be reported including radians as a unit. However, we do not recommend that one change the common practice of writing expressions like $\cos\omega t$ to the more pedantic form of $\cos\{\omega t\}$. It would be too disruptive to make it a requirement of the SI to distinguish between an angle and its numerical value.

For units involved in counting, the prevailing practice is to include them in expressions for such quantities. This is in contrast to the current SI, where they are omitted. Here a consistent formulation for the use of such quantities is provided.

With regard to fundamental constants, publications of the CODATA Task Group on Fundamental Constants are based on the current SI [10]. We recommend that future listings of values of the fundamental constants give complete units, including radians and counting units in order to provide a guide for a consistent use of the constants, particularly by computer programs that include units. Users of the constants may still choose to omit either radians or counting units, but including them in the listed values would encourage users to use them coherently if they choose to.

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