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LETTER TO THE EDITOR

Resistance fluctuation at the mobility edge

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Abstract. Following a Migdal–Kadanoff-type bond moving procedure, we derive the renormalisation-group equations for the characteristic function of the full probability distribution of resistance (conductance) of a three-dimensional disordered system. The resulting recursion relations for the first two cumulants, κ_1 the mean resistance and κ_2, the mean-square deviation of resistance exhibit a mobility edge dominated by large dispersion, i.e., $\kappa_2^{1/2}/\kappa_1 \simeq 1$, suggesting inadequacy of the one-parameter scaling *ansatz*.

The non-self-averaging nature of the quantum ohmic resistance of a disordered 1-dimensional resistor has been the subject of several recent investigations (Abrikosov 1981, Mel'nikov 1980, Kumar 1985, Kumar and Mello 1985). These statistical fluctuations of residual resistance over the ensemble of macroscopically identical samples have been treated exactly for a strictly 1-dimensional system, i.e., for the 1-dimensional 1-channel case. But the results can presumably be generalised to the case of a physically thin wire that should correspond to the 1-dimensional n-channel case, with large n. It is the series randomness of 1-space dimensionality that seems to be the determining property. However, the question as to whether or not these fluctuations get harnessed by the multiple connectivity of higher space dimensions ($d > 1$) is still open, and may be crucial to the proper understanding of the physics at the mobility edge. Indeed, the results of the recent numerical analysis on a finite 3-dimensional lattice with off-diagonal disorder show a large dispersion of physical quantities such as the conductivity and the participation ratio at the mobility edge, indicating a breakdown of the *ansatz* of one-parameter scaling (Ioffe *et al* 1985) suggested earlier by Abrahams *et al* (1979). This has prompted us to report some of our recent analytical results on this problem that lend support to this conclusion. We find that for the 3-dimensional case with extremely anisotropic disorder, the variance dominates the mean value of resistance on the insulating side of the mobility edge as expected. But on the 'metallic' side too the mean resistance and its dispersion remain comparable, even in the infinite-sample limit. We call this the 'meta-metallic' phase. The full probability distribution of resistance in this case has no non-trivial unstable fixed point. For the isotropic case we do have an unstable fixed point for the full distribution (the mobility edge) but that too is dominated by dispersion.

Our starting point is the evolution equation for the probability distribution $W_\rho^{(1)}(\rho, l)$ of resistance ρ of a 1-dimensional disordered resistor of length l (Kumar 1985),

$$\frac{\partial W_\rho^{(1)}}{\partial l} = \rho(\rho+1)\frac{\partial^2 W_\rho^{(1)}}{\partial \rho^2} + (1+2\rho)\frac{\partial W_\rho^{(1)}}{\partial \rho}. \tag{1}$$

Here, the resistance ρ is measured in units of $\pi\hbar/e^2$ and the length l in units of a microscopic length, namely the backscatter mean-free path calculated in the Born approximation. Defining the associated moment generating function $\chi_\rho^{(1)}(x, l)$ and the characteristic function (the cumulant generating function, or the 'free energy') $K_\rho^{(1)}(x, l)$ as

$$\chi_\rho^{(1)}(x,l) \equiv \int_0^\infty e^{-x\rho} W_\rho^{(1)}(\rho, l)\,d\rho$$

$$K_\rho^{(1)}(x,l) \equiv \ln \chi_\rho^{(1)}(x,l) \tag{2a}$$

we readily derive from (1)

$$\frac{\partial \chi_\rho^{(1)}}{\partial l} = x^2\frac{\partial^2 \chi_\rho^{(1)}}{\partial x^2} + x(2-x)\frac{\partial \chi_\rho^{(1)}}{\partial x} - x\chi_\rho^{(1)} \tag{2b}$$

and

$$\frac{\partial K_\rho^{(1)}}{\partial l} = x^2\frac{\partial^2 K_\rho^{(1)}}{\partial x^2} + x^2\left(\frac{\partial K_\rho^{(1)}}{\partial x}\right)^2 + x(2-x)\frac{\partial K_\rho^{(1)}}{\partial x} - x. \tag{2c}$$

We have in particular for the first moment $\rho_1^{(1)}$, ($\equiv$cumulant $\kappa_1^{(1)}$)

$$\rho_1^{(1)} \equiv \langle \rho^{(1)} \rangle \equiv \kappa_1^{(1)} = \tfrac{1}{2}(e^{2l} - 1), \tag{3}$$

where $\langle\ldots\rangle$ denotes the ensemble average.

Consider now a d-dimensional hypercubic lattice each site of which is occupied by a random 'elementary' scatterer with $2d$ incoming and $2d$ outgoing channels. Following Shapiro (1982), we partition the lattice into equal hypercubic blocks of edge b (the scale factor). The Migdal–Kadanoff procedure now involves singling out arbitrarily the direction of current flow and then cutting the bonds in the $(d-1)$ transverse directions. Thus each block now consists of $b^{(d-1)}$ chains in parallel, each chain b scatterers long. Proceeding as in Shapiro (1982) but with a difference now that for a given realisation of randomness each chain has a different resistance in general, we can write for a block

$$(\rho_{(b)}^{(d)})^{-1} = \sum_{i=1}^{b(d-1)} (\rho_i^{(1)}(b))^{-1} \tag{4a}$$

where $\rho_i^{(1)}(b)$ is the resistance of the ith 1-dimensional chain of length b in the d-dimensional block and $\rho^{(b)}(b)$ is the resultant block resistance.

We will first consider the relatively simple, albeit somewhat unrealistic case of extreme anisotropy wherein the randomness evolves only along the chosen direction of current flow. For this case, we have $\rho_i^{(1)}(b) = \rho^{(1)}(b)$ (independent of the chain index i) and the law of harmonic combination in (4a) simplifies to

$$\rho^{(d)}(b) = b^{(d-1)}\rho^{(1)}(b) \tag{4b}$$

giving

$$\chi^{(d)}(x,b) = \langle \exp(-xb^{(d-1)}\rho^{(1)}(b))\rangle. \tag{5}$$

To derive the recursion relation for the probability distribution in the differential form, we set the scale factor $b = 1 + \mathrm{d}\zeta$ and let $\mathrm{d}\zeta \to 0^+$. We get at once

$$\frac{\partial \chi_\rho^{(d)}}{\partial \ln l} = -(d-1)x\frac{\partial}{\partial x}\chi_\rho^{(d)} + \left(\frac{\partial \chi^{(1)}}{\partial l}\right)_d \tfrac{1}{2}\ln(1+2\rho_l^{(d)}). \tag{6}$$

Here $(\partial\chi_\rho^{(1)}/\partial l)_d$ now signifies $(\partial\chi_\rho^{(1)}/\partial l)$ as given by (2*b*), but with $\chi_\rho^{(1)}$ on the right-hand side of (2*b*) re-interpreted as $\chi_\rho^{(d)}$ in the sense of iteration. Here l is the actual size of the block at the present stage of length scaling. Also we have recalled the iterational relations $\mathrm{d}l = l\,\mathrm{d}\zeta$ (or $\zeta = \ln l$). Here ρ_l^d is the mean resistance of the block at the present length scale.

From (2*b*), (2*c*) and (6), we get the equation for the cumulant generating function

$$\begin{aligned}\frac{\partial K_\rho^{(d)}}{\partial \ln l} = -(d-1)x\frac{\partial K_\rho^{(d)}}{\partial x} + \tfrac{1}{2}\Bigg[x^2\frac{\partial^2 K_\rho^{(d)}}{\partial x^2} + x^2\left(\frac{\partial K_\rho^{(d)}}{\partial x}\right)^2 \\ + x(2-x)\frac{\partial K_\rho^{(d)}}{\partial x} - x\Bigg]\ln(1+2\rho_1^{(d)}).\end{aligned} \tag{7}$$

This equation contains full information on the problem in question. The fixed points of the probability distribution, if any, are obtained by setting $\partial K_\rho^{(d)}/\partial \ln l = 0$. The resulting second-order differential equation in x as independent variable is to be solved subject to the boundary conditions $K_\rho^{(d)}(x=0, l) = 0$ and $(-\partial K_\rho^{(d)}(x,l)/\partial x)_{x=0} = K_1^{(d)}$ $(\equiv \rho_1^{(d)}$, the mean resistance). The solution, however, also contains $\rho_1^{(d)}$ parametrically and the probability distribution so obtained must, therefore, reproduce $\rho_1^{(d)}$ self-consistently. This fixes the value $\rho_1^{(d)*}$ as the fixed-point. More explicitly, we obtain from (7) the equations for the first two cumulants by series expansion of $K_\rho^{(d)}$ on both sides of (7) in powers of x and equating the coefficients of x and x^2. We get

$$\frac{\partial \kappa_{1\rho}^{(d)}}{\partial \ln l} = -(d-1)\kappa_{1\rho}^{(d)} + \tfrac{1}{2}(1+2\kappa_{1\rho}^{(d)})\ln(1+2\kappa_{1\rho}^{(d)}) \tag{8}$$

$$\frac{\partial \kappa_{2\rho}^{(d)}}{\partial \ln l} = -2(d-1)\kappa_{2\rho}^{(d)} + (\kappa_{1\rho}^{(d)2} + \kappa_{1\rho}^{(d)} + 3\kappa_{2\rho}^{(d)})\ln(1+2\kappa_{1\rho}^{(d)}). \tag{9}$$

Equation (8) is, of course, the same as the one obtained by Shapiro (his equation (4), the unit of resistoner being $2\pi\hbar/e^2$ and not $\pi\hbar/e^2$ as here). The new information about dispersion is contained in (9). Straightforward analysis shows that for $d=3$, $(\partial\kappa_{1\rho}^{(d)}/\partial \ln l) = 0$ gives $\rho_1^{(d)*} = 1.96$ which together with $(\partial\kappa_{2\rho}^{(d)}/\partial \ln l) = 0$ gives $\kappa_2^{(d)*} < 0$ indicating that there is no non-trivial fixed for the full distribution! However, it is readily seen that for $\rho_1^{(d)} > \rho_1^{(d)*}$, both the average resistance ($\kappa_{1\rho}^{(d)}$ and its variance $(\kappa_{2\rho}^{(d)})^{1/2}$ grow with the size l leading to the insulating phase with the variance exponentially dominating the mean value. More interestingly, for $\rho_l^{(d)} < \rho_l^{(d)*}$ both the mean and the variance flow to the 'metallic' fixed-point $\kappa_{1\rho}^{(d)*} = 0 = \kappa_{2\rho}^{(d)*}$. In this regime we have $\kappa_{1\rho}^{(d)}, \kappa_{2\rho}^{(d)} \ll 1$ and (8) and (9) lead to

$$\mathrm{d}\kappa_{1\rho}^{(d)2}/\mathrm{d}\ln l \simeq -2\kappa_{1\rho}^{(d)2} \tag{10}$$

$$\mathrm{d}\kappa_{2\rho}^{(d)}/\mathrm{d}\ln l \simeq -4\kappa_{2\rho}^{(d)} + 2\kappa_{1\rho}^{(d)2}. \tag{11}$$

This gives $(\mathrm{d}\ln\kappa_{1\rho}^{(d)2}/\mathrm{d}\ln\kappa_{2\rho}^{(d)})_{l=\infty} = 1$ showing that the mean and the variance remain comparable in the metallic regime even in the infinite-sample limit. For this reason we prefer the term 'metal-metallic' for this phase.

We now pass on to the physically more relevant case of isotropic disorder. Here the equation (4*a*) must be retained as such since $\rho_i^{(1)}(b)$, for $i = 1, 2, 3, \ldots b^{(d-1)}$, are no longer equal, but identically distributed independent random variables. It is convenient to re-write (4*a*) in terms of the conductance ($g_i^{(1)}$), rather than the resistances ($\rho_i^{(1)}$), as

$$g^{(d)}(b) = \sum_{i=1}^{b^{(d-1)}} g_i^{(1)}(b). \tag{12}$$

Correspondingly, we can re-write our starting equation (1) in terms of $W_g^{(1)}(g, l)$ and proceed as before. There is, however, a technical problem arising from the circumstance that $W_g^{(1)}(g, l)$ associated with (1) gives infinite algebraic moments $g^\nu \equiv \langle g^\nu \rangle$ for $\nu > \frac{1}{2}$. While in principle one could still carry out the above programme, as the associated cumulant generating function $K_g^{(1)}(l, x)$ always exists, albeit now non-analytic at $x = 0$, certain intermediate mathematical operations cannot be carried out as we shall presently see. It will be quite apt to resort to a mathematical artifice of terminating the 1-dimensional chains so as to include a non-zero boundary resistance. Indeed this is inherent in the treatment of Abrikosov (1981) who gets instead of (1) the equation

$$\frac{\partial W_{\rho'}^{(1)}}{\partial l} = \rho'(1+\rho')\frac{\partial^2 W_{\rho'}^{(1)}}{\partial \rho'^2} + (2\rho' - 1)\frac{\partial W_{\rho'}^{(1)}}{\partial \rho'}, \tag{13}$$

that differs from (1) by the replacement $\rho' = \rho + 1$. With (13) one has

$$\rho_1^{(1)'} \equiv \langle \rho' \rangle = \tfrac{1}{2}(e^{2l} + 1) = \rho_1^{(1)} + 1. \tag{14}$$

Thus $\rho_l^{(1)}$ is the 'internal' resistance *sans* boundary. Here $\rho_l^{(1)} \to 1$ as $\lambda \to 0$. This clearly displays the boundary resistance, and makes all moments of conductance finite. The associated probability distribution $W_{g'}^{(1)}$ of conductance g' obeys

$$\frac{\partial W_{g'}^{(1)}}{\partial l} = g'^2(1-g')\frac{\partial W_{g'}^{(1)}}{\partial g'^2} + (4g' - 5g'^2)\frac{\partial W_{g'}^{(1)}}{\partial g'} + (2 - 4g')W_{g'}^{(1)}. \tag{15}$$

Now we introduce respectively, the moment and the cumulant generating functions $\chi_{g'}^{(1)}$ and $K_{g'}^{(1)}$ as before and get

$$\frac{\partial K_{g'}^{(1)}}{\partial l} = x^2\left[\left(\frac{\partial K_{g'}^{(1)}}{\partial x}\right)^3 + 3\frac{\partial K_{g'}^{(1)}}{\partial x}\frac{\partial^2 K_{g'}^{(1)}}{\partial x^2} + \frac{\partial^3 K_{g'}^{(1)}}{\partial x^3}\right]$$
$$+ (x^2 + x)\left[\left(\frac{\partial K_{g'}^{(1)}}{\partial x}\right)^2 + \frac{\partial^2 K_{g'}^{(1)}}{\partial x^2}\right]. \tag{16}$$

Steps leading to (16) from (15) involve taking the Laplace transform on both sides of (15) and interchanging the order of integration and differentiation, e.g.,

$$\int_0^\infty e^{-xg'} g'^2 \frac{\partial^2 W_{g'}^{(1)}}{\partial g'^2}\, dg' = \frac{d^2}{dx^2}\left(x^2 \int_0^\infty e^{-xg'} W_{g'}^{(1)}\, dg'\right). \tag{17}$$

It is precisely this operation that becomes inadmissible when the moments do not exist as noted above.

Proceeding now as before, we get the recursion relation for $K_g^{(d)}$ in the differential form

$$\frac{\partial K_g^{(d)}}{\partial \ln l} = -(d-1)K_g^{(d)} + \tfrac{1}{2}\left\{x^2\left[\left(\frac{\partial K^{(d)}}{\partial x}\right)^3 + 3\frac{\partial K_g^{(d)}}{\partial x}\frac{\partial^2 K_g^{(d)}}{\partial x^2} + \frac{\partial^3 K_q^{(d)}}{\partial x^3}\right] + (x^2+x)\left[\left(\frac{\partial K_g^{(d)}}{\partial x}\right)^2 + \frac{\partial^2 K_g^{(d)}}{\partial x^2}\right]\right\}\ln(1+2\rho_1^{(d)}). \tag{18}$$

An essential complexity of (18) for the cumulant generating function $W_g^{(d)}(x, l)$ of conductance g is that it involves parametrically the mean resistance $\rho_1^{(d)}$. We can still carry out an approximate analysis by writing the equations for the first two cumulants $K_{1g}^{(d)}$ and $K_{2g}^{(d)}$, from (18). We have

$$\frac{\partial \kappa_{1g}^{(d)}}{\partial \ln l} = (d-1)\kappa_{1g}^{(d)} - \tfrac{1}{2}(\kappa_{1g}^{(d)2} + \kappa_{2g}^{(d)})\ln(1+2\rho_1^{(d)}) \tag{19a}$$

$$\frac{\partial \kappa_{2g}^{(d)}}{\partial \ln l} = (d-1)\kappa_{2g}^{(d)} - (\kappa_{1g}^{(d)3} + 3\kappa_{1g}^{(d)}\kappa_{2g}^{(d)} + 2\kappa_{3g}^{(d)} - \kappa_{1g}^{(d)2} - \kappa_{2g}^{(d)})\ln(1+2\rho_1^{(d)}). \tag{19b}$$

Above we have replaced ρ_1' by $1+\rho_1$ where ρ_1 tunes the intrinsic disorder. We could approach the fixed point from the metallic side, i.e., when $\rho_1^{(d)}$ is small. Neglecting $\kappa_{3g}^{(d)}$ and higher order cumulants, and approximating very crudely $\rho_1^{(d)} \simeq (\kappa_{1g}^{(d)})^{-1}$, we get from (19*a*)

$$\frac{\partial \ln \kappa_{1g}^{(d)}}{\partial \ln l} = (d-2) - \frac{1}{\kappa_{1g}^{(d)}}\kappa_{2g}. \tag{20}$$

Equation (20) clearly shows that a β-function ($\equiv \partial \ln \kappa_{1g}^{(d)}/\partial \ln l$) depending only on the conductance $\kappa_{1g}^{(d)}$ does not exist even in the metallic regime, (Abrahams *et al* 1979). The approximate fixed point for (19), neglecting the $\kappa_{3g}^{(d)}$ and assuming $\rho_1^{(d)} \simeq (\kappa_{1g})^{-1}$ turns out to be $\kappa_{1g}^{(d)*} \simeq 0.92$ and $\kappa_{2g}^{(d)*} \simeq 2.35$. Again $(\kappa_{2g}^{(d)*})^{1/2}/\kappa_{1g}^{(d)*} \simeq 1.67$ suggesting large statistical dispersion of conductance. This fixed point turns out to be a saddle point, unstable along the $\kappa_{1g}^{(d)}$-like eigen-direction and stable along the $\kappa_{2g}^{(d)}$-like eigen-direction. Detailed numerical analysis of (18) and (19) and comparison with the numerical results of Ioffe *et al* (1985), McKinnon and Kramer (1983) and of McMillan (1985) will be reported elsewhere.

In conclusion we have shown that the dispersion of resistance remains comparable with its mean value even in the 'metallic' regime for the 3-dimensional anisotropically disordered conductor. For the isotropic case this remains true at the mobility edge indicating inadequacy of the one-parameter scaling *ansatz*.

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