

COMBINED EFFECTS OF OBLATENESS AND RADIATION ON THE NONLINEAR STABILITY OF L_4 IN THE RESTRICTED THREE-BODY PROBLEM

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ABSTRACT

The nonlinear stability of the triangular libration point, L_4 , when both of the primaries are oblate spheroids as well as sources of radiation has been studied. It is found that L_4 is stable for all mass ratios in the range of linear stability except for three mass ratios depending upon oblateness coefficients and mass reduction factors.

Key words: celestial mechanics

1. INTRODUCTION

The restricted three-body problem describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. It is originally formulated due to the approximately circular motion of the planets around the sun, and the small masses of the asteroids and the satellites of the planets compared to the planets' masses.

This classical restricted three-body problem is not valid when at least one of the interacting bodies is an intense emitter of radiation. In this connection, it is reasonable to modify the model by superposing a light repulsion field whose source coincides with the source of the gravitational field provided by the radiating body on the gravitational field of the main bodies. According to Radzievsky (1950, 1953), the problem in such a statement is called the photogravitational problem. He discussed it for three specific bodies: the sun, a planet, and a dust particle. It was found that an allowance for direct solar radiation pressure results in a change in the positions of the libration points.

In certain stellar dynamics problems it is altogether inadequate to consider solely the gravitational interaction force. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure. Since a large fraction of all stars belong to binary systems, the particle motion in the field of a double star offers special interest. Probably the simplest dynamical model of such a system is the restricted circular photogravitational three-body problem. Among the possible motions of a particle, the equilibrium states at the libration points in a reference frame sharing the orbital motion of the stars were studied by Schuerman (1980) and Kunitsyn & Tureshbaev (1985).

In stellar systems numerous examples are available where a body is moving under the gravitational field of two radiating bodies. Simmons et al. (1985) obtained a complete solution of the restricted three-body problem. They discussed the existence and linear stability of the equilibrium points for all values of radiation pressures of both luminous bodies and all values of mass ratios. The participating bodies in the classical restricted three-body problem are strictly spherical in shape, but in actual situations we find that several heavenly bodies, such as Saturn and Jupiter, are sufficiently oblate. The minor planets and meteoroids have irregular shapes. In these cases, on account of the small dimensions of the bodies in comparison with their

distances from the primaries, they are considered to be point masses, but in many cases the dimensions of the bodies are larger than the distances from their respective primaries. Thus, the above assumption is not justified, and the results obtained are far from a realistic approach. The lack of sphericity, or the oblateness, of the planet causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. This enables many researchers to study the restricted problem by taking into account the shapes (oblateness) of the bodies (Subba Rao & Sharma, 1975; Bhatnagar & Chawla, 1979; Elife & Ferrer, 1985; Elife 1992; Markellos et al. 1996; Chandra & Kumar, 2004).

An investigation of the positions of libration points, when the more massive primary is a source of radiation and the smaller one is an oblate spheroid, was carried out by, e.g., Sharma (1987). He showed that the triangular points are linearly stable for the mass parameter $0 < \mu < \mu_{\text{crit}}$ and the critical mass value μ_{crit} decreases with the increase in oblateness and radiation force.

The effect of oblateness and radiation pressure forces of the primaries on the location and the linear stability of the triangular points in the restricted three-body problem was analyzed by Singh & Ishwar (1999). They considered both primaries as sources of radiation as well as oblate spheroids, and observed that these points are stable for $0 < \mu < \mu_{\text{co}}$ and unstable for $\mu_{\text{co}} < \mu < 1/2$, where μ_{co} is the critical value of the mass parameter and depends on the radiating and oblateness coefficients. The same problem under the influence of small perturbations in the Coriolis and the centrifugal forces was studied by AbdulRaheem & Singh (2006). Dealing with the overall effect they observed that the range of stability of triangular points decreases.

Many mathematicians and astronomers have been interested in the study of the stability of an equilibrium point for all time and all the orders of the terms in the expansion of the Hamiltonian. The nonlinear stability of the triangular libration points in the classical restricted three-body problem was investigated by Deprit & Deprit-Bartholome (1967). Bhatnagar & Hallan (1983) discussed the effect of perturbations in the Coriolis and centrifugal forces on the nonlinear stability of equilibrium points. Later, analytical studies of the nonlinear stability of L_4 under different aspects were also carried out by Niedzielska (1994), Subba Rao & Sharma (1997), Hallan et al. (2000), Gozdziowski (2003), and Chandra & Kumar (2004).

Our aim is to study the combined effects of oblateness and radiation of the primaries on the nonlinear stability of the libration point L_4 as it has been found that they produce

significant changes in the location and stability (Singh & Ishwar 1999) of the triangular points. We also know that the inclusion of the nonlinear terms sometimes changes the entire pattern of the stability and, hence, we intend to study the stability in the nonlinear sense as well. By applying Lyapunov's theorem (Lyapunov 1956) to the linear stability results in Singh & Ishwar (1999) mentioned earlier, we can state that L_4 for $\mu_{co} < \mu < 1/2$ is unstable in the nonlinear sense also. So, we need to study the nonlinear stability of L_4 for $0 < \mu < \mu_{co}$. For this, we will apply Arnold's theorem (Arnold 1961) and follow the procedure adopted by Bhatnagar & Hallan (1983). Arnold (1961) proved that if

1. $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs (k_1, k_2) of rational integers where ω_1, ω_2 are the basic frequencies for the linear dynamical system and
2. The determinant $D \neq 0$, where $D = \det(b_{ij})$ ($i, j = 1, 2, 3$), $b_{ij} = \left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right)_{I_i=I_j=0}$ ($i, j = 1, 2$), $b_{i3} = b_{3i} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i=I_i=0}$ ($i, j = 1, 2$),

$b_{33} = 0$ and $H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1 I_2 + CI_2^2) + \dots$ is the normalized Hamiltonian with I_1, I_2 as the action momenta coordinates, and A, B, C as second-order coefficients in the frequencies, then on each energy manifold $H = h$ in the neighborhood of equilibrium, there exists invariant tori of quasi-periodic motions which divide the manifolds, and consequently, the equilibrium is stable. This is valid for a system with two degrees of freedom, which is the case under consideration. Moser (1962) showed that Arnold's theorem is true if the first condition of the theorem is replaced by $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs (k_1, k_2) of rational integers such that $|k_1| + |k_2| \leq 4$.

Markellos et al. (1996), Subba Rao & Sharma (1997), and Chandra & Kumar (2004) studied the nonlinear stability of L_4 when the bigger primary is an oblate spheroid. We consider here the case where both primaries are oblate spheroids as well as sources of radiation. We use A_i ($i = 1, 2$) for the oblateness coefficients of the bigger and smaller primaries, respectively, such that $0 < A_i \ll 1$ (McCuskey 1963). We denote the radiation factors as q_i ($i = 1, 2$) for the bigger and smaller primaries and these are given by $F_p = F_g(1 - q_i)$ such that $0 < 1 - q_i \ll 1$ (Radzievsky 1950). For simplicity we further substitute $q_1 = 1 - \epsilon$, $q_2 = 1 - \epsilon'$, $|\epsilon| \ll 1$, $|\epsilon'| \ll 1$ and restrict ourselves to linear terms in A_i, ϵ , and ϵ' . Then the coefficients of A_1 throughout the paper are exactly the same as worked out in the paper of Markellos et al. (1996), Subba Rao & Sharma (1997), Chandra & Kumar (2004).

This paper should be read in conjunction with the papers by Bhatnagar & Hallan (1983) and Subba Rao & Sharma (1997) as, to save space, we are not mentioning the values of various variables given in those papers, although they are used in this paper.

2. EQUATIONS OF MOTION

Using dimensionless variables and a synodic coordinate system (x, y) as Szebehely (1967) and Singh & Ishwar (1999) did, the equations of motion of the infinitesimal mass can be written as

$$\ddot{x} - 2n\dot{y} = U_x, \quad \ddot{y} + 2n\dot{x} = U_y, \quad (1)$$

where

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{q_1(1 - \mu)}{r_1} + \frac{\mu q_2}{r_2} + \frac{A_1 q_1(1 - \mu)}{2r_1^3} + \frac{A_2 q_2 \mu}{2r_2^3}$$

$$r_1^2 = (x - \mu)^2 + y^2, \quad r_2^2 = (x + 1 - \mu)^2 + y^2, \quad (2)$$

where r_1 and r_2 are the distances of the infinitesimal mass from the primaries and μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \leq 1/2$. The perturbed mean motion of the primaries, n , is given by

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2). \quad (3)$$

The coordinates of the triangular point L_4 , obtained by Singh & Ishwar (1999), are

$$x_0 = -\frac{1}{2} \left[\gamma + \frac{2}{3}(q_1 - q_2) + A_1 q_1 - A_2 q_2 \right] \text{ with } \gamma = 1 - 2\mu$$

$$y_0 = \left(\frac{\sqrt{3}}{2} \right) \left[1 - \frac{2}{3}(A_1 + A_2) + \frac{1}{3}(A_1 q_1 + A_2 q_2) - \frac{2}{9}(1 - q_1) - \frac{2}{9}(1 - q_2) \right]. \quad (4)$$

3. LINEAR STABILITY

The Lagrangian function of the problem can be written as

$$L = 1/2(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + (n^2/2)(x^2 + y^2) + (1 - \mu)q_1/r_1 + \mu q_2/r_2 + A_1 q_1(1 - \mu)/2r_1^3 + A_2 q_2 \mu/2r_2^3.$$

Shifting the origin to L_4 and expanding in power series of x and y , we have

$$L_2 = (\dot{x}^2 + \dot{y}^2)/2 + n(x\dot{y} - \dot{x}y) + (n^2/2) \times (x^2 + y^2) - Ex^2 - Fy^2 - Gxy,$$

$$L_3 = T_1 x^3/3 + 1/2(T_2 x^2 y) + T_3 x y^2 + 1/3(T_4 y^3) \quad (5)$$

$$L_4 = N_1 x^4 + N_2 x^3 y + N_3 x^2 y^2 + N_4 x y^3 + N_5 y^4.$$

Consequently,

$$H_2 = (P_x^2 + P_y^2)/2 + n(yP_x - xP_y) + Ex^2 + Fy^2 + Gxy,$$

$$H_2 = (\dot{x}^2 + \dot{y}^2)/2 - n^2(x^2 + y^2)/2 + Ex^2 + Fy^2 + Gxy, \quad (6)$$

$$H_3 = -L_3, \quad H_4 = -L_4$$

where

$$E = [2 - 3A_1 - 12A_1\gamma - 3A_2 + 12A_2\gamma - 2\epsilon + 6\epsilon' - 2\epsilon' - 6\epsilon'\gamma]/16$$

$$F = -[10 + 21A_1 + 21A_2 - 2\epsilon - 2\epsilon' + 6\epsilon' - 6\epsilon'\gamma]/16,$$

$$G = \sqrt{3}[6A_1 + 13A_1\gamma - 6A_2 + 13A_2\gamma - 2\epsilon + 2\epsilon' + 2\epsilon'\gamma/3 + 2\epsilon'\gamma/3 + 6\gamma]/8,$$

$$T_1 = [-24\gamma + 18A_1 - 75A_1\gamma + 79A_2/2 - 35A_2\gamma/2 + 21\epsilon + 21\epsilon'\gamma/3 - 21\epsilon'\gamma/3 + 21\epsilon'\gamma]/13,$$

$$T_2 = -\sqrt{3}[6 + 43A_1 + 60A_1\gamma + 43A_2 - 60A_2\gamma - 3\epsilon - 3\epsilon'\gamma - 3\epsilon' + 3\epsilon'\gamma]/16$$

$$T_3 = [33\gamma/4 + 33A_1/4 + 195A_1\gamma/8 - 33A_2/4 + 195A_2\gamma/8 - 33\epsilon/8 - 33\epsilon'\gamma/8 + 33\epsilon'\gamma/8 - 33\epsilon'\gamma/8]/4$$

$$T_4 = -\sqrt{3}[18 + 69A_1 - 41A_2 - 9\epsilon - 9\epsilon' + \gamma(-110A_2 - 9\epsilon + 9\epsilon')]/32.$$

$$N_1 = -37/128 - 285A_1/256 - 25A_1\gamma/32 - 36A_2/64 + 1545A_2\gamma/256 - 29\epsilon/128 + 113\epsilon'\gamma/384 - 29\epsilon'/128 - 113\epsilon'\gamma/384.$$

$$\begin{aligned}
N_2 &= \sqrt{3}[25\gamma/32 - 45A_1/32 + 265A_1\gamma/192 \\
&\quad + 135A_2/96 + 265A_2\gamma/192 \\
&\quad + 735\epsilon/384 + 925\epsilon'\gamma/1152 \\
&\quad - 35\epsilon'/32 - 5\epsilon'\gamma/288] \\
N_3 &= 12364 + 255A_1/32 + 1215A_1\gamma/128 + 1245A_2/128 \\
&\quad - 525A_2\gamma/64 - 583\epsilon/128 \\
&\quad + 467\epsilon\gamma/128 - 303\epsilon'/128 + 863\epsilon'\gamma/128 \\
N_4 &= -\sqrt{3}[45\gamma/32 + 45A_1/32 + 355A_1\gamma/64 + 435A_2/64 \\
&\quad - 85A_2\gamma/32 + 35\epsilon/32 \\
&\quad + 25\epsilon\gamma/96 + 105\epsilon'/64 - 475\epsilon'\gamma/192] \\
N_5 &= 3/128 - 225A_1/256 - 105A_1\gamma/64 + 435A_2/64 \\
&\quad - 85A_2\gamma/32 + 35\epsilon/32 + 25\epsilon\gamma/96 \\
&\quad + 105\epsilon'/64 - 475\epsilon'\gamma/192
\end{aligned} \tag{7}$$

with considerations

$$q_1 = 1 - \epsilon, \quad q_2 = 1 - \epsilon', \quad \|\epsilon\| \ll 1, \quad |\epsilon'| \ll 1.$$

All these values are different from the classical case due to the radiation and oblateness of both primaries. When $A_2 = \epsilon = \epsilon' = 0$, these values agree with those found by Subba Rao & Sharma (1997).

The characteristic equation corresponding to Hamiltonian H_2 is given by

$$\lambda^4 + 2(E + F + n^2)\lambda^2 + 4EF - G^2 + n^4 - 2n^2(E + F) = 0. \tag{8}$$

The roots of Equation (8) are purely imaginary if its discriminant $\Delta > 0$ and thus it is a necessary condition for the stability, in the linear sense, around the L_4 point. The solution of the equation $\Delta = 0$ for μ gives the critical mass value μ_{co} of the mass parameter.

This implies that (Singh & Ishwar 1999)

$$\begin{aligned}
\mu_{co} &= 0.03852089 \dots - 0.28500178 \dots A_1 - 0.00891747 \dots \epsilon \\
&\quad - 0.0627795 \dots A_2 - 0.00891747 \dots \epsilon'.
\end{aligned}$$

For $A_2 = \epsilon = \epsilon' = 0$ the value of μ_{co} agrees with that given by Subba Rao & Sharma (1975), Markellos et al. (1996), and Chandra & Kumar (2004). The range of stability, in the linear sense, can be written as $0 < \mu < \mu_{co}$. If Equation (8) has four imaginary roots, $\pm i\omega_1, \pm i\omega_2$ where ω_1, ω_2 are the basic frequencies, then we have

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= (1/16)[16 + \gamma(24A_2 - 24A_1)], \\
0 &< \omega_2 < 1/\sqrt{2} < \omega_1 < 1 \\
\omega_1^2\omega_2^2 &= (1/16)27 + 117A_1 + 117A_2 + 6\epsilon + 6\epsilon' \\
&\quad - \gamma^2(27 + 117A_1 + 117A_2 + 6\epsilon + 6\epsilon'), \tag{9}
\end{aligned}$$

which give

$$\begin{aligned}
\gamma^2 &= (16\omega_1^4 - 16\omega_1^2 + 27)/27 + \{208\omega_1^2(1 - \omega_1^2)/81 \\
&\quad + 8\gamma\omega_1^2/9\}A_1 + \{208\omega_1^2(1 - \omega_1^2)/81 \\
&\quad - 8\gamma\omega_1^2/9\}A_2 + \{32\omega_1^2(1 - \omega_1^2)/243\} \epsilon \\
&\quad + \{32\omega_1^2(1 - \omega_1^2)/243\} \epsilon'. \tag{10}
\end{aligned}$$

The results given by Equations (9) and (10) in the case $A_2 = \epsilon = \epsilon' = 0$ agree with those found by Subba Rao & Sharma (1997).

4. FIRST-ORDER NORMALIZATION

Following the method given in Whittaker (1965), we use a canonical transformation from the phase space x, y, P_x, P_y into the phase space of the angles (ϕ_1, ϕ_2) and the actions (I_1, I_2) so that H_2 be normalized:

$$X = JT \tag{11}$$

where

$$X = \begin{pmatrix} x \\ y \\ P_x \\ P_y \end{pmatrix}, \quad J = (J_{ij}) \quad 1 \leq i, j \leq 4, \quad T = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}$$

$Q_j = (2I_j/\omega_j)^{1/2} \sin \phi_j$, $P_j = (2I_j/\omega_j)^{1/2} \cos \phi_j$ ($j = 1, 2$), and the elements of the dyadic J can be obtained by adopting the method of Breakwell & Pringle (1966). Elements of J necessary for further analysis are given in Appendix A.

The transformation changes the second-order part of the Hamiltonian into the normal form $H_2 = \omega_1 I_1 - \omega_2 I_2$.

5. SECOND-ORDER NORMALIZATION

For transforming the Hamiltonian H to the Birkhoff's normal forms, we have utilized Henrard's method (Deprit & Deprit-Bartholome 1967). We have expanded the coordinates (x, y) of the infinitesimal body in double d'Alembert series:

$$x = \sum_{j \geq 1} B_j^{1,0}(\phi_1, \phi_2, I_1, I_2), \quad y = \sum_{j \geq 1} B_j^{0,1}(\phi_1, \phi_2, I_1, I_2) \tag{12}$$

where the homogeneous components $B_j^{1,0}$ and $B_j^{0,1}$ are of degree j , I_1 and I_2 are to be taken as constants of integration, while ϕ_1 and ϕ_2 are to be determined as linear functions of time such that

$$\dot{\phi}_1 = \omega_1 + \sum_{j \geq 1} f_{2j}(I_1, I_2), \quad \dot{\phi}_2 = -\omega_2 + \sum_{j \geq 1} f_{2j}(I_1, I_2).$$

The first-order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y obtained by Equation (11). Proceeding as in Deprit & Deprit-Bartholome (1967), we observe that the second-order components $B_2^{1,0}$ and $B_2^{0,1}$ are solutions of the partial differential equations

$$\Delta_1 \Delta_2 B_2^{1,0} = \Phi_2, \quad \Delta_1 \Delta_2 B_2^{0,1} = -\Psi_2, \tag{13}$$

with

$$\begin{aligned}
\Delta_i &= D^2 + \omega_i^2, \quad i = 1, 2; \quad D = \omega_1(\partial/\partial\phi_1) - \omega_2(\partial/\partial\phi_2), \\
\Phi_2 &= (D^2 + S_1)X_2 + (S_2D + S_3)Y_2, \\
\Psi_2 &= (S_2D - S_3)X_2 - (D^2 + S_4)Y_2, \quad S_1 = 2F - n^2, \\
S_2 &= 2[1 + (3/4)(A_1 + A_2)], \quad S_3 = -G, \quad S_4 = n^2 - 2E.
\end{aligned}$$

X_2, Y_2 are obtained by substituting respectively in $\partial L_3/\partial x, \partial L_3/\partial y$ the first-order components for x and y .

Equation (13) can be solved for $B_2^{1,0}$ and $B_2^{0,1}$ by using the formula

$$\frac{1}{\Delta_1 \Delta_2} \begin{Bmatrix} \cos(m\phi_1 + n\phi_2) \\ \text{or} \\ \sin(m\phi_1 + n\phi_2) \end{Bmatrix} = \frac{1}{\Delta_{m,n}} \begin{Bmatrix} \cos(m\phi_1 + n\phi_2) \\ \text{or} \\ \sin(m\phi_1 + n\phi_2) \end{Bmatrix}$$

where

$$\Delta_{m,n} = [\omega_1^2 - (m\omega_1 - n\omega_2)^2][\omega_2^2 - (m\omega_1 - n\omega_2)^2]$$

provided $\Delta_{m,n} \neq 0$.

Since $\Delta_{1,0} = \Delta_{0,1} = 0$, the terms $\cos \phi_1$, $\sin \phi_1$, $\cos \phi_2$, and $\sin \phi_2$ are the critical terms. Φ_2 and Ψ_2 are free from such terms. By condition (1) of Moser's theorem (Moser 1962), none of the divisors $\Delta_{2,0}$, $\Delta_{0,2}$, $\Delta_{1,1}$, $\Delta_{1,-1}$, is zero. The second-order components $B_2^{1,0}$ and $B_2^{0,1}$ are as follows:

$$\begin{aligned} B_2^{1,0} &= r_1 I_1 + r_2 I_2 + r_3 I_1 \cos 2\phi_1 + r_4 I_2 \cos 2\phi_2 + r_5 I_1 \sin 2\phi_1 \\ &\quad + r_6 I_2 \sin 2\phi_2 + r_7 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) \\ &\quad + r_8 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) + r_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) \\ &\quad + r_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) \\ B_2^{0,1} &= s_1 I_1 + s_2 I_2 + s_3 I_1 \cos 2\phi_1 + s_4 I_2 \cos 2\phi_2 \\ &\quad + s_5 I_1 \sin 2\phi_1 + s_6 I_2 \sin 2\phi_2 \\ &\quad + s_7 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + s_8 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ &\quad + s_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) \\ &\quad + s_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) \end{aligned}$$

The coefficients r_k and s_k for $k = 1, 2, \dots, 10$ are given in Appendix B.

6. SECOND-ORDER COEFFICIENTS IN THE FREQUENCIES

Following Henrard's method, we find that the third-order components $B_3^{1,0}$ and $B_3^{0,1}$ in the coordinates x, y and the second-order polynomials f_2 and g_2 in the frequencies ϕ_1 and ϕ_2 satisfy the partial differential equations:

$$\begin{aligned} \Delta_1 \Delta_2 B_3^{1,0} &= \Phi_3 - 2f_2 P - 2g_2 Q, \\ \Delta_1 \Delta_2 B_3^{0,1} &= \psi_3 - 2f_2 U - 2g_2 V, \end{aligned} \quad (14)$$

with

$$\begin{aligned} \Phi_3 &= [D^2 - \{1 + 3(A_1 + A_2)/2 - 2F\}]X_3 \\ &\quad + [2\{1 + 3(A_1 + A_2)/4\}D - G]Y_3, \\ \psi_3 &= [D^2 - \{1 + 3(A_1 + A_2)/2 - 2E\}]Y_3 \\ &\quad - [2\{1 + 3(A_1 + A_2)/4\}D + G]X_3 \\ P &= (\partial/\partial\phi_1)[\{\omega_1^2 \partial^2/\partial\phi_1^2 - 1 - 3(A_1 + A_2)/2 \\ &\quad + 2F\}\{\omega_1 \partial B_1^{1,0}/\partial\phi_1 - (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\} \\ &\quad + \{2(1 + 3A_1/4 + 3A_2/4)\omega_1 \partial/\partial\phi_1 - G\} \\ &\quad \times \{\omega_1 \partial B_1^{0,1}/\partial\phi_1 + (1 + 3A_1/2 + 3A_2/2)B_1^{0,1}\}] \\ Q &= (\partial/\partial\phi_2)[\{\omega_2^2 \partial^2/\partial\phi_2^2 - 1 - 3(A_1 + A_2)/2 + 2F\} \\ &\quad \times \{-\omega_2 \partial B_1^{1,0}/\partial\phi_2 - (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\} \\ &\quad + \{-2(1 + 3A_1/4 + 3A_2/4)\omega_2 \partial/\partial\phi_2 - G\} \\ &\quad \times \{-\omega_2 \partial B_1^{0,1}/\partial\phi_2 + (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\}] \\ U &= (\partial/\partial\phi_1)[\{\omega_1^2 \partial^2/\partial\phi_1^2 - 1 - 3A_1/2 - 3A_2/2 + 2E\} \\ &\quad \times \{\omega_1 \partial B_1^{1,0}/\partial\phi_1 + (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\} \\ &\quad - \{2(1 + 3A_1/4 + 3A_2/4)\omega_1 \partial/\partial\phi_1 + G\}\{\omega_1 \partial B_1^{0,1}/\partial\phi_1 \\ &\quad - (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\}] \end{aligned}$$

$$\begin{aligned} V &= (\partial/\partial\phi_2)[\{\omega_2^2 \partial^2/\partial\phi_2^2 - 1 - 3A_1/2 - 3A_2/2 + 2E\} \\ &\quad \times \{-\omega_2 \partial B_1^{1,0}/\partial\phi_2 + (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\} \\ &\quad + \{-2(1 + 3A_1/4 + 3A_2/4)\omega_2 \partial/\partial\phi_2 + G\} \\ &\quad \times \{-\omega_2 \partial B_1^{0,1}/\partial\phi_2 + (1 + 3A_1/4 + 3A_2/4)B_1^{0,1}\}], \end{aligned}$$

X_3, Y_3 are the homogeneous components of order 3 obtained by substituting, respectively,

$$\begin{aligned} x &= B_1^{1,0} + B_2^{1,0}, \quad y = B_1^{0,1} + B_2^{0,1}, \\ &\text{into } \frac{\partial}{\partial x}(L_3 + L_4), \quad \frac{\partial}{\partial y}(L_3 + L_4). \end{aligned}$$

We do not require to find out the components $B_3^{1,0}$ and $B_3^{0,1}$. We find the coefficients of $\cos \phi_1$, $\sin \phi_1$, $\cos \phi_2$, and $\sin \phi_2$ on the right-hand sides of Equation (14). They are the critical terms. We eliminate these terms by properly choosing the coefficients in the polynomials

$$f_2 = f_{2,0}I_1 + f_{0,2}I_2, \quad \text{and} \quad g_2 = g_{2,0}I_1 + g_{0,2}I_2.$$

Further, we find that

$$\begin{aligned} f_{2,0} &= \frac{1 \text{ (coefficient of } \cos \phi_1 \text{ in } \Phi_3)}{2 \text{ (coefficient of } \cos \phi_1 \text{ in } P)} = A, \\ f_{2,0} = g_{2,0} &= \frac{1 \text{ (coefficient of } \cos \phi_2 \text{ in } \Phi_3)}{2 \text{ (coefficient of } \cos \phi_2 \text{ in } Q)} = B, \\ g_{2,0} &= \frac{1 \text{ (coefficient of } \cos \phi_2 \text{ in } \Psi_3)}{2 \text{ (coefficient of } \cos \phi_2 \text{ in } Q)} = C. \end{aligned}$$

with

$$\begin{aligned} A &= A_{1,1} + (A_{1,2} + A_{1,3}\gamma)A_1 + (A_{1,4} + A_{1,5}\gamma)A_2 \\ &\quad + (A_{1,6} + A_{1,7}\gamma) \in + (A_{1,8} + A_{1,9}\gamma) \in', \\ B &= B_{1,1} + (B_{1,2} + B_{1,3}\gamma)A_1 + (B_{1,4} + B_{1,5}\gamma)A_2 \\ &\quad + (B_{1,6} + B_{1,7}\gamma) \in + (B_{1,8} + B_{1,9}\gamma) \in', \\ C &= C_{1,1} + (C_{1,2} + C_{1,3}\gamma)A_1 + (C_{1,4} + C_{1,5}\gamma)A_2 \\ &\quad + (C_{1,6} + C_{1,7}\gamma) \in + (C_{1,8} + C_{1,9}\gamma) \in', \end{aligned}$$

where the coefficients A, B, C are given in Appendix C.

7. STABILITY

Since condition (1) of Moser's theorem (Moser 1962) is applicable, Birkhoff's normalization up to third order can be obtained. Condition (1) of Moser's theorem is satisfied in the interval $0 < \mu < \mu_{c0}$ if the mass ratio does not take the critical values:

$$\begin{aligned} \mu_{c1} &= 0.0242938 \dots - 0.1790727 \dots A_1 - 0.0368505 \dots A_2 \\ &\quad - 0.0055364 \dots \in - 0.0055364 \dots \in' \\ \mu_{c2} &= 0.013516 \dots - 0.099383 \dots A_1 - 0.019383 \dots A_2 \\ &\quad - 0.0030452 \dots \in - 0.0030452 \dots \in' . \end{aligned}$$

These results, in the case $A_2 = \in = \in' = 0$, agree with those found by Markellos et al. (1996), Subba Rao & Sharma (1997), and Chandra & Kumar (2004).

Normalized Hamiltonian up to fourth order is

$$H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}(A I_1^2 + 2B I_1 I_2 + C I_2^2) + \dots,$$

where the coefficients A, B, C are given in Appendix C.

Calculating the determinant D occurring in condition (2) of the Moser (1962) theorem, we have

$$D = -(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2).$$

Putting the values of A , B , and C , we have

$$\begin{aligned} D = & [(644u^4 - 541u^2 + 36)/8(4u^2 - 1)(25u^2 - 4)] \\ & + (D_2 + D_3\gamma)A_1 + (D_4 + D_5\gamma)A_2 \\ & + (D_6 + D_7\gamma)\epsilon + (D_8 + D_9\gamma)\epsilon' \end{aligned}$$

with

$$\begin{aligned} D_2 = & [(39176u^4 - 14359u^2 + 492)/[48(4u^2 - 1)(25u^2 - 4)], \\ D_3 = & [3(1593600u^{10} + 2122096u^8 - 13052000u^6 + 5408175u^4 \\ & - 840076u^2 + 23616)]/[16(4u^2 - 1)^2 \\ & \times (25u^2 - 4)^2(16u^2 + 117)], \\ D_4 = & [59944u^6 + 14269689u^4 - 12215187u^2 \\ & + 166284]/[1962(4u^2 - 1)2(25u^2 - 4)^2], \\ D_5 = & [55977008u^{10} + 16915980u^8 - 694540170u^6 \\ & + 171574880u^4 - 31023250u^2 + 732927]/ \\ & [208(4u^2 - 1)^2(25u^2 - 4)^2(16u^2 + 117)], \\ D_6 = & [195560u^6 - 549558u^4 + 24575u^2 \\ & + 48]/[432(4u^2 - 1)^2(25u^2 - 1)], \\ D_7 = & [48947712u^{12} + 617901664u^{10} - 454674128u^8 \\ & + 245406416u^6 - 73692732u^4 + 8118909u^2 - 71472]/ \\ & [96(4u^2 - 1)^3(25u^2 - 4)^2(16u^2 + 117)], \\ D_8 = & [19440u^8 - 578504u^6 + 272091u^4 - 33970u^2 \\ & + 216]/[654(4u^2 - 1)^2(25u^2 - 4)], \\ D_9 = & [252817792u^{12} + 2444733120u^{10} - 1258751060u^8 \\ & - 45728424u^6 + 32733556u^4 - 5336499u^2 + 631382]/ \\ & [48(4u^2 - 1)^3(25u^2 - 4)^2(16u^2 + 117)]. \end{aligned}$$

Moser's second condition is violated for the unperturbed problem, i.e., for $A_1 = A_2 = \epsilon = \epsilon' = 0$ when $\mu_0 = 0.0109136 \dots$. When $A_1, A_2 \in, \epsilon' \neq 0$, we take $\mu = \mu_0 + \alpha A_1 + \alpha' A_2 + \alpha'' \in + \alpha''' \epsilon' +$ such that $D = 0$. It is seen that the second condition of Moser's theorem is satisfied, that is $D \neq 0$ if in the interval $0 < \mu < \mu_{c0}$, the mass ratio does not take the value

$$\mu_{c3} = \mu_0 + \alpha A_1 + \alpha' A_2 + \alpha'' \in + \alpha''' \epsilon',$$

where

$$\begin{aligned} \alpha = & -0.1036878 \dots, \quad \alpha' = 0.5456468 \dots, \\ \alpha'' = & -0.4748319 \dots, \quad \alpha''' = -0.1432604 \dots \end{aligned}$$

8. CONCLUSION

By taking both the primaries as oblate spheroids as well as sources of radiation, it has been seen that, in the nonlinear sense, the triangular point L_4 is stable for all mass ratios in the range of linear stability except for three mass ratios,

$$\begin{aligned} \mu_{c1} = & 0.0242 \dots - 0.1790 \dots A_1 - 0.0368 \dots A_2 \\ & - 0.0055 \dots \in - 0.0055 \dots \epsilon', \\ \mu_{c2} = & 0.0135 \dots - 0.0993 \dots A_1 - 0.093 \dots A_2 \\ & - 0.0030 \dots \in - 0.0030 \dots \epsilon', \\ \mu_{c3} = & 0.0109 \dots - 0.1036 \dots A_1 + 0.5456 \dots A_2 \\ & - 0.4748 \dots \in - 0.1432 \dots \epsilon', \end{aligned}$$

at which the Moser (1962) theorem fails.

When $A_1 A_2 = \epsilon = \epsilon' = 0$, the values of μ_{ci} ($i = 1, 2, 3$) agree with those found by Deprit & Deprit-Bartholome (1967).

When $A_2 = \epsilon = \epsilon' = 0$, the values of μ_{ci} are the same as those found by Markellos et al. (1996), Subba Rao & Sharma (1997), and Chandra & Kumar (2004).

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APPENDIX A

$$\begin{aligned} J_{1,3} = & l_1 [1 + 33A_1/4l_1^2 - 3A_1\gamma/4k_1^2 + 33A_2/4l_1^2 \\ & + 3A_2\gamma/4k_1^2 - (3\omega_1^2 - 7) \in / 2l_1^2 k_1^2 \\ & - (3\omega_1^2 - 7) \in \gamma / 2l_1^2 k_1^2 - (3\omega_1^2 - 7) \epsilon' / 2l_1^2 k_1^2 \\ & + (7\omega_1^2 + 2) \epsilon' \gamma / 2l_1^2 k_1^2] / 2\omega_1 k_1 \\ J_{2,1} = & -4\omega_1 [l + 6k_1^2 A_1 / 4l_1^2 - 3A_1\gamma / 4k_1^2 + 6k_1^2 A_2 / 4l_1^2 \\ & + 3A_2\gamma / 4k_1^2 + (7\omega_1^2 + 2) \in / 2l_1^2 k_1^2 \\ & + (7\omega_1^2 + 2) \in \gamma / 2l_1^2 k_1^2 + (7\omega_1^2 + 2) \epsilon' / 2l_1^2 k_1^2 \\ & - (3\omega_1^2 - 7) \epsilon' \gamma / 2l_1^2 k_1^2] / l_1 k_1 \\ J_{2,3} = & \sqrt{3} [6\gamma + (-16\omega_1^4 + 88\omega_1^2 - 63) A_1 / 6k_1^2 \\ & + (104\omega_1^2 + 135) A_1\gamma / 2l_1^2 \\ & - (-16\omega_1^4 + 88\omega_1^2 - 63) A_2 / 6k_1^2 \\ & + (104\omega_1^2 + 135) A_2\gamma / 2l_1^2 + (112\omega_1^6 - 136\omega_1^4 \\ & - 22l\omega_1^2 - 189) \in / 9k_1^2 l_1^2 \\ & - (24\omega_1^4 + 2l\omega_1^2 - 33) \in \gamma / l_1^2 k_1^2 - (112\omega_1^6 + 136\omega_1^4 \\ & - 22l\omega_1^2 - 189) \epsilon' / 9k_1^2 l_1^2 \\ & - (24\omega_1^4 + 2l\omega_1^2 - 33) \epsilon' \gamma / l_1^2 k_1^2], \end{aligned}$$

with

$$l_j^2 = \omega_1^2 + 9(J = 1, 2) \quad k_1^2 = 2\omega_1^2 - 1, \quad k_2^2 = 1 - 2\omega_2^2.$$

The values of $J_{1,4} J_{2,2} J_{2,4}$ can be obtained respectively from $J_{1,3}, -J_{2,1}, J_{2,3}$ by replacing ω_1 by ω_2 , l_1 by l_2 , k_1 by k_2 , k_1^2 by $-k_2^2$ wherever they occur keeping $A_1, A_2, \in, \epsilon'$ unchanged.

APPENDIX B

$$\begin{aligned} \mathbf{r}_i = & r_{i,1}\gamma + r_{i,2}A_1 + r_{i,3}A_1\gamma + r_{i,4}A_2 + r_{i,5}A_2\gamma + r_{i,6} \in \\ & + r_{i,7} \in \gamma + r_{i,8} \epsilon' + r_{i,9} \epsilon' \gamma \\ \mathbf{r}_j = & r_{j,1} + r_{j,2}A_1 + r_{j,3}A_1\gamma + r_{j,4}A_2 + r_{j,5}A_2\gamma + r_{j,6} \in \\ & + r_{j,7} \in \gamma + r_{j,8} \epsilon' + r_{j,9} \epsilon' \gamma, \\ \mathbf{s}_i = & s_{i,1} + s_{i,2}A_1 + s_{i,3}A_1\gamma + s_{i,4}A_2 + s_{i,5}A_2\gamma + s_{i,6} \in \\ & + s_{i,7} \in \gamma + s_{i,8} \epsilon' + s_{i,9} \epsilon' \gamma, \\ \mathbf{s}_j = & s_{j,1}\gamma + s_{j,2}A_1 + s_{j,3}A_1\gamma + s_{j,4}A_2 + s_{j,5}A_2\gamma + s_{j,6} \in \\ & + s_{j,7} \in \gamma + s_{j,8} \epsilon' + s_{j,9} \epsilon' \gamma, \end{aligned}$$

for $i = 1, 2, 3, 4, 7, 8$ and $j = 5, 6, 9, 10$.

The values of all $r_{i,1}, r_{j,1}, r_{i,2}, r_{j,2}, r_{i,3}, r_{j,3}, s_{i,1}, s_{j,1}, s_{i,2}, s_{j,2}, s_{i,3}, s_{j,3}$ are the same as those in the paper by Subba Rao &

Sharma (1997):

$$\begin{aligned}
\mathbf{r}_{1,4} &= [-958464\omega_1^{10} - 1709568\omega_1^8 + 14119744\omega_1^6 \\
&\quad + 15690052\omega_1^4 - 9561825\omega_1^2 \\
&\quad + 3082941]/27648\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2) \\
\mathbf{r}_{1,5} &= [-612352\omega_1^8 - 780840\omega_1^6 - 25430134\omega_1^4 \\
&\quad + 2831380\omega_1^2 + 1700271]/1024\omega_1^3 I_1^4 k_1^2 (1 - \omega_1^2) \\
\mathbf{r}_{1,6} &= [200704\omega_1^{12} - 2838528\omega_1^{10} - 9627904\omega_1^8 \\
&\quad + 4520136\omega_1^6 - 5069434\omega_1^4 - 2258551\omega_1^2 \\
&\quad + 9372528]/13824\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{r}_{1,7} &= -[12672\omega_1^{10} + 29520\omega_1^8 + 32568\omega_1^6 + 95792\omega_1^4 \\
&\quad + 168210\omega_1^2 - 130491]/64\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2) \\
\mathbf{r}_{1,8} &= [1024\omega_1^{12} + 1964312\omega_1^{10} - 391680\omega_1^8 \\
&\quad - 5436512\omega_1^6 + 2663658\omega_1^4 - 3674889\omega_1^2 \\
&\quad + 256962]/6912\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2) \\
\mathbf{r}_{1,9} &= [8448\omega_1^{10} - 79008\omega_1^8 - 65455\omega_1^6 - 186064\omega_1^4 \\
&\quad - 89964\omega_1^2 + 88938]/128\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{r}_{3,4} &= [20353\omega_1^{10} - 45342\omega_1^8 + 52839\omega_1^6 - 73059\omega_1^4 \\
&\quad + 25394\omega_1^2 - 1145]/(288\omega_1 I_1^2 k_1^4 z_1^2), \\
\mathbf{r}_{3,5} &= [1755\omega_1^6 - 5809\omega_1^4 - 98766\omega_1^2 - 2544]/ \\
&\quad (16\omega_1 I_1^4 k_1^2 z_1), \\
\mathbf{r}_{3,6} &= [16535\omega_1^{12} - 203943\omega_1^{10} + 17528\omega_1^8 + 23197\omega_1^6 \\
&\quad - 889120\omega_1^4 - 336119\omega_1^2 + 786267]/(144\omega_1^4 I_1^4 k_1^4 z_1^2), \\
\mathbf{r}_{3,8} &= r_{3,6}, \mathbf{r}_{3,9} = -r_{3,7} \\
\mathbf{r}_{5,4} &= \sqrt{3}[2890240\omega_1^{12} + 589184\omega_1^{10} + 1462400\omega_1^8 \\
&\quad - 5521120\omega_1^6 + 2246062\omega_1^4 - 680835\omega_1^2 + 105705]/ \\
&\quad (864\omega_1^2 I_1^4 k_1^4 z_1^2), \\
\mathbf{r}_{5,5} &= \sqrt{3}[1800\omega_1^8 - 4496\omega_1^6 - 2731\omega_1^4 \\
&\quad + 1890\omega_1^2 - 243]/(6\omega_1^2 I_1^2 k_1^4 z_1^2), \\
\mathbf{r}_{5,6} &= -\sqrt{3}[-56320\omega_1^{14} - 576128\omega_1^{12} + 620624\omega_1^{10} \\
&\quad - 17304\omega_1^8 - 1638348\omega_1^6 + 1669932\omega_1^4 - 902583\omega_1^2 \\
&\quad + 124659]/(288\omega_1^4 I_1^4 k_1^4 z_1^2), \\
\mathbf{r}_{5,7} &= -\sqrt{3}[-160896\omega_1^{12} + 8192\omega_1^{10} - 271752\omega_1^8 \\
&\quad - 531644\omega_1^6 + 7594492\omega_1^4 - 664848\omega_1^2 + 104976]/ \\
&\quad (288\omega_1^4 I_1^4 k_1^4 z_1^2), \\
\mathbf{r}_{5,8} &= r_{5,6} \quad \mathbf{r}_{5,9} = -r_{5,7} \\
\mathbf{r}_{7,4} &= -[156u^5 - 536325u^4 - 81902u^3 \\
&\quad + 15809u^2 - 34119u - 16140]/8uz_3(5u - 2)^2 \\
\mathbf{r}_{7,5} &= 3[6156u^4 + 42688u^3 + 24721u^2 \\
&\quad + 145069u + 27438]/8z_3(5u - 2)(16u^2 + 117), \\
\mathbf{r}_{7,6} &= [184u^5 + 36964u^4 + 83154u^3 - 20520u^2 \\
&\quad + 48117u + 16560]/96uz_3(5u - 2)^2, \\
\mathbf{r}_{7,7} &= 3[1296u^4 + 52996u^3 + 72849u^2 + 168072u \\
&\quad + 53848]/64z_3(5u - 2)(16u^2 + 117), \\
\mathbf{r}_{7,8} &= r_{7,6}, \mathbf{r}_{7,9} = -r_{7,7} \\
\mathbf{r}_{9,4} &= \sqrt{3}[32774u^4 - 3568u^3 - 24u^2 + 36882u \\
&\quad + 85368]\sqrt{(1 - 2u)/12z_3(5u - 2)(16u^2 + 117)} \\
\mathbf{r}_{9,5} &= 3\sqrt{3}[1284u^4 + 4055u^3 + 1650u^2 \\
&\quad - 1924u + 27]/16uz_3(5u - 2)2\sqrt{(1 - 2u)}, \\
\mathbf{r}_{9,6} &= \sqrt{3}[1012u^4 - 5468u^3 - 84u^2 + 53972u \\
&\quad + 48956]\sqrt{(1 - 2u)/8z_3(5u - 2)(16u^2 + 117)}, \\
\mathbf{r}_{9,7} &= 3\sqrt{3}[5462u^4 + 6412u^3 + 2436u^2 \\
&\quad - 184u + 81]/8uz_3(5u - 2)2\sqrt{(1 - 2u)} \\
\mathbf{r}_{9,8} &= r_{9,6}, \mathbf{r}_{9,9} = -r_{9,7} \\
\mathbf{s}_{1,4} &= \sqrt{3}[853234\omega_1^{10} + 313266\omega_1^8 + 11536480\omega_1^6 \\
&\quad + 20235825\omega_1^4 - 9284569\omega_1^2 + 1032645]/ \\
&\quad 6912\omega_1^3 I_1^2 k_1^4 (1 - \omega_1^2), \\
\mathbf{s}_{1,5} &= \sqrt{3}[324860\omega_1^8 - 12158552\omega_1^6 + 13235261\omega_1^4 \\
&\quad - 2413835\omega_1^2 - 2300032]/(2048\omega_1^3 I_1^4 k_1^2 (1 - \omega_1^2)), \\
\mathbf{s}_{1,6} &= \sqrt{3}[100802\omega_1^{12} + 1535328\omega_1^{10} - 322955458\omega_1^8 \\
&\quad + 19231146\omega_1^6 - 8256628\omega_1^4 + 3381658\omega_1^2 \\
&\quad - 8727857]/6912\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{s}_{1,7} &= \sqrt{3}[24686\omega_1^{10} - 92130\omega_1^8 + 65323\omega_1^6 - 59944\omega_1^4 \\
&\quad - 452820\omega_1^2 + 222465]/32\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{s}_{1,8} &= \sqrt{3}[1256\omega_1^{12} + 8553678\omega_1^{10} - 134730\omega_1^8 + 3162025\omega_1^6 \\
&\quad - 865856\omega_1^4 + 7489056\omega_1^2 - 852967]/ \\
&\quad 3456\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{s}_{1,9} &= \sqrt{3}[5616\omega_1^{10} + 82809\omega_1^8 - 73525\omega_1^6 + 825292\omega_1^4 \\
&\quad + 16568 - 63598]/256\omega_1^3 I_1^4 k_1^4 (1 - \omega_1^2), \\
\mathbf{s}_{3,4} &= \sqrt{3}[16881\omega_1^{10} - 64001\omega_1^8 + 632153\omega_1^6 \\
&\quad + 16593\omega_1^4 - 11657\omega_1^2 + 6352]/(288\omega_1 I_1^2 k_1^4 z_1^2), \\
\mathbf{s}_{3,5} &= \sqrt{3}[1656\omega_1^6 + 5701\omega_1^4 - 23459\omega_1^2 + 2862]/ \\
&\quad (32\omega_1 I_1^4 k_1^2 z_1) \\
\mathbf{s}_{3,6} &= \sqrt{3}[24954\omega_1^{12} - 669953\omega_1^{10} + 63527\omega_1^8 + 32286\omega_1^6 \\
&\quad - 465130\omega_1^4 + 118316\omega_1^2 - 956210]/(288\omega_1^4 I_1^4 k_1^4 z_1^2), \\
\mathbf{s}_{3,7} &= \sqrt{3}[-502810\omega_1^{10} + 115652\omega_1^8 - 15418\omega_1^6 + 468880\omega_1^4 \\
&\quad - 365710\omega_1^2 + 325795]/(144\omega_1^4 I_1^4 k_1^4 z_1^2), \\
\mathbf{s}_{3,8} &= s_{3,6}, \mathbf{s}_{3,9} = -s_{3,7} \\
\mathbf{s}_{5,4} &= (1658\omega_1^8 + 2853\omega_1^6 + 5258\omega_1^4 \\
&\quad - 1628\omega_1^2 + 1768)/(54I_1^2 k_1^4 z_1^2), \\
\mathbf{s}_{5,5} &= (2816\omega_1^6 + 3639\omega_1^4 - 1159\omega_1^2 + 5308)/(8I_1^4 k_1^2 z_1), \\
\mathbf{s}_{5,6} &= (8752\omega_1^8 + 1153\omega_1^6 - 8816\omega_1^4 \\
&\quad + 5353\omega_1^2 + 1844)/(12I_1^2 k_1^2 z_1^2), \\
\mathbf{s}_{5,7} &= (6751\omega_1^6 - 8559\omega_1^4 + 6353\omega_1^2 - 1008)/(8I_1^4 k_1^2 z_1), \\
\mathbf{s}_{5,8} &= s_{5,6}, \mathbf{s}_{5,9} = -s_{5,7} \\
\mathbf{s}_{7,4} &= -\sqrt{3}(1180u^5 - 45081u^4 + 15932u^3 + 283362u^2 \\
&\quad + 148157u + 638264)/72(5u - 2)(16u^2 + 117)z_3 \\
\mathbf{s}_{7,5} &= -\sqrt{3}(84u^4 + 8212u^3 - 1654u^2 \\
&\quad + 1553u - 3372)/16u(54 - 2)^2 z_3
\end{aligned}$$

$$\begin{aligned}
s_{7,6} &= -\sqrt{3}(114u^5 - 39368u^4 + 80456u^3 - 10820u^2 \\
&\quad + 34168u - 33582)/36(5u - 2)(16u^2 + 117)z_3 \\
s_{7,7} &= -\sqrt{3}(3496u^4 - 82756u^3 - 92346u^2 \\
&\quad + 148052u + 91637)/64(5u - 2)^2z_3 \\
s_{9,4} &= (20278u^5 + 16643u^4 - 30128u^3 + 14286u^2 \\
&\quad + 4802u + 4563)/48u(5u - 2)^2z_3 \\
s_{9,5} &= 3(3641u^3 + 1814u^2 + 5263u + 7209)/ \\
&\quad 8(5u - 2)(16u^2 + 117)z_3, \\
s_{9,6} &= (72082u^5 + 68809u^4 - 23756u^3 - 28270u^2 \\
&\quad + 32883u - 3609)/36u(5u - 2)^2z_3\sqrt{(1 - 2u)}, \\
s_{9,7} &= 3(3786u^3 + 3606u^2 + 3949u - 8008)\sqrt{(1 - 2u)}/ \\
&\quad 8(5u - 2)(16u^2 + 117)z_3, \\
s_{9,8} &= s_{9,6}, s_{9,9} = -s_{9,7} \\
z_1 &= 1 - 5\omega_1^2, z_2 = 1 - 5\omega_2^2 \\
z_3 &= l_1l_2k_1k_2\sqrt{u}, u = \omega_1\omega_2.
\end{aligned}$$

The values of r_k, s_k , for $k = 2, 4, 6$ can be obtained respectively from those for $k = 1, 3, 5$ by replacing ω_1 by $-\omega_2$, l_1 by $l_2k_1^2$ by $-k_2^2z_1$ by z_2 wherever they occur and the values of r_k, s_k for $k = 8, 10$ can be obtained respectively from those for $k = 7, 9$ by replacing u by $-u$, keeping z_3 unchanged, wherever they occur.

APPENDIX C

The values of all $A_{1,1}, A_{1,2}, A_{1,3}, B_{1,1}, B_{1,2}, B_{1,3}, C_{1,1}, C_{1,2}$, and $C_{1,3}$ are the same as those in the paper by Subba Rao & Sharma (1997):

$$\begin{aligned}
A_{1,4} &= (1319 - 12639\omega_1^2 + 18275\omega_1^4 - 1598\omega_1^6)/ \\
&\quad 436(1 - 2\omega_1^2)^2(1 - 5\omega_1^2), \\
A_{1,5} &= (57 + 525\omega_1^2 - 950\omega_1^4 + 3118\omega_1^6 + 1132\omega_1^8)/ \\
&\quad 52(1 - 2\omega_1^2)^3(1 - 5\omega_1^2) \\
A_{1,6} &= (24 - 624\omega_1^2 + 416\omega_1^4 - 2004\omega_1^6 + 856\omega_1^8)/ \\
&\quad 864(1 - 2\omega_1^2)^2(1 - 5\omega_1^2) \\
A_{1,7} &= (32 - 842\omega_1^2 - 608\omega_1^4 - 1628\omega_1^6 + 1856\omega_1^8)/ \\
&\quad 64(1 - 2\omega_1^2)^3(1 - 5\omega_1^2)^2 \\
A_{1,8} &= (72 - 884\omega_1^2 - 204\omega_1^4 + 1600\omega_1^6 + 648\omega_1^8)/ \\
&\quad 872(1 - 2\omega_1^2)^2(1 - 5\omega_1^2) \\
A_{1,9} &= (36 - 432\omega_1^2 + 448\omega_1^4 + 1208\omega_1^6 + 896\omega_1^8)/ \\
&\quad 64(1 - 2\omega_1^2)^3(1 - 5\omega_1^2)^2, \\
B_{1,4} &= \omega_1\omega_2(2223 - 6817\omega_1^2\omega_2^2)/36(1 - 2\omega_1^2) \\
&\quad \times (1 - 2\omega_2^2)(1 - 5\omega_1^2)(1 - 5\omega_2^2)
\end{aligned}$$

$$\begin{aligned}
B_{1,5} &= (89211 - 2042998\omega_1^2\omega_2^2 + 10285776\omega_1^4\omega_2^4 \\
&\quad - 16052098\omega_1^6\omega_2^6 - 1215804\omega_1^8\omega_2^8)/32\omega_1\omega_2(4\omega_1^2 + 9) \\
&\quad \times (4\omega_2^2 + 9)(1 - 2\omega_1^2)(1 - 2\omega_2^2)(1 - 5\omega_1^2)^2(1 - 5\omega_2^2)^2 \\
B_{1,6} &= \omega_1\omega_2(1854 - 3256\omega_1^2\omega_2^2)/48(1 - 2\omega_1^2) \\
&\quad \times (1 - 2\omega_2^2)(1 - 5\omega_1^2)(1 - 5\omega_2^2) \\
B_{1,7} &= (40332 - 2456280\omega_1^2\omega_2^2 + 1603200\omega_1^4\omega_2^4 \\
&\quad - 12140864\omega_1^6\omega_2^6 - 1043208\omega_1^8\omega_2^8)/48\omega_1\omega_2(4\omega_1^2 + 9) \\
&\quad \times (4\omega_2^2 + 9)(1 - 2\omega_1^2)(1 - 2\omega_2^2)(1 - 5\omega_1^2)^2(1 - 5\omega_2^2)^2 \\
B_{1,8} &= \omega_1\omega_2(1264 - 4868\omega_1^2\omega_2^2)/48(1 - 2\omega_1^2) \\
&\quad \times (1 - 2\omega_2^2)(1 - 5\omega_1^2)(1 - 5\omega_2^2) \\
B_{1,9} &= (16004 + 1284430\omega_1^2\omega_2^2 - 1216028\omega_1^4\omega_2^4 \\
&\quad - 81442042\omega_1^6\omega_2^6 - 8034956\omega_1^8\omega_2^8)/48\omega_1\omega_2(4\omega_1^2 + 9) \\
&\quad \times (4\omega_2^2 + 9)(1 - 2\omega_1^2)(1 - 2\omega_2^2)(1 - 5\omega_1^2)^2(1 - 5\omega_2^2)^2.
\end{aligned}$$

$C_{1,i}$ ($i = 1, 2, 3, \dots, 9$) can be obtained respectively from $A_{1,i}$ by replacing ω_1 by ω_2 .

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