

AN IMPROVED SEMIANALYTICAL SPHERICAL COLLAPSE MODEL FOR NONLINEAR DENSITY EVOLUTION

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ABSTRACT

We derive a semianalytical extension of the spherical collapse model of structure formation that takes account of the effects of deviations from spherical symmetry and shell crossing, which are important in the nonlinear regime. Our model is designed so that it predicts a relation between the peculiar velocity and density contrast that agrees with the results of N -body simulations in the region where such a comparison can sensibly be made. Prior to turnaround, when the unmodified spherical collapse model is expected to be a good approximation, the predictions of the two models coincide almost exactly. The effects of a late time dominating dark energy component are also taken into account. The improved spherical collapse model is a useful tool when one requires a good approximation not just to the evolution of the density contrast but also its trajectory. Moreover, the analytical fitting formulae presented is simple enough to be used anywhere where the standard spherical collapse might be used but with the advantage that it includes a realistic model of the effects of virialisation.

Subject headings: cosmology: miscellaneous — cosmology: theory — galaxies: general — relativity

Online material: color figures

1. INTRODUCTION

The spherical collapse model (SCM) developed by Gunn & Gott (1972) is perhaps the simplest model for the evolution of nonlinear structure, and yet it has been shown to be remarkably successful when correctly interpreted. However, despite the SCM's many success, it is ultimately flawed since it predicts that any overdensity of matter collapses to a singularity in a finite time. In addition, making the assumption of spherical symmetry, while simplifying, means abandoning the many interesting and important aspects of structure formation that result from deviations from spherical symmetry. Indeed, these deviations play a crucial role in ultimately halting the collapse of the overdensity and the formation of virialized structures.

The usual approach to virialisation in the SCM is to put it in *by hand*; the collapse is simply halted once the virial radius has been reached. This procedure, within an Einstein–de Sitter universe, leads to the result that bound structures are formed when the nonlinear overdensity is about 178 or, equivalently, the linear overdensity is approximately 1.68.³ Despite the fact that these figures are not too far away from what is actually observed in N -body simulations, the ad hoc nature of this approach means the SCM cannot be used to predict the precise manner in which the overdensity evolves; moreover, the SCM's prediction for the peculiar velocity becomes virtually useless shortly after turnaround.

Engineer et al. (2000) proposed a different and better motivated way in which the SCM could be extended to include virialisation. Their idea was to alter the standard SCM evolution equation for the mean density contrast, $\bar{\delta}$, in such a way that stable structures would form. Their modified evolution equation was constructed by adding a Taylor series in $1/\bar{\delta}$ to the standard equation: the idea being that these additional terms would encode all the effects due

to shell-crossing and deviations from spherical symmetry that occur for large $\bar{\delta}$. The coefficients of the terms in Taylor series were chosen so as to provide a good approximation to the statistical density-contrast found from N -body simulations. While their improved SCM was found to agree fairly well with data from N -body simulations for $\bar{\delta} \gtrsim 15$, their solution for $\bar{\delta}$ is inaccurate in the linear regime, where spherical symmetry and hence also the unmodified SCM are expected to be good approximations.

In this article we take a similar approach to Engineer et al. (2000) and add terms to the standard SCM evolution equation so that for large $\bar{\delta}$ the evolution of the density contrast and the peculiar velocity agrees with the data from N -body simulations. Importantly, however, the analytical solutions that we find remain valid as $\bar{\delta} \rightarrow 0$ and agree almost exactly with the unmodified SCM prior to turnaround.

2. MODELING NONLINEAR STRUCTURES

We consider a spherical overdensity embedded in a background universe described by a Friedman–Robertson–Walker (FRW) metric. In a spherically symmetric system and in the absence of shell-crossing, the mass, M , inside each comoving spherical shell, with physical radius $R(t)$, remains constant. The mean density contrast inside shell therefore scales as

$$1 + \bar{\delta} = \lambda \frac{a^3(t)}{R(t)^3}, \quad (1)$$

where $a(t)$ is the scale factor of the background FRW universe and λ is constant on each shell. Equation (1) can also be used to define $R(t)$ for shells of constant M when deviations from spherical symmetry occur. In these cases, however, the physical meaning of $R(t)$ is less clear. Following Engineer et al. (2000) we continue to think of $R(t)$, as defined by equation (1), as the effective “radius” of each shell of constant M .

The peculiar velocity, h_{SC} , is defined by

$$h_{\text{SC}} \equiv \frac{1}{3} \frac{d \ln(1 + \bar{\delta})}{d \ln a} = \left(1 - \frac{\dot{R}}{HR}\right), \quad (2)$$

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³ In dark energy scenarios these values change slightly depending on the model (Percival 2005; Debnath et al. 2006; Bagla et al. 2005; Weinberg & Kamionkowski 2003).

where $H = \dot{a}/a$ is the Hubble parameter. In a matter-dominated universe the unmodified SCM predicts $h_{SC} = h_{SC}(\bar{\delta})$. Our key assumption in what follows is that this remains the case even when deviations from spherical symmetry are taken into account. This assumption is supported by numerical studies of structure formation such as that conducted by Hamilton et al. (1991), which we discuss below.

For the moment we take the background universe to be matter dominated; however, in § 5 we generalize our results to nonlinear overdensities in more realistic universes, which have recently transitioned to an epoch of dark energy domination. We define a new coordinate $\eta(\bar{\delta})$ by

$$1 + \bar{\delta} = \frac{9}{2} \frac{T(\eta)^2}{(1 - \cos \eta)^3}, \quad (3)$$

for some function $T(\eta)$. Now M is constant on each shell and so, taking $t = t_i$, $R = R_i$, and $\bar{\delta} = \delta_i$ at some instance, we have

$$M = \frac{2R_i^3(1 + \delta_i)}{9t_i^2} = \frac{2R^3(1 + \bar{\delta})}{9t^2}.$$

By considering this relation together with equation (3) we see that

$$R = \frac{R_i(1 + \delta_i)}{2\delta_i j(\eta, R_i)} [1 - \cos(\eta)],$$

$$t = \frac{3t_i(1 + \delta_i)}{4\delta_i^{3/2} j^{3/2}(\eta, R_i)} T(\eta),$$

where $j(\eta, \delta_i)$ is some function of η and δ_i . If $h_{SC} = h_{SC}(\bar{\delta})$, then $j(\eta, R_i) = j(\eta)$, and without loss of generality we can set $j(\eta) = 1$. This gives

$$h_{SC} = 1 - \sqrt{\frac{1 + \bar{\delta}}{2}} \frac{\sin \eta}{(1 - \cos \eta)^{1/2}} \frac{d\tau}{Td}, \quad (4)$$

where $\tau(\eta) = \eta - \sin \eta$, and

$$R = R(\eta) = \frac{R_i(1 + \delta_i)}{2\delta_i} (1 - \cos \eta), \quad (5)$$

$$t = t(\eta) = \frac{3t_i(1 + \delta_i)}{4\delta_i^{3/2}} T(\eta). \quad (6)$$

We reduce to the standard SCM in the limit where $T = \tau(\eta) = \eta - \sin \eta$ (Padmanabhan 2002), and it is clear that $h_{SC} = h_{SC}(\bar{\delta})$ in the unmodified SCM. It is important to stress that if $h_{SC} = h_{SC}(\bar{\delta})$, then equations (5) and (6) are the most general solutions for R and t .

The radial acceleration of each shell is found to be

$$\frac{d^2 R}{dt^2} = -\frac{GM}{R^2} \left[T'^{-2} + \sin \eta (1 - \cos \eta) \frac{T''}{T'^3} \right], \quad (7)$$

where $T' = dT/d\tau$ and $T'' = d^2 T/d\tau^2$. As should be expected, when $T' = 1$, we reduce to the standard SCM equation for R_{sc} .

When deviations from spherical symmetry and the leading order effects of a gradient in the velocity dispersion are taken into account, Engineer et al. (2000) showed that one should have

$$\begin{aligned} \frac{d^2 R}{dt^2} &= -\frac{GM}{R^2} - \frac{H^2 R}{3} S \\ &= -\frac{GM}{R^2} \left[1 + \frac{2}{3(1 + \bar{\delta})} S \right], \end{aligned} \quad (8)$$

where $S(a, x) = a^2(\sigma^2 - 2\Omega^2) + f(a, x)$; σ^2 and Ω^2 , respectively, quantify the shear and rotation of the fluid; $f(a, x)$ contains the lowest order contribution from velocity dispersion terms. Importantly, no matter what form $S(a, x)$ takes, if, as we have assumed $h_{SC} = h_{SC}(\bar{\delta})$, then we must have $S = S(\bar{\delta})$ and by comparing equations (7) and (8) we can clearly see that

$$S(\bar{\delta}) = \frac{3(1 + \bar{\delta})}{2} \left[T'^{-2} - 1 + \sin \eta (1 - \cos \eta) \frac{T''}{T'^3} \right].$$

In principle, the form of both $S(\bar{\delta})$ and hence $T(\eta)$ can be found using the results of N -body simulations. Unfortunately, however, making the required comparison with simulations is not as straightforward as one might expect it to be. This is because the results of such simulations are given in terms of the statistical properties of the matter distribution rather than in terms of the mean density contrast, $\bar{\delta}$, and peculiar velocity, h_{SC} . The statistical properties in question are the averaged two-point correlation function, $\bar{\xi}$, and the averaged pair velocity, $h(a, x)$, which are given as defined by

$$\bar{\xi} = \frac{3}{r^3} \int_0^r \xi(x, a) x^2 dx, \quad h(a, x) = -\frac{\langle v(a, x) \rangle}{\dot{a}x}, \quad (9)$$

where ξ is the two-point correlation function and is defined to be the Fourier transform of the power spectrum, $P(k)$. The assumption that $h(a, x)$ depends on a and x only through ξ , i.e., $h(a, x) = h[\xi(a, x)]$ is common in the literature (see Engineer et al. 2000; Jain et al. 1995), and it appears to have been confirmed by numerical simulations (see Hamilton et al. 1991; Peacock & Dodds 1996).

The results of, for example, Hamilton et al. (1991) can be used to construct the fitting formula for $h(\xi)$ (Engineer et al. 2000). However, before we can make use of such a formula, we must relate the statistical quantities $\bar{\xi}$ and $h(\bar{\xi})$ to $\bar{\delta}$ and $h_{SC}(\bar{\delta})$. It is well known that (Padmanabhan 1996, 2002; Peebles 1980; Padmanabhan & Engineer 1998; Del Popolo et al. 2001; Padmanabhan & Ray 2006), on scales smaller than the size of the collapsing objects and around high density peaks,

$$\rho \simeq \rho_b(1 + \bar{\xi}). \quad (10)$$

It follows that, in the nonlinear regime, we have $\bar{\delta} \approx \bar{\xi}$. This relationship between $\bar{\delta}$ and $\bar{\xi}$ was also used by Engineer et al. (2000), although it was, as it is here, the weakest part of the whole analysis. We only require that $\bar{\delta} \approx \bar{\xi}$ hold where it is expected to be a good approximation i.e., $\bar{\delta} \gtrsim 15$. As $\bar{\delta} \rightarrow \infty$ we assume that $\bar{\delta} \sim \bar{\xi}$. Peebles (1980) showed that $h(\bar{\xi})$ satisfies

$$h = \frac{1}{3} \frac{1}{1 + \bar{\xi}} \frac{d\bar{\xi}}{d \ln a}; \quad (11)$$

thus, if $\bar{\xi} \approx \bar{\delta}$, then by comparing equations (2) and (11), we see that $h \approx h_{SC}$. It must be stressed that this second relation between h_{SC} and h is only valid if $\bar{\xi} \approx \bar{\delta}$; however, whenever it does hold it implies that h_{SC} is given by a function of $\bar{\delta}$ alone, i.e., $h_{SC} = h_{SC}(\bar{\delta})$.

The assumption that $\bar{\xi} \approx \bar{\delta}$ breaks down for small $\bar{\delta}$. Fortunately, when $\bar{\delta}$ is small, the unmodified SCM provides an accurate model. For times $t \gg t_i$, the unmodified SCM predicts that $h_{SC} = h_{SC}(\bar{\delta})$. Our key assumption that $h_{SC} = h_{SC}(\bar{\delta})$ is therefore expected to hold for almost all $\bar{\delta}$. The assumption does break down for $t \sim t_i$; however, this is entirely due to the decaying mode in $\bar{\delta}$, which is negligible for $t \gg t_i$.

3. CONSTRUCTING AN IMPROVED SCM

If $h_{SC} = h_{SC}(\bar{\delta})$, then all properties of a modified SCM are encoded in a single function $T(\eta)$. Our aim is to combine the unmodified SCM and data from N -body simulations to find a fitting formula for $T(\eta)$ that results in accurate predictions for $h_{SC}(\bar{\delta})$ in all regimes.

Prior to turnaround we expect the standard SCM to be accurate and hence $T(\eta) \approx \eta - \sin \eta$. Furthermore, for the unmodified SCM to be accurate, at leading order, in the linear regime we must have $T(\eta) \sim \eta - \sin \eta + o(\eta^5)$ for small η . When $\bar{\delta} \gtrsim 15$, we expect $h_{SC}(\bar{\delta}) \approx h(\xi)$ and $\xi \approx \bar{\delta}$, and we may use $h(\xi)$ to extract the large $\bar{\delta}$ form of $T(\eta)$. We describe how this is done below. In a matter-dominated universe, the linearly extrapolated mean two-point correlation function, ξ_{lin} , scales as $\xi_{lin} \propto a^2 \propto T^{4/3}$, therefore as $\bar{\delta} \rightarrow \infty$,

$$(1 + \bar{\xi}) \approx (1 + \bar{\delta}) = \frac{9}{2} \frac{T(\eta)^2}{[1 - \cos(\eta)]^3}, \quad (12)$$

and so

$$(1 - \cos \eta) \approx (1 - \cos \eta_{sim}) \equiv \frac{A \bar{\xi}_{lin}^{1/2}}{(1 + \bar{\xi})^{1/3}}, \quad (13)$$

where A is a constant and we treat it as a parameter to be fitted. Hamilton et al. (1991) found the following fitting formula for $\bar{\xi}_{lin}(\bar{\xi})$:

$$\bar{\xi}_{lin} = \bar{\xi} \left(\frac{1 + 0.0158 \bar{\xi}^2 + 0.000115 \bar{\xi}^3}{1 + 0.926 \bar{\xi}^2 - 0.0743 \bar{\xi}^3 + 0.0156 \bar{\xi}^4} \right)^{1/3}. \quad (14)$$

We define $\eta_\infty = \lim_{\bar{\delta} \rightarrow \infty} \eta$. By taking $\bar{\xi} \rightarrow \infty$ we find that

$$A = 2.2668(1 - \cos \eta_\infty) = 2.2668 \left(\frac{2R_{vir}}{R_{ta}} \right), \quad (15)$$

where we have used $\cos \eta_\infty = 1 - 2R_{vir}/R_{ta}$, which follows from equation (5); R_{vir} is the radius of the shell at virialisation, and R_{ta} is its radius at turnaround. If spherical symmetry is assumed, then the virial theorem in an Einstein–de Sitter universe gives $R_{vir} = R_{ta}/2$. This relation is generally used when virialisation is placed by hand into the SCM. For comparison, Hamilton et al. (1991) found that $R_{ta}/R_{vir} \approx 1.8$ from their simulations. We treat R_{vir}/R_{ta} as a fitting parameter. Equation (13) provides $\eta = \eta(\bar{\xi} \approx \bar{\delta})$, and $T(\eta)$ may now be found using equation (12):

$$T[\eta(\bar{\xi})] \approx T_{sim} \equiv \left(\frac{2}{9} \right)^{1/2} A^{3/2} \bar{\xi}_{lin}^{3/4}(\bar{\xi}), \quad (16)$$

with $\tau \approx \tau_{sim} = \eta_{sim} - \sin \eta_{sim}$; η_{sim} is defined by equation (13).

We fit for the parameter A (or equivalently, R_{ta}/R_{vir}) by considering some important physical constraints on the behavior of $T(\eta)$. Since the effect of deviations from spherical symmetry is to slow down the collapse of the overdensity, it follows from equation (7) that

$$T'^2 - \sin \eta (1 - \cos \eta) \frac{T''}{T'} \geq 1. \quad (17)$$

Furthermore, the unaltered SCM should be a good approximation up to around turnaround (when $\eta = \pi$) and so we must have

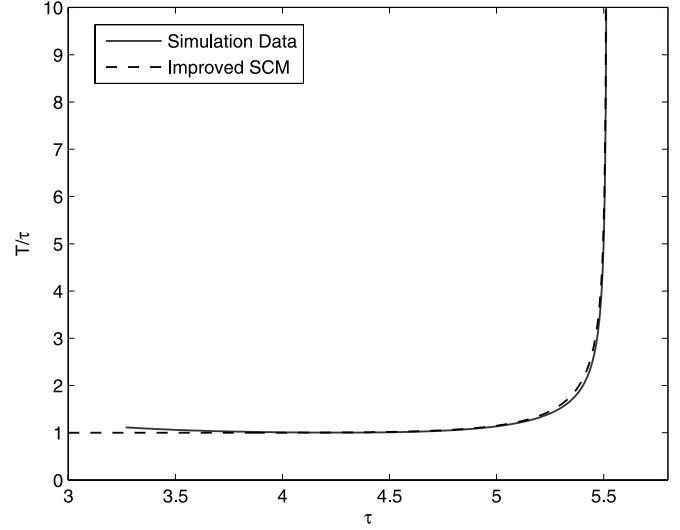


FIG. 1.—Plot of T/τ vs. τ . The solid line is $T/\tau = T_{sim}/\tau_{sim}$ against $\tau = \tau_{sim}$, whereas the dashed line shows $T(\tau)/\tau$ in our improved SCM and uses the fitting formula for $T(\tau)$, eq. (18), to evaluate T/τ . As required, we see that in the region $\tau \gtrsim 4.4 \Rightarrow \bar{\delta} \gtrsim 15$, our improved SCM is a very good approximation to the simulation data. For small values of τ , we see that $T/\tau \rightarrow 1$, as is required, in our improved SCM. [See the electronic edition of the Supplement for a color version of this figure.]

that $T' \approx 1$, $T'' \ll T'$ for $\eta \lesssim \pi$. As $\tau \rightarrow \eta_\infty - \sin \eta_\infty$, we must have $T \rightarrow \infty$ and so $T' > 0$ as $\eta \rightarrow \eta_\infty$. Equation (17) implies that, for $\eta > \pi$, we cannot have $0 < T' < 1$ and $T'' < 0$, and so it follows that $T' \geq 1$ for $\eta > \pi$. It follows that $T \geq \tau$ everywhere. This condition implies that we should choose R_{vir}/R_{ta} , and hence A , so that T_{sim} is always greater than $\tau_{sim} \equiv \eta_{sim} - \sin \eta_{sim}$ for $\bar{\xi} \gtrsim 15$, which roughly corresponds to $\eta_{sim} \gtrsim 3.8$, $\tau_{sim} \gtrsim 4.4$. Moreover, since we want $T \rightarrow \tau$ as $\eta \rightarrow 0$, we choose R_{vir}/R_{ta} so that, at the minimum of T_{sim} , $T_{sim} = \tau_{sim}$. This requirement gives

$$\frac{R_{vir}}{R_{ta}} = 0.5896.$$

We now use T_{sim} to find a fitting formula for $T(\eta)$ that agrees with the unmodified SCM at early times (i.e., $T \sim \eta - \sin \eta$ for small η).

4. RESULTS

We find that the fitting formula,

$$T[\eta(\tau)] = \tau + \frac{3.468(\tau_f - \tau)^{-1/2} \exp\{-[15(\tau_f - \tau)]/\tau\}}{[1 + 0.8(\tau_f - \tau)^{1/2} - 0.4(\tau_f - \tau)]}, \quad (18)$$

where $\tau_f = 5.516$, provides an excellent fit to the form of $T[\eta(\tau)]$ derived from the simulations of Hamilton et al. (1991) i.e., $T \approx T_{sim}$, in the range $\bar{\delta} \gtrsim 15$. It also provides an evolution of $\bar{\delta}$ that matches up smoothly to the one predicted by the standard SCM in the region where we expect it to provide an accurate approximation, i.e., prior to turnaround $\bar{\delta} \lesssim 5$.

In Figure 1 we plot T_{sim} against τ_{sim} , and our fitting formula for $T[\eta(\tau)]$ against τ . We see that, in the range $\bar{\delta} \gtrsim 15$, $\tau \gtrsim 4.4$, where we expect $\bar{\xi} \approx \bar{\delta}$ and hence $T_{sim} \approx T$, the fit is indeed very good. The parameters in the formula for $T[\eta(\tau)]$ have been chosen

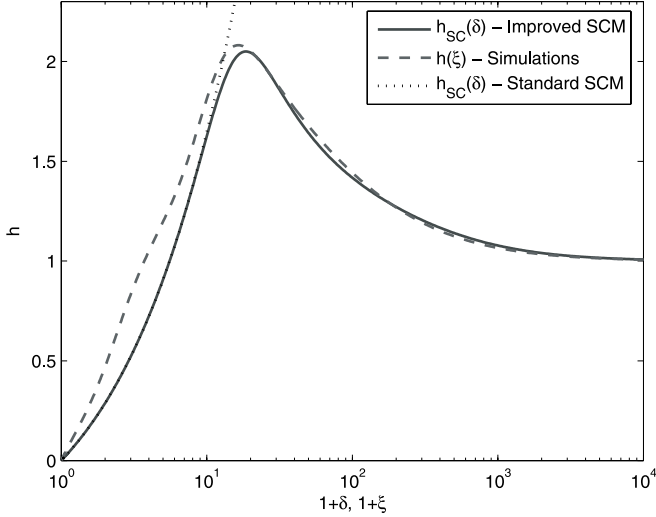


FIG. 2.— Relationship between h and $\bar{\delta}$ in different models. The solid line shows $h_{\text{SC}}(\bar{\delta})$ in our improved spherical collapse model. The dotted line is $h_{\text{SC}}(\bar{\delta})$ in the standard SCM, and the dashed line shows $h(\xi)$ as seen in simulations (Hamilton et al. 1991). As required, our improved SCM gives an h_{SC} that agrees with $h(\xi)$ very well in the region where we expect $\bar{\delta} \approx \xi$ i.e., $\bar{\delta} \gtrsim 15$. We also note that $h_{\text{SC}}(\bar{\delta})$, in the improved SCM, has the same asymptotic behavior, to leading order, as $h(\xi \approx \bar{\delta})$ in the limit $\bar{\delta} \rightarrow \infty$. Importantly, our improved SCM model also agrees with the standard SCM in the region where it is expected to provide a very good approximation i.e., prior to the epoch of turnaround, $\bar{\delta} \lesssim 5.6$. [See the electronic edition of the Supplement for a color version of this figure.]

so that as $\tau \rightarrow \tau_f$, the leading order terms in the asymptotic expansions of both $T[\eta(\tau)]$ and $T_{\text{sim}}[\eta_{\text{sim}}(\tau)]$ match.

In Figure 2 we use equations (3) and (4) to plot $h_{\text{SC}}(\bar{\delta})$ and the fitting formula found by Hamilton et al. (1991) to plot $h(\xi)$. We also plot the unmodified SCM prediction for $h_{\text{SC}}(\bar{\delta})$. It is clear from this plot that our fitting formula for T gives an $h_{\text{SC}}(\bar{\delta})$ that is an excellent approximation to $h(\xi)$ (provided $\xi \approx \bar{\delta}$) in the region $\bar{\delta} \gtrsim 20$. As $\bar{\delta}, \xi \rightarrow \infty$, the curves $h_{\text{SC}}(\bar{\delta})$ and $h(\xi)$ have the same leading order asymptotic behavior. In the region $\bar{\delta} \gtrsim 15$, our model gives an $h_{\text{SC}}(\bar{\delta})$ that is always within 3% of the fitting formula for $h(\xi)$ derived from simulations (Hamilton et al. 1991). In addition, our fitting formula for $h_{\text{SC}}(\bar{\delta})$ agrees almost exactly with the predictions of the unmodified SCM prior to turnaround $\bar{\delta} < 5.6$.

We found a best-fit value of $R_{\text{vir}}/R_{\text{ta}} = 0.5896$, which is fairly close to the value of 0.5 that is generally used in the spherical collapse model, despite the fact that we did not constrain it to be so. For comparison, the improved spherical collapse developed by Engineer et al. (2000) gave $R_{\text{vir}}/R_{\text{ta}} \approx 0.65$, and N -body simulations have been found to support $R_{\text{vir}}/R_{\text{ta}} \approx 0.56$ (Hamilton et al. 1991). Our value of $R_{\text{vir}}/R_{\text{ta}}$ is therefore a better approximation to the value found from simulations than that generally used in the unmodified SCM and that found in the model developed by Engineer et al. (2000).

In our improved SCM the linear density contrast is given by

$$\delta_{\text{lin}} = \frac{3}{5} \delta_i \left(\frac{t}{t_i} \right)^{2/3} = \frac{3}{5} \left(\frac{3}{4} \right)^{2/3} T(\eta)^{2/3}.$$

When $\delta_{\text{lin}} = 1.6865$, which corresponds to the instant of collapse in the unmodified SCM, we find $\bar{\delta} = 54.65$ rather than the SCM's value of 178. We find that $\bar{\delta} = 200$ corresponds to $\delta_{\text{lin}} = 2.286$. We compare the form of $\bar{\delta}(\bar{\delta}_{\text{lin}})$ found in our model to that predicted by the unmodified SCM model, with virialisation put in by hand, in Figure 3. We also show the divergent behavior of $\bar{\delta}(\bar{\delta}_{\text{lin}})$

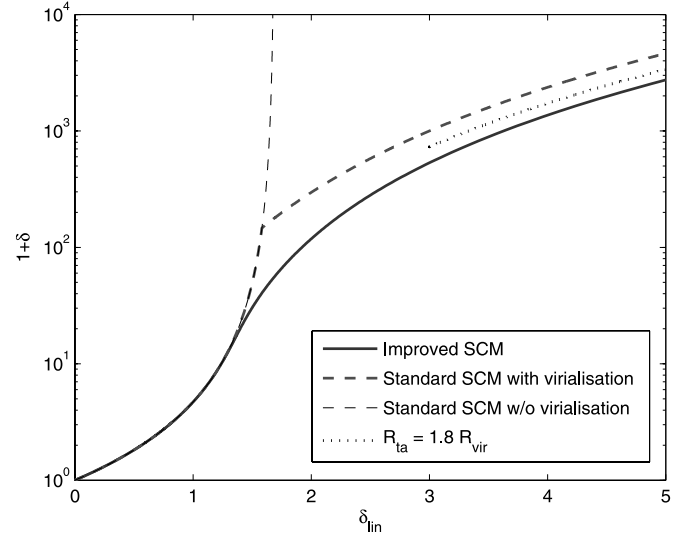


FIG. 3.— Plot of how the nonlinear mean density contrast, $\bar{\delta}$, depends on the linear one, δ_{lin} . The solid line is the improved SCM developed here, the thick dashed line is the standard SCM with virialisation put in by hand, and the thin dashed line is the standard SCM without virialisation. The dotted line shows the asymptotic behavior of $\bar{\delta}(\bar{\delta}_{\text{lin}})$ that we would expect if $R_{\text{ta}}/R_{\text{vir}} = 1.8$ as seen in the simulations of Hamilton et al. (1991). [See the electronic edition of the Supplement for a color version of this figure.]

in the standard SCM without virialisation, and the asymptotic behavior of $\bar{\delta}(\bar{\delta}_{\text{lin}})$ if $R_{\text{ta}} = 1.8 R_{\text{vir}}$ as suggested by simulations (Hamilton et al. 1991). The two major advantages of our improved SCM over the unmodified version are clearly visible in this plot:

1. At late times the improved SCM provides a better approximation to the behavior seen in N -body simulations than the unmodified SCM does.
2. $\bar{\delta}(\bar{\delta}_{\text{lin}})$ is smooth in the improved SCM. This is *not* the case in the unmodified SCM when virialisation is put in by hand. The improved SCM therefore provides not only a good approximation to the evolution of density contrast, $\bar{\delta}$, but also to that of $d\bar{\delta}/dt$ and hence to that of the peculiar velocity.

5. INCLUDING DARK ENERGY

We have constructed an improved, semianalytical SCM whose prediction for the dependence of peculiar velocity on the mean density contrast concurs with that extrapolated from simulations in the nonlinear regime. However, as mentioned above, we have done this for a matter-dominated universe. It is well known, however, that the universe today is not matter dominated and a significant fraction of its total content is believed to be in the form of dark energy. Fortunately, however, the effects of this dark energy on the background universe are generally believed to only have become non-negligible relatively recently at $z \lesssim 1.8$. This means the local evolution of overdensities with $\bar{\delta} \gtrsim 100$ today has been matter dominated, to a very good approximation, right up until the epoch of matter-radiation equality. Therefore, the evolution of the density, ρ , and radial velocity, v , of a shell of matter that is well into the nonlinear regime today is, to a good approximation, the same in a matter-dominated background as it is in a background where today $\Omega_m \approx 0.27$ and $\Omega_{\text{de}} \approx 0.73$ for which the equation of state parameter for the dark energy, w , satisfies the current astronomical bound of $w = -1 \pm 0.1$ for $z < 1$ (Riess et al. 2007). It is therefore a fairly straightforward task to generalize our formulae for $\bar{\delta}$ and h_{SC} to include background universes where

$\Omega_m \approx 0.27$, provided that these are only applied to overdensities that are large today. We then find that

$$1 + \bar{\delta}(\Omega_m, t) = \frac{9f(a)T(\eta)^2}{2(1 - \cos \eta)^3}, \quad (19)$$

$$h_{SC}(\Omega_m, t) = 1 - \Omega_m^{0.5} \sqrt{\frac{1 + \bar{\delta}}{2(1 - \cos \eta)}} \sin \eta \frac{d\tau}{dT}, \quad (20)$$

where $\tau = \eta - \sin \eta$ as before, $T(\eta)$ is still given by equation (18), and

$$f(a) = \frac{4}{9t^2\Omega_m H^2}.$$

The equations for $R(\eta)$ and $t(\eta)$ remain unchanged.

If dark energy has behaved similarly to a cosmological constant in the recent past ($z < 1.8$), then

$$f(a) \approx \frac{1 - \Omega_m}{\Omega_m \left\{ \sinh^{-1} \left[\sqrt{(1 - \Omega_m)/\Omega_m} \right] \right\}^2} \approx \Omega_m^{-0.4}.$$

Equations (19) and (20) combined with the fitting formula equation (18) provide a very good approximation to the evolution of overdensities of matter in a realistic universe provided that $\bar{\delta} \gtrsim 100$ today. For smaller values of $\bar{\delta}$ our results are only accurate for $\Omega_m \approx 1$. However, for $\bar{\delta} \lesssim 15$ the effects of deviations from spherical symmetry, which have been our primary concern in this article, are small enough to be ignored and the results derived using the unmodified SCM can be used with confidence. Notice also that such fitting formulae and assumptions are strictly speaking only applicable to dark energy models that are not coupled to matter. If such a matter coupling is allowed, then the whole

process of structure formation, both in the linear and nonlinear regimes, may change (see, e.g., Brookfield et al. 2006; Lahav et al. 1991; Mota & van de Bruck 2004; Maor & Lahav 2005; Nunes & Mota 2006; Mota & Shaw 2006, 2007). This said, in many dark energy models, the matter coupling is constrained by experiments to be very small (relative to the coupling between matter and gravity), and as a result any alterations to the process of structure formation are similarly small.

6. CONCLUSIONS

In this article we have extended the spherical collapse model so that it takes account of the effects of deviations from spherical symmetry and shell crossing, which are important in the nonlinear regime. The key assumptions that we used when constructing our model was that $h_{SC} = h_{SC}(\bar{\delta})$ and that for $\bar{\delta} \gtrsim 15$, $\bar{\delta} \approx \bar{\xi}$. The latter assumption is probably the weakest link in the whole analysis, although both assumptions are found commonly in the literature (see Engineer et al. 2000; Jain et al. 1995; Peebles 1980). Our improved SCM predicts a form for $h_{SC}(\bar{\delta})$ that is consistent with the results of N -body simulations in the regime where a comparison can sensibly be made ($\bar{\delta} \gtrsim 15$) and with the unmodified SCM prior to turnaround. Analytical formulae for $\bar{\delta}$ and h_{SC} in the improved model have been presented, and they essentially differ from the comparable formulae in the unmodified SCM only by the replacement of $\tau = \eta - \sin \eta$ with $T(\eta)$, which is given by equation (18). The improved SCM is therefore simple enough to be used anywhere where the unmodified SCM might be used but with the advantage that it includes a realistic model of the effects of virialisation.

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