The Astrophysical Journal, 666: L69–L72, 2007 September 10

SCALING RELATIONS OF COMPRESSIBLE MHD TURBULENCE

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Received 2007 May 14; accepted 2007 July 23; published 2007 August 30

ABSTRACT

We study scaling relations of compressible and strongly magnetized turbulence using isothermal numerical simulations with resolution 512^3. We find a good correspondence of our results with the Fleck model of compressible hydrodynamic turbulence. In particular, we find that the density-weighted velocity, i.e., $\boldsymbol{v} \equiv \rho^{1/3}\boldsymbol{u}$, proposed by Kritsuk and coworkers obeys the Kolmogorov scaling, i.e., $\mathcal{E}_v(k) \sim k^{-5/3}$, for the high Mach number turbulence. Similarly, we find that the exponents of the third-order structure functions for $\boldsymbol{v}$ stay equal to unity for all Mach numbers studied. The higher order correlations obey the She-Lévêque scalings corresponding to the two-dimensional dissipative structures, and this result does not change with the Mach number either. In contrast to velocity $\boldsymbol{u}$, which exhibits different scaling parallel and perpendicular to the local magnetic field, the scaling of $\boldsymbol{v}$ is similar in both directions. In addition, we find that the peaks of density create a hierarchy in which both physical and column densities decrease with the scale in accordance to the Fleck predictions. This hierarchy can be related to ubiquitous small ionized and neutral structures (SINS) in the interstellar gas. We believe that studies of statistics of the column density peaks can provide both a consistency check for the turbulence velocity studies and insight into supersonic turbulence, when the velocity information is not available.

Subject headings: ISM: structure — MHD — turbulence

Online material: color figures

1. INTRODUCTION

The interstellar medium (ISM) is a highly compressible turbulent, magnetized fluid, exhibiting density fluctuations on all observable scales. It has been long realized by many researchers that an incompressible hydrodynamic (HD) description is inadequate for such a medium (see Elmegreen & Scalo 2004 for review).

Attempts to include effects of compressibility into the interstellar turbulence description can be dated as far back as the work by von Weizsäcker (1951). There a simple model based on a hierarchy of clouds was presented, in which every large cloud consists of smaller clouds, which contain even smaller clouds. For such a model von Weizsäcker (1951) proposed a relation between subsequent levels of hierarchy:

$$\rho_\nu/\rho_{\nu-1} = (l_\nu/l_{\nu-1})^{-3\alpha}, \tag{1}$$

where ρ_ν is the average density inside a cloud at level ν, l_ν is the mean size of that cloud, 3 is the number of dimensions, and α is a constant that reflects the degree of compression at each level ν.

The Kolmogorov energy spectrum ($\sim k^{-5/3}$) follows from the assumption of a constant specific energy transfer rate $\epsilon \sim u^2/(l/u)$. Lighthill (1955) pointed out that in a compressible fluid, the volume energy transfer rate is constant in a statistical steady state:

$$\epsilon_V = \rho\epsilon \sim \rho u^2/(l/u) = \rho u^3/l. \tag{2}$$

In an important but not sufficiently appreciated work, Fleck (1996, hereafter F96) incorporated the above hierarchical model with energy transfer in compressible fluid. By combining equations (1) and (2) F96 presented the following set of scaling relations in terms of the degree of compression α:

$$\rho_l \sim l^{-3\alpha},\ N_l \sim l^{1-3\alpha},\ M_l \sim l^{3-3\alpha},\ u_l \sim l^{1/3+\alpha}, \tag{3}$$

where N_l and M_l are, respectively, the column density of the fluctuation with the scale l and the mass of the cloud of size l. This entails the spectrum of velocities $E(k) \sim k^{-5/3-2\alpha}$.

In the spirit of the F96 model, Kritsuk et al. (2007, hereafter KNPW07) proposed to use the density-weighted velocity $\boldsymbol{v} \equiv \rho^{1/3}\boldsymbol{u}$ as a new quantity, for which the Kolmogorov scaling for second-order structure functions (SFs) can be restored in compressible HD turbulence. Their HD simulations provided the spectrum for $\boldsymbol{v}$ close to $-5/3$, and they showed that in supersonic HD turbulence the SFs of $\boldsymbol{v}$ scale linearly with separation.

Will the F96 model be valid for compressible *strongly magnetized* turbulence? This is the major question that we address in this Letter.

2. NUMERICAL MODELING

We used a second-order-accurate essentially nonoscillatory (ENO) scheme (see Cho & Lazarian 2002; Kowal et al. 2007 for details) to solve the ideal isothermal magnetohydrodynamic (MHD) equations in a periodic box maintaining the $\nabla \cdot \boldsymbol{B} = 0$ constraint. The rms velocity δu is approximately unity, so that $\boldsymbol{u}$ and $\boldsymbol{B}/(4\pi\rho)^{1/2}$ are expressed in units of δu. The time t is in units of the large eddy turnover time ($\sim L/\delta u$) and the length in units of L, the size of the box. The magnetic field consists of the uniform background and fluctuating parts: $\boldsymbol{B} = \boldsymbol{B}_{\rm ext} + \boldsymbol{b}$. Initially $\boldsymbol{b} = \boldsymbol{u} = 0$. We use units in which the Alfvén speed $v_A = B_{\rm ext}/(4\pi\rho)^{1/2} = 1$ and $\rho = 1$ initially. We assume that $\delta B \sim B_{\rm ext}$ and $v_A \sim \delta u$. Then, the sound speed is the controlling parameter. We consider two regimes: supersonic (low β, where $\beta \equiv p_{\rm gas}/p_{\rm mag}$) and subsonic (high β).

We drove the turbulence at wave scale $k_f \simeq 2.5$ (the injection scale) using a random solenoidal large-scale driving acceler-

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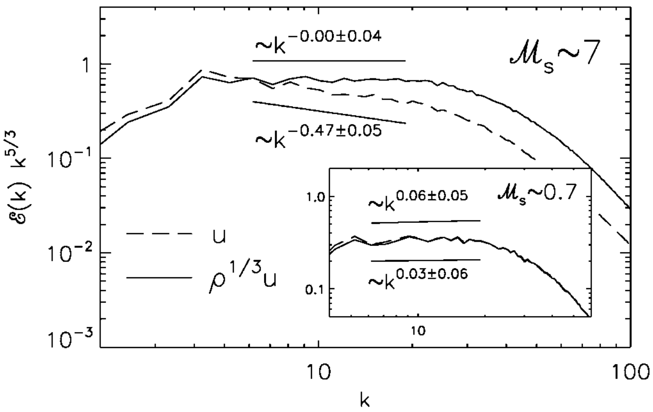

FIG. 1.—Spectra of $\boldsymbol{u}$ and $\boldsymbol{v}$ (*dashed and solid lines, respectively*) for super- and subsonic models (*big and small plots, respectively*). Here $\mathcal{M}_A \sim 0.7$. Spectra are compensated by $k^{5/3}$.

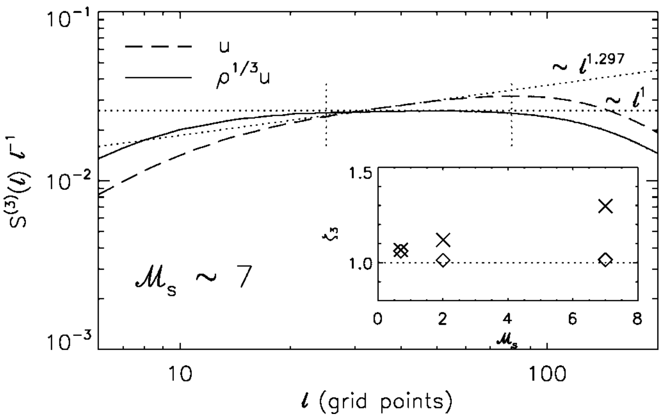

FIG. 2.—SFs of the third order for $\boldsymbol{u}$ (*dashed line*) and $\boldsymbol{v}$ (*solid line*) compensated by l^{-1} for supersonic model. Here $\mathcal{M}_A \sim 0.7$. Dotted lines show the best fit within the inertial range. Two dotted vertical lines bound the inertial range. The inset shows the dependence of scaling exponents ζ_3 for $\boldsymbol{u}$ (*crosses*) and $\boldsymbol{v}$ (*diamonds*) on $\mathcal{M}_s$. [*See the electronic edition of the Journal for a color version of this figure.*]

ation. The scale at which the dissipation starts to act k_ν is about 30 (for resolution 512^3). The limited inertial range between k_f and k_ν produces the bottleneck effect, resulting in flattened spectra of velocity (see KNPW07).

We present results for 3D numerical experiments of compressible MHD turbulence with a strong magnetic field for sonic Mach numbers $\mathcal{M}_s \equiv \langle |u|/c_s \rangle$ between 0.6 and 7. The Alfvénic Mach number $\mathcal{M}_A \equiv \langle |u|/c_A \rangle \sim 0.7$. To study effects of magnetization we also performed experiments with $\mathcal{M}_A \sim 2$. All models were calculated with the resolution 512^3 for six dynamical times.

3. RESULTS

3.1. *Kolmogorov Scalings for Supersonic Flows*

In Figure 1 we present spectra for velocity $\boldsymbol{u}$ and density-weighted velocity $\boldsymbol{v} \equiv \rho^{1/3}\boldsymbol{u}$ for two strongly magnetized models: subsonic ($\beta \sim 2$) and supersonic ($\beta \sim 0.02$). Naturally, for the subsonic model the differences between spectra for $\boldsymbol{u}$ and $\boldsymbol{v}$ are marginal and both spectra correspond to Kolmogorov's $k^{-5/3}$ (see Fig. 1, *inset*). However, for the supersonic case the velocity spectrum is steeper. The steepening corresponds to $\alpha \approx 0.23$ (from $\mathcal{E}_u \sim k^{-5/3-2\alpha}$). At the same time, the spectrum of $\boldsymbol{v}$ matches well the Kolmogorov slope.

In the original Kolmogorov theory (Kolmogorov 1941) it was shown that the spectral index of the third-order SF, e.g., the SF of $\boldsymbol{u}$, $S_u^{(3)}(l) \equiv \langle |\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{l}) - \boldsymbol{u}(\boldsymbol{r})|^3 \rangle \sim l^{\zeta_3}$, should be equal to 1, i.e., $\zeta_3 = 1$. In Figure 2 we show SFs of the third order for velocity and the density-weighted velocity for supersonic model. We checked that for the subsonic case for both $\boldsymbol{u}$ and $\boldsymbol{v}$ the index ζ_3 is indeed close to unity. For the supersonic case, ζ_3 grows with $\mathcal{M}_s$ for $\boldsymbol{u}$, but remains unity for $\boldsymbol{v}$ (see Fig. 2). This suggests that the Kolmogorov universality is preserved for supersonic MHD turbulence when the density weighting is applied.

3.2. *She-Lévêque Intermittency Model*

A proper description of turbulence requires higher moments (see Lazarian 2006 for review). Those characterize intermittency, which in the original model of Kolmogorov (1941) is not accounted for. Substantial progress in understanding turbulence intermittency is related to a discovery by She & Lévêque (1994), who found a simple form for the scaling of exponents ζ_p of higher order longitudinal correlations $S^{(p)}(l) \equiv \langle | [\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{l}) - \boldsymbol{u}(\boldsymbol{r})] \cdot \hat{\boldsymbol{l}}|^p \rangle \sim l^{\zeta_p}$. While in the model of Kolmogorov (1941) $\zeta_p \equiv p/3$, She & Lévêque (1994) provide (after modifications to more general form by Müller & Biskamp 2000)

$$\zeta_p = \frac{p}{g}(1-x) + C[1-(1-x/C)^{p/g}], \tag{4}$$

where g is related to the scaling of the velocity $u_l \sim l^{1/g}$, x is related to the energy cascade rate $t_l^{-1} \sim l^{-x}$, and C is the codimension of the very high intensity structures. In HD incompressible turbulence $g = 3$ and $x = 2/3$. Müller & Biskamp (2000) introduced the dimension of the most singular dissipative structures, $D = 3 - C$. For MHD turbulence the dissipation happens in two-dimensional dissipative structures such as current sheets, corresponding to $D = 2$ (Müller & Biskamp 2000). Thus, for subsonic MHD turbulence we expect $\zeta_p = p/9 + 1 - (1/3)^{p/3}$ for both $\boldsymbol{u}$ and $\boldsymbol{v}$. This is what we actually observe in Figure 3 (*inset*). The same scaling, however, is preserved for $\boldsymbol{v}$ for *supersonic* magnetized turbulence.

3.3. *Anisotropies Induced by Magnetic Field*

Magnetic field is known to induce anisotropies in compressible MHD turbulence (see Higdon 1984). Anisotropy increasing

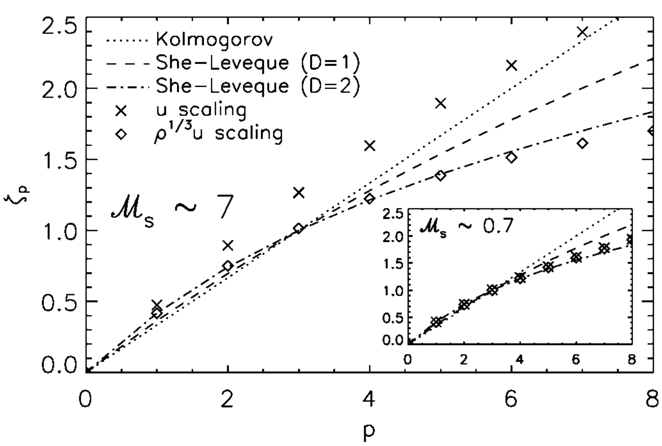

FIG. 3.—Scaling exponents for $\boldsymbol{u}$ (*crosses*) and $\boldsymbol{v}$ (*diamonds*) for supersonic and subsonic (*inset*) models. Here $\mathcal{M}_A \sim 0.7$. The plots show unnormalized values of the scaling exponents obtained directly from SFs by fitting the relation $S^{(p)}(l) = al^{\zeta_p}$ within the inertial range, i.e., without using the extended self-similarity (Benzi et al. 1993).

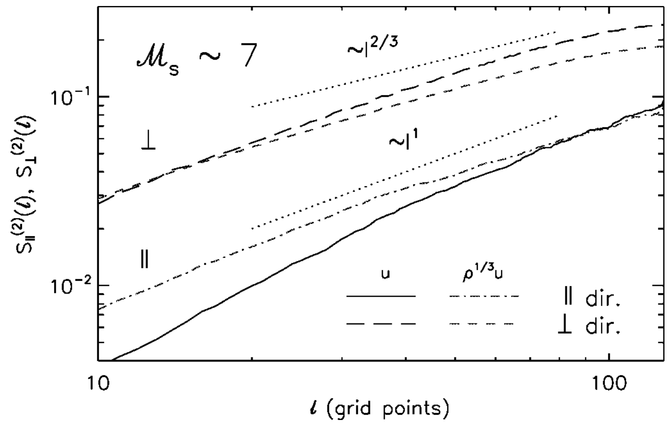

FIG. 4.—SFs of the second order for $\boldsymbol{u}$ and $\boldsymbol{v}$ in the local reference frame for supersonic experiment. Here $\mathcal{M}_A \sim 0.7$. SFs for $\boldsymbol{u}$ scale as $\sim l^{1.22}$ and $\sim l^{0.87}$ for $\parallel$ and $\perp$ directions, respectively. SFs for $\boldsymbol{v}$ scale as $\sim l^{0.88}$ and $\sim l^{0.74}$ for $\parallel$ and $\perp$ directions, respectively. [*See the electronic edition of the Journal for a color version of this figure.*]

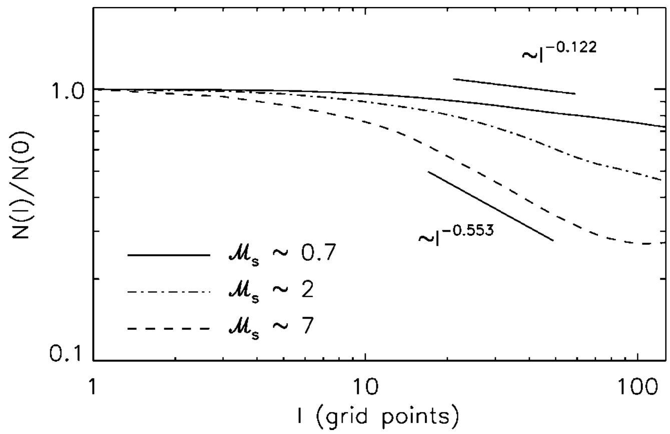

FIG. 5.—Scaling relations for the column density N for three models: subsonic ($\mathcal{M}_s \sim 0.7$) and supersonic ($\mathcal{M}_s \sim 2$ and 7). Here $\mathcal{M}_A \sim 0.7$.

with the decrease of scale was predicted for Alfvénic motions by Goldreich & Sidhar (1995; see also Lithwick & Goldreich 2001) and confirmed numerically for compressible MHD in Cho & Lazarian (2002, 2003).

For supersonic motions Figure 4 shows that the SFs for $\boldsymbol{u}$ are much steeper in both directions than those predicted by the model of Goldreich & Sidhar (1995; 1.22 and 0.87 for $\parallel$ and $\perp$ directions to the local magnetic field, respectively). However, the anisotropy is still close to the predictions of Goldreich & Sidhar (1995; $l_\parallel \sim l_\perp^{0.72}$), which is indicative of the dominance of the Alfvénic ("incompressible") motions. Note that in Figure 4 the SFs are obtained in the system of reference of the *local* magnetic field, i.e., the field on scales of the fluctuations under study. The terms $S_\parallel^{(2)}$ and $S_\perp^{(2)}$ denote the second-order SFs parallel and perpendicular to the local magnetic field, respectively.

The SFs for $\boldsymbol{v}$ are significantly shallower (0.88 and 0.74 for $\parallel$ and $\perp$ directions to the local magnetic field, respectively). The SF of $\boldsymbol{v}$ in the $\perp$ direction scales more like in incompressible motions, i.e., $S_\perp^{(2)} \sim l_\perp^{2/3}$. The slope of $S_\parallel^{(2)}(l)$ for $\boldsymbol{v}$ is smaller than the corresponding one for $\boldsymbol{u}$, resulting in the reduced degree of anisotropy ($l_\parallel \sim l_\perp^{0.84}$). Intuitively, this can be understood in terms of dense clumps strongly distorting magnetic field as they move with respect to magnetized fluid.

3.4. *Statistics of Column Density Peaks*

Results for velocity from our simulations of strongly magnetized turbulence provide $\alpha \simeq 0.23$ for $\mathcal{M}_s \sim 7$. The spectrum of density fluctuations $\mathcal{E}_\rho \sim k^{-1+6\alpha}$ follows from the scaling relation of density (see eq. [3]) according to the F96 model. This suggests the existence of a rising spectrum of density fluctuations within the hierarchy of density clumps when $\alpha > 1/6$. We try to make our study more related to *observations*, which usually measure densities integrated along the line of sight, i.e., column densities, or alternatively study the hierarchy of observed clump masses (see eq. [3]).

The F96 model assumes the existence of an infinitely extended hierarchy. In our computations the structures are generated by turbulence at scales smaller than the scale of the computational box. Therefore the F96 scaling relations (see eq. [3]) should be modified as follows:

$$N_l \sim L \times l^{-3\alpha} \sim l^{-3\alpha} \text{ and } M_l \sim L \times l^{2-3\alpha} \sim l^{2-3\alpha}. \quad (5)$$

Our procedure of obtaining the scaling relation from column density maps is similar to that in KNPW07, with the difference that they dealt with 3D data, while we deal with 2D data. First, we seek for a local maximum of column density. Then we calculate the average column density within concentric boxes with gradually increasing size l. In the case of determining the relation for M_l, instead of averaging we apply the integration over the boxes. Naturally, the results correspond to each other.

In Figure 5 we present an example of the scaling relation for column density for three models of turbulence with $\mathcal{M}_s \sim 0.7$, 2, and 7. One can note that the relation becomes steeper with the sonic Mach number within the inertial range. The fractal dimensions can be calculated from the relation $D_m = 3 + \gamma$ (see KNPW07), where $\gamma \equiv -3\alpha$ is a slope estimated from the plot within the inertial range. For our models the fractal dimension ranges from $D_m \simeq 2.5$ for the highly supersonic models to $D_m \simeq 2.9$ for the subsonic model. Respectively, the compressibility coefficients for the presented models range from $\alpha \simeq 0.04$ for $\mathcal{M}_s \sim 0.7$ to $\alpha \simeq 0.19$ for $\mathcal{M}_s \sim 7$. The latter is roughly consistent with the α-value obtained from the SF of velocity in § 3.1. The differences are probably due to insufficient statistics of rather rare high-density peaks. In general, the filling factor of a peak decreases with the maximum density of this peak, which means that the higher maximum density that peak has, the smaller the space it occupies.

3.5. *Variations of Scalings Induced by Fluid Magnetization*

What is the effect of magnetic field on the $\boldsymbol{v}$-scaling? The spectra, the third and higher moments of correlations obtained for our super-Alfvénic simulations with $\mathcal{M}_A \sim 2$, happen to be very similar to the case of strongly magnetized turbulence. Our results indicate that unlike velocity, $\boldsymbol{v}$ is much less affected by magnetic field. Naturally, in super-Alfvénic turbulence the anisotropies induced by magnetic field are not observed at larger scales within the inertial range (see the last paragraph of § 4).

4. ASTROPHYSICAL IMPLICATIONS

Dependence of α on the extend of inertial range.—If we combine several facts together, namely, (1) that α is a function of $\mathcal{M}_s$, (2) that the maximum of density corresponds to the dissipation scale, e.g., shock thickness scale $l_{\rm diss}$, (3) that the amplitude of density in peaks scales as the mean density times $\mathcal{M}_s^2$, then we have to conclude that as the inertial range from the injection scale $l_{\rm inj}$ to $l_{\rm diss}$ increases, for a given Mach number, α should decrease. Connecting these facts we get the relation $\rho_{\rm peak} \sim \mathcal{M}_s^2 \sim (l_{\rm inj}/l_{\rm diss})^{3\alpha}$, which results in a dependence of α

on $l_{\rm inj}/l_{\rm diss}$ and $\mathcal{M}_s$, namely, $\alpha \sim \log \mathcal{M}_s / \log(l_{\rm inj}/l_{\rm diss})$. An interesting consequence of this would be a prediction of Kolmogorov scaling for supersonic *velocities* when the injection and dissipation ranges are infinitely separated. Consequently, the steeper velocity spectra reported in Padoan et al. (2007) can be interpreted as an indication of a limited inertial range. Further research justifying such conclusions is required. In KNPW07 the authors have noticed that the scaling in equation (3) could have a break at the sonic scale if the inertial range is wide enough. This feature we plan to study elsewhere.

SINS of supersonic turbulence.—Ubiquitous small ionized and neutral structures (SINS) are observed in the ISM (see Heiles & Stinebring 2007). Their nature is extremely puzzling if one thinks in terms of Kolmogorov scalings for density fluctuations. The fact that the spectrum of fluctuations of density in supersonic turbulence is shallower that the Kolmogorov one is well known (see Kowal et al. 2007 and references therein). However, just the difference in slope cannot explain the really dramatic variations in observed column densities. The present Letter provides a different outlook at the problem of SINS. We see that while the low-amplitude density fluctuations exhibit Kolmogorov scaling (Beresnyak et al. 2005; Kowal et al. 2007), high peaks of density correspond to a rising spectrum of fluctuations. Thus, observing supersonic turbulence at small scales, we shall most frequently observe small-amplitude fluctuations corresponding to a Kolmogorov-like spectrum of density fluctuations. Occasionally, but inevitably, one will encounter isolated high-density peaks. An alternative mechanism for getting infrequent large density fluctuations over small scales is presented in Lazarian (2007) and related to current sheets in the viscosity-damped regime of MHD turbulence.

Clumps and star formation.—The ISM is known to be clumpy. Frequently the clumps in molecular clouds are associated with the action of gravity. Our study shows that supersonic turbulence tends to produce small and very dense clumps. If such clumps happen to attain the Jean's mass, they can form stars. Therefore, star formation is inevitable in supersonic turbulence. However, the efficiency of star formation is expected to be low, as the filling factor of peaks decreases with the increase of the peak height. Inhibition of star formation via shearing may dominate in terms of influencing star formation efficiencies. We consider a strongly magnetized case, where the magnetic field is dynamically important and dominant. We see some analogy with the weakly magnetized cases discussed in Padoan et al. (2007).

5. SUMMARY

In this Letter we have studied the scaling of supersonic MHD turbulence. We found the following:

1. The Fleck (1996) model is applicable to strongly magnetized compressible turbulence.
2. The spectra and structure functions of density-weighted velocities are consistent with predictions of the Kolmogorov theory.
3. Intermittency of density-weighted velocity can be well described by the She-Lévêque model with the dimension of dissipative structures equal to 2 (Müller & Biskamp 2000).
4. Strongly magnetized supersonic turbulence demonstrates a lower degree of anisotropy if described using the density-weighted velocity.
5. The high peaks of column densities exhibit the increase of the mean values of column densities with the decrease of scale, which may be relevant to the explanation of SINS.

The research of Grzegorz Kowal and Alex Lazarian is supported by NSF grant AST 03-07869 and the Center for Magnetic Self-Organization in Astrophysical and Laboratory Plasmas. We thank Alexei Kritsuk for valuable discussion and suggestions. We also thank Andrey Beresnyak for discussion about SFs.

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