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ON THE GROWTH OF PERTURBATIONS AS A TEST OF DARK ENERGY AND GRAVITY

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ABSTRACT

The strongest evidence for dark energy at present comes from geometric techniques such as the supernova distance-redshift relation. By combining the measured expansion history with the Friedmann equation, one determines the energy density and its time evolution and hence the equation of state of dark energy. Because these methods rely on the Friedmann equation, which has not been independently tested, it is desirable to find alternative methods that work for both general relativity and other theories of gravity. Assuming that sufficiently large patches of a perturbed Robertson-Walker spacetime evolve like separate Robertson-Walker universes, that shear stress is unimportant on large scales, and that energy and momentum are locally conserved, we derive several relations between long-wavelength metric and matter perturbations. These relations include generalizations of the initial-value constraints of general relativity. For a class of theories including general relativity we reduce the long-wavelength metric, density, and velocity potential perturbations to quadratures including curvature perturbations, entropy perturbations, and the effects of nonzero background curvature. When combined with the expansion history measured geometrically, the long-wavelength solution provides a test that could distinguish modified gravity from other explanations of dark energy.

Subject headings: cosmology: theory — gravitation

1. INTRODUCTION

Current evidence for dark energy is based on two key assumptions.

1. The cosmological principle holds; i.e., on large scales the matter distribution and its expansion are homogeneous and isotropic, and the spacetime geometry is Robertson-Walker.
2. The cosmic expansion scale factor $a(t)$ obeys the Friedmann equation, which can be written

$$\mathcal{H}^2 \equiv \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = a^2H^2 = \frac{8\pi G a^2}{3}\rho(a) - K, \qquad (1)$$

where K is the spatial curvature with units of inverse length squared (we set $c = 1$) and τ is conformal time, related to cosmic proper time t by $dt = a(\tau)\,d\tau$. Usually the Friedmann equation is applied indirectly through an expression for the angular-diameter or luminosity distance; these formulae depend crucially on the expression for $\tau(a)$ obtained by integrating equation (1).

The cosmological principle is more general than general relativity (GR) and is amenable to direct observational test through measurements of distant objects such as galaxies, Type Ia supernovae, and the microwave background radiation. The Friedmann equation, however, is equivalent to one of the Einstein field equations of GR applied to the Robertson-Walker metric, and, so far at least, it has not been independently tested. Instead, the Friedmann equation is used by astronomers in effect to deduce $\rho(a)$ including dark energy through measurements of the Hubble expansion rate $H(a)$.

If the evidence for dark energy is secure, there are four possible explanations.

1. The dark energy is a cosmological constant or, equivalently, the energy density and negative pressure of the vacuum (Gliner 1966; Zel'dovich 1967).
2. The dark energy is some other source of stress-energy, for example, a scalar field with large negative pressure (Ratra & Peebles 1988; Steinhardt et al. 1999).
3. General relativity needs to be modified and we cannot use it to deduce the existence of either a cosmological constant or an exotic form of energy (Dvali et al. 2000; Lue et al. 2004).
4. General relativity is correct but our understanding of it is not; e.g., long-wavelength density perturbations modify the Friedmann equation (Kolb et al. 2005, 2006).

It would be interesting to find observational tests that can distinguish among these possibilities. The first case can be tested by measuring the equation of state parameter $w \equiv p/\rho$ on large scales; a cosmological constant has $w = -1$. This test can be made using methods such as the supernova distance-redshift relation (Riess et al. 1998; Perlmutter et al. 1999) and baryon acoustic oscillations (Eisenstein et al. 2005), whose interpretation relies on the Friedmann equation. Such methods are called geometric (they measure the large-scale geometry of spacetime) or kinematic (they rely on energy conservation and on the first time derivative of the expansion scale factor).

It is more difficult to test the third possibility, namely, that dark energy represents a modification of GR rather than (or in addition to) a new form of mass-energy. Redshift-distance tests invoke the Friedmann equation of GR or its equivalent in other theories in order to determine the abundance and equation of state of dark energy. Such tests do not work without a formula relating $a(\tau)$ to the cosmic energy density and pressure. While specific alternative models can be tested, it would be desirable to have general cosmological tests independent of the Friedmann equation.

The evolution of density perturbations has been proposed as an independent test of dark energy and general relativity (Linder 2005; Ishak et al. 2005). For example, weak gravitational lensing measurements, the evolution of galaxy clustering on large scales, and the abundance of rich galaxy clusters all have some sensitivity to the gravitational effects of dark energy at redshifts $z < 1$.

These methods can be called dynamic because the evolution equations for the perturbations are at least second-order in time.

This paper examines what the linear growth of metric, density, and velocity perturbations can tell us about gravity. By generalizing previous work on evolution of separate universes (Wands et al. 2000; Gordon 2005 and references therein), we show that in GR, on length scales larger than the Jeans length (more generally, on scales large enough that spatial gradients can be neglected in the equations of motion), the evolution of the metric and density perturbations of a background Robertson-Walker universe can be determined from the Friedmann equation and local energy-momentum conservation. Generalizing this to arbitrary theories of gravity, under certain conditions explained in this paper, the evolution of Robertson-Walker spacetimes (assuming they are solutions of the gravitational field equations) combined with local energy-momentum conservation is sufficient to determine the evolution of long-wavelength perturbations of the metric and matter variables.

Because GR is so fully integrated into most treatments of cosmological perturbation theory, and because we wish to test gravitation more generally, it is worth recalling its basic elements.

1. Spacetime is describable as a classical four-dimensional manifold with a metric locally equivalent to Minkowski. This is generalizable to higher dimensions, where matter fields reside on a three-dimensional spatial brane.
2. Special relativity holds locally. In particular, energy-momentum is locally conserved. This is more general than GR.
3. The weak equivalence principle holds; i.e., freely falling bodies follow spacetime geodesics. This is more general than GR.
4. The metric is the solution to the Einstein field equations subject to appropriate initial and boundary conditions. This is uniquely true in GR.

The first three ingredients are assumed by most viable theories of gravitation applied on cosmological scales. They are assumed to be correct throughout this paper. The Einstein field equations are not assumed to hold except where explicitly stated below. We assume throughout that the universe is approximately (or, for some calculations, exactly) Robertson-Walker.

2. ROBERTSON-WALKER SPACETIMES AND THEIR PERTURBATIONS

The general Robertson-Walker spacetime is specified by giving a spatial curvature constant K (with units of inverse length squared) and a dimensionless scale factor $a(\tau)$ (normalized so that $a = 1$ today). Let us assume that the evolution of $a(\tau)$ depends on K and on the properties of the matter and energy filling the universe. Applying the cosmological principle, the matter must behave as a perfect fluid at rest in the comoving frame. The pressure of a perfect fluid can be written $p(\rho, S)$, where ρ is the proper energy density and S is the comoving entropy density in the fluid rest frame. The line element can thus be written

$$ds^2 = a^2(\tau, K, S)\left[-d\tau^2 + d\chi^2 + r^2(\chi, K)\, d\Omega^2\right], \quad (2)$$

where $r(\chi, K) = K^{-1/2} \sin(K^{1/2}\chi)$ for $K > 0$ (and is analytically continued for $K \leq 0$) and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$. Initially we make no assumption about the dynamics except that the evolution of the scale factor depends only on the geometry (K) and composition (S) of the $(3+1)$-dimensional universe and not, for example, on parameters describing extra dimensions.[1] Only below is the Friedmann equation assumed to determine the exact form of $a(\tau, K, S)$.

2.1. *Curvature and Coordinate Perturbations*

Metric perturbations are obtained by comparing two slightly different spacetimes. We consider two homogeneous and isotropic Robertson-Walker spacetimes differing only by their (spatially homogeneous) spatial curvature. The first spacetime has spatial curvature K; the second one has spatial curvature $K(1+\delta_K)$, where δ_K is a small constant. We write the metric of the second spacetime as a perturbation of the first, as follows. First, the angular radius can be Taylor-expanded to first order in δ_K to give

$$r^2(\chi, K + K\delta_K) = r^2(\chi, K)(1-\delta_K) + \chi r(\chi, 4K)\delta_K, \quad (3)$$

while $a(\tau, K + K\delta_K, S) = a(\tau, K, S) + \delta_K(\partial a/\partial \ln K)$. To simplify the appearance of the line element we change variables $\tau \to \tau + \alpha(\tau)$ and $\chi \to \chi(1-\kappa)$, where α and κ are assumed to be first-order in δ_K. In these new coordinates, to first order in δ_K the second spacetime has line element

$$\begin{aligned} ds^2 = a^2(\tau, K, S)&\left(1 + 2\mathcal{H}\alpha + 2\delta_K \frac{\partial \ln a}{\partial \ln K}\right) \\ &\times \Big\{ -(1+2\dot{\alpha})\, d\tau^2 + (1-2\kappa)\, d\chi^2 \\ &\quad + \left[(1-\delta_K) r^2(\chi, K) + (\delta_K - 2\kappa)\chi r(\chi, 4K)\right] d\Omega^2 \Big\}, \end{aligned} \quad (4)$$

where $\mathcal{H}(\tau, K, S) \equiv \partial \ln a/\partial\tau$. This line element describes a perfectly homogeneous and isotropic Robertson-Walker spacetime with spatial curvature $K(1+\delta_K)$. However, for appropriate choices of α and κ (i.e., the appropriate coordinate transformation), it takes precisely the same form as a perturbed Robertson-Walker spacetime with background spatial curvature K and perturbation Ψ,

$$\begin{aligned} ds^2 = a^2&(\tau, K, S) \\ &\times \left\{-(1+2\Psi)\, d\tau^2 + (1-2\Psi)\left[d\chi^2 + r^2(\chi, K)\, d\Omega^2\right]\right\}, \end{aligned} \quad (5)$$

provided that the following three conditions hold:

$$\kappa = \frac{1}{2}\delta_K, \quad 2\Psi = \dot{\alpha} + \kappa, \quad \Psi = \kappa\left(1 - 2\frac{\partial \ln a}{\partial \ln K}\right) - \mathcal{H}\alpha. \quad (6)$$

In other words, a perturbed Robertson-Walker spacetime whose metric perturbation $\Psi(\tau)$ is spatially homogeneous is identical to an unperturbed Robertson-Walker spacetime represented by a perturbed coordinate system.[2]

This equivalence is significant because it suggests that long-wavelength curvature perturbations (for which Ψ is effectively independent of spatial position) should evolve like patches of a Robertson-Walker spacetime (Wands et al. 2000; Gordon 2005 and references therein), whose dynamics is more general than general relativity. So far we have assumed that the cosmological principle holds, but we have not assumed the validity of the Einstein field equations of general relativity. However, if we know how a perfect Robertson-Walker spacetime evolves, the evolution of long-wavelength curvature perturbations follows.

[1] It would be straightforward to add such parameters to the argument list of $a(\tau, K, S)$ and then perturb them.

[2] In § 4 we generalize eq. (5) to the case of two distinct potentials for the time and space parts of the metric.

To make further progress let us assume the form of $a(\tau, K, S)$, which requires specifying a theory of gravity. The simplest choice is to assume the validity of the Friedmann equation, which gives

$$\tau(a, K, S) = \int_0^a \left[\frac{8\pi}{3} G\tilde{a}^4 \rho(\tilde{a}, S) - K\tilde{a}^2\right]^{-1/2} d\tilde{a}. \quad (7)$$

Using this, one finds

$$\left(\frac{\partial \ln a}{\partial \ln K}\right)_{\tau, S} = -\frac{(\partial\tau/\partial \ln K)_{a,S}}{(\partial\tau/\partial \ln a)_{K,S}} = -\frac{1}{2} K\mathcal{H} \int^{\tau} \frac{d\tau'}{\mathcal{H}^2(\tau', K, S)}. \quad (8)$$

We also require conservation of energy, which can be written

$$\frac{\partial}{\partial \ln a} \rho(a, S) = -3[\rho + p(\rho, S)]. \quad (9)$$

When combined with the Friedmann equation, this implies

$$\gamma \equiv 4\pi G a^2 (\rho + p) = \mathcal{H}^2 + K - \dot{\mathcal{H}} = \frac{3}{2}(1 + w)\left(\mathcal{H}^2 + K\right), \quad (10)$$

where $\dot{\mathcal{H}} = \partial\mathcal{H}(\tau, K, S)/\partial\tau$. Combining equations (6), (8), and (10) yields a relation between the long-wavelength potential Ψ and the curvature perturbation[3] κ:

$$\kappa = \frac{\mathcal{H}^2}{\gamma a^2} \frac{\partial}{\partial \tau}\left(\frac{a^2 \Psi}{\mathcal{H}}\right). \quad (11)$$

For long wavelengths κ can depend on the wavenumber k but cannot depend on τ, so $\partial\kappa/\partial\tau = 0$, which implies

$$\begin{aligned} \frac{\gamma}{\mathcal{H}} \frac{\partial}{\partial \tau}\left[\frac{\mathcal{H}^2}{\gamma a^2} \frac{\partial}{\partial \tau}\left(\frac{a^2 \Psi}{\mathcal{H}}\right)\right] &= \ddot{\Psi} + 3\left(1 + c_w^2\right)\mathcal{H}\dot{\Psi} + 3\left(c_w^2 - w\right)\mathcal{H}^2\Psi \\ &\quad - \left(2 + 3w + 3c_w^2\right)K\Psi \\ &= 0, \end{aligned} \quad (12)$$

where

$$w \equiv \frac{p(\rho, S)}{\rho}, \quad c_w^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S. \quad (13)$$

By comparing two different Friedmann-Robertson-Walker models, we have arrived at a second-order differential equation for the metric perturbation Ψ. The only dynamical equations assumed have been the Friedmann equation and energy conservation in a homogeneous and isotropic universe. We have not made use of the perturbed Einstein or fluid equations. Nevertheless, as we show next, equation (12) is identical with the dynamical evolution equation for long-wavelength curvature perturbations obtained using the perturbed fluid and Einstein equations of general relativity.

2.2. *Linear Cosmological Perturbations in General Relativity*

Cosmological perturbation theory has been well studied (e.g., Lifshitz 1946; Bardeen 1980; Kodama & Sasaki 1984; Hwang & Noh 2002). Nonetheless, the curvature and entropy variables relevant for long-wavelength density perturbations differ from those given previously in the literature, so a brief summary is presented here.

In the conformal Newtonian gauge (Mukhanov et al. 1992; Ma & Bertschinger 1995), the metric of a perturbed Robertson-Walker spacetime with scalar perturbations can be written as a generalization of equation (5),

$$ds^2 = a^2(\tau)\left\{-(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)\left[d\chi^2 + r^2(\chi, K)\, d\Omega^2\right]\right\}. \quad (14)$$

Here $\Phi(x^i, \tau)$ and $\Psi(x^i, \tau)$ are small-amplitude gravitational potentials. The dependence of the scale factor on K and S is suppressed because we are considering now a single universe with unique values of these parameters and are introducing perturbations only through the potentials.

For scalar perturbations the stress-energy tensor components can be written in terms of spatial scalar fields $\delta\rho$, u, and π, as follows:

$$T^0{}_0 = -(\bar{\rho} + \delta\rho), \quad (15a)$$

$$T^0{}_i = -(\bar{\rho} + \bar{p})\nabla_i u, \quad (15b)$$

$$T^i{}_j = \delta^i{}_j(\bar{p} + \delta p) + \frac{3}{2}(\bar{\rho} + \bar{p})\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i{}_j\Delta\right)\pi, \quad (15c)$$

where ∇_i and ∇^i are the three-dimensional covariant derivative for the spatial line element $d\chi^2 + r^2\, d\Omega^2$, while $\Delta = \nabla^i\nabla_i$. Unless stated otherwise, all variables refer to the total stress-energy summed over all components. The unperturbed energy density and pressure are $\bar{\rho}(\tau)$ and $\bar{p}(\tau)$, respectively, while $\delta\rho$ and δp are the corresponding perturbations measured in the coordinate frame.

In the scalar mode, all perturbations to the metric and the stress-energy tensor arise from spatial scalar fields and their spatial gradients. For example, the energy flux is a potential field with velocity potential $u(x^i, \tau)$. Similarly, the shear stress follows from a shear stress (viscosity) potential $\pi(x^i, \tau)$ (defined as in Bashinsky & Seljak 2004). For an ideal gas, $\pi = 0$. Equations (15a)–(15c) are completely general for the scalar mode. Stress-energy perturbations that arise from divergenceless (transverse) vectors contribute only to the vector mode, while divergenceless, trace-free tensors contribute only to the tensor mode. Vector and tensor modes are ignored in this paper.

Combining the perturbed Einstein field equations (Bertschinger 1996) with equations (10) and (13) gives the following second-order partial differential equation for the linear evolution of the potential Ψ:

$$\begin{aligned} &\frac{\gamma}{\mathcal{H}} \frac{\partial}{\partial \tau}\left[\frac{\mathcal{H}^2}{\gamma a^2} \frac{\partial}{\partial \tau}\left(\frac{a^2}{\mathcal{H}}\Psi\right)\right] - c_w^2 \Delta\Psi \\ &= \gamma\left[\frac{\delta p - c_w^2 \delta\rho}{\bar{\rho} + \bar{p}} + \frac{3}{\mathcal{H}} \frac{\partial}{\partial \tau}\left(\mathcal{H}^2 \pi\right) + \Delta\pi\right]. \end{aligned} \quad (16)$$

Equation (16) is exact in linear perturbation theory of general relativity and is fully general for scalar perturbations. It was given in another form by Hwang & Noh (2002). Aside from the right-hand side and the sound speed term $c_w^2\Delta\Psi$, it agrees exactly with equation (12), whose derivation did not assume the validity of the perturbed Einstein field equations. The right-hand side of equation (16) includes entropy perturbations proportional to $\delta p - c_w^2\delta\rho$ and shear stress perturbations, both of which were absent in the Robertson-Walker models of § 2.1.

[3] For spatially uniform, isentropic perturbations with vanishing shear stress, κ reduces to the ζ variable of Bardeen et al. (1983). In other cases the two variables differ, with κ being simpler in both its dynamics and interpretation.

In a perfectly homogeneous and isotropic universe the shear stress must vanish, and a correct treatment of shear stress requires going beyond the Friedmann and energy conservation equations. In the universe at low redshift, the shear stress due to photons, neutrinos, and gravitationally bound structures is orders of magnitude smaller than the mass density perturbations. Unless the dark energy is a peculiar substance with large shear stress, we can neglect π in the equation of motion for Ψ.

The sound wave term represents the effect of pressure forces in resisting gravitational instability. For a perturbation of comoving wavenumber k, $-c_w^2\Delta\Psi = k^2 c_w^2\Psi$; for comparison the time derivative terms in equation (16) are $\sim\mathcal{H}^2\Psi$. Thus, for wavelengths much longer than the comoving Jeans length $\lambda_{\rm J}$ defined by $\pi/\lambda_{\rm J} = \mathcal{H}/c_w$, the sound wave term can be neglected. In the standard cosmology, the Jeans length at $z < 100$ is less than about 20 Mpc.

The entropy source term in equation (16) can also be obtained by comparing separate Robertson-Walker spacetimes with slightly different entropies, allowing us to reduce the long-wavelength evolution entirely to quadratures, as we show next.

2.3. *Entropy Perturbations*

Consider two Robertson-Walker spacetimes with identical spatial curvature K but with entropies S and $S + \delta S$, respectively, where δS is a small constant. At a given expansion factor a (a^3 plays the role of volume), the density $\rho(a, S)$ and pressure $p(\rho, S)$ will differ slightly in the two spacetimes. Writing the pressure as $p(\rho(a, S), S)$, from equations (9) and (13) we obtain

$$\left(\frac{\partial p}{\partial \ln a}\right)_S = -3c_w^2(\rho + p), \quad \left(\frac{\partial p}{\partial S}\right)_a = \left(\frac{\partial p}{\partial S}\right)_\rho + c_w^2\left(\frac{\partial \rho}{\partial S}\right)_a. \tag{17}$$

Using equations (9) and (17), we find

$$\left(\frac{\partial p}{\partial S}\right)_\rho = -\left(\frac{\rho + p}{3}\right)\frac{\partial}{\partial \ln a}\left[\frac{1}{\rho + p}\left(\frac{\partial \rho}{\partial S}\right)_a\right]. \tag{18}$$

Using this result, we define a fractional entropy perturbation variable,

$$\begin{aligned}\sigma(a, S) &\equiv \frac{\delta p - c_w^2\delta\rho}{\rho + p} = \frac{\delta S}{\rho + p}\left(\frac{\partial p}{\partial S}\right)_\rho \\ &= -\frac{\delta S}{3}\frac{\partial}{\partial \ln a}\left[\frac{1}{\rho + p}\left(\frac{\partial \rho}{\partial S}\right)_a\right].\end{aligned} \tag{19}$$

This is a formal result because for a multicomponent imperfect fluid one would not evaluate $\rho(a, S)$ but would instead characterize density and pressure perturbations for the individual components, as we show below in § 3.1. For now, we assume that σ can be determined, and we use it to derive a quadrature for the isocurvature modes.

The scale factor in the second Robertson-Walker spacetime is $a(\tau, K, S + \delta S) = a(\tau, K, S) + (\partial a/\partial S)\delta S$. Changing time variable $\tau \to \tau + \alpha(\tau)$ (with no change in spatial coordinates), the line element for the second spacetime becomes

$$\begin{aligned}ds^2 &= a^2(\tau, K, S)\left(1 + 2\mathcal{H}\alpha + 2\delta S\frac{\partial \ln a}{\partial S}\right) \\ &\quad\times\left[-(1 + 2\dot{\alpha})\,d\tau^2 + d\chi^2 + r^2(\chi, K)\,d\Omega^2\right].\end{aligned} \tag{20}$$

This line element describes a perfectly homogeneous and isotropic Robertson-Walker spacetime with entropy $S + \delta S$. However, for an appropriate choice of α (i.e., the appropriate coordinate transformation), it takes precisely the same form as a perturbed Robertson-Walker spacetime with background entropy S, given by equation (5), provided that the following two conditions hold:

$$2\Psi = \dot{\alpha}, \quad \Psi = -\delta S\frac{\partial \ln a}{\partial S} - \mathcal{H}\alpha. \tag{21}$$

So far we have assumed nothing about gravity. To proceed further we assume that equation (7) is valid, giving

$$\begin{aligned}\left(\frac{\partial \ln a}{\partial S}\right)_{\tau,K} &= -\frac{(\partial\tau/\partial S)_{a,K}}{(\partial\tau/\partial \ln a)_{K,S}} \\ &= \frac{4\pi G\mathcal{H}}{3}\int^\tau \frac{a^2(\tau', K, S)}{\mathcal{H}^2(\tau', K, S)}\left(\frac{\partial\rho}{\partial S}\right)_{a'} d\tau'.\end{aligned} \tag{22}$$

Combining equations (10), (19), (21), and (22) gives

$$\begin{aligned}\frac{1}{a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2\Psi}{\mathcal{H}}\right) &= -\delta S\frac{\partial}{\partial\tau}\left[\frac{1}{\mathcal{H}}\left(\frac{\partial \ln a}{\partial S}\right)_{\tau,K}\right] \\ &= -\frac{4\pi G a^2\delta S}{3\mathcal{H}^2}\left(\frac{\partial\rho}{\partial S}\right)_a \\ &= \frac{\gamma}{\mathcal{H}^2}\int^\tau \sigma(a(\tau', K, S), S)\mathcal{H}(\tau', K, S)\,d\tau'.\end{aligned} \tag{23}$$

This result agrees exactly with equation (16) for long-wavelength entropy perturbations in GR with vanishing shear stress potential π. Therefore, in GR the dynamical evolution of long-wavelength entropy perturbations in a Robertson-Walker spacetime follows directly from the Friedmann and energy conservation equations without requiring the perturbed Einstein field equations. Equation (23) implies equation (12) when the entropy perturbation vanishes.

2.4. *General Solution for Long-Wavelength Perturbations in General Relativity*

Equations (12) and (23) have a simple exact solution. The homogeneous solution with $\sigma = 0$ (curvature perturbations) is (suppressing the dependencies on K and S, which are held fixed)

$$\begin{aligned}&\Psi(\tau) = \kappa\Psi_+(\tau) + C\Psi_-(\tau), \\ &\Psi_+(\tau) = \frac{\mathcal{H}}{a^2}\int^\tau \frac{\gamma(\tau')a^2(\tau')}{\mathcal{H}^2(\tau')}\,d\tau', \quad \Psi_-(\tau) = \frac{\mathcal{H}}{a^2}.\end{aligned} \tag{24}$$

Here κ is the curvature perturbation of equation (11), which gives the amplitude of the "growing mode" of density perturbations; because the lower limit of integration for $\Psi_+(\tau)$ is unspecified, one can add any constant multiple C of the "decaying mode" solution $\Psi_-(\tau) = \mathcal{H}a^{-2}$. The decaying mode is a gauge mode that can be eliminated using the coordinate transformation $\tau \to \tau + Ca^{-2}$. The particular solution with $\Psi = \dot{\Psi} = 0$ at $\tau = 0$ (isocurvature perturbations) and $\sigma \neq 0$ (again suppressing the dependencies on K and S) is

$$\Psi(\tau) = \int_0^\tau [\Psi_+(\tau)\Psi_-(\tau') - \Psi_+(\tau')\Psi_-(\tau)]\sigma(\tau')a^2(\tau')\,d\tau'. \tag{25}$$

The general solution of equation (16) with $\pi = \Delta\Psi = 0$ is given by adding equations (24) and (25). We see that it is equivalent to integration of

$$2\Psi = \dot{\alpha} + \kappa, \quad \Psi = -\mathcal{H}\alpha + \kappa\left(1 - 2\frac{\partial \ln a}{\partial \ln K}\right) - \delta S\frac{\partial \ln a}{\partial S} \tag{26}$$

when the Friedmann equation governs the background expansion. By comparing the evolution of separate Robertson-Walker universes we have solved equation (26) by the quadratures of equations (24) and (25). The solution requires only that the background evolution obeys the Friedmann and energy conservation equations and that spatial gradient terms are negligible. Later we drop the assumption of the Friedmann equation to obtain quadratures for any theory of gravity.

Although we have set $k^2 = -\Delta = 0$, the solution is valid for all wavelengths much greater than the Jeans length. One simply allows κ and δS to depend on wavevector $\boldsymbol{k}$ as determined by initial conditions. At short wavelengths (shorter than the Jeans length, for example) κ and δS, as obtained using equations (11) and (19), depend on time.

3. DENSITY AND VELOCITY PERTURBATIONS AND EINSTEIN CONSTRAINTS

In addition to the metric perturbations, the density and velocity perturbations of the matter can also be reduced to quadratures in the long-wavelength limit assuming local energy-momentum conservation. The density and velocity perturbations then obey initial-value constraint equations that relate them to the metric perturbations.

Under the coordinate transformations given in the preceding sections, the value of $T^0{}_0$ does not change to first order in the perturbations. However, comparing the density field of two different Robertson-Walker spacetimes at the same coordinate values (τ, x^i) gives a density perturbation:[4]

$$\begin{aligned}\delta(\tau) \equiv \frac{\delta\rho}{\bar{\rho}+\bar{p}} &= -3\left[\mathcal{H}\alpha + 2\kappa\left(\frac{\partial \ln a}{\partial \ln K}\right)_{\tau,S} + \delta S\left(\frac{\partial \ln a}{\partial S}\right)_{\tau,K}\right] \\ &\quad + \frac{\delta S}{\bar{\rho}+\bar{p}}\left(\frac{\partial\rho}{\partial S}\right)_a \\ &= 3(\Psi - \kappa) + \frac{\delta S}{\bar{\rho}+\bar{p}}\left(\frac{\partial\rho}{\partial S}\right)_a \\ &= 3(\Psi-\kappa) - 3\int_0^\tau \sigma(\tau')\mathcal{H}(\tau')\,d\tau' + A,\end{aligned} \tag{27}$$

where A is an integration constant. Equation (27) also follows from energy conservation in a perturbed Robertson-Walker spacetime, which gives the perturbed continuity equation

$$\dot{\delta} + 3\mathcal{H}\sigma = 3\dot{\Psi} + \Delta u, \tag{28}$$

where u is the velocity potential. For long wavelength perturbations of wavenumber $k \to 0$, $\Delta u = -k^2 u$ is generally negligible compared with the other terms in the equation.

Determining the velocity perturbations requires considering spatial variations of the velocity potential, which do not exist in a perfectly homogenous Robertson-Walker spacetime. Thus, we consider a perturbed Robertson-Walker spacetime with metric (14) and energy-momentum tensor (15). Evaluating $\nabla_\mu T^\mu{}_i = 0$ gives the first-order result

$$\dot{u} + \left(1 - 3c_w^2\right)\mathcal{H}u = c_w^2\delta + \sigma + \Phi + (\Delta + 3K)\pi. \tag{29}$$

If the shear stress can be neglected, we can integrate to obtain

$$\begin{aligned}u(\tau) = \frac{1}{(\rho+p)a^4}\int_0^\tau &\Big\{\left[c_w^2(\tau')\delta(\tau') + \sigma(\tau') + \Phi(\tau')\right] \\ &\times [\rho(\tau') + p(\tau')]a^4(\tau')\,d\tau'\Big\} + \frac{B}{(\rho+p)a^4},\end{aligned} \tag{30}$$

where B is an integration constant.

Equations (27)–(30) do not assume the Einstein field equations—they hold for any theory of gravity provided it is consistent with local energy-momentum conservation. However, the integral forms of these equations are rarely used. Instead, given the potential Ψ on long wavelengths, the usual procedure in general relativity is to evaluate the density and velocity potential using initial-value constraints, of which the Poisson equation is one. That these constraint equations are more general is shown next.

We define the following combinations of metric and energy-momentum perturbations:

$$C_1 = (\Delta + 3K)\Psi - \gamma(\delta + 3\mathcal{H}u), \tag{31a}$$

$$C_2 = \dot{\Psi} + \mathcal{H}\Phi - \gamma u, \tag{31b}$$

$$C_3 = \Psi - \Phi - 3\gamma\pi. \tag{31c}$$

Here $\delta + 3\mathcal{H}u$ is the number density perturbation on a hypersurface in which $T^0{}_i$ vanishes, i.e., the number density perturbation in the local fluid rest frame.

In general relativity each of the constraints vanishes ($C_1 = C_2 = C_3 = 0$); however, we do not assume this to be automatically true. Differentiating the constraints and using equations (10), (28), and (29) gives

$$\begin{aligned}&\frac{1}{a}\frac{\partial}{\partial\tau}(aC_1) - (\Delta + 3K - 3\gamma)C_2 - \mathcal{H}(\Delta+3K)C_3 \\ &= 3\gamma\left(\mathcal{H}^2 + K - \dot{\mathcal{H}} - \gamma\right)u,\end{aligned} \tag{32a}$$

$$\begin{aligned}&\frac{\gamma}{a}\frac{\partial}{\partial\tau}\left(\frac{a}{\gamma}C_2\right) - c_w^2 C_1 + \frac{\gamma}{\mathcal{H}}\frac{\partial}{\partial\tau}\left(\frac{\mathcal{H}^2}{\gamma}C_3\right) - KC_3 \\ &= \frac{\gamma}{\mathcal{H}}\frac{\partial}{\partial\tau}\left[\frac{\mathcal{H}^2}{\gamma a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2}{\mathcal{H}}\Psi\right)\right] - c_w^2\Delta\Psi - \gamma\left[\sigma + \frac{3}{\mathcal{H}}\frac{\partial}{\partial\tau}\left(\mathcal{H}^2\pi\right) + \Delta\pi\right] \\ &\quad + \left[\Psi\left(\frac{1}{\mathcal{H}}\frac{\partial}{\partial\tau} + 1 + 3c_w^2\right) - \Phi\right]\left(\dot{\mathcal{H}} + \gamma - \mathcal{H}^2 - K\right).\end{aligned} \tag{32b}$$

Here we set $\gamma \equiv 4\pi G a^2(\bar{\rho} + \bar{p})$ and have assumed nothing about gravity. If we assume that the Einstein field equations hold, the right-hand sides of equations (32a) and (32b) vanish. If we make the weaker assumption that the background is governed by the Friedmann equation and that $\Delta\Psi$ and π (shear stress) can be neglected on large scales so that equation (23) holds, the right-hand sides still vanish. In keeping with the previous treatment we also assume $\Psi = \Phi$ so that $C_3 = 0$. With these assumptions we can integrate equations (32a) and (32b) to obtain

$$\begin{aligned}C_1 &= 3\mathcal{H}C_2 - A\gamma, \\ C_2 &= -\frac{A}{a^2}\int_0^\tau c_w^2(\tau')\gamma(\tau')a^2(\tau')\,d\tau' - \frac{4\pi GB}{a^2},\end{aligned} \tag{33}$$

[4] Note the unusual definition of δ; for a gas of particles it gives the relative number density perturbation rather than the relative energy density perturbation. The equations simplify with this choice.

where A and B are integration constants. Comparison with equations (27), (30), and (31a)–(31c) shows that these are exactly the same initial-value constants obtained from integrating the equations of energy-momentum conservation. In other words, they correspond to changing δ and u without changing Ψ. The third initial-value constant C appearing in equation (24) contributes nothing new to the constraints because the pure decaying mode has $\delta = 3\Psi = -3\mathcal{H}u$; the decaying mode can be eliminated by a change of the time coordinate.

In general relativity, the Einstein field equations give $C_1 = C_2 = C_3 = 0$ implying $A = B = 0$. However, in other theories of gravity the constraints can be nonzero without contradicting equations (32a) and (32b). The initial-value constants follow from

$$A = \lim_{a\to 0} \frac{T\delta S}{a^3(\bar\rho + \bar p)}, \quad B = \lim_{a\to 0} a^4(\bar\rho + \bar p)u, \tag{34}$$

where T is the temperature. While nonzero values A and B are testable, in principle, by comparing the density and velocity fields of galaxies with the gravitational potential implied by gravitational lensing or hydrostatic equilibrium in clusters of galaxies, much more stringent tests obtain at high redshift, where the A and B terms contribute relatively more to the energy-momentum tensor. It would be interesting to place limits on these constants using measurements of the cosmic microwave background anisotropy.

Assuming that A and B are small compared with δ and $a^4(\bar\rho + \bar p)u$, respectively, at low redshift, and assuming furthermore that the Friedmann equation and energy-momentum conservation are valid, then the density and velocity fields in a perturbed Robertson-Walker spacetime whose metric takes the form of equation (5) must on large scales obey the same constraint equations as in general relativity, namely, $C_1 = C_2 = C_3 = 0$.

Although the linearized Einstein field equations give three constraints, the first two ($C_1 = C_2 = 0$) are initial-value constraints with no dynamical content. If $C_1 = C_2 = 0$ on an initial time slice, energy-momentum conservation combined with the other Einstein equations is enough to ensure $C_1 = C_2 = 0$ for all times. The third constraint, $C_3 = 0$, is a true dynamical constraint because its time derivative is not forced to vanish as a result of the Einstein field equations and energy-momentum conservation.

We see below that initial-value constraints exist not only in GR but in any gravity theory yielding a long-wavelength perturbed Robertson-Walker solution with local energy-momentum conservation. The key distinguishing features of general relativity are then seen to be the Friedmann equation plus the dynamical constraint $C_3 = 0$.

3.1. *Multicomponent Fluids*

Equations (25), (27), and (30) are true quadratures only if $\sigma(\tau)$ is known. While equation (19) gives σ from $p(\rho, S)$ or $\rho(a, S)$, the entropy depends on the internal degrees of freedom of a fluid.

Consider a multicomponent imperfect fluid whose density and pressure are described by a set of parameters $\{X\}$ that can vary with position at fixed expansion factor a. We replace the single entropy S by as many parameters as are necessary to characterize spatial variations in the equation of state of the fluid. For example, a system of noninteracting fluids has

$$\rho(a, X) = \sum_i \tilde\rho_i r_i(a), \quad p(a, X) = \sum_i w_i \tilde\rho_i r_i(a),$$
$$r_i(a) = \exp\left[3\int_a^1 (1 + w_i)\, d\ln a\right], \tag{35}$$

where the $\tilde\rho_i$ are independent of a. The w_i generally are not spatially varying; for example, $w = 0$ for cold dark matter ("dust") and $w = \frac{1}{3}$ for a relativistic ideal gas ("radiation"). In this case the parameters are the abundances of each fluid component, $\{X\} = \{\tilde\rho_1, \tilde\rho_2, \ldots\}$. The same procedure will work regardless of what parameters characterize the multicomponent fluid, but the set $\{X\}$ includes only those that can vary with position at a fixed value of a (a playing the role of volume).

The net equation of state and sound speed parameters are $w \equiv p(a, X)/\rho(a, X)$ and $c_w^2 \equiv (\partial p/\partial a)/(\partial\rho/\partial a)$. The entropy perturbation follows in a manner similar to equation (19):

$$\sigma = \sum_X \frac{\delta X}{\rho + p}\left(\frac{\partial p}{\partial X} - c_w^2 \frac{\partial\rho}{\partial X}\right) = -\frac{1}{3}\frac{\partial\epsilon}{\partial\ln a},$$
$$\epsilon(a, X) \equiv \sum_X \frac{\delta X}{\rho + p}\left(\frac{\partial\rho}{\partial X}\right)_a. \tag{36}$$

The symbol ϵ, which equals the number density perturbation at fixed expansion factor a, is introduced here to avoid confusion with the number density perturbation δ measured at fixed τ. For the example of equation (35),

$$\sigma = \sum_i f_i\left(w_i - c_w^2\right)\epsilon_i = \sum_i f_i w_i(\epsilon_i - \epsilon), \tag{37}$$

where

$$f_i(a) \equiv \frac{(\tilde\rho_i + \tilde p_i) r_i(a)}{\rho + p}, \quad \epsilon_i \equiv \frac{\delta\tilde\rho_i}{\tilde\rho_i + \tilde p_i}, \quad \epsilon = \sum_i f_i \epsilon_i. \tag{38}$$

Here $\tilde p_i = w_i \tilde\rho_i$ and ϵ_i are both independent of time and f_i is the enthalpy fraction for component i, normalized so that $\sum_i f_i = 1$. We see that the entropy perturbation of a superposition of ideal gases arises entirely from differences in the particle number density of the different species measured at fixed expansion a. For example, for cold dark matter (subscript m) and radiation (subscript r) with $y(a) \equiv \rho_m/\rho_r$, $f_m = 3y/(3y + 4)$, $f_r = 4/(3y + 4)$, and $\sigma = \frac{1}{3} f_r f_m(\epsilon_r - \epsilon_m)$. For a superposition of N fluids there are $N - 1$ independent entropy modes corresponding to the $N - 1$ independent differences of the number density perturbations ϵ_i.

In the general case in which the individual components have a time-varying w_i and can also have entropy perturbations,

$$\sigma = \sum_i f_i\left[\sigma_i + \left(c_i^2 - c_w^2\right)\epsilon_i\right] = \sum_i f_i\left[\sigma_i + c_i^2(\epsilon_i - \epsilon)\right], \tag{39}$$

where $c_i^2 \equiv [\partial(w_i r_i)/\partial a]/(\partial r_i/\partial a)$ and σ_i are the sound speed squared and entropy perturbation for each component, respectively. As an example, consider a tightly coupled plasma of photons and nonrelativistic baryons with $w = [3(1 + y_b)]^{-1}$, where $y_b(a) \equiv \rho_b/\rho_r$ (subscript b denotes baryons and r denotes photons). In this case, $w = f_r/(4 - f_r)$ and $c_w^2 = \frac{1}{3} f_r$, where $f_r = 4/(3y_b + 4)$. Although this plasma behaves like a single fluid, it can have a nonzero entropy perturbation arising from initial variations in the baryon-to-photon ratio, $\sigma = \frac{1}{3} f_r f_b(\epsilon_r - \epsilon_b)$, where $\epsilon_r - \epsilon_b = \delta\ln(n_r/n_b)$, the fractional perturbation in the photon-to-baryon number density ratio, is constant in time. Thus, by examining the composition of each fluid component, we can write the time-dependent total entropy perturbation σ in terms of a set of time-independent constants ϵ_i. By doing so, we are able to fully reduce to quadratures the metric perturbation as shown in § 2.4.

Using the same methods we obtain exact results for the density of individual fluid components. Here one must be careful to evaluate the perturbations at fixed (τ, χ) after the coordinate transformations $\tau \to \tau + \alpha(\tau)$ and $\chi \to \chi(1-\kappa)$ discussed in § 2.1. The density perturbation of component i at fixed τ is

$$\delta_i(\tau) = \frac{\rho_i(a(\tau+\alpha, K+\delta K, X+\delta X), X+\delta X) - \rho_i(a, X)}{(\rho+p)_i} = 3(\Psi - \kappa) + \epsilon_i. \quad (40)$$

The ϵ_i comes from the initial number density perturbation at fixed a, while $3(\kappa - \Psi)$ is the fractional change in volume introduced by shifting the initial hypersurface of constant τ. Averaging over all components with weights f_i gives the net density perturbation

$$\delta(\tau) = 3(\Psi - \kappa) + \epsilon(\tau), \quad (41)$$

where $\epsilon(\tau) \equiv \epsilon(a(\tau, K, X), X)$ is given by equation (36) and we are holding K and X fixed to lowest order in perturbation theory. By comparing equations (27) and (41) and using $\partial\epsilon/\partial \ln a = -3\sigma$ we see that the constant A has been determined.

The velocity potential of individual components follows from equation (30) with subscript i applying to all quantities except a, τ, and Φ. For N uncoupled fluids there are N constants B_i. The net velocity potential is the average over the fluids, $u = \sum_i f_i u_i$.

We have succeeded in reducing the long-wavelength density, velocity, and entropy perturbations of all fluid components to quadratures. This presentation works for both general relativity and alternative theories of gravity. In the former case, equations (24) and (25) reduce the problem entirely to quadratures specified by a set of constants $(\kappa, \epsilon_1, \ldots, \epsilon_N, B_1, \ldots, B_N)$. We explore next the reduction of the metric perturbation to quadratures for alternative theories of gravity.

4. QUADRATURES FOR ALTERNATIVE THEORIES OF GRAVITY

Much of the treatment given so far assumes the validity of the Friedmann equation or GR. It is straightforward to redo the calculations of equations (4)–(11) and (22)–(23) without making these assumptions. The expansion rate $\mathcal{H}$ can depend only on the scale factor a and curvature K of the background Robertson-Walker spacetime and the parameters $\{X\}$ describing the equation of state as discussed in § 3.1: $\mathcal{H} = \mathcal{H}(a, K, X)$. In this case equation (7) is replaced by some function $\tau(a, K, X) = \int_0^a [a\mathcal{H}(a, K, X)]^{-1}\, da$. In addition, for an arbitrary theory of gravity the scalar mode has two distinct gravitational potentials, equation (14). Repeating the derivations of §§ 2.1 and 2.3 with the metric of equation (14) with $\Phi \neq \Psi$ and with an arbitrary background expansion rate $\mathcal{H}(a, K, X)$ gives the following result for long-wavelength metric perturbations with $\pi = 0$:

$$\begin{aligned}\frac{1}{a^2}\frac{\partial}{\partial\tau}\left(\frac{a^2\Psi}{\mathcal{H}}\right) + (\Phi - \Psi) &= \frac{\kappa}{a}\frac{\partial}{\partial\tau}\left(\frac{a}{\mathcal{H}}\right) + \mathcal{H}\frac{\partial}{\partial \ln a}\left[2\kappa\left(\frac{\partial\tau}{\partial \ln K}\right)_{a,X} + \sum_X \delta X\left(\frac{\partial\tau}{\partial X}\right)_{a,K}\right] \\ &= \frac{\kappa}{a}\frac{\partial}{\partial\tau}\left(\frac{a}{\mathcal{H}}\right) - 2\kappa\left(\frac{\partial \ln\mathcal{H}}{\partial \ln K}\right)_{a,X} - \sum_X \delta X\left(\frac{\partial \ln\mathcal{H}}{\partial X}\right)_{a,K}. \quad (42)\end{aligned}$$

The quantities κ and δX are (for long wavelengths) independent of time.

Equation (42) provides one differential equation for two functions, Φ and Ψ. Without a second relation, provided by a law of gravity, we can determine neither Φ nor Ψ.

Even without a full theory of gravity, however, we can still obtain interesting and useful results if we assume energy-momentum conservation. In this case we have quadratures (30) and (41). By combining these equations with equation (42), one can derive the following results, which generalize the energy-momentum constraints C_1 and C_2 of general relativity in the limit of vanishing shear stress ($\pi = 0$) and long wavelengths (the spatial Laplacian $\Delta = 0$ when applied to all perturbations):

$$\delta + 3\mathcal{H}u = \frac{3\mathcal{H}B'}{a^4(\rho+p)}, \quad (43a)$$

$$\dot{\Psi} + \mathcal{H}\Phi = -a\frac{\partial}{\partial\tau}\left(\frac{\mathcal{H}}{a}\right)u + \frac{\mathcal{H}}{a^2(\rho+p)}\frac{\partial}{\partial a}\left(\frac{\mathcal{H}B'}{a}\right), \quad (43b)$$

where

$$B'(a) \equiv \int_0^a \left\{a^2(\rho+p)\left[-2\kappa\left(\frac{\partial \ln\mathcal{H}}{\partial \ln K}\right)_{a,X} - \sum_X \delta X\left(\frac{\partial \ln\mathcal{H}}{\partial X}\right)_{a,K} + \frac{\epsilon}{3a}\frac{\partial}{\partial\tau}\left(\frac{a}{\mathcal{H}}\right)_{K,X}\right]\frac{a\, da}{\mathcal{H}}\right\} + B. \quad (44)$$

The constant B is essentially the same integration constant as in equation (30). Without loss of generality we can set $B = 0$ in equation (44).

Equations (41)–(43b) are not all independent. Any three of the four imply the remaining one. Thus, we can describe the perturbations by equations (41), (43a), and (43b). To reduce the system fully to quadratures requires one more equation equivalent to a relation between Φ and Ψ.

It is worth emphasizing that the reduction to quadratures is based on only a few assumptions.

1. The spacetime is nearly Robertson-Walker with perturbation amplitude small enough for linear perturbation theory to apply.
2. Local energy-momentum conservation holds: $\nabla_\mu T^{\mu\nu} = 0$.
3. In the conformal Newtonian gauge, spatial gradients of fields are small enough to be neglected in all equations of motion.
4. Shear stress perturbations are negligible.
5. A theory of gravity provides some relation between Φ and Ψ and possibly the matter perturbations. This assumption is needed only for a complete reduction to quadratures.

There are no restrictions on the equation of state of matter and radiation fields. There are no restrictions on the geometry of perturbations; in particular, there is no assumption of spherical symmetry. The neglect of spatial gradients becomes invalid on scales over which nongravitational forces act so as to modify local energy-momentum conservation. By definition, spatial gradients of pressure are important and modify our results on scales less than the Jeans length.

Constraints (43a) and (43b) are particularly useful when $B' = 0$. For any theory having a flat Robertson-Walker solution (including GR), $\lim_{K\to 0}(\partial \ln\mathcal{H}/\partial \ln K) = 0$. When there are no entropy perturbations, $\epsilon = 0$. Thus, in a flat universe with initially isentropic perturbations, the only possible contribution to B' comes from the $\sum_X \delta X(\partial \ln\mathcal{H}/\partial \ln X)$ term. If the only dependence of $\mathcal{H}$ on equation of state parameters X is through the density ρ, then

$$\sum_X \delta X \left(\frac{\partial \ln \mathcal{H}}{\partial X}\right)_{a,K} = \epsilon(\rho + p)\frac{\partial \ln \mathcal{H}}{\partial \rho}, \tag{45}$$

which vanishes for isentropic initial conditions $\epsilon = 0$. Thus, for a broad class of theories, $B' = 0$ for curvature perturbations in a flat universe implying the long-wavelength constraints $\delta + 3\mathcal{H}u = 0$ and $\dot{\Psi} + \mathcal{H}\Phi = \gamma u$, where $\gamma = -a\,\partial(\mathcal{H}/a)/\partial\tau$ for $K = 0$. From equation (41) we also have $\delta = 3(\Psi - \kappa)$ for long wavelengths and isentropic initial conditions. In this case we have three relations between the four time-dependent functions (Φ, Ψ, δ, u). In the general case we have an additional function, B', whose specification requires a theory for the homogeneous expansion.

Thus, without knowing anything about the underlying gravity theory except that it is consistent with local energy-momentum conservation and that it admits a Robertson-Walker solution, we have deduced initial-value constraints similar to those of GR under the conditions of long wavelengths and negligible shear stress. One is not free to specify arbitrary metric and matter perturbations on long wavelengths, at least under the assumption that long-wavelength perturbations evolve like separate universes.

However, the initial-value constraints (43a) and (43b) combined with equations (36) and (41) are insufficient to fully determine the growth of perturbations. We need a third constraint, a relation between Φ and Ψ given by some theory of gravity. Such a dynamical constraint would allow equation (42) to be integrated and, with the initial-value constraints, reduce (Φ, Ψ, δ, u) fully to quadratures for long-wavelength perturbations with vanishing shear stress.

The simplest case of a dynamical constraint on the gravitational fields is $\Phi = \Psi$ for $\pi = 0$ as in GR (although this is more general than GR, since we do not assume the Friedmann equation). Under this assumption, equation (42) can be integrated completely, giving (suppressing the dependencies on K and X and using a as the time variable)

$$\Psi(a) = \kappa\Psi_+(a) + \sum_X (\delta X)\Psi_X(a) + C\Psi_-(a) \quad \text{if } \Phi = \Psi, \tag{46}$$

where

$$\Psi_+(a) = \frac{\mathcal{H}}{a^2}\int_0^a \left[1 - \left(\frac{\partial \ln \mathcal{H}}{\partial \ln a}\right)_{K,X} - 2\left(\frac{\partial \ln \mathcal{H}}{\partial \ln K}\right)_{a,X}\right]\frac{a\,da}{\mathcal{H}},$$
$$\Psi_-(a) = \mathcal{H}a^{-2}, \tag{47}$$

$$\Psi_X(a) = -\frac{\mathcal{H}}{a^2}\int_0^a \left(\frac{\partial \ln \mathcal{H}}{\partial X}\right)_{a,K}\frac{a\,da}{\mathcal{H}}. \tag{48}$$

The potential has been reduced entirely to quadratures depending only on the expansion rate $\mathcal{H}$ as a function of expansion, curvature, and equation of state parameters.

Considering again a flat universe with isentropic initial conditions and assuming equation (45) holds, but now with the added condition $\Phi = \Psi$, we see that $\Psi = \kappa\Psi_+$ is determined completely by the expansion rate $\mathcal{H}(a, 0, X)$ with constant values of the equation of state parameters X. The density and velocity potential perturbations follow from $\delta = 3(\Psi - \kappa) = -3\mathcal{H}u$. In this case the growth of long-wavelength perturbations is completely determined by the expansion history $\mathcal{H}(a)$ with fixed curvature $(K = 0)$ and composition $(\delta X = 0)$.

For example, in the DGP brane-world theory of Dvali et al. (2000), the Friedmann equation is modified by replacing ρ by $[(\rho + \rho_{r_c})^{1/2} + (\rho_{r_c})^{1/2}]^2$, where ρ_{r_c} is a constant. If $\Phi = \Psi$, equation (11) remains valid for the long-wavelength curvature mode provided that γ changes to

$$\gamma' \equiv \mathcal{H}^2 + K - \dot{\mathcal{H}} = \frac{3}{2}(1 + w)\left(\mathcal{H}^2 + K\right)\left(\frac{\sqrt{\rho + \rho_{r_c}}}{\sqrt{\rho + \rho_{r_c}} + \sqrt{\rho_{r_c}}}\right). \tag{49}$$

The isocurvature mode is also modified slightly, with $\gamma \to \gamma'\rho/(\rho + \rho_{r_c})$ in the last line of equation (23). Equations (27) and (30) are unchanged.

The solutions implied here are only valid as $k \to 0$, so spatial gradient terms can be neglected in the equations of motion. The DGP model has a length scale $r_c = (32\pi G\rho_{r_c}/3)^{-1/2}$ comparable to H_0^{-1}. The long-wavelength limit becomes $kr_c \ll 1$ suggesting that the formal solution given here is not applicable for wavelengths shorter than the present Hubble length. Other theories of gravity will give different ranges of applicability. In general, any long-range departures from general relativity will modify the evolution of patches of a Robertson-Walker spacetime.

The results of this section highlight the importance of testing the dynamical constraint $C_3 = \Psi - \Phi - 3\gamma\pi = 0$ in general relativity. The relation between Φ and Ψ, combined with the background expansion rate $\mathcal{H}(a, K, X)$, is the key to long-wavelength perturbations of a Robertson-Walker cosmology.

5. COMPARING GEOMETRIC AND DYNAMIC METHODS FOR TESTING GRAVITY AND DARK ENERGY

As we have seen, the evolution of long-wavelength perturbations requires two ingredients: (1) a relation between the two gravitational potentials Φ and Ψ (possibly involving auxiliary fields) and (2) a generalization of the Friedmann equation for the background expansion rate. Without the first ingredient, the long-wavelength perturbations cannot be predicted. Let us assume that a dynamical constraint exists relating Φ and Ψ so that equation (42) can be integrated. Then the evolution of perturbations is determined by the expansion rate of the universe, the very same information that determines the redshift-distance relation used by geometric methods. Does this mean that dynamic and geometric methods measure the same thing?

Not necessarily. First, the dependence of perturbation (dynamic) methods on the expansion rate necessarily depends on the value of $\Phi - \Psi$, which can be nonzero for theories of gravity other than GR. Second, dynamic methods depend not only on $\mathcal{H}(a, K, X)$ and its variation with a but also on $(\partial \ln \mathcal{H}/\partial \ln K)_{a,X}$ (for curvature perturbations) or $(\partial \ln \mathcal{H}/\partial X)_{a,K}$ (for entropy perturbations; here X are parameters describing the fluid composition or equation of state). Geometric methods, by comparison, depend only on $\mathcal{H}(a, K, X)$ with fixed K and X. If the universe is nonflat or if there exist entropy perturbations, dynamic methods have the potential to reveal information about the dependence of $\mathcal{H}$ on K and X that is not present in the expansion history for one universe with fixed K and X. Finally, the quadrature results for perturbations are valid only in the limit of long wavelengths; long-range forces different from Einstein gravity can lead to spatial gradient terms that modify the quadratures on currently observable scales.

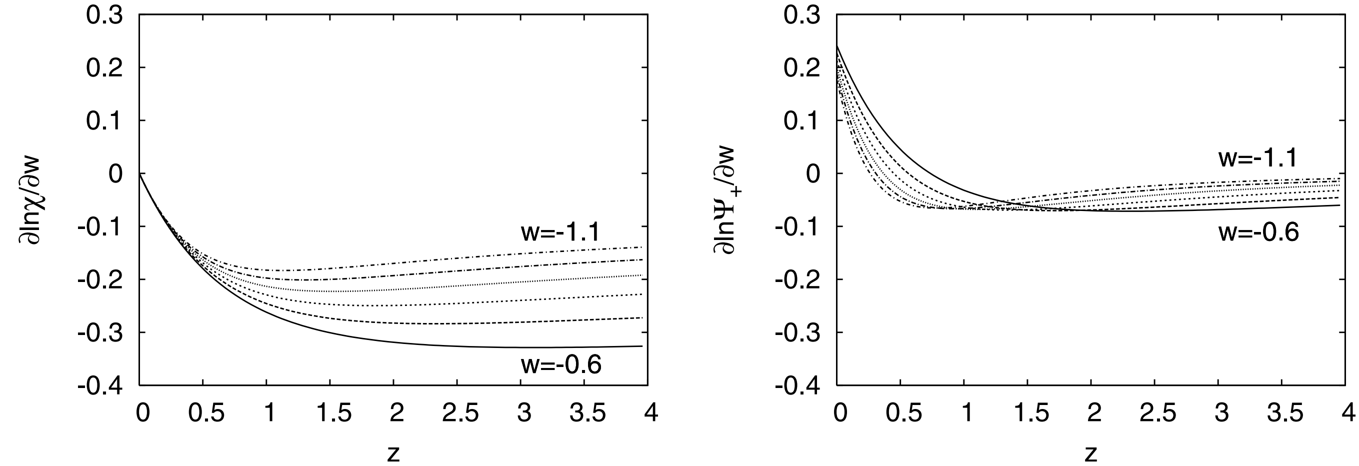

FIG. 1.—Sensitivity of geometric methods (*left*; characterized by the comoving distance χ to redshift z) and dynamic methods (*right*; characterized by the curvature perturbation Ψ_+ at redshift z) to the equation of state parameter w. A flat model with $\Omega_m = 0.3$ was assumed. Curves are shown for $w = -0.6, -0.7, \ldots, -1.1$. Dynamic (perturbation) methods are insensitive to dark energy at high redshift but are more sensitive than geometric methods at low redshift.

Geometric tests are generally cast in terms of the luminosity distance or angular-diameter distance but in fact depend strictly on the redshift-distance relation for radial null geodesics,

$$\chi(z) = \tau_0 - \tau(a, K, X) = \int_0^z \frac{dz'}{H(z')}, \tag{50}$$

where $\tau_0 = \tau(1, K, X)$ and $a = (1+z)^{-1}$. The redshift dependence of the long-wavelength curvature perturbation $\Psi_+(z)$ requires specifying a theory of gravity. If $\Phi = \Psi$, for both general relativity and DGP brane worlds the same result holds on very long wavelengths:[5]

$$\Psi_+(z) = (1+z)H(z) \int_z^\infty \left[\frac{K}{H^2(z')} + \frac{1}{(1+z')H(z')}\frac{dH}{dz'}\right]\frac{dz'}{H(z')}. \tag{51}$$

More generally, let us make no assumptions about $\Phi - \Psi$ but consider a flat universe with negligible shear stress and initially isentropic curvature fluctuations as predicted by the simplest inflationary universe models. In this case, equation (42) reduces to the following relation between $H(z)$, $\Psi(z)$, and $\Phi(z)$ for long wavelengths:

$$\frac{\Phi(z)}{1+z} = \frac{d\Psi}{dz} + \frac{(\kappa - \Psi)}{H}\frac{dH}{dz}. \tag{52}$$

Here κ is independent of redshift on large scales, although κ, Ψ, and Φ will vary with position or spatial wavenumber. In principle, measurements of $H(z)$ from geometric methods and $\Psi(z)$ from perturbations could (for isentropic perturbations of a flat universe with negligible shear stress) determine $\Phi(z)$ up to an additive term proportional to $d \ln H/d \ln(1+z)$ (since the relation between κ and Ψ is unknown if the theory of gravity is unspecified). In particular, such measurements could enable GR or alternative theories of gravity to be tested without any assumptions about the density and pressure of mass-energy in the universe.

This kind of test is difficult to imagine carrying out because of the difficulty of measuring $\Psi(z)$.[6] Alternatively, if one assumes that GR is valid, dynamic methods can be used to provide additional constraints on dark energy because the geometric and dynamic methods have a different dependence on the equation of state of dark energy.

Geometric measurements of $\chi(z)$ at a given redshift depend on the expansion history at smaller redshifts; dynamic measurements of $\Psi(z)$ depend on $H(z)$ at higher redshifts. Because the effects of dark energy typically decline with increasing redshift, perturbation methods are at a disadvantage at high redshift but may be superior to geometric methods at low redshift.

To assess their relative merits for measuring dark energy (assuming GR is the correct gravity theory), the methods were compared using a flat model with $\Omega_m = 0.3$ and a dark energy component with constant w independent of z. While unphysical, this model is commonly used to compare theories with data. The logarithmic derivatives of equations (50) and (51) with respect to w at fixed z were evaluated numerically to produce the results shown in Figure 1. A large value of $|\partial \ln \chi/\partial w|$ indicates that the geometric method is relatively powerful—a given measurement error in $\ln \chi$ translates into a smaller uncertainty in w for larger values of $|\partial \ln \chi/\partial w|$. The same is true for dynamic methods using $\ln \Psi_+$.

As expected, perturbation methods are insensitive to the dark energy parameter w at high redshift. At low redshift, dark energy dominates more rapidly for smaller w leading to a greater suppression of linear growth and hence a larger variation with w. Geometric methods, on the other hand, are insensitive to w at small redshift, where the lookback time is much less than the Hubble time. For the standard Friedmann equation and $w \approx -1$, dynamic methods are more sensitive than geometric methods for $z < 0.2$. At high redshift the comoving distance remains sensitive to w despite the declining importance of dark energy because $\chi(z)$ is an integral over the past light cone. The expansion history at low redshift affects the redshift-distance relation at high redshift.

The theoretical sensitivities shown in Figure 1 must be combined with the measurement uncertainties of the two methods

[5] This does not mean that the two theories predict identical perturbations on scales larger than the Jeans length. The DGP model has a much longer length scale, r_c, below which eq. (51) could be invalid.

[6] Observations of density perturbations, e.g., from galaxy redshift surveys, are easier than observations of the gravitational potential perturbations. The relation between density perturbations and Ψ depends on the theory of gravity and may require specifying more than just $H(z)$.

before a reliable estimate can be made of their relative merits. The perturbation methods face a severe challenge—to achieve a discriminating power of 0.1 in w requires 2% accuracy in measurement of Ψ_+ at low redshift.

6. SUMMARY AND OUTLOOK FOR COSMOLOGICAL TESTS OF GRAVITY

Long-wavelength cosmological perturbations involve at least two metric perturbations (Φ, Ψ) and two matter perturbations (δ, u) (density and velocity potential, all in conformal Newtonian gauge). One might think that without a theory of gravity, no predictions can be made about relations between these variables.

In fact, we have shown that with minimal assumptions, on large scales there are automatically three independent relations between (Φ, Ψ, δ, u). For an arbitrary theory of gravity these are any three of the four equations (41), (42), (43a), and (43b). Equations (43a) and (43b) enforce local energy-momentum conservation. They generalize the initial-value constraints of general relativity. GR also has a dynamical constraint on $\Phi - \Psi$ enabling a complete reduction to quadratures. These relations depend on the expansion rate of the background spacetime and on the dependence of the expansion rate on spatial curvature (if $K \neq 0$) and on the composition or entropy of the matter filling space (if there are composition or entropy perturbations).

Thus, the key ingredients needed to specify long-wavelength perturbations are (1) a relation between the two gravitational potentials Φ and Ψ and (2) a relation between the expansion rate, density, and pressure (e.g., the Friedmann equation). Given these two ingredients, we have shown how to reduce (Φ, Ψ, δ, u) to quadratures by introducing conserved curvature and entropy variables. Explicit expressions for the time dependence of the metric perturbations were given for Einstein gravity, equations (24) and (25), or equations (46)–(48).

In some circumstances (e.g., isentropic fluctuations in a flat universe with $\Phi = \Psi$) the long-wavelength growth of perturbations is determined completely by the expansion history $H(z)$. In such cases measurements of perturbation growth cannot be combined with geometric measurements (e.g., supernova distances) to discern whether dark energy is a new form of mass-energy or a failure of the Friedmann equation. A loophole exists in this argument if $\Phi \neq \Psi$ or if there are significant long-range nongravitational forces. For a flat universe with curvature fluctuations, the argument can be inverted to yield information about $\Phi(z)$ from measurements of $H(z)$ and $\Psi(z)$ thereby providing a test of gravity theories independently of assumptions about the energy density and pressure.

The long-wavelength limit corresponds to wavelengths larger than any relevant spatial scales, so spatial gradients can be neglected in the equations of motion. Pressure forces modify the dynamics on scales smaller than the Jeans length; the dark energy could plausibly be a scalar field with a Jeans length comparable to the Hubble length. By measuring the wavelength dependence of the linear growth rate on scales greater than 1 Gpc, one might measure the time dependence of the dark energy Jeans length and thereby constrain its intrinsic properties. This measurement is exceedingly difficult because the amplitude of density perturbations in the dark energy is expected to be less than about 10^{-4}.

In addition to providing tests of general relativity, perturbation measurements can provide constraints on dark energy under the assumption that general relativity is valid. By comparing the dependence of curvature perturbations and the redshift-distance relation on the dark energy equation of state parameter, we verified quantitatively the expected conclusion that perturbation methods are most useful at small redshift when the accelerated expansion begins to suppress curvature perturbations. A changing gravitational potential generates cosmic microwave background anisotropy through the integrated Sachs-Wolfe effect. Measurement of this effect (Padmanabhan et al. 2005) has the potential to further constrain dark energy (Pogosian et al. 2005).

The analysis in this paper shows the importance of testing the equality of the two Newtonian gauge gravitational potentials, $\Phi = \Psi$, in equation (14). While this equality holds in general relativity (in the absence of large shear stress), it might not be true for other theories of gravity. In addition to tests combining geometric and perturbation methods using equation (52), one can in principle measure $\Phi - \Psi$ by comparing the deflection of light by gravitational lenses (an effect proportional to $\Psi + \Phi$) with the nonrelativistic motion of galaxies (an effect proportional to Φ). A similar test exists on solar system scales (or in binary pulsars), where the deflection (or Shapiro delay) of light is compared with Newtonian dynamics. Thus, for testing modified gravity as an alternative to GR it is important to extend tests of the parameterized post-Newtonian (PPN) parameter $\gamma_{\rm PPN} \equiv \Psi/\Phi$ (Will 2006). Stringent limits on $|\gamma_{\rm PPN} - 1|$ apply on solar system scales. It would be worthwhile to improve the limits on megaparsec and larger scales by combining weak gravitational lensing and galaxy peculiar velocity measurements or by adding $\gamma_{\rm PPN}$ to the parameters used in analyzing cosmic microwave background anisotropy.

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