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THE SURFACE DENSITY DISTRIBUTION IN THE SOLAR NEBULA

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ABSTRACT

The commonly used minimum-mass power-law representation of the early solar nebula is reanalyzed using a new cumulative mass model. This model is a first integral of the planetary data and predicts a smoother surface density approximation compared with methods based on direct computation of surface density. The density is quantified using two independent analytical formulations. First, a best-fit transcendental function is applied directly to the basic planetary data. Next, a solution to the time-dependent disk evolution equation is parametrically adapted to the solar nebula data. The latter model is shown to be a good approximation to the finite-size early solar nebula and, by extension, to extrasolar protoplanetary disks.

Subject headings: accretion, accretion disks — planetary systems: protoplanetary disks — solar system: formation

1. INTRODUCTION

Surface density models are routinely used as basic building blocks for predicting complex physical processes in protoplanetary disks. For example, simulations of molecular species distribution require such models to determine the background temperature and number density environment. These models are used in either one-dimensional (Aikawa et al. 1996) or vertical structure treatments (Aikawa & Herbst 2001). Another example is the prediction of infrared emission from protoplanetary disks where radiation transport also depends on surface density modeling (Dullemond et al. 2001; Chiang & Goldreich 1997; Nomura 2002). Such investigations routinely use power-law surface density models, probably the most popular being the Hayashi (1981) minimum-mass model with a radial decay proportional to $r^{-3/2}$. Other investigations using detailed vertical structure models conclude that computed surface densities do not fit the power-law models (D'Alessio et al. 1998). The issue concerning the relevance of a "minimum-mass" solar nebula has been argued repeatedly (Cameron 1988), but its basic appeal lies in its simplicity; therefore, it is commonly used as a primary tool in theoretical studies. It is in the spirit of retaining a simple model for the surface density in a solar (or extrasolar) nebula that the current development is aimed.

In this Letter, the basic Hayashi minimum-mass model is compared with an approximate transcendental function and an analytical disk evolution model from Davis (2003), each of which does not admit a simple power-law representation. The analytical disk evolution model predicts radial distributions with approximate inner disk decay rates of $r^{-1/2}$ followed by a rapid exponential decay. Conventional power-law models, which are reasonable approximations in the intermediate region of the solar nebula, possess too much surface density in the inner region, extend the surface density too far, and have infinite masses. Unlike power-law approximations, these surface density models predict a sharp outer edge to the nebula as discussed by Levison et al. (2004). The evolutionary model further predicts that the surface density depends primarily on the initial angular momentum of the disk and on the mass of the central star. It should be borne in mind in applying these models that no allowance is taken of irreversible evolutionary processes. Only a simple augmentation of present-day masses of planets in their current orbits is considered in predicting nebular surface densities.

In the following sections, the early solar nebula is examined by augmenting the planet's present-day mass following the prescriptions of Cameron (1962), Kusaka et al. (1970), and Weidenschilling (1977). These three prescriptions give different augmentation factors but are all reasonable approximations considering the many uncertainties. In contrast to the method used in the cited papers, the current approach avoids an a priori choice of a surface area over which to apply the local augmented planetary masses. Rather than using a more or less arbitrary annular space over which the augmented planetary mass acts, an approximate analytical curve is fit to cumulative augmented planetary masses. Due to the unavoidable coarseness of the augmented planetary masses and the scatter in the planetary data, great precision is not to be expected, but important trends regarding radial surface density distributions are revealed. The cumulative mass is a monotonically increasing function that is essentially a first integral (smoothing process) of the surface density distribution. The surface density is computed directly from the quotient of the gradient of either the empirical curve or the analytical expression with the disk's area variation.

2. SURFACE DENSITY APPROXIMATION FROM PLANETARY MASSES

A useful standard for solar nebula modeling is the Hayshi minimum-mass nebula (Hayashi 1981). It is an approximation to the gas-dominated nebula, taking into account the primitive composition of the solar disk. It expresses as a simple power law $\Sigma(r) = 1700r^{-3/2}$, where r is measured in AU. The total mass of the nebula is infinite in this formulation, so inner and outer boundaries must be specified. For example, a value of 0.013 solar masses ($M_\odot$) represents a power-law disk mass between 0.35 and 36 AU.

Converting planetary data to surface densities is discussed by Cameron (1962), Kusaka et al. (1970), and Weidenschilling (1977). The mass of each planet is augmented by a ratio of condensable mass to total mass representing the primitive composition of the preplanetary disk, although the implementation is different in each case. In this manner an equivalent mass is computed at each planet's radial location. Next, a choice must be made concerning the range of radii associated with each planet. The ratio of this mass to an appropriate annular area is the local surface density. However, such a procedure is critically

TABLE 1
Basic Planetary Data and Cumulative Mass

Planet	Radius (AU)	Mass ($M_\oplus$)	Cumulative Mass ($M_\oplus$)	Augumented Mass ($M_\oplus$)	Augmented Cumulative Mass ($M_\oplus$)
Mercury	0.3871	0.0553	0.0553	18.433	18.433
Venus	0.7233	0.815	0.8703	271.66	290.09
Earth	1	1	1.8702	333.33	623.43
Mars	1.5327	0.1074	1.9776	35.8	659.233
Jupiter	5.2028	317.894	319.87	3178.94	3838.17
Saturn	9.5388	95.185	415.05	951.85	4790.02
Uranus	19.1914	14.537	429.59	855.11	5645.14
Neptune	30.0611	17.132	446.72	1007.76	6652.90
Pluto/Kuiper	40	0.1499	446.87	15	6667.90

dependent on the choice of radial annulus over which the augmented planetary mass is distributed.

The new approach used here is to compute planetary data as a monotonically increasing augmented mass starting from the central star. Relevant solar system data are shown in Table 1 using data from Kusaka et al. (1970) as an example. The third and fourth columns are the mass and cumulative mass of the planets as currently constituted, except that now Pluto is considered along with the Kuiper Belt (Jewitt & Luu 2000). The cumulative solar system mass is 446.9 $M_\oplus$ (2.669×10^{30} g or 0.00134 $M_\odot$), representing the mass of the entire solar system excluding the (small) asteroid belt. The fifth column is the augmented mass following Kusaka et al. (1970), and the final column is the cumulative mass in the primitive solar system. It is 6668 $M_\oplus$ (0.0199 $M_\odot$) or about 15 times the current planetary mass. The angular momentum of the existing planetary system is 3.148×10^{50} g cm^2 s^{-1} (see Cox 2000). The augmented system possesses an angular momentum of 5.349×10^{51} g cm^2 s^{-1} (0.379 $M_\odot$ AU^2 yr^{-1}) or an augmentation factor of approximately 17.

The augmented cumulative mass in the last column of Table 1 and similar data from Cameron (1962) and Weidenschilling (1977) will be used in two ways. First, it will be fit to an approximating transcendental function and, second, to an analytical solution of the nebula evolution equations. These monotonically increasing curves are shown in Figure 1 (the triangles and circles represent data from Table 1 and Cameron 1962, respectively, and error bar estimates are from Weidenschilling 1977). The only clear trend is the saturation effect at large distances from the central star.

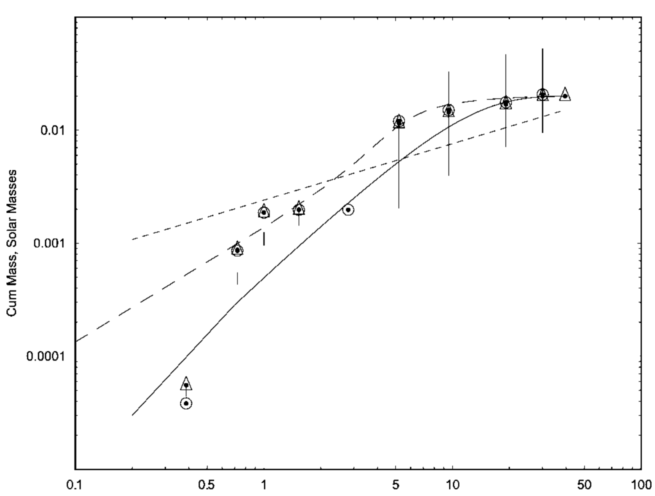

Fig. 1.—Distribution of cumulative mass growth in an early solar nebula model. The triangles represent data from Kusaka et al. (1970), the circles data from Cameron (1962), the short-dashed line the power-law nebula, the long-dashed line the empirical transcendental curve $M(r) = 0.00667[\pi/2 - \tan^{-1}(5/r - r/5)]$, and the solid line the solution of the evolution equation.

In the first approach, the cumulative mass was approximately fit to a monotonically increasing function. Considering the slow initial and final growth tendencies (a classical growth/saturation effect not apparent in this logarithmic plot), an arc tangent function was chosen with parameters that reasonably fit the data. Such a curve is shown in Figure 1 as the long-dashed line and is taken as $M(r) = 0.00667[\pi/2 - \tan^{-1}(5/r - r/5)]$. The short-dashed line is the integrated 3/2 power law that over- and underestimates the cumulative mass in the inner and outer nebula. The terrestrial planets have the largest deviations (which can possibly be associated with unknown migration factors), but the transcendental curve seems to fit the gaseous planets better than the minimum-mass model. Considering the approximate nature of these augmented masses, this is probably a reasonable indicator of the radial mass growth in the early solar system. The associated surface density is easily found from

$$\Sigma(r) = \frac{dM}{dA} = \frac{dM/dr}{dA/dr} = \frac{dM/dr}{2\pi r}. \qquad (1)$$

In the second approach, analytical solutions to the evolution equation are used to approximate the surface density. In Davis (2003), a solution for the evolving nebula is

$$\Sigma(r, t) = \frac{81 GM^{3/2} M_0^4 \Gamma_{7/3}^3}{256\pi J_0^3 \sqrt{r}(1 - 3\dot{M}_0 t/M_0)^{4/3}} \times \exp\left[-\frac{27 GM^{3/2} M_0^3 \Gamma_{7/3}^3 r^{3/2}}{64 J_0^3 (1 - 3\dot{M}_0 t/M_0)}\right], \qquad (2)$$

where GM is the product of the solar mass and the universal gravitational constant (in astronomical units $GM = 4\pi^2$ for the solar nebula); M_0, $\dot{M}_0$ (a negative quantity), and J_0 are initial nebula conditions representing mass, accretion rate, and angular momentum, respectively; and Γ is the complete Gamma function of the indicated order. This formula is derived based on the so-called β-viscosity method that was shown in the aforementioned reference to be generally equivalent to the often used α-viscosity formulation. The numerical values of α or β are empirical constants constrained by observational data regarding the lifetimes of protoplanetary disks; β is of the order

10^{-6} to 10^{-5}. Using the relation between $\dot{M}_0$ and β from Davis (2003), the surface density can be written in terms of β instead of $\dot{M}_0$:

$$\Sigma(r, t) = \frac{81GM^{3/2}M_0^4\Gamma_{7/3}^3}{256\pi J_0^3\sqrt{r}[1 + (729GM^2M_0^3\Gamma_{7/3}^3/256J_0^3)\beta t]^{4/3}} \times \exp\left\{-\frac{27GM^{3/2}M_0^3\Gamma_{7/3}^3 r^{3/2}}{64J_0^3[1 + (729GM^2M_0^3\Gamma_{7/3}^3/256J_0^3)\beta t]}\right\}. \quad (3)$$

Note that the time evolution in equation (3) appears only as a βt product, so the evolution time scales inversely to β. Further simplifications are possible. According to Ruden & Lin (1986), the early nebula quickly adjusts itself to a unique structure. This structure is computed by expanding the above equation for large βt to obtain

$$\Sigma(r, t) = \frac{4J_0 2^{2/3}\exp(-4r^{3/2}/27\sqrt{GM}\beta t)}{81GM^{7/6}\pi\sqrt{r}(\beta t)^{4/3}\Gamma_{7/3}}, \quad (4)$$

which does not depend on the initial mass M_0 but is directly proportional to the initial angular momentum J_0. Further simplify equation (4) by using the values of GM for the solar nebula and the mass augmented angular momentum $J_0 = 0.379\ M_\odot\ \mathrm{AU}^2\ \mathrm{yr}^{-1}$ results in

$$\Sigma(r, t) = \frac{0.000109\exp(-2r^{3/2}/27\pi\beta t)}{\sqrt{r}(\beta t)^{4/3}}. \quad (5)$$

The surface density now depends only on the parameter βt, but its value can be fixed using the augmented mass of the nebula in Table 1. The analytical expression for the instantaneous mass is $M_0(1 - 3\dot{M}_0 t/M_0)^{-1/3}$, which, after substituting for $\dot{M}_0$, is expanded for large βt and becomes $0.01937\beta t$. Equating this quantity to the value 0.0199 $M_\odot$ (the last entry in the last column of Table 1 in terms of solar mass) shows that $\beta t = 0.969$. (This value is sufficiently large so that the approximation resulting in eq. [4] is still valid.) Values of β quoted above imply that the nebula is being modeled at $t \sim 10^5$–10^6 yr in its evolution. The final universal surface density is taken as equation (5) with βt as indicated. The cumulative mass $M(r)$ is easily found by integrating over elementary surface areas. Such a curve superposed on the planetary data is shown in Figure 1 as the solid line. This is a reasonable average overall fit. (Note that other distributions can be obtained if the total mass parameter is allowed to vary.) Also note that the cumulative mass for the Hayashi power-law nebula increases as the square root of r.

Finally, Figure 2 summarizes the surface density (in cgs units) from the three sources described above. They are compared with empirical surface density data (*circles and triangles*) from Cameron (1988) and Kusaka et al. (1970). The latter paper used Cameron's data but doubled the final surface density (Neptune is an exception since it had a different annular surface area) and included Pluto. Weidenschilling (1977) gives a range of surface densities using vertical bars. The surface density computed from the transcendental fit curve in Figure 1 is shown as the long-dashed line in Figure 2. It indicates the distribution to be relatively flat to about 5 AU, after which it decays rapidly. The Hayashi minimum-mass nebula (*dashed line*) is actually a moderately good power-law fit to this albeit imprecise data, but overpredicts the surface mass in the inner nebula, underpredicts it in the outer nebula, and, as previously mentioned, does not have a finite disk mass. Overall, the distribution of surface density is clearly not a power-law fit, and the nebula seems to have a sharp edge. In this regard Levison et al. (2004) postulate an abrupt edge to the Kuiper Belt at 48 AU. The analytical curve from equation (5) (the solid line in Fig. 2) also generally follows the transcendental curve with an inner region decay rate of $r^{-1/2}$ and a subsequent exponential decay.

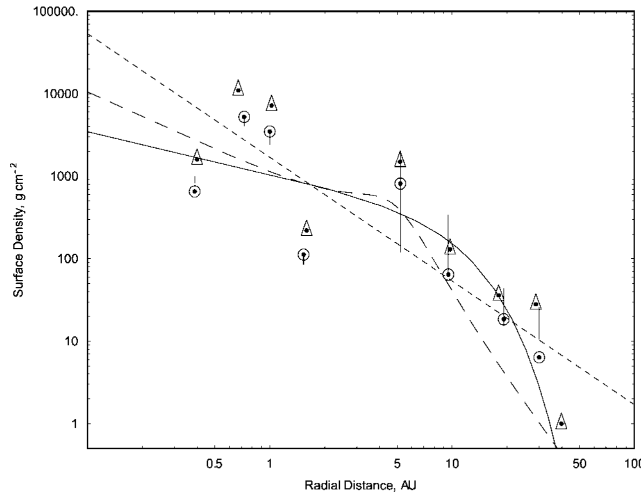

FIG. 2.—Surface density distributions in the early solar nebula. The symbols and curves are the same as in Fig. 1.

In summary, the simple formula of equation (5), being a bone fide solution of the disk evolution equation, is a reasonable alternative for use in protoplanetary disk investigations. Not only is it shown to be a good representation of an approximate solar nebula, but its full form represented by equation (3) may be useful for extrasolar nebulae since the only parameters required are the disk's initial angular momentum and the mass of the central star.

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