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INTERACTING DARK MATTER AND DARK ENERGY

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ABSTRACT

We discuss models for the cosmological dark sector in which the energy density of a scalar field approximates Einstein's cosmological constant and the scalar field value determines the dark matter particle mass by a Yukawa coupling. A model with one dark matter family can be adjusted so the observational constraints on the cosmological parameters are close to but different from what is predicted by the ΛCDM model. This may be a useful aid to judging how tightly the cosmological parameters are constrained by the new generation of cosmological tests that depend on the theory of structure formation. In a model with two families of dark matter particles the scalar field may be locked to near zero mass for one family. This can suppress the long-range scalar force in the dark sector and eliminate evolution of the effective cosmological constant and the mass of the nonrelativistic dark matter particles, making the model close to ΛCDM, until the particle number density becomes low enough to allow the scalar field to evolve. This is a useful example of the possibility for complexity in the dark sector.

Subject heading: cosmology: theory

1. INTRODUCTION

The striking success of the ΛCDM model in fitting the precision *WMAP* measurements of the anisotropy of the 3 K thermal cosmic background radiation and the other cosmological tests (Bennett et al. 2003 and references therein) shows that this cosmology is a useful approximation to the physics of the dark matter (DM) and dark energy (DE). However, it is not difficult to imagine more complicated physics in the dark sector. If the physics differs from ΛCDM enough to matter, it will be manifest as anomalies in the fits to the observations. It is prudent to anticipate this possibility, by exploring models for more complicated physics in the dark sector.

The starting idea for the physics under discussion in this paper is that the DM particle mass can be determined by its interaction with a scalar field whose energy density is the DE. The notion that a particle gets its mass from interacting with a scalar field is of course familiar from the standard model for visible sector matter, in which the quark and lepton masses are due to their interaction with the Higgs field. If DM and DE inhabit a decoupled "dark sector" or brane, the scenario we envisage would be very natural, apart from the ever-present puzzle of the numerical value of the DE today. We do not consider here the question of whether such a scenario can emerge naturally and without untoward consequences in models (such as supersymmetry or axions) where the DM interacts with ordinary matter.

We explore the physics and astrophysics of models for this extension of ΛCDM under the simplifying assumptions of general relativity theory, standard physics in the visible sector, and, in the dark sector, a Yukawa coupling of the DE field to the DM particles. Much of the physics, as summarized in § 3, is in the literature, but we have not seen it all collected and applied to the astrophysics. The example application in § 4, which assumes a single DM family, allows parameter choices that make the model predictions viable but different from ΛCDM. The example in § 5, with two DM families, allows an interesting mixture of near equivalence to ΛCDM at early times and complicated departures from this model at late times.

The line of ideas in this topic has a long history, which informs assessments of where we are now. In § 2 we present our selection of the main steps in the historical development.

For convenient reference we write down here the forms we will be considering for the action in the dark sector, as a sum of two terms. The first is the familiar DE model,

$$S_{\rm DE} = \int d^4x\sqrt{-g}\left[\tfrac{1}{2}\phi_{,\nu}\phi^{,\nu} - V(\phi)\right], \qquad (1)$$

where the function $V(\phi)$ of the classical real DE scalar field ϕ is chosen so the field stress-energy tensor approximates the effect of the cosmological constant Λ in Einstein's field equation. In numerical examples we use the power-law potential,

$$V(\phi) = \frac{K}{\phi^\alpha}, \qquad (2)$$

where K is a positive constant and the constant α may be positive or negative. The DM term, in the form used in much of our discussion, is

$$S_{\rm DMf} = \int d^4x\sqrt{-g}\left[i\bar{\psi}\gamma\partial\psi - y(\phi - \phi_*)\bar{\psi}\psi\right]. \qquad (3)$$

The subscript "DMf" indicates that this is the action written in terms of the wave function ψ for a spin-$\frac{1}{2}$ DM field. In the Yukawa interaction term, y is a dimensionless constant and the constant ϕ_* has units of energy (with $\hbar = 1 = c$). If ϕ_* in equation (3) is negligibly small, the entire particle mass is due to its interaction with the field ϕ. This seems particularly attractive because we may need this field anyway, to account for the DE. The interaction between the DM and DE allows the DM particle mass to be variable, producing a long-range nongravitational interaction in the dark sector. Both effects can be suppressed by the presence of a second DM family with a different value of ϕ_*, as we discuss in § 5, or by suitable choices of ϕ_* and $V(\phi)$ for one family (§ 4).

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For completeness one might also consider the analog of equation (3) for scalar DM particles. When ϕ_* is negligibly small, the analogous form is[3]

$$S_{\rm DMb} = \int d^4x\sqrt{-g}\Big[\tfrac{1}{2}\chi_{,\nu}\chi^{,\nu} - \tfrac{1}{2}y^2(\phi - \phi_*)^2\chi^2\Big]. \quad (4)$$

In the limiting situation that is thought to apply to cosmology, where the DM particle de Broglie wavelengths are much smaller than the characteristic length scale of variation of the DE field, both DM actions (eqs. [3] and [4]) are equivalent to the model of a classical gas of pointlike particles with action[4]

$$S_{\rm DMp} = -\sum_i \int y|\phi(x_i) - \phi_*|\, ds_i. \quad (5)$$

The invariant interval along the path $x_i^\mu(t)$ of the ith particle is $ds_i = (g_{\mu\nu}\, dx_i^\mu\, dx_i^\nu)^{1/2}$. The DE particle mass is $m_{\rm eff} = y|\phi - \phi_*|$. The absolute value in equation (5) has to be a prescription (along with $y > 0$), but as we discuss in § 3, it is not needed in equation (3) or equation (4). Equation (5) is a convenient form for analyses of structure formation.

2. REMARKS ON THE HISTORY OF IDEAS

The particle action in equation (5), with a variable effective mass, appears in Nordström's (1912) scalar field model for gravity in Minkowski spacetime, in the form $L_i \propto e^{-\phi/\phi_*}\, ds_i$ (in our notation). This form reappears in Misner, Thorne, & Wheeler (1973), and it still is favored by many, in part because the exponential is suggested by superstring theory. The functional form does not much affect the force generated by the particle-field interaction, but it affects the cosmic evolution of particle masses and the force between them. In our preliminary examination of possible alternatives to the ΛCDM model we prefer the linear form in equation (5) because it translates to the familiar and simple Yukawa interaction in the field action model in equation (3). This linear coupling appears also (explicitly or as a particular case) in many recent discussions of the possible interaction of DM and DE (e.g., Casas, Garcia-Bellido, & Quiros 1992; Anderson & Carroll 1997; Bean 2001 and references therein).

The particle action with variable mass appears in the scalar-tensor gravity theory considered by Jordan (1955, 1959) and Brans & Dicke (1961), when expressed in units chosen so that the action for gravity is the Einstein form (Fierz 1956; Dicke 1965). The units in this theory can be rescaled to standard local physics with constant masses and a generalized action for gravity, which is the form Jordan (1955, 1959) and Brans & Dicke (1961) used to implement Dirac's (1938) idea that the strength of the gravitational interaction may be small because it is rolling toward zero.

Superstring scenarios led to the thought that particle mass ratios (and other dimensionless constants) may be variable, and in particular gravity physics and local physics in the visible sector may be close to standard while particle masses in the dark sector are more significantly variable (Damour, Gibbons, & Gundlach 1990). This allows interesting departures from standard cosmology within the tight constraints on gravity physics from precision tests in the visible sector.

Damour et al. (1990) work with a scalar-tensor theory for gravity physics, a route taken in many subsequent papers. For clarity in our exploratory discussion of the dark sector physics we adopt general relativity theory, standard physics in the visible sector, and a single scalar field that fixes particle masses in the dark sector.

The scalar field model for the DE in equation (1) was introduced by Wetterich (1988) and Peebles & Ratra (1988) (as reviewed in Peebles & Ratra 2003). This DE model allows one to imagine that the effective cosmological constant is evolving to its "natural" value, $\Lambda = 0$, and is small now because the universe is old, a natural extension of the ideas of Dirac (1938), Jordan (1955), and Dicke (1964). Wetterich (1995) seems to have been the first to propose that the scalar field in this model for the DE may also fix the DM particle mass. The idea has since been discussed in a considerable variety of contexts (e.g., Damour, Piazza, & Veneziano 2002, references therein, and the references in the following discussion).

A scalar interaction present only in the dark sector makes the accelerations of visible and dark matter test particles different, an effect we call a "fifth force." The empirical constraints on this kind of fifth force are considerably weaker than the constraints from the Eötvös experiment in the visible sector, as was recognized from the beginning of the modern discussions, in Damour et al. (1990). The first numerical example we have seen of the effect of this kind of fifth force on the growth of mass density fluctuations in the expanding universe is in Amendola (2000). Amendola & Tocchini-Valentini (2002) point out that the fifth force in the dark sector might have a substantial effect on the relative distributions of baryonic and dark matter. However, we now have convincing evidence (Bennett et al. 2003 and references therein) that structure grew out of primeval adiabatic departures from homogeneity, as well as good evidence that the growth of the mass density fluctuations is not very different from the ΛCDM prediction, from the consistency within this model between the power spectra of the present distributions of galaxies and the 3 K thermal cosmic background radiation (CBR). Thus, one is interested in DM-DE interaction models that can be adjusted so that the fifth force and the evolution of the DE field are weak enough to fit the now demanding observational constraints but strong enough to make an observationally interesting departure from ΛCDM.

The search for models has been influenced by the attractor concept, that the physics may have the property that the astrophysics is insensitive to initial conditions. Peebles & Ratra (1988) introduced the DE power-law potential in equation (2) because it has this attractor property. The physics need not have an attractor, of course, as in the example of França & Rosenfeld (2002), which is based on Wetterich's (1988) potential and suitably chosen initial conditions. We discuss models of this kind in § 4.

In the attractor model considered by Anderson & Carroll (1997) and more recently by Comelli, Pietroni, & Riotto (2003), the potential of the DE field is the sum of a term linear in the field and proportional to the DM particle number density (as in eqs. [3] and [5]) and a power-law self-interaction term with $\alpha > 0$ in equation (2). The field is assumed to have been attracted to the minimum of the total potential. Within

[3] The constant ϕ_* in eqs. (3) and (4) is the sum of "bare" and renormalization parts. For generality one would add a constant to $y^2(\phi - \phi_*)^2$ in eq. (4). We ignore considerations of naturalness in the choice of these constants. Issues of naturalness and renormalization plague other aspects of cosmology and fundamental particle physics, as in the meaning of eq. (2) within quantum field theory, and most notably the value of the vacuum energy density.

[4] It will be recalled that the exclusion principle affects initial conditions (the occupation numbers in single particle phase space) but not the equation of motion in the particle limit.

the models considered in § 4 this attractor case is unacceptable because the fifth force is too large.

Attractor solutions may also be relevant to the dilaton potential. In the scenario considered by Damour & Polyakov (1994) the masses of all particles have minima as a function of the dilaton field at a universal value ϕ_m; near this minimum the fifth force scales as $(\phi - \phi_m)^2$. They show that comic evolution can cause the present value ϕ of the dilaton field to be close to ϕ_m, thus suppressing the fifth force.

In the class of models we are considering, the contribution to the potential energy of the DE coming from its interaction with the DM has a minimum at zero DM particle mass. This minimum value certainly is not acceptable for the DM in galaxies, but one can imagine that there are two families of DM particles, with different values of ϕ_* in equations (3) and (5). At large enough DM particle number densities the DE field would be locked to the zero of the particle mass for the preponderant family, making these particles relativistic (and with a mass density that can be acceptably small). This has the effect of suppressing the fifth force and eliminating the evolution of the massive DM particle mass and the evolution of the DE density. This resembles Damour & Polyakov's (1994) "least coupling," but with the difference that in an expanding universe the particle number density must eventually become low enough to release the field.

We see in this history of ideas a conservative aspect of theoretical physics. The Nordström action, embodying a variable particle mass, was under discussion before Einstein had completed his general relativity theory of gravity. It reappeared in the 1950s and 1960s, in scalar-tensor generalizations of general relativity expressed in terms of the Einstein action. The scalar-tensor theories were developed to explore Dirac's idea that the strength of the gravitational interaction may be variable, and these theories served also as a guide to the work of developing precision tests of gravity physics. The current reappearance of this action is motivated in part by superstring scenarios and in part by the continuing fascination with variable parameters of nature. Moreover, the Nordström action in the form of equation (5) is equivalent to a Yukawa interaction with a classical scalar field, the form of which was introduced for very different purposes in meson and weak interaction physics. To be discovered is whether this convergence of ideas from particle and gravity physics is a useful guide to observationally significant aspects of the physics of the dark sector.

3. BASIC RELATIONS

We begin with the physics of the DM particle model in equation (5). Many of the results summarized here have appeared in one or more of the papers cited above, but we have not seen them all collected or applied. The relation of the particle model to the field model in equation (3), which is discussed in § 3.4, might be considered self-evident, but it should be checked in the present context.

3.1. *Particle and Field Equations*

We simplify notation in this subsection by setting ϕ_* to zero [which has the effect of shifting the minimum of $V(\phi)$] and taking ϕ to be positive. Throughout y is positive.

The particle action in equation (5) gives the equation of motion

$$\frac{d}{ds} y\phi g_{\mu\nu} \frac{dx^\nu}{ds} = \frac{y\phi}{2} \frac{\partial g_{\rho\sigma}}{\partial x^\mu} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} + \frac{\partial y\phi}{\partial x^\mu}. \tag{6}$$

We leave the constant y in this equation because it is useful to note that the particle four-momentum is $p^\mu = y\phi a\, dx^\mu/ds$. When spacetime curvature fluctuations can be neglected, the equation of motion is

$$\frac{da\boldsymbol{p}}{dt} = \frac{d}{dt} \frac{ay\phi \boldsymbol{v}}{\sqrt{1-v^2}} = -y\sqrt{1-v^2} \frac{\partial \phi}{\partial \boldsymbol{x}}, \tag{7}$$

where $a(t)$ is the cosmological expansion factor as a function of the proper world time t and the proper peculiar velocity is $\boldsymbol{v} = a\, d\boldsymbol{x}/dt$. When the spatial variation of the DE field ϕ can be neglected, the momentum is conserved; if in addition the proper peculiar velocity is nonrelativistic, the velocity scales as $v \propto 1/[a(t)\phi(t)]$.

The DM that is bound to galaxies and clusters of galaxies has to be nonrelativistic. These systems are well described by the weak-field limit of gravity, where the gravitational potential satisfies

$$\frac{\nabla^2 \Phi}{a^2} = 4\pi G \rho_b(t) \delta(\boldsymbol{x}, t). \tag{8}$$

The mean (background) nonrelativistic mass density is $\rho_b(t)$, and $\delta = \delta\rho/\rho_b$ is the mass density contrast. We write the DE field as

$$\phi = \phi_b(t) + \phi_1(\boldsymbol{x}, t), \tag{9}$$

where the mean background field $\phi_b(t)$ is a function of world time t and the departure ϕ_1 from homogeneity can be treated in linear perturbation theory. (We also assume $|\phi_1| \ll |\phi_b - \phi_*|$, so that there is no risk that the field value passes through zero, which would make the DM transiently relativistic.) In linear theory we can also neglect the term proportional to $\boldsymbol{v} \cdot \boldsymbol{\nabla}\phi$ in equation (6). With all these approximations equation (6) becomes

$$\frac{d\boldsymbol{v}}{dt} + \left(\frac{\dot a}{a} + \frac{\dot\phi_b}{\phi_b}\right)\boldsymbol{v} = -\frac{1}{a}\boldsymbol{\nabla}\left[\Phi + \frac{\phi_1}{\phi_b(t)}\right], \tag{10}$$

where the dot denotes the derivative with respect to time t. The expansion of the universe produces the familiar slowing of the peculiar velocity in the second term of this equation, while the evolution of the DE field value produces a term that may increase or decrease the peculiar velocities. The spatial variation of the DE field produces the fifth force term on the right-hand side of the equation. This force tends to move DM particles so as to minimize their masses $y\phi$.

The DE field equation from the action in equations (1) and (5) is

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} + \frac{dV}{d\phi} + \frac{dV_I}{d\phi} = 0. \tag{11}$$

The term from the interaction with the DM is

$$\begin{aligned} \frac{dV_I}{d\phi} &= y \sum_i \frac{ds_i}{dt} \frac{\delta(\boldsymbol{x} - \boldsymbol{x}_i)}{\sqrt{-g}} \\ &\simeq y \sum \sqrt{1 - v_i^2} \frac{\delta(\boldsymbol{x} - \boldsymbol{x}_i)}{a^3}. \end{aligned} \tag{12}$$

The last expression neglects the effect of spacetime curvature fluctuations on the DM source term for the DE.

One sees that when the particles are relativistic, $v_i \to 1$, the source term $dV_I/d\phi$ vanishes. Damour & Polyakov (1994) express this in terms of an equation of state. We find it convenient to use instead the proper DM particle number density,

$$n(\boldsymbol{x}, t) = \sum_i \frac{\delta(\boldsymbol{x} - \boldsymbol{x}_i)}{a^3} = \sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i), \qquad (13)$$

where $\delta \boldsymbol{r} = a\, \delta \boldsymbol{x}$ is a proper relative position. Thus, the source term can be written as

$$\frac{dV_I}{d\phi} = yn(\boldsymbol{x}, t)\left\langle\sqrt{1 - v^2}\right\rangle. \qquad (14)$$

The angle brackets signify the mean of the reciprocal Lorentz factor, γ, for the DM particles. The pressure in this homogeneous gas of DM particles is $p = \gamma y \phi n v^2/3$ and the energy density is $\rho = \gamma y \phi n$, so another form for the source term is $dV_I/d\phi = (\rho - 3p)/\phi$. This is the form derived by Damour & Polyakov (1994).

3.2. *The Fifth Force*

There is a fifth force in the dark sector, which we analyze for a single nonrelativistic DM family model. The DM in galaxies and clusters of galaxies is well described by the approximations of equations (8) and (10), which ignore relativistic corrections and the effect of spacetime curvature fluctuations on the DE field equation. Returning now to the general case when ϕ_* is nonzero, we can write the spatial mean of the field equation as

$$\frac{d^2\phi_b}{dt^2} + 3\frac{\dot{a}}{a}\frac{d\phi_b}{dt} + \left\langle\frac{dV(\phi)}{d\phi}\right\rangle \pm yn_b = 0, \qquad (15)$$

where ϕ_b is the mean field (eq. [9]) and $n_b(t)$ is the mean particle number density.

Here and below, the last term has the sign of $\phi - \phi_*$. The departure ϕ_1 from the homogeneous part of the field satisfies the equation

$$\frac{d^2\phi_1}{dt^2} + 3\frac{\dot{a}}{a}\frac{d\phi_1}{dt} - \frac{1}{a^2}\nabla^2\phi_1 + \frac{d^2V}{d\phi^2}\phi_1 \pm yn_b\delta = 0. \qquad (16)$$

The DM number density contrast $\delta n/n$ has been replaced by the mass contrast $\delta = \delta\rho/\rho$ because, as we argue below, in situations of interest the fractional perturbation to the field ϕ is small compared to $\delta n/n$.

For the analysis of structure formation at modest redshifts we are interested in density fluctuations on scales small compared to the Hubble length, which means that the time derivatives in equation (16) are small compared to the space derivatives. The term $d^2V/d\phi^2$ is also small in many cases of interest, so a useful approximation to the field equation is

$$\frac{\nabla^2\phi_1}{a^2} = \pm yn_b\delta. \qquad (17)$$

It follows from equations (8), (10), and (17) with $\rho_b = y|\phi_b - \phi_*|n_b$ that the ratio of the fifth force to the gravitational force in the dark sector is

$$\beta \equiv \frac{|\boldsymbol{\nabla}\phi_1|}{|\phi_b - \phi_*|\boldsymbol{\nabla}\Phi} = \frac{1}{4\pi G(\phi_b - \phi_*)^2}. \qquad (18)$$

The evolution of the mass density contrast in linear perturbation theory satisfies

$$\frac{\partial^2\delta}{\partial t^2} + \left[2\frac{\dot{a}}{a} + \frac{\dot{\phi}_b}{(\phi_b - \phi_*)}\right]\frac{\partial\delta}{\partial t} = 4\pi G\rho_b(1 + \beta)\delta. \qquad (19)$$

This differs from the usual expression by the factor $1 + \beta$, which takes into account the fifth force, and the part $\dot{\phi}_b/(\phi_b - \phi_*)$, from the evolving DM particle mass (as has been discussed by Amendola & Tocchini-Valentini 2002; Matarrese, Pietroni, & Schimd 2003; and others).

Now we can check that $\delta n/n \simeq \delta\rho/\rho$. Equation (17) applied to a particle concentration with contrast δ and size r says

$$\frac{\delta m}{m} = \frac{\phi_1}{(\phi_b - \phi_*)} \sim \frac{-yn_b r^2\delta}{|\phi_b - \phi_*|}, \qquad (20)$$

where H is the Hubble parameter. The sign of the mass shift $\delta m/m$ is opposite to the sign of δ, consistent with the attractive nature of the fifth force. Using the condition that the DM mass density is not greater than the total, we find that the fractional mass shift is

$$\frac{|\delta m|}{m} \lesssim \beta(Hr)^2|\delta|. \qquad (21)$$

This is small because β cannot be much larger than unity, and we are interested in density fluctuations on scales small compared to the Hubble length.

3.3. *A Relativistic Dark Matter Family*

The DM bound to galaxies and clusters of galaxies has to be nonrelativistic, with $\langle(1 - v^2)^{1/2}\rangle$ close to unity, so the source term given by equation (14) due to the scalar field interaction with such DM is simply proportional to the particle number density. However, there may be more than one DM family, and ϕ may be drawn close to the zero of the source term belonging to one of the families, causing that family to be relativistic. Here we consider the behavior of this piece of $dV_I/d\phi$ as a function of the field value, under the simplifying assumption that the scalar field is spatially homogeneous. In § 5, on a model with two DM families, we consider the response of ϕ to the source term and the effects of inhomogeneities.

The velocity of a DM particle in this relativistic family is

$$v = \frac{k/a}{\sqrt{y^2\phi^2 + k^2/a^2}}, \qquad (22)$$

where the comoving wavenumber of a DM particle is k and the proper peculiar momentum is k/a. The initial distribution of comoving wavenumbers is determined by the DM particle production process. We discuss the evolution of ϕ in § 5. For the purpose of this subsection we are concerned with the case $y|\phi| \lesssim k/a \ll H^{-1}$. The first inequality means that the DM particles are relativistic; the second means that the discussion of the evolution of the mass distribution can ignore time derivatives of ϕ and the expansion of the universe. In this approximation the energy of a DM particle is conserved as it propagates through space and encounters DE field gradients. Therefore, the local value of the DM particle velocity changes in response to changes in the local value of the DM particle mass, as determined by the local value of the DE field. With this in mind, we can rewrite equation (14) as

$$\frac{dV_I}{d\phi} = yn(t)\left\langle y\phi\left(y^2\phi^2 + \frac{k^2}{a^2}\right)^{-1/2}\right\rangle_k, \qquad (23)$$

where the average is over the distribution of particle wavenumbers in the relativistic family. When ϕ passes through zero (or more generally, through the zero of the particle mass) in a close to homogeneous way, it does not greatly affect the distribution of momenta k/a, but it makes the DM peculiar motions relativistic. That causes the source term given by equation (23) to vary smoothly from $dV_I/d\phi = yn$ at $y\phi \gg k_{\rm eff}/a$ to $dV_I/d\phi = -yn$ at $y\phi \ll -k_{\rm eff}/a$, where the effective comoving wavenumber $k_{\rm eff}$ is defined by the average in equation (23).

As a final remark, we note that, according to the action principle, the equation of motion is derived by extremizing the action given by equation (4) with the particle orbits held fixed. Thus, although V_I can be written as a function of ϕ, n, and the momenta k/a, the source term $dV_I/d\phi$ is not the same as the derivative of V_I with respect to ϕ at fixed n and k/a. In the derivation of the field equations from the Lagrangian given by equation (3), which is discussed in § 3.4, the independent field ψ is held fixed as ϕ is varied, leading to equation (14).

3.4. *The Field Action*

Here we consider, in the limit where the particle de Broglie wavelengths are small compared to the length scale of variation of ϕ, the equivalence of the particle action in equation (5) to the spin-$\frac{1}{2}$ fermion field action in equation (3) and the boson field action in equation (4), for the purpose of deriving particle orbits and the source term for ϕ. For brevity we again take ϕ to be positive and suppress ϕ_*.

If the parameters y and ϕ in the Yukawa interaction term in equation (3) are positive, the particles created by the field ψ manifestly have positive mass $y\phi$. A chiral rotation by π changes the sign of $\bar{\psi}\psi$ without changing the kinetic part of the field equation, so when $y\phi$ is negative, the chiral rotation yields the usual sign for the mass in the Dirac equation. Alternatively, we can leave the "wrong" sign for the mass when ϕ is negative and note by the chiral transformation argument that the solutions to the field equation in this case make $\bar{\psi}\psi$ negative, meaning that $y\phi\bar{\psi}\psi$ is never negative. This condition leads to the prescription for the absolute field value in the particle action given by equation (5).

In the field action (eq. [3]) the source term for the DE field is

$$\frac{dV_I}{d\phi} = y\bar{\psi}\psi = yn\sqrt{1 - v^2}, \qquad (24)$$

when y and ϕ are positive. The last expression follows because $\bar{\psi}\psi$ and $n(1 - v^2)^{1/2}$ both are scalars, and we know that they are the same (up to the sign) in the nonrelativistic limit.[5] This agrees with the source term in equation (14) in the particle model.

[5] One can check these arguments by writing down plane wave solutions to the Dirac equation. The DE source term in eq. (24) can be derived from the free quantum field operator for ψ, apart from the standard problem with the zero-point contribution to the particle number operator.

The second step in demonstrating the equivalence of the actions given by equations (3) and (5) is to check the equation of motion of wave packets. An easy way to proceed uses the commutator of the particle momentum operator with the Dirac Hamiltonian $H = \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{p} + \hat{\boldsymbol{\beta}} y\phi$,

$$[\boldsymbol{p}, H] = -iy\hat{\boldsymbol{\beta}}\boldsymbol{\nabla}\phi. \qquad (25)$$

It follows that the time derivative of the expectation value of the momentum is

$$\frac{d\boldsymbol{p}}{dt} = -y\boldsymbol{\nabla}\phi \int d^3r\,\psi^\dagger\hat{\boldsymbol{\beta}}\psi. \qquad (26)$$

The last factor is the integral over $\bar{\psi}\psi$, which we know is the reciprocal of the Lorentz factor for a single particle wave packet, as in equation (24). Equation (26) thus agrees with the rate of change of momentum in the particle model in equation (7).

For completeness let us check the relation between the momentum and velocity of the wave packet. An easy way is to use a WKB approximation. In considering the motion of the wave packet we can ignore the time evolution of ϕ. The interesting part of the spatial variation in one dimension of a wave function with energy ϵ is

$$\psi \sim \exp i\left(\int^x \sqrt{\epsilon^2 - y^2\phi^2}\,dx - \epsilon t\right), \qquad (27)$$

which means that the momentum defined by the gradient operator is

$$p = \sqrt{\epsilon^2 - y^2\phi^2}, \qquad (28)$$

as usual. The velocity of a wave packet constructed as a linear combination of these energy eigenstates follows from the stationary point of the exponential: $v = p/\epsilon$, again as usual. These two results give the standard relation between momentum and velocity of a particle of mass $y\phi$. On can also check that, when the time evolution of ϕ and the expansion parameter $a(t)$ can be neglected, the time derivative of equation (28) agrees with the rate of change of momentum in equation (7).

Similar arguments show that in the limit of short de Broglie wavelengths the boson DM model in equation (4) also reduces to the point particle model (apart from the problem of the zero-point contributions to ϕ^2 and χ^2), with DM particle mass $y|\phi - \phi_*|$.

4. MODELS WITH ONE DARK MATTER FAMILY

In the models presented in this section, the DE potential is the power-law form in equation (2), with positive or negative power-law index α, and there is one family of nonrelativistic DM particles with $\phi_* = 0$, meaning that the DM particle mass is y times the DE field value. For the purpose of this preliminary exploration we neglect the mass in baryons, so the matter density parameter is

$$\Omega_m H_0^2 = \tfrac{8}{3}\pi G\rho_b(t_0) = \tfrac{8}{3}\pi G y\phi_b(t_0)n_b(t_0). \qquad (29)$$

Here and below the subscript 0 means the present value. The Hubble parameter at the present world time t_0 is H_0.

Throughout we use $\Omega_m = 0.3$ and assume that space curvature vanishes.[6]

We use the dimensionless variables

$$\tau = H_{0t}, \quad f = G^{1/2}\phi_b, \tag{30}$$

in terms of which equation (15) for the mean field ϕ_b is

$$\frac{d^2f}{d\tau^2} + \frac{3}{a}\frac{da}{d\tau}\frac{df}{d\tau} = \frac{\alpha\kappa}{f^{\alpha+1}} - \frac{3}{8\pi}\left(\frac{a_0}{a}\right)^3\frac{\Omega_m}{f_0}, \tag{31}$$

and the Friedmann equation for the expansion rate is

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \Omega_m\frac{f}{f_0}\left(\frac{a_0}{a}\right)^3 + \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{df}{dt}\right)^2 + \frac{\kappa}{f^\alpha}\right], \tag{32}$$

where the dimensionless parameter representing the constant K in the power-law potential $V(\phi)$ is

$$\kappa = \frac{KG^{1+\alpha/2}}{H_0^2}. \tag{33}$$

The present values f_0 and a_0 of the field and expansion parameter appear in the combination f_0/a_0^3, which is the unknown final condition for given initial conditions.

At high redshift the first term on the right-hand side of equation (31), which represents $dV/d\phi$, is relatively small. When this term can be neglected, the first integral of equation (31) is, apart from the decaying term,

$$\frac{df}{d\tau} = -\frac{3\Omega_m\tau}{8\pi f_0}\left(\frac{a_0}{a}\right)^3. \tag{34}$$

At $z > z_{\rm eq}$, where the expansion is dominated by radiation, the expansion factor varies $a \propto \tau^{1/2}$, and equation (34) says the departure from the initial value of f grows as $\tau^{1/2}$. At lower redshift where the expansion is matter dominated, $a \propto \tau^{2/3}$, the departure grows as $\log\tau$ in the approximation of equation (34). At still lower redshifts $dV/d\phi$ may be important, and we need a numerical solution.

We commence the numerical solution at a fixed initial time corresponding to equality of mass densities in matter and radiation in the ΛCDM model. We start with an arbitrary choice for the initial field value $f_i = G^{1/2}\phi_i$. The initial value of $df/d\tau$ is taken from equation (34). The final field value (and hence energy density) has to be consistent with $da/d\tau = a$ at the present epoch. We achieve this by iteratively adjusting κ. (We found this more convenient than choosing κ and seeking the initial field value.) Having solved numerically for $f(\tau)$, we can find the epoch $z_{\rm eq}$ at equal mass densities in matter and radiation.

Table 1 lists parameters and present values of some quantities of interest for solutions with three choices for the value of α, omitting numbers that are so far off the ΛCDM model prediction as to seem uninteresting. The second column is the initial field value, expressed in units of the Planck mass. The third column is the value of κ required for a consistent solution. The fourth column is the ratio of the field value at $z_{\rm eq}$ to the present value. Because we are assuming $\phi_* = 0$, this is the ratio of DM particle masses then and now. The redshift at equality scales in proportion to this ratio.

TABLE 1
Numerical Results for One DM Family

α (1)	$G^{1/2}\phi_i$ (2)	κ (3)	$\phi_{\rm eq}/\lvert\phi_0\rvert$ (4)	$\delta_0/\delta_{\Lambda\rm CDM}$ (5)	$l_{\rm peak}/l_{\Lambda\rm CDM}$ (6)
−2	0.95	1.5E+01	. . .	. . .	. . .
	1.00	1.4E+00	. . .	. . .	. . .
	2.00	3.0E−02	1.22	1.18	1.18
	4.00	5.7E−03	1.04	1.04	1.04
4................	0.50	4.7E−03	. . .	. . .	. . .
	1.00	3.5E−03	2.17	2.88	2.02
	2.00	6.8E−01	1.19	1.18	1.17
	4.00	1.8E+01	1.04	1.04	1.04
6................	0.50	1.0E−02	. . .	. . .	. . .
	1.00	1.3E−02	1.33	1.37	1.29
	2.00	2.0E+00	1.18	1.17	1.16
	4.00	2.8E+02	1.04	1.03	1.04

Figures 1 and 2 show the evolution of the DE field in solutions with $\alpha = -2$ and 6. The latter look much like the solutions for $\alpha = 4$ entered in Table 1. Since we have set $\phi_* = 0$, the DE field in solutions with $\alpha < 0$ is drawn toward zero. At $\alpha = -2$ and the smallest initial field value (listed in the first line of Table 1 and shown as the dotted curve in Fig. 1) the field has passed through zero slightly before the present epoch. Among other undesirable consequences, this would have made the DM transiently relativistic, driving the DM out of the halos of galaxies. The slightly larger initial field value in the second row of the table, with the appropriate adjustment of

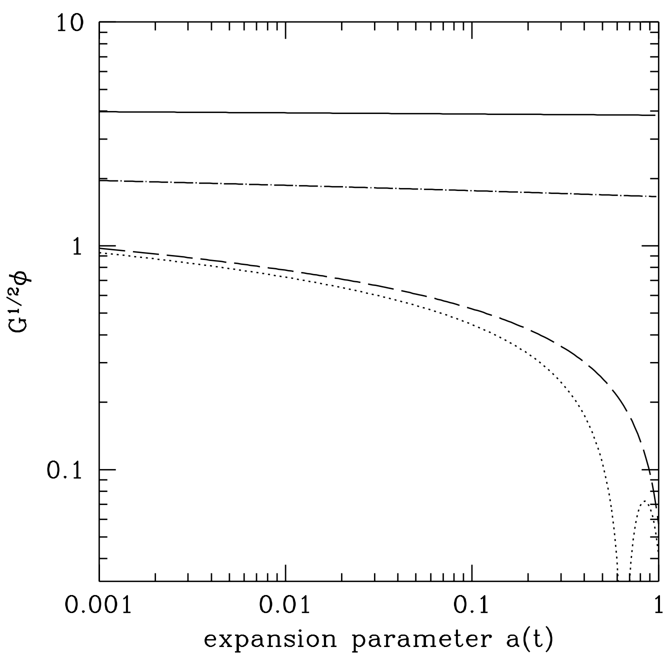

Fig. 1.—Evolution of the DE field in models with the power-law exponent $\alpha = -2$ in the potential (eq. [2]). The solutions are fixed by the initial value of $f = G^{1/2}\phi$ listed in the first four rows of Table 1. The initial values are close to the field values at the left side of the plot.

[6] For simplicity in this exploratory discussion of observational constraints we do not adjust the value of Ω_m to take account of the fifth force. In a theory with a fifth force β enters different measures of Ω_m in different ways. For instance, the relative motions of DM halos depend on the product $\Omega_m(1+\beta)$, whereas weak lensing depends on the fifth force only indirectly, through whatever effect the scalar field has on the angular size distance. The dynamics of ordinary matter is not directly affected by the fifth force in the dark sector.

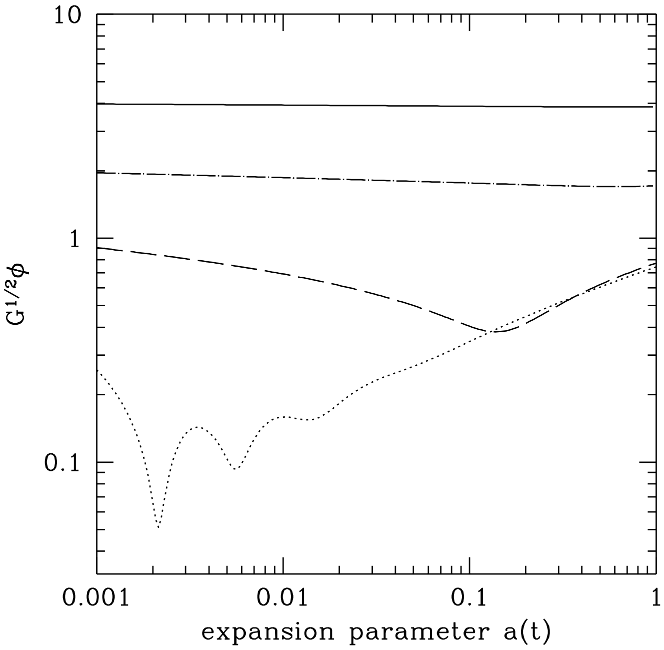

FIG. 2.—Same as Fig. 1, but for solutions with $\alpha = 6$. The initial values of ϕ are listed in the last four entries in Table 1.

κ, removes this problem: in this case, shown as the dashed curve in Figure 1, the DM particle mass has not yet passed through zero, but that will soon happen and the halos will be disrupted.

When α is positive, the potential for ϕ has a minimum away from zero. If the initial value of ϕ is small enough, the field relaxes to this minimum by the present epoch. This is seen in our solution with the smallest initial field value, listed in the fourth entry from the bottom of Table 1 and plotted as the dotted curve in Figure 2. In this case the field oscillates about and approaches the minimum of the potential. This solution is unacceptable, however, because the relatively small field values produce a large fifth force on the DM, which substantially enhances the growth of mass density fluctuations, as we discuss next. At the two largest initial field values in Table 1, the solutions for $\alpha = 6$ are well away from the minimum of the potential. They look much like the solutions for $\alpha = -2$ and, as we show next, produce only a modest effect on the evolution of mass density fluctuations.[7]

Figures 3 and 4 show numerical solutions to equation (19) for the evolution of the mass density contrast $\delta(t)$ in linear perturbation theory. The solutions are normalized to a common initial value at redshift $z = 1300$, roughly the epoch of decoupling, and they are multiplied by the redshift factor $1 + z = a_0/a(t)$ to scale out the main trend of the evolution. The solution for the ΛCDM model, with the same value of Ω_m, is plotted as the short-dashed curves in both figures. The solution for $\alpha = -2$ and $f_i = G^{1/2}\phi_i = 0.95$ is not plotted because

[7] Anderson & Carroll (1997) and Comelli et al. (2003) consider the case that the scalar field sits at the minimum of the effective potential. They adopt the same particle coupling and potential (with $\alpha > 0$) as in the examples in this section. However, when the scalar field is at the minimum of the effective potential, as in an attractor scenario, there is an unacceptably rapid evolution of the DM particle mass subsequent to decoupling. This evolution, as well as the fifth force, can be suppressed by choosing a large value of $\phi_* \sim m_{\rm Pl}$ in eq. (18). Our acceptable-looking cases are not attractor solutions: they are sensitive to the initial value of ϕ, and their success relies in part on the assumption that ϕ is far from its value at the minimum of the potential.

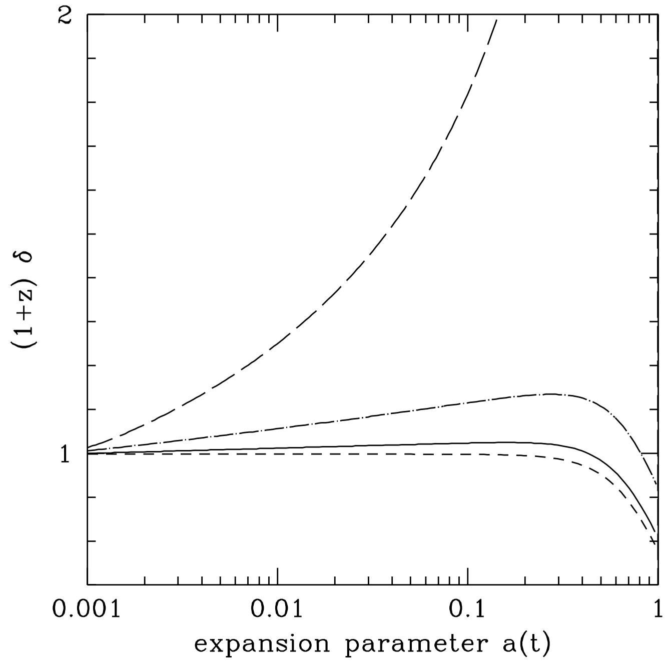

FIG. 3.—Evolution of the mass density contrast in linear perturbation theory in models with $\alpha = -2$. The density contrast has been multiplied by the redshift factor $1 + z$ to scale out the evolution when the expansion is matter dominated. The short-dashed curve is the solution for the ΛCDM model. The line types of the other curves match Fig. 1, where the initial field values are close to what is plotted at the left side of the figure.

equation (19) does not take account of the transient relativistic motions of the DM particles. The fifth column of Table 1 lists the ratio of the growth factor since decoupling, $\delta_0/\delta_{\rm dec}$, to the prediction of the ΛCDM model.

At the two largest initial field values and all three choices of α, the growth of density fluctuations is close to the ΛCDM prediction. In the solution for $f_i = 1$ and $\alpha = -2$, plotted as the long-dashed curves in Figures 1 and 3, the growth of the

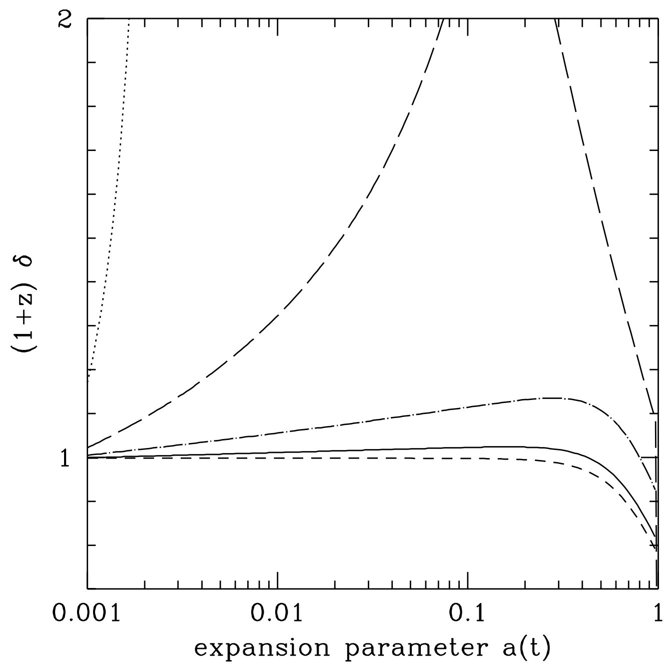

FIG. 4.—Same as Fig. 3, but for solutions with $\alpha = 6$

density contrast is about 4 times that of the ΛCDM model. That is ruled out by the consistency of the CBR temperature anisotropy and the large-scale fluctuations in the galaxy distribution within the ΛCDM model. For $f_i = 1$ and $\alpha = 6$ the density fluctuation growth factor is more than a factor of 2 different from ΛCDM at redshift $z = 10$ but happens to be fairly close at the present epoch (as one sees by comparing the long- and short-dashed curves in Fig. 4). This solution is challenged by the position of the peak of the CBR fluctuation spectrum, however, as we now discuss.

We estimate the angular scale of the peak of the CBR temperature anisotropy power spectrum as follows. Since the physical wavelength of the mode that produces the peak of the fluctuation spectrum is set by the Hubble length at $z_{\rm eq}$, the wavenumber at the peak varies with the model parameters as

$$\frac{k_{\rm peak}}{a_0} \sim \frac{a_{\rm eq}}{a_0}\frac{1}{t_{\rm eq}} \propto \frac{a_0}{a_{\rm eq}} \propto \frac{f_{\rm eq}}{f_0}. \tag{35}$$

The second step follows because the expansion time is inversely proportional to the square of the temperature at $z_{\rm eq}$, that is, $t_{\rm eq} \propto a_{\rm eq}^2$, and the last step follows because the redshift at equal mass densities in radiation and DM varies as $f_{\rm eq}/f_0$ through the evolution of the DM particle mass. The peak of the angular power spectrum of the CBR temperature is at spherical harmonic number $l_{\rm peak} \sim k_{\rm peak} r$, where the angular size distance $r = \int dt/a$ is integrated from decoupling to the present epoch. Thus, the ratio of the spherical harmonic index $l_{\rm peak}$ in the model to the predicted index at the peak in the ΛCDM model with the same cosmological parameters is

$$\frac{l_{\rm peak}}{l_{\Lambda\rm CDM}} \simeq \frac{f_{\rm eq}}{f_0}\frac{r}{r_{\Lambda\rm CDM}}. \tag{36}$$

This ratio is listed in the last column of Table 1.

The model with $\alpha = 6$ and $G^{1/2}\phi_i = 1$, whose present density fluctuations happen to be close to the ΛCDM prediction, puts the peak of the CBR temperature fluctuation spectrum at angular scale ∼30% smaller than ΛCDM, which likely is unacceptable. At $G^{1/2}\phi_i = 2$ the peak is shifted from ΛCDM by about 16%, which may be tolerable within the uncertainties allowed by the other cosmological parameters that determine the value of $l_{\rm peak}$. A closer analysis of the joint distribution of allowed values of ϕ_i and the cosmological parameters seems inappropriate in this preliminary exploration. The point we wish to demonstrate is that there is a range of initial field values that produce a significant but acceptable departure from the behavior of the ΛCDM model.

The evolution of the DE density nowadays is characterized by an effective DE equation of state. We define the effective pressure $p_{\rm eff}$ by the expression for local energy conservation,

$$\frac{d}{dt}(\rho_{\rm DM} + \rho_{\rm DE}) = -3\frac{\dot a}{a}(\rho_{\rm DM} + \rho_{\rm DE} + p_{\rm eff}). \tag{37}$$

The ratio of the effective pressure to the DE density is

$$w = \frac{p_{\rm eff}}{\rho_{\rm DE}} = -\frac{V - \dot\phi^2/2 + (\nabla\phi)^2/2}{V + \dot\phi^2/2 + (\nabla\phi)^2/2}. \tag{38}$$

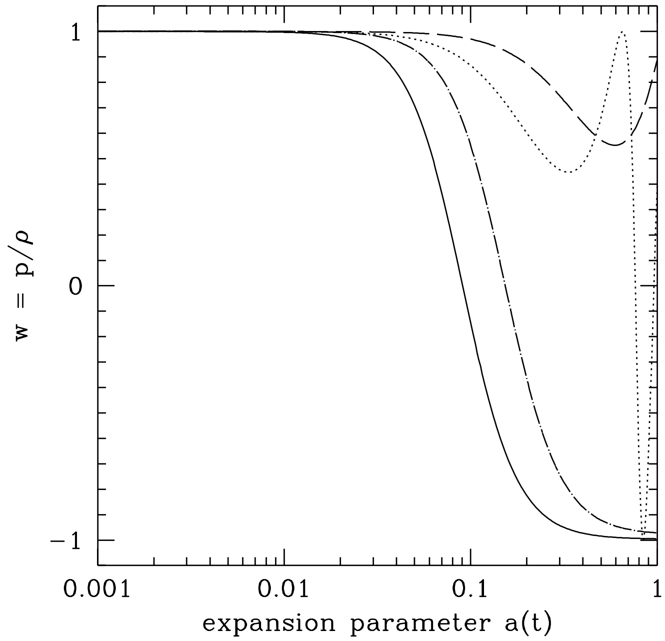

Fig. 5.—Evolution of the equation-of-state parameter (eq. [38]) in solutions with $\alpha = -2$. The line types match Fig. 1.

Figures 5 and 6 show the evolution of the equation-of-state parameter w in our numerical solutions, which neglect the gradient energy density in equation (38). The complicated behavior of w in the solutions with the two smallest initial field values is of no interest because the models are not viable. At the two larger initial field values the parameter is close to constant at $w \simeq -1$ in the range of redshifts reached by the Type Ia supernova observations. At higher redshifts $w \simeq +1$ because in these solutions the DE energy is dominated by $\dot\phi^2/2$, but at high redshift the DE density is well below the DM mass density.

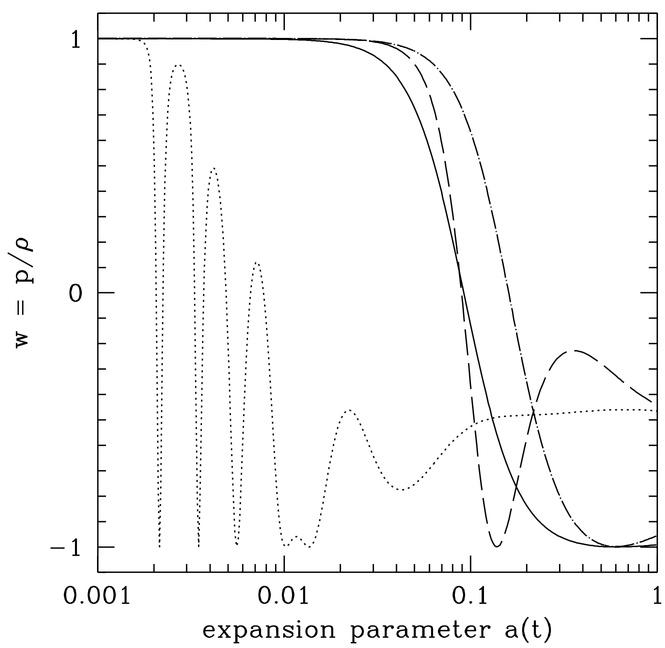

Fig. 6.—Evolution of the equation-of-state parameter in solutions with $\alpha = 6$. The line types match Fig. 2.

5. COSMOLOGIES WITH TWO DARK MATTER FAMILIES

Here our field Lagrangian is

$$L = \frac{\phi_{,\nu}\phi^{,\nu}}{2} - V(\phi + \phi_s) + i\bar{\psi}\gamma\partial\psi + i\bar{\psi}_s\gamma\partial\psi_s - y(\phi_* - \phi)\bar{\psi}\psi - y_s\phi\bar{\psi}_s\psi_s, \quad (39)$$

where for definiteness we choose y, y_s, ϕ_s, and ϕ_* to be positive constants. The mean number densities are $\bar{n}$ in the first family, with wave functions ψ, and $\bar{n}_s$ in the second family. If ϕ is never close to zero or to ϕ_*, then there are two nonrelativistic DM families, a situation that can be equivalent to what we discussed in the previous section. A new possibility offered by this model is that at high particle number density the DE field is "locked" to $\phi \simeq 0$ to minimize the effective potential of the family with the larger value of the Yukawa coupling constant times the number density, making that family relativistic. If $y_s n_s > yn$ in equation (39) and $V(\phi)$ is subdominant, the particle mass in the massive DM family is fixed to $m = y\phi_*$, and the DE density is fixed to $\rho_{\rm DE} = V(\phi_s)$, so the DE behaves like Einstein's cosmological constant. We show in § 5.1 that the locking can also substantially suppress the fifth force. In § 5.2 we comment on the complicated behavior when the number density in the second family becomes small enough to allow the DE field to evolve.

5.1. *A Relativistic Family in the Dark Sector*

We make several approximations that simplify the analysis of the fifth force when the DE field is close to the minimum of the potential of one family.

First, we assume that the length scale of the density fluctuations of interest is much smaller than the Hubble length, so the DM particles can relax to near time-independent equilibrium. Second, we assume that at high redshift, when the DM number densities are large, the scalar field relaxes to the minimum of the particle potential. We take $y_s n_s \gg y\bar{n}$, so the mean value of the scalar field is close to zero. This means that there is a nonrelativistic DM family with mean mass density $y\phi_*\bar{n}$ together with a relativistic second family. Third, it is reasonable to assume that the space distribution of the relativistic family is close to homogeneous. We can see how this comes about by considering the distribution of positions and momenta $p = [\epsilon^2 - y^2\phi(\boldsymbol{x})^2]^{1/2}$ in single particle phase space. If the distribution has relaxed to become nearly independent of time and a function only of the energy ϵ, then the space number density distribution for the second-family particles with energy ϵ is

$$n_s(\boldsymbol{x}) \propto \int d^3p\, \delta\left(\epsilon - \sqrt{\phi^2 + p^2}\right) \propto \frac{p}{\epsilon} = v. \quad (40)$$

This must be averaged over the distribution of energies. However, we see the key and familiar point, that when the second family is relativistic, that is, v is close to unity, the space distribution is close to homogeneous.

To get a viable model, we have to choose parameters so the energy density in the relativistic family is small enough to avoid spoiling light-element production. The typical energy $\epsilon_{\rm eff}$ of a relativistic second-family particle is dominated by its momentum, which scales with the expansion of the universe as $a(t)^{-1}$, because the expansion stretches the de Broglie wavelengths. Thus, as long as the second family remains relativistic, its mean energy density is $\bar{\rho}_s = n_s\epsilon_{\rm eff} \propto a(t)^{-4}$, as usual for relativistic particles. The ratio of the mean energy densities in the two DM families is

$$\frac{\bar{\rho}_s}{\bar{\rho}} \sim \frac{\epsilon_{\rm eff} n_s}{y\phi_*\bar{n}} = \frac{y_s n_s}{y\bar{n}}\frac{\epsilon_{\rm eff}}{y_s\phi_*}. \quad (41)$$

The first factor in the last expression must be larger than unity, say, by a factor of 100, to keep ϕ locked to the second family even as density concentrations in the first family develop. To avoid affecting the standard model for the origin of the light elements, we want the energy density in the relativistic second family to be small compared to the thermal background radiation, meaning that the present density is $(\bar{\rho}_s/\bar{\rho})_0 \lesssim 10^{-4}$. Thus, initial conditions for the particle momenta must be such that $\epsilon_{\rm eff} \sim k_{\rm eff}/a_0 \lesssim 10^{-6}y_s\phi_*$.

Under the above assumptions and neglecting $dV/d\phi$ for the moment, the DE field equation when ϕ is near zero is

$$\frac{\nabla^2\phi}{a^2} = \frac{y_s^2 n_s|\phi|}{\epsilon_{\rm eff}} - y\bar{n}(1+\delta). \quad (42)$$

In the first source term we have written the relativistic correction (eq. [14]) in terms of the effective mean particle energy, $\epsilon_{\rm eff}$, as in equation (23):

$$\left\langle\sqrt{1 - v^2}\right\rangle \equiv \frac{y_s|\phi(\boldsymbol{x})|}{\epsilon_{\rm eff}}. \quad (43)$$

We have dropped the time derivatives of ϕ because we are assuming that the field is locked to a value near zero. The number density n_s in the second (relativistic) family is nearly independent of position. The density contrast in the nonrelativistic DE is $\delta(\boldsymbol{x}, t)$. On scales small compared to the Hubble length the second-family particles see a nearly static potential, so the particle energy $\epsilon_{\rm eff}$ is conserved and thus independent of position along its trajectory.

The space average of equation (42), which neglects $dV/d\phi$, gives the mean field value,

$$\phi_b = \frac{\epsilon_{\rm eff} y\bar{n}}{y_s^2 n_s}. \quad (44)$$

This relation in equation (43) reproduces the condition that the mean inverse Lorentz factor for the second family satisfies

$$\left\langle\sqrt{1 - v^2}\right\rangle = \frac{y\bar{n}}{y_s n_s} \ll 1. \quad (45)$$

The departure from the mean of equation (42) (with $\phi_b > 0$ since we are taking $dV/d\phi = 0$ for the moment) is

$$\frac{\nabla^2\phi_1}{a^2} = \frac{y_s^2 n_s\phi_1}{\epsilon_{\rm eff}} - y\bar{n}\delta. \quad (46)$$

The Fourier transform is

$$\phi_1(\boldsymbol{k}) = \frac{y\bar{n}\delta(\boldsymbol{k})}{k^2/a^2 + y_s^2 n_s/\epsilon_{\rm eff}}. \quad (47)$$

Green's function thus has a Yukawa form, $\propto r^{-1}\exp(-r/r_5)$, with cutoff length $r_5 = (\epsilon_{\rm eff}/y_s^2 n_s)^{1/2}$. This scales with time as $r_5 \propto a(t)$, so the comoving cutoff length is constant.

If the Hubble parameter H is dominated by the mass density $y\bar{n}\phi_*$ in the nonrelativistic DM, the cutoff length satisfies

$$(Hr_5)^2 \sim \frac{G\epsilon_{\rm eff}\phi_* y\bar{n}}{y_s^2 n_s} \sim \frac{1}{\beta}\frac{\bar{\rho}_s}{\bar{\rho}}\left(\frac{y\bar{n}}{y_s n_s}\right)^2. \quad (48)$$

As argued above, the last two factors are at most 10^{-4} and $(10^{-2})^2$, so that

$$(Hr_5)_0 \lesssim 10^{-4}\beta^{-1/2}. \quad (49)$$

To produce $r_5(a_0)$ of order the Hubble radius would require $\beta \sim 10^{-8}$ and thus $\phi_* \sim 10^4 m_{\rm Pl}$, which is disagreeably large on theoretical grounds. On the other hand, $\beta = 1$ implies $r_5(a_0) \lesssim 1$ Mpc, which is well within the nonlinear clustering length and so requires closer analysis to test.

5.2. *Late Time Transition*

The behavior of ϕ when the number densities become small enough to allow the field to evolve depends on the self-interaction potential $V(\phi)$. In linear perturbation theory for the field, the condition that the second-family particle number density is large enough to hold ϕ constant and close to zero is

$$\frac{dV}{d\phi} = \pm y_s n_s\left\langle\sqrt{1-v^2}\right\rangle + y\bar{n}(1+\delta). \quad (50)$$

This assumes $dV/d\phi > 0$ at $\phi \simeq 0$. The negative sign in the first term on the right-hand side applies when $\phi > 0$, and the positive sign applies when the particle number densities are small enough to allow $dV/d\phi$ to pull ϕ to a slightly negative value. Because the second family is relativistic, n_s is nearly homogeneous, as we have discussed. This means that the reciprocal Lorentz factor (eq. [43]) must be a function of position, balancing the irregular DM mass distribution in the first family represented by the number density contrast $\delta(\boldsymbol{x}, t)$. We check below that even when the mass density contrast is nonlinear, equation (50) is a good approximation for the perturbation to the field.

We first consider the case where $dV/d\phi$ can be neglected, so equation (50) is

$$y_s n_s\left\langle\sqrt{1-v^2}\right\rangle = y\bar{n}(1+\delta). \quad (51)$$

There comes a time when the value $\delta_{\rm max}$ of the density contrast within the strongest concentrations of the massive DM family is large enough to satisfy $y_s n_s = y\bar{n}(1+\delta_{\rm max})$. This forces the second family to become nonrelativistic in the neighborhood of $\delta_{\rm max}$. Further expansion of the universe increases the density contrasts, causing ϕ in the vicinity of a DM mass concentration to increase. When $y_s\phi > \epsilon_{\rm eff}$, second-family particles are pushed out of the regions of first-family concentrations because their energy is not sufficient to allow them to have such a large mass. If this rearrangement is happening on length scales much smaller than the Hubble length, the DE field equation is dominated by the spatial derivatives, and we have

$$\frac{\nabla^2\phi}{a^2} = y_s n_s(\boldsymbol{x}, t)\left\langle\sqrt{1-v^2}\right\rangle - y\bar{n}(1+\delta), \quad (52)$$

for the case that $dV/d\phi$ can be neglected. As indicated, the second-family number density n_s is now a function of position because less energetic particles are excluded from the concentrations of the first family. The spatial mean of equation (52) says that the inverse Lorentz factor averaged over all second-family particles satisfies

$$\left\langle\left\langle\sqrt{1-v^2}\right\rangle\right\rangle = \frac{y\bar{n}}{y_s\bar{n}_s}. \quad (53)$$

Since we are assuming that the ratio on the right-hand side is smaller than unity, the pools of relativistic second-family particles in the regions between the concentrations of nonrelativistic DM are always able to hold the field value close to zero in the voids between the concentrations of galaxies.

We have been assuming that the field value within a concentration of the first family is less than ϕ_*, so these particles are nonrelativistic. To estimate the extent of the mass shift of first-family particles, consider a concentration of $N \sim \bar{n}R^3$ nonrelativistic DM particles drawn from an initially homogeneous patch of size R into a concentration with density contrast δ over a size r. The typical shift in the DE field value averaged over this concentration is $\phi_r \sim yN/r$. Assuming that the expansion rate is mainly due to the mass density in this family so that the Hubble parameter satisfies $H^2 \sim Gy\bar{n}\phi_*$, we get

$$\frac{\phi_r}{\phi_*} \sim \frac{(HR)^2}{\phi_*^2}\frac{R}{r} \sim \beta(Hr)^2\delta. \quad (54)$$

The fractional shift in the scalar field value is largest in the largest mass concentrations. For example, the density contrast in a large galaxy is about $\delta = 10^6$ at $r = 10$ kpc, which gives $\phi_r/\phi_* \approx 10^{-5}\beta$, and in a rich cluster at $r = 2$ Mpc where $\delta \simeq 100$, $\phi_r/\phi_* \approx 10^{-4}\beta$. That is, as long as β is not large, the first-family particle masses are only slightly perturbed by the spatial variation of ϕ, and the force on massive DM particles from the gradient of ϕ is well approximated as β times the gravitational attraction (eq. [18]).

Finally, let us briefly consider what happens when $dV/d\phi$ in equation (50) is positive and large enough to pull ϕ to negative values. This makes the second family first become nonrelativistic in the voids, where the first-family number density is low. When this happens, the field in the voids moves toward the minimum of V, at $\phi = -\phi_s$. If $y_s\phi_s \gg \epsilon_{\rm eff}$, then the DE field pushes the second family out of the voids. Since the mean particle number densities are decreasing as the universe expands, the potential V must eventually pull the DE field away from zero everywhere, producing a second family of nonrelativistic DM.

When the DE field is no longer locked to the zero of mass for the second family, there is a fifth force, but it need not parallel the peculiar gravitational attraction of the DM because the space distributions of the two families may differ.

A model in which the lock on the DE field has broken before the present epoch and produced a large fifth force within concentrations of galaxies would not be acceptable, but a model with a large fifth force in the voids between the concentrations of large galaxies might be quite interesting, as a way to understand why the voids are so empty.

6. CONCLUDING REMARKS

We have explored physical processes in models where the DM particles have a Yukawa coupling to a scalar field that can be the source of the DE. Our central conclusion is that parameters and initial conditions in such models can be chosen so that the model is viable but significantly different from the standard ΛCDM cosmology. It might be useful to follow this up by considering whether the interesting range of initial field values, just somewhat larger than the Planck mass for the models in § 4, could naturally follow from models for the very early universe. It would also be useful to know whether the number density of DM particles required to give the observed value of Ω_m can be naturally understood, for example, by gravitational production.

As illustrated in § 4 for a single DM family, the constraints on cosmological parameters derived by fitting the model for the dark sector to the observations can differ from what is obtained from fitting to the standard model. Such alternative models are therefore useful as foils to ΛCDM, for the purpose of evaluating the empirical constraints on the cosmological parameters.

We have also considered two-family models that can lead to a rich and interesting cosmology. As discussed in § 5, initial conditions can be chosen so the DE field is locked to the zero of mass for the more numerous family and remains so up to the present epoch. This removes the evolution of the DE field and the evolution of the mass of the nonrelativistic DM particles, and it can suppress the fifth force in the dark sector. This is an example of how a dynamical model for the DE can be observationally indistinguishable from Einstein's cosmological constant. The situation changes when the DE particle number density becomes low enough to free the DE field. When this happens, the behavior can become quite different from the standard model.

Models of the type studied here, in which one or possibly several families of DM have masses set by their interaction with a dynamical scalar field, are a useful cautionary example of another point: the empirical evidence on how the universe has been evolving up to now may be a rather deceptive guide to its physics or to its future.

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