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ENERGETIC CHARGED PARTICLE TRANSPORT AND ENERGIZATION IN DYNAMIC TWO-DIMENSIONAL TURBULENCE

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ABSTRACT

Quasi-linear theory has been used extensively to study the interaction of energetic particles with MHD fluctuations in the solar wind. However, in recent years there has developed a view that solar wind MHD turbulence can be modeled approximately as a dominating incompressible two-dimensional turbulence component combined with a minor one-dimensional parallel-propagating Alfvén wave component (a one-dimensional slab component in the static limit). Here a quasi-linear theory is developed to investigate the effect of dynamical two-dimensional MHD turbulence in the solar wind on low-energy charged particle pitch-angle scattering and momentum diffusion. Stochastic acceleration by transverse two-dimensional turbulence electric field fluctuations is also considered, yielding finite momentum diffusion coefficients. We find significant effects by energy-containing–scale, dynamic two-dimensional turbulence on low-energy particles in the vicinity of 1 keV energies, and overall dominance of parallel-propagating Alfvén waves at higher energies.

Subject headings: acceleration of particles — MHD — scattering — solar wind — turbulence

1. HISTORICAL PERSPECTIVE

The landmark study of Belcher & Davis (1971) and other similar studies have established a strong presence of Alfvén waves in solar wind MHD fluctuations up to a level of 90% of the total energy density in the magnetic field fluctuations in some solar wind regions. Some other studies suggested that the Alfvén waves are one-dimensional and outward propagating along the large-scale magnetic field (Daily 1973; Chang & Nishida 1973). Belcher & Davis (1971) also discovered that variations of magnetic field magnitude are small compared to the energy density of the magnetic field fluctuations. This suggested that the magnetoacoustic wave component is smaller (<10%) than the Alfvénic component. The strong presence of Alfvén waves in the solar wind implied that the solar wind medium is approximately incompressible. This lack of compressibility is consistent with expected kinetic damping of all plasma waves except the Alfvén mode (Barnes 1979).

At the same time another perspective of MHD fluctuations in the solar wind evolved. Coleman (1968) presented observations that show extended power-law spectra in the energy density of solar wind Alfvénic fluctuations, suggesting an interpretation in terms of MHD turbulence theory. Lacking detailed information, for a number of years this turbulent cascade was assumed by default to be isotropic, as, for example, was done in theoretical work of similar vintage (Kraichnan 1965).

These considerations led to an approach in which quasi-linear kinetic theories (QLTs) for energetic particle transport in solar wind MHD turbulence were developed that treated the turbulence as having either a "slab" geometry inspired by the observations of one-dimensional outward-propagating Alfvén waves or an isotropic geometry, as in the theoretical work of Kraichnan. Application of the slab model of QLTs led to the well-known problem that the parallel diffusion mean free path of cosmic rays is predicted to be about a factor of 10 lower than expected from observations (Bieber et al. 1994).

More recently, it was realized that the isotropic cascade in wavenumber space for incompressible MHD turbulence should be revised (Montgomery & Turner 1981; Shebalin, Matthaeus, & Montgomery 1983; Sridhar & Goldreich 1994; Oughton, Priest, & Matthaeus 1994; Goldreich & Sridhar 1995; Montgomery & Matthaeus 1995; Matthaeus et al. 1998). Many of the underlying ideas of this revision were discussed by Montgomery & Turner (1981). According to Montgomery & Turner (1981), in the presence of a strong large-scale magnetic field, the incompressible MHD turbulence spectrum should have two components, an idea that was applied later (Matthaeus, Goldstein, & Roberts 1990), in a somewhat different form, to solar wind observations. The first component, also included in the solar wind two-component model, is viewed as the dominant or most robust form of incompressible nonlinear MHD activity. Its velocity and magnetic fluctuations, as well as its wavenumbers, are all perpendicular or nearly perpendicular to the background magnetic field $\boldsymbol{B}_0$. The so-called two-dimensional MHD component with perpendicular wavenumbers $k_\perp$ may be highly dynamic as a result of nonlinear effects but is also thought of as having "zero frequency." This terminology derives from the wave frequency ω obtained from the shear Alfvén wave dispersion relation, $\omega = V_\mathrm{A}|k_\parallel|$, where $k_\parallel$ is the wavenumber parallel to $\boldsymbol{B}_0$ and V_A is the Alfvén speed. Having purely perpendicular wavenumbers ($k_\parallel = 0$), two-dimensional turbulence has zero frequency or equivalently is "nonpropagating." The second minor component, determined by the linear terms of the incompressible MHD equations, describes the shear Alfvén waves.

Montgomery & Turner (1981) predicted that most of the energy should reside in the two-dimensional turbulence component. The possibility of energy transfer from a two-dimensional mode to two Alfvén waves exists, but the rate of transfer is expected to be much lower than between three two-dimensional modes. This suggests that an energy cascade should preferentially occur perpendicular to $\boldsymbol{B}_0$ in the two-dimensional turbulence component and suppressed along $\boldsymbol{B}_0$ for the shear Alfvén wave component (Matthaeus et al. 1998). Consequently, the turbulence correlation length should be much longer along $\boldsymbol{B}_0$ than transverse to $\boldsymbol{B}_0$. This anisotropy also makes sense if one considers that it is much more difficult to bend and stretch magnetic field lines along the background magnetic field $\boldsymbol{B}_0$ if $\boldsymbol{B}_0$ is strong compared with the fluctuating magnetic field. Consistent with this view are the incompressible three-dimensional MHD simulations of Cho & Lazarian (2002), which suggest that the fluctuation energy density spectrum $E^{\delta B}(k_\perp)$ for perpendicular wavenumber $k_\perp$ is a Kolmogorov power law given by $E^{\delta B}(k_\perp) \propto k_\perp^{-5/3}$, while for parallel wavenumber $k_\parallel$ they found $E^{\delta B}(k_\parallel) \propto k_\parallel^{-2}$, approximately.

There is evidence from more recent solar wind data studies that a quasi–two-dimensional incompressible MHD turbulence viewpoint might have validity. The two-point correlation function of magnetic field fluctuations, assumed axisymmetric, has a two-component structure, suggesting the presence of both a one-dimensional or "slab" turbulence component with a long correlation length perpendicular to $\boldsymbol{B}_0$ and a quasi–two-dimensional turbulence component with a long correlation length along $\boldsymbol{B}_0$ (Matthaeus et al. 1990). A complementary direct analysis of solar wind magnetic spectra, assuming at the onset a composite two-dimensional plus slab turbulence model, revealed that as much as $\sim$85% of the fluctuation energy resides in the two-dimensional component (Bieber, Wanner, & Matthaeus 1996).

The advantage of viewing MHD turbulence in the solar wind as consisting of a superposition of a dominating two-dimensional component and a minor slab component became apparent when Bieber et al. (1996) revised the parallel mean free path predicted by slab QLT for cosmic rays and found good agreement with observations if $\sim$15% of the fluctuation energy is in the static slab modes. This result is based on the finding from QLT that static two-dimensional turbulence does not contribute to cosmic-ray scattering (Bieber et al. 1994).

Now we discuss the issue of incompressibility. A detailed study of solar wind density fluctuations using high-resolution *Voyager* data found fractional density fluctuations to be on average small ($\sim$10%; Matthaeus et al. 1991) so that solar wind is nearly incompressible in a statistical sense. A mathematical theory of nearly incompressible MHD (NIMHD) turbulence (Zank & Matthaeus 1992, 1993) explains how low compressibility fits into the quasi–two-dimensional picture. According to the NI theory, as long as the plasma $\beta_p \leq 1$ approximately, the leading-order and underlying description of turbulence in the presence of $\boldsymbol{B}_0$ is determined by the incompressible two-dimensional MHD equations that are valid in planes perpendicular to $\boldsymbol{B}_0$. A higher order description allows in addition one-dimensional Alfvén waves propagating along $\boldsymbol{B}_0$, while three-dimensional density fluctuations propagate isotropically as high-frequency magnetosonic waves when $\beta_p \approx 1$. When $\beta_p \ll 1$, low-frequency acoustic waves propagate along $\boldsymbol{B}_0$, while high-frequency two-dimensional density fluctuations are convected by the incompressible two-dimensional MHD fluctuations in a plane perpendicular to $\boldsymbol{B}_0$. The NIMHD turbulence description thus corresponds closely to the incompressible MHD description by yielding a dominant incompressible two-dimensional turbulence component and a minor Alfvén wave component, but it also allows for additional minor compressible wave and density fluctuation components. Interestingly, the theory supports the existence of a parallel-propagating Alfvén wave mode as suggested by solar wind observations (Matthaeus et al. 1990).

2. PURPOSE OF THIS PAPER

Based on the discussion above, we thus assume that MHD turbulence in the solar wind can be described by a composite, two-component model: a superposition of a dominant two-dimensional turbulence component with a perpendicular correlation length $l_{c\perp}$ and a minor one-dimensional incompressible Alfvén wave component propagating along $\boldsymbol{B}_0$ with a parallel correlation length $l_{c\parallel}$. The QLT for particle transport through parallel-propagating one-dimensional Alfvén wave turbulence has been worked out before (e.g., Schlickeiser 1989), and only the results are shown.

In the past, QLT was almost exclusively used to study resonant interaction of energetic charged particles with waves rather than turbulence. The notable exception is the work by Bieber et al. (1994), in which a QLT for pitch-angle scattering by dynamic two-dimensional MHD magnetic field turbulence is presented but little detail is given (see also Shalchi & Schlickeiser 2003). In this paper a detailed discussion is presented of QLT from the perspective of turbulence. In an extension of the Bieber et al. (1994) work, which treats only dynamic two-dimensional magnetic turbulence fluctuations, transverse two-dimensional electric fields are also included in this theory. This enables us to study the role of dynamic two-dimensional turbulence in the stochastic acceleration of energetic particles.

In the case of magnetostatic turbulence, two-dimensional magnetic field fluctuations do not contribute to particle pitch-angle scattering (Bieber et al. 1994, 1996) so that one-dimensional parallel-propagating Alfvén waves, or, in the magnetostatic limit, nonpropagating "slab" turbulence, are mainly responsible for this process. The static assumption is good for typical cosmic-ray energies when the particle speed is much larger than the rms convective speed of the turbulence (particle rigidities $\sim$100 MV and higher).

However, at lower super-Alfvénic particle speeds, particles are affected by time variations in the turbulence, and dynamic two-dimensional turbulence will contribute to pitch-angle scattering (Bieber et al. 1994). In this paper our emphasis will be on the effect of dynamic two-dimensional MHD turbulence and parallel-propagating Alfvén waves on particle pitch-angle scattering, as well as stochastic acceleration in the solar wind at super-Alfvénic but sub–cosmic-ray energies. Specifically, the relative contributions of dynamic two-dimensional turbulence and propagating Alfvén waves to pitch-angle diffusion and momentum diffusion will be compared as a function of energy at different radial distances in the ecliptic plane from near the Sun to 5 AU during quiet solar wind conditions.

Our main finding is that the dynamic two-dimensional turbulence component causes significant pitch-angle and momentum diffusion of low-energy protons with super-Alfvénic particle speeds ($\sim$1 keV protons) in the ecliptic plane of the solar wind. However, the prediction is that parallel-propagating Alfvén waves will still dominate these diffusion processes in spite of the greater energy density assumed to be in the two-dimensional component. The dominance of Alfvén waves in pitch-angle scattering and momentum diffusion is expected to increase with increasing heliocentric distance from the Sun and with increasing particle energy.

3. QUASI-LINEAR THEORY OF PARTICLE PITCH-ANGLE SCATTERING AND MOMENTUM DIFFUSION IN DYNAMIC TWO-DIMENSIONAL TURBULENCE

We followed a standard QLT approach to derive a Fokker-Planck kinetic transport equation for the diffusion of charged particles in pitch angle and momentum space during interaction with the dynamic two-dimensional turbulence component of MHD solar wind turbulence. The details of the QLT for two-dimensional turbulence can be found in the Appendix. The main assumptions of the QLT are the following:

1. Small-amplitude electromagnetic fluctuations. Particles follow undisturbed helical orbits on a particle correlation timescale t_c^p, which indicates the typical time that elapses before the particle starts to see incoherent or random fluctuations along its undisturbed trajectory.
2. The particle correlation time $t_c^p \ll t_\mu$, where t_μ is the particle pitch-angle scattering timescale. The pitch-angle timescale represents the characteristic time needed before the particle experiences random deviations in its pitch angle. The condition $t_c^p \ll t_\mu$ ensures that an undisturbed helical orbit is maintained on a particle correlation timescale that is much shorter than the timescale over which particle orbits are distorted by pitch-angle scattering on the two-dimensional turbulence.
3. The ensemble-averaged distribution function is assumed to be gyrotropic, which rules out the contributions of perpendicular spatial diffusion and gradient and curvature drifts to particle transport. This implies that the particle gyroradius $r_g \ll l_{c\perp}$, so that for typical solar wind parameters near Earth ($l_{c\perp} = 0.01$ AU), the derivation is valid for energetic particles with rigidities $R \ll 2.5$ GV. However, even if perpendicular diffusion and drifts are important (Giacalone & Jokipii 1999), the theory should still provide a good description of pitch-angle scattering and stochastic acceleration.
4. It is implicit in the theory that particles have to propagate many turbulence correlation lengths $l_{c\perp}$ along the background magnetic field before they undergo pitch-angle scattering, which results in spatial diffusion along the field so that $l_{c\perp} \ll \lambda_\parallel$, where $\lambda_\parallel$ is the mean free path for spatial diffusion along the large-scale magnetic field $\boldsymbol{B}_0$.
5. It is also assumed that the large-scale magnetic field is nonuniform only on the largest scale of the system L. Thus, the ordering of length scales for QLT is given by $r_g \ll l_{c\perp} \ll \lambda_\parallel \ll L$.
6. The QLT was derived under the usual simplified conditions of homogeneous stationary two-dimensional turbulence that is axisymmetric around the large-scale magnetic field. The assumptions here are not required to hold on spatial scales exceeding the turbulence correlation length and timescales longer than the particle correlation time.

The equation for particle transport during interaction with the transverse random magnetic and electric field fluctuations (see eqs. [A5] and [A6]) in the dynamic two-dimensional component of MHD turbulence (Appendix, eq. [A49]) is given by

$$\frac{\partial f_0}{\partial t} + v\mu\frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_0}{\partial \mu} + D_{\mu p}\frac{\partial f_0}{\partial p}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D_{p\mu}\frac{\partial f_0}{\partial \mu} + D_{pp}\frac{\partial f_0}{\partial p}\right), \quad (1)$$

where $f_0(z, p, \mu, t)$ is the ensemble-averaged particle distribution function, which is a function of position z along the large-scale uniform magnetic field $\boldsymbol{B}_0$, particle momentum p, the cosine of the particle pitch angle μ, and time t. The expressions for the Fokker-Planck diffusion coefficients in momentum vector space are

$$D_{\mu\mu} = \Omega^2(1-\mu^2)T_c^p\left[\mu^2 r_{\rm A}\left(\frac{V_{\rm A}}{v}\right)^2 + \mu\sigma_c(r_{\rm A}+1)\left(\frac{V_{\rm A}}{v}\right) + 1\right],$$

$$D_{pp} = \Omega^2(1-\mu^2)\left(\frac{pV_{\rm A}}{v}\right)^2 T_c^p r_{\rm A},$$

$$D_{\mu p} = D_{p\mu} = -\Omega^2(1-\mu^2)\left(\frac{pV_{\rm A}}{v}\right)T_c^p \times\left[\mu\left(\frac{V_{\rm A}}{v}\right)r_{\rm A} + \frac{1}{2}\sigma_c(r_{\rm A}+1)\right], \quad (2)$$

where $D_{\mu\mu}$ is the coefficient for particle diffusion in pitch-angle space, D_{pp} is the coefficient for particle diffusion in momentum magnitude space, $D_{p\mu} = D_{\mu p}$ are the coefficients in combined μ and p space, $r_{\rm A}$ is the Alfvén ratio, Ω is the particle gyrofrequency, T_c^p is a time integral along the undisturbed particle orbit and is associated with the particle decorrelation time, $V_{\rm A}$ is the Alfvén speed, and σ_c is the cross helicity. The Alfvén ratio is $r_{\rm A} = \langle\delta U_\perp^2\rangle/\langle\delta V_{\rm A}^2\rangle$, where $\delta V_{\rm A}^2 = \langle\delta B_\perp^2\rangle/(4\pi\rho)$, $\langle\delta U_\perp^2\rangle$ is the ensemble-averaged energy in the transverse two-dimensional velocity fluctuations, $\langle\delta B_\perp^2\rangle$ is the ensemble-averaged energy density in the transverse two-dimensional magnetic field fluctuations, and ρ is the mass density of the solar wind plasma, while the cross helicity is defined as $\sigma_c = 2\langle\delta\boldsymbol{U}_\perp\cdot\delta\boldsymbol{V}_{\rm A}\rangle/(\langle\delta U_\perp^2\rangle + \langle\delta V_{\rm A}^2\rangle)$. For the case of stationary homogeneous axisymmetric two-dimensional turbulence the time integral T_c^p has the expression

$$T_c^p = \int_0^\infty d(\Delta t)\left[\cos(\Omega\Delta t)R_{xx}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) - \sin(\Omega\Delta t)R_{xy}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t)\right], \quad (3)$$

where $R_{xx}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \langle\delta B_x(0,\ 0)\delta B_x(\Delta\boldsymbol{r}(\Delta t), \Delta t)\rangle$ and $R_{xy}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \langle\delta B_x(0,\ 0)\delta B_y(\Delta\boldsymbol{r}(\Delta t), \Delta t)\rangle$ are the two-point two–time correlation functions involving transverse components of the two-dimensional turbulence magnetic field fluctuations. In equation (3), the argument $\Delta\boldsymbol{r}(\Delta t)$ of the correlation functions R_{ij} contains the undisturbed particle helical trajectory information, which in Cartesian coordinates has the components

$$\Delta x(\Delta t) = -r_g[\cos(\Omega\Delta t) - 1],$$
$$\Delta y(\Delta t) = +r_g\sin(\Omega\Delta t),$$
$$\Delta z(\Delta t) = v\cos\theta(\Delta t). \quad (4)$$

An approximate expression for T_c^p is developed in the Appendix, leading to specific forms for the diffusion coefficients in equation (2).

It has been shown before that the time integral $T_c^p = 0$ in QLT for the case of static two-dimensional turbulence (Bieber et al. 1994, 1996; le Roux, Zank, & Matthaeus 2001). To gain more insight in this issue, we investigated the properties of T_c^p. The time integral T_c^p describes how particles experience incoherent two-dimensional magnetic field fluctuations over a particle correlation time t_c^p during its undisturbed orbit along $\boldsymbol{B}_0$. In the limit of static (noninteracting) turbulence, this integral is singular, as we now discuss. Specifically,

$$T_c^p = \lim_{T\to\infty} F(T), \quad F(T) = \int_0^T f(t)\,dt, \tag{5}$$

where $f(t)$ is a periodic function of time, with period $2\pi/\Omega$. We show below that $F(T)$ itself is a periodic function of T and that $F(T) = 0$ if T is a harmonic of $2\pi/\Omega$, which implies that T_c^p either converges to zero or does not converge at all. As far as we know, M. A. Forman (2003, private communication) and G. Qin, J. W. Bieber, & W. H. Matthaeus (2003, private communication) were the first to point out this problem independently and to show that the net result of gyration over one unperturbed particle orbit in two-dimensional static turbulence implies no particle diffusion.

To illustrate that $F(T) = 0$ if T is a harmonic of $2\pi/\Omega$, it is useful to express the two-dimensional magnetic field fluctuations in terms of their magnetic vector potential on the basis of the solenoidal condition that $\nabla\cdot\delta\boldsymbol{B}_\perp = 0$:

$$\delta\boldsymbol{B}_\perp = \nabla\times A\boldsymbol{e}_z, \tag{6}$$

where the magnetic vector potential, given by $\boldsymbol{A} = A\boldsymbol{e}_z$, is specified along the direction of the background magnetic field $\boldsymbol{B} = B_0\boldsymbol{e}_z$. From equation (6) it follows that the transverse two-dimensional magnetic field components can be expressed as

$$\delta B_x = \frac{\partial A}{\partial y}, \quad \delta B_y = -\frac{\partial A}{\partial x}. \tag{7}$$

Because the two-dimensional turbulence component is a function of spatial coordinates (x, y) transverse to $\boldsymbol{B}_0$, the magnetic field correlation function along the undisturbed particle orbit only depends on the particle motion in the two-dimensional plane perpendicular to $\boldsymbol{B}_0$. Thus, for static two-dimensional turbulence we specify $R_{ij}^{\delta B}(\Delta\boldsymbol{r}_\perp(\Delta t))$, where $\Delta\boldsymbol{r}_\perp(\Delta t) = (\Delta x(\Delta t), \Delta y(\Delta t))$ with the components given by equation (4). The integral $F(T)$ in equation (5) over an arbitrary integer number m of undisturbed particle gyro-orbits $T = m2\pi/\Omega$ can then be expressed as

$$\begin{aligned} F\left(m\frac{2\pi}{\Omega}\right) =& \frac{2}{r_g\Omega}\int_0^{m(2\pi/\Omega)} d(\Delta t) \\ &\times\left\langle \delta B_x(\Delta\boldsymbol{r}_\perp(\Delta t=0))\frac{d\delta A}{d\Delta t}(\Delta\boldsymbol{r}_\perp(\Delta t))\right\rangle \\ =& \frac{2}{r_g\Omega}\left\langle \delta B_x(\Delta\boldsymbol{r}_\perp(\Delta t=0)) \right. \\ &\left. \times\left[A\left(\Delta\boldsymbol{r}_\perp\left(\Delta t = m\frac{2\pi}{\Omega}\right)\right) - A(\boldsymbol{r}_\perp(\Delta t=0))\right]\right\rangle \\ =& 0, \end{aligned} \tag{8}$$

where $m > 0$ is an integer.

However, this symmetry can be broken by allowing the particle orbit to be distorted during a particle correlation time by the neglected nonlinear terms in an extended QLT (Dupree 1966; Jones, Kaiser, & Birmingham 1973; Volk 1973; Owens 1974) or by specifying that the fluctuations have a spatial dependence along $\boldsymbol{B}_0$. A third option, which is the one we explore here, is to introduce dynamic turbulence effects (Bieber et al. 1994).

4. A SIMPLE ESTIMATE OF THE PITCH-ANGLE AND MOMENTUM DIFFUSION COEFFICIENTS FOR DYNAMIC TWO-DIMENSIONAL TURBULENCE

Before we present the results for the time integral T_c^p following the standard Fourier transform approach in wavenumber space, it is instructive to first estimate this integral in real space. This provides physical insight that is more difficult to acquire with the more complicated but also more precise Fourier transform approach.

Within the framework of QLT and the corresponding approximate evaluation of forces along unperturbed orbits, we assume that the two-point two–time particle correlation function decays with time as a result of two effects: (1) the particle gyromotion in the two-dimensional plane perpendicular to the large-scale magnetic field $\boldsymbol{B}_0$ (there is no spatial variation along $\boldsymbol{B}_0$ and correspondingly no spatial decorrelation) and (2) the random convective motions of two-dimensional turbulence relative to the particle orbit.

Accordingly, the temporal decay of the correlation functions of magnetic field fluctuations $R_{ij}^{\delta B\,\delta B}$ along the undisturbed particle orbit is modeled as

$$\begin{aligned} R_{xx}^{\delta B\,\delta B}(\Delta\boldsymbol{r}_\perp(\Delta t),\Delta t) &= R_{xx}^{\delta B\,\delta B}(\Delta\boldsymbol{r}_\perp(\Delta t))e^{-\Delta t/t_c^p}, \\ R_{xy}^{\delta B\,\delta B}(\Delta\boldsymbol{r}_\perp(\Delta t),\Delta t) &= R_{xy}^{\delta B\,\delta B}(\Delta\boldsymbol{r}_\perp(\Delta t))e^{-\Delta t/t_c^p}, \end{aligned} \tag{9}$$

where $\Delta\boldsymbol{r}_\perp(\Delta t)$ represents the spatial separation in the two-dimensional plane perpendicular to $\boldsymbol{B}_0$ and the timescale t_c^p represents the correlation time due to dynamic turbulence decorrelation effects modeled as an exponential decay (Bieber et al. 1994). This is in the spirit of, but simpler than, the Fourier transform approach where a similar separation of spatial and temporal decorrelation takes place (see eq. [A37]). The maximum spatial separation in the two-dimensional plane is $|\Delta\boldsymbol{r}_\perp| \approx r_g$. However, since in the present development we restrict ourselves to $r_g \ll l_{c\perp}$, it follows that $R_{ij}^{\delta B\,\delta B}(\boldsymbol{r}_\perp(\Delta t)) \approx R_{ij}^{\delta B\,\delta B}(0) = \langle\delta B_i\,\delta B_j\rangle$. Accordingly,

$$\begin{aligned} R_{xx}^{\delta B\,\delta B} &= \langle\delta B_x^2\rangle e^{-\Delta t/t_c^p}, \\ R_{xy}^{\delta B\,\delta B} &= \langle\delta B_x\delta B_y\rangle e^{-\Delta t/t_c^p}, \end{aligned} \tag{10}$$

and the dominant decorrelation effect is due to intrinsic time variation of the turbulence.

For the case of axisymmetric turbulence $\langle\delta B_x^2\rangle = \frac{1}{2}\langle\delta B_\perp^2\rangle$, where $\langle\delta B_\perp^2\rangle$ denotes the total energy density in the two-dimensional magnetic field fluctuations, and $\langle\delta B_x\delta B_y\rangle = 0$. Accordingly,

$$\begin{aligned} R_{xx}^{\delta B\,\delta B} &= \tfrac{1}{2}\langle\delta B_\perp^2\rangle e^{-\Delta t/t_c^p}, \\ R_{xy}^{\delta B\,\delta B} &= 0. \end{aligned} \tag{11}$$

Straightforward integration of the time integral T_c^p leads to the result that

$$T_c^p = \frac{1}{2}\left[\frac{1/t_c^p}{(1/t_c^p)^2+\Omega^2}\right]\frac{\langle\delta B_\perp^2\rangle}{B_0^2}. \tag{12}$$

Further simplification can be achieved by taking the limit $\Omega \gg 1/t_c^p$, which will hold for the important contributions to diffusion associated with the energy-containing eddies, which we refer to in § 4.1 as energy-containing–scale turbulence.

4.1. *Energy-Containing–Scale Two-dimensional Turbulence*

With energy-containing–scale two-dimensional fluctuations we imply fluctuations with wavenumbers restricted to $k_\perp \ll 1/r_g$. Within the QLT conceptual framework, a particle with an undisturbed helical orbit along $\boldsymbol{B}_0$ cannot experience decorrelated energy-containing–scale two-dimensional fluctuations as a result of its gyromotion in the two-dimensional plane perpendicular to $\boldsymbol{B}_0$ because $r_g \ll 1/k_\perp$. Decorrelation due to particle motion along $\boldsymbol{B}_0$ is also not possible because the two-dimensional component is not a function of distance along $\boldsymbol{B}_0$. Within the context of QLT decorrelation can only occur as a result of the random convective motions of the turbulence relative to the undisturbed particle orbit (dynamic turbulence). Thus, the particle correlation time

$$t_c^p(\Delta t) = \frac{l_{c\perp}}{\delta U(l_{c\perp})} \approx \frac{l_{c\perp}}{\delta U_\perp}, \tag{13}$$

where the assumption of equipartition in the energy of the two-dimensional velocity and magnetic field fluctuations means that $\delta U_\perp = (\delta B_\perp / B_0) V_{\rm A}$ (see paragraph below eq. [A43]). This assumption is reasonable for solar wind conditions (Goldstein, Roberts, & Matthaeus 1995) and is also supported by MHD turbulence simulations in the presence of a sufficiently strong background magnetic field (Oughton, Matthaeus, & Ghosh 1998). For typical heliospheric parameters $\Omega \gg 1/t_c^p$ so that T_c^p in equation (12) becomes

$$\begin{aligned} T_c^p &= \frac{1}{2}\frac{1}{t_c^p \Omega^2}\frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \\ &= \frac{1}{2}\left(\frac{\delta U_\perp}{l_{c\perp}}\right)\frac{1}{\Omega^2}\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}. \end{aligned} \tag{14}$$

By inserting equation (14) into equation (2) and assuming $v \gg V_{\rm A}$ and $\sigma_c \neq 0$, we get

$$\begin{aligned} D_{\mu\mu} &= \frac{1}{2}\left(1-\mu^2\right)\left(\frac{\delta U_\perp}{l_{c\perp}}\right)\frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \\ &= \frac{1}{2}\left(1-\mu^2\right)\left(\frac{1}{t_c^p}\right)\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}, \\ D_{pp} &= D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)^2 r_{\rm A}, \\ D_{p\mu} &= D_{\mu p} = -D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)\left[\frac{1}{2}\sigma_c(r_{\rm A}+1)\right]. \end{aligned} \tag{15}$$

5. DIFFUSION COEFFICIENTS FOR DYNAMIC TWO-DIMENSIONAL TURBULENCE BASED ON THE FOURIER-TRANSFORMED TIME INTEGRAL

Evaluation of the time integral T_c^p in equation (3) is carried out in the Appendix. It is found that, for small-amplitude fluctuations and super-Alfvénic particles that are resonant in the inertial range, the decorrelation integral T_c^p simplifies to

$$T_c^p \approx \frac{8\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}\frac{1}{\Omega^2}\int_{r_g/l_{c\perp}}^{\infty} dx\,\frac{1}{x^3}\sum_{n=1}^{\infty} J_n^2(x) \tag{16}$$

(see also Shalchi & Schlickeiser 2003). The principal additional approximations regarding the turbulence employed in arriving at this form are a two-dimensional Kolmogorov inertial range spectrum for the wavenumber dependence and dynamical effects modeled with a $k_\perp$-dependent exponential decay in time at a rate given by the local (in wavenumber) nonlinear eddy turnover timescale, along with incompressibility, axisymmetry, and structural similarity of correlation functions.

In terms of Fourier transforms the expression for T_c^p is complicated. Nonetheless, it is clear that in the limit of static (noninteracting) two-dimensional turbulence ($\delta U_\perp = 0$ in eq. [16]), T_c^p and therefore the diffusion coefficients involving transverse two-dimensional fluctuations are zero for any wavenumber, which is consistent with previous QLT theories for static two-dimensional turbulence (Bieber et al. 1994, 1996; le Roux et al. 2001). This result is also consistent with the fact that the time integral $F(T) = 0$ if T is a harmonic of $2\pi/\Omega$ whereby T_c^p either converges to zero or does not converge at all in the limit of static two-dimensional turbulence (see discussion below eq. [4]). Whereas the QLT for static slab turbulence leads to a gyroresonance for wavelengths of the order of the particle gyroradius, no such resonances for any wavenumber occur in the QLT for static two-dimensional turbulence. In fact, the particles are not affected at all.

Since the integral in T_c^p cannot be computed analytically, further simplification is needed. Referring to the Appendix (eq. [A43]), we see that the approximated expression for T_c^p in equation (16) involves an integration in which the Bessel function contributions are weighted by the fluctuation spectrum. Therefore, it is convenient to think of T_c^p as consisting of contributions from various scales of turbulence. For this purpose a useful parameter is the argument $x = k_\perp r_g$ of the Bessel functions. In §§ 5.1, 5.2, and 5.3 we calculate separately the contribution to T_c^p due to three ranges of scale (wavenumber), corresponding to ranges of x. We show that in the contribution associated with small-scale turbulence $x \gg 1$, gyroscale turbulence ($x = 1$), and energy-containing–scale turbulence $x \ll 1$, it is possible to obtain analytically simplified expressions for the diffusion coefficients.

5.1. *Energy-Containing–Scale Two-dimensional Turbulence*

Simplification of T_c^p in equation (16) follows if we take the Bessel functions in the limit $x = k_\perp r_g \ll n$. Then $J_n(x) \approx (x/2)^n/n!$. Application of the condition for energy-containing–scale turbulence, $x = k_\perp r_g \ll 1$, yields the result that

$$\sum_{n=1}^{\infty}\frac{J_n^2(x)}{x^2} \approx \frac{1}{4}. \tag{17}$$

Thus,

$$T_c^p \approx \frac{2\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}\frac{1}{\Omega^2}\int_{r_g/l_{c\perp}}^{a} dx\,\frac{1}{x}, \tag{18}$$

where $a \ll 1$ is a constant ensuring that for the upper integration limit $x \ll 1$. After integration it follows that

$$T_c^p = \frac{2\pi}{3}\frac{1}{\Omega^2}\left(\frac{\delta U_\perp}{l_{c\perp}}\right)\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}\ln\left(\frac{l_{c\perp}a}{r_g}\right). \tag{19}$$

The substitution of equation (19) into equation (2) and the assumption of fast particles $v \gg V_{\rm A}$ and finite cross helicity σ_c

result in the following simplified expressions for Fokker-Planck diffusion coefficients:

$$D_{\mu\mu} = \frac{2\pi}{3}\left(1-\mu^2\right)\left(\frac{\delta U_\perp}{l_{c\perp}}\right)\frac{\langle\delta B_\perp^2\rangle}{B_0^2}\ln\left(\frac{al_{c\perp}}{r_g}\right),$$
$$D_{pp} = D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)^2 r_{\rm A},$$
$$D_{p\mu} = D_{\mu p} = -D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)\left[\frac{1}{2}\sigma_c(r_{\rm A}+1)\right], \quad (20)$$

where $\delta U_\perp(l_{c\perp}) = (\delta B_\perp/B_0)V_{\rm A}$. Comparison of expressions of the diffusion coefficients in equation (20) with the first estimate of those coefficients in equation (15) reveals that the expressions are the same except for an additional logarithmic factor in equation (20). This difference arises from integrating over a power law for the spectral magnetic field fluctuation energy density in wavenumber over a finite wavenumber interval determined by the condition for energy-containing–scale fluctuations and the correlation scale.

We investigate whether the QLT condition $t_c^p \ll t_\mu$ is fulfilled by the expression for $D_{\mu\mu}$ in the case of the Fourier transform approach. Since particle decorrelation for energy-containing–scale turbulence is determined by the dynamics of the turbulence (see § 4.1 and eq. [13]), we calculate the particle correlation time t_c^p by specifying $t_c^p = 1/[k_\perp \delta U_\perp(k_\perp)] = 1/\gamma(k_\perp)$, where $\gamma(k_\perp)$ is given by equation (A44). After averaging t_c^p over all the wavenumbers fulfilling the condition $k_\perp r_g \ll 1$, we find

$$t_c^p = 3\left(\frac{l_{c\perp}}{\delta U_\perp}\right)\left(\frac{r_g}{al_{c\perp}}\right)^{2/3}, \quad (21)$$

where $\delta U_\perp = (\delta B_\perp/B_0)V_{\rm A}$. After approximation of $D_{\mu\mu}$ in equation (20) as $D_{\mu\mu} = 1/t_\mu$, the ratio t_c^p/t_μ becomes

$$\frac{t_c^p}{t_\mu} = 2\pi\left(1-\mu^2\right)\frac{\langle\delta B_\perp^2\rangle}{B_0^2}\left(\frac{r_g}{al_{c\perp}}\right)^{2/3}\ln\left(\frac{al_{c\perp}}{r_g}\right)$$
$$\approx 0.007\left(\frac{v}{U_0}\right)^{2/3}\left|\ln 5.3 - \ln\left(\frac{v}{U_0}\right)\right|, \quad (22)$$

where standard solar wind parameters at 1 AU were assumed [$l_{c\perp} = 0.01$ AU, $\delta U_\perp = (\delta B_\perp/B_0)V_{\rm A} = 0.2\times 40$ km s^{-1} = 2 km s^{-1}, the solar wind flow speed $U_0 = 400$ km s^{-1}, $r_g \approx 5\times 10^{-6}$ AU for 1 keV protons]. The constant a was chosen to be an upper limit of $a = 0.1$, and $\mu = 0$ to ensure maximum values for the ratio corresponding to the worst-case scenario. According to equation (22), $t_c^p/t_\mu \leq 0.1$ approximately for a wide range of particle energies, and it is only for proton energies of $\sim E \gg 90$ MeV or so that the ratio becomes unacceptably large.

In the case of energy-containing–scale fluctuations, $\Omega \gg 1/t_c^p$ so that the particle sees slowly time-varying two-dimensional turbulence fluctuations. This implies that the particle's magnetic moment μ_m should approximately be conserved. In the absence of fluctuations the magnetic moment of the particle following an undisturbed helical trajectory is

$$\mu_{m0} = \frac{(1/2)mv_{\perp 0}^2}{B_0}. \quad (23)$$

We find for $\delta B_\perp/B_0 \ll 1$ that the deviation in the magnetic moment produced by QLT in the presence of slowly varying two-dimensional fluctuations is determined by the expression

$$\mu_m = \frac{(1/2)mv_{\perp 0}^2}{B_0+\delta B_\perp} \approx \mu_{m0}\left(1-\frac{\delta B_\perp}{B_0}\right). \quad (24)$$

For $\delta B/B_0 \approx 0.2$, appropriate for 1 AU, we find that the magnetic moment deviates by $\sim$20%. This raises questions about the accuracy of the diffusion coefficients given by equation (20). However, the accuracy should improve at smaller radial distances from the Sun, where $\delta B/B_0$ is expected to become smaller because, according to the Parker spiral magnetic field model, $B_0 \propto r^{-2}$ close to the Sun, while $\delta B \propto r^{-3/2}$ approximately close to the Sun according to observations and MHD turbulence transport theory (Zank, Matthaeus, & Smith 1996).

Assuming that QLT does conserve magnetic moment sufficiently well in the presence of slowly varying two-dimensional turbulence, it is clear that pitch-angle scattering will occur because random changes in the magnitude of the total magnetic field result in random changes in the velocity component perpendicular to the $\boldsymbol{B}_0$ while the particle speed is conserved. Slowly changing two-dimensional transverse fluctuating electric fields $\delta\boldsymbol{E}_\perp = -\delta\boldsymbol{U}_\perp \times \boldsymbol{B}_0$ leads to a polarization drift motion in the direction of the electric fields in the plane perpendicular to $\boldsymbol{B}_0$. This causes random particle acceleration, resulting in particle momentum magnitude diffusion.

5.2. *Small-Scale Two-dimensional Turbulence*

A useful identity to apply for the case of small-scale two-dimensional turbulence $x = k_\perp r_g \gg 1$ (Arfken 1985, p. 584; Shalchi & Schlickeiser 2003) is obtained from

$$\sum_{n=1}^{\infty} J_n^2(x) = \tfrac{1}{2} - \tfrac{1}{2}J_0^2(x). \quad (25)$$

In the limit $x \gg 1$,

$$\sum_{n=1}^{\infty} J_n^2(x) \approx \tfrac{1}{2}. \quad (26)$$

Consequently, T_c^p (eq. [16]) becomes

$$T_c^p \approx \frac{4\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle\delta B_\perp^2\rangle}{B_0^2}\frac{1}{\Omega^2}\int_{1/a}^{\infty} dx\,\frac{1}{x^3}, \quad (27)$$

where the lower integration limit ensures that the smallest value of $x \gg 1$ because $a \ll 1$. After integration T_c^p can be expressed as

$$T_c^p = \frac{2\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle\delta B_\perp^2\rangle}{B_0^2}\frac{a^2}{\Omega^2}. \quad (28)$$

When equation (28) is substituted in equation (2), we find in the limit of super-Alfvénic particle speeds ($v \gg V_{\rm A}$) and for the case of finite cross helicity that the diffusion coefficients simplify to

$$D_{\mu\mu} = \frac{2\pi}{3}a^2\left(1-\mu^2\right)\left(\frac{\delta U_\perp}{l_{c\perp}}\right)\frac{\langle\delta B_\perp^2\rangle}{B_0^2},$$
$$D_{pp} = D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)^2 r_{\rm A},$$
$$D_{p\mu} = D_{\mu p} = -D_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)\frac{1}{2}\sigma_c(r_{\rm A}+1). \quad (29)$$

The basic requirement of QLT, that the particle correlation time $t_c^p \ll t_\mu$, where t_μ is the timescale for pitch-angle scattering, is easily fulfilled because

$$\frac{t_c^p}{t_\mu} = \frac{2\pi}{3} a^2 \left(\frac{\delta U_\perp}{v}\right) \left(\frac{\langle \delta B_\perp^2 \rangle}{B_0^2}\right) \left(\frac{a r_g}{l_{c\perp}}\right) \ll 1, \tag{30}$$

since all the factors in parentheses are much less than 1 for 1 keV protons in solar wind conditions.

In the case of small-scale turbulence $x = k_\perp r_g \gg 1$, simplification of T_c^p is also possible by taking the large argument limit of the Bessel function $J_n(x)$ ($x = k_\perp r_g \gg n$) in equation (26). Then $J_n(x) \approx [2/(\pi x)]^{1/2} \cos^2 x$ and T_c^p can be written as

$$T_c^p \approx \frac{8}{3} \frac{\delta U_\perp}{l_{c\perp}} \frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \frac{1}{\Omega^2} \sum_{n=1}^{\infty} \int_{n/a}^{\infty} dx \, \frac{1}{x^4}, \tag{31}$$

after assuming that one can approximately replace the integral over $\cos^2 x$ by the average $\langle \cos^2 x \rangle = \frac{1}{2}$. The lower integration limit reflects the assumption that $x \gg n$ (the constant $a \ll 1$). After integration we find that

$$T_c^p = \frac{8}{9} \frac{a^3}{\Omega^2} \frac{\delta U_\perp}{l_{c\perp}} \frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \zeta(3), \tag{32}$$

where $\zeta(3) \approx 1.2$ is the Riemann zeta function.

However, since we are interested in the case $x = k_\perp r_g \gg 1$, the $x = k_\perp r_g \gg n$ condition is too restrictive, leading to an underestimation of the importance of small-scale fluctuations (compare eq. [28] with eq. [32]).

5.3. *Gyroscale Two-dimensional Turbulence*

A remaining case of interest is to compute the contributions to T_c^p due to turbulence with scale lengths between energy-containing scale and small scale, that is, scale lengths of the order of the particle gyroradius $k_\perp r_g \approx 1$.

For simplicity we assumed that $k_\perp r_g = 1$ and specified the differential energy density of the two-dimensional magnetic field fluctuations as follows:

$$E^{\delta B}(k_\perp) = E_0 \delta\left(k_\perp - \frac{1}{r_g}\right), \tag{33}$$

where E_0 is the differential energy density of the magnetic field fluctuations at wavenumber $k_\perp = 1/r_g$. Integration of $E^{\delta B}(k_\perp)$ for a Kolmogorov spectrum given by equation (A45) and for the alternative spectrum given by equation (33) over all wavenumbers $k_\perp \geq 1/r_g$ produces the relationship

$$E_0 = r_g \langle \delta B_\perp^2 \rangle \left(\frac{r_g}{l_{c\perp}}\right)^{2/3}. \tag{34}$$

By substituting equations (33) and (34) into the time integral T_c^p (eq. [A43]), one finds that

$$T_c^p = 4\pi \frac{1}{\Omega^2} \left(\frac{\delta U_\perp}{l_{c\perp}}\right) \frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \sum_{n=1}^{\infty} \frac{n^2 J_n^2(1)}{y^2 + n^2}, \tag{35}$$

where y^2 is given by equation (A47). As before we take the limit $y \ll 1$ and T_c^p simplifies to

$$T_c^p = 4\pi \frac{1}{\Omega^2} \left(\frac{\delta U_\perp}{l_{c\perp}}\right) \frac{\langle \delta B_\perp^2 \rangle}{B_0^2} \sum_{n=1}^{\infty} J_n^2(1). \tag{36}$$

As was done in § 5.2, we use

$$\sum_{n=1}^{\infty} J_n^2(x) = \tfrac{1}{2} - \tfrac{1}{2} J_0^2(x). \tag{37}$$

By taking the first two terms of a series representation of $J_n(x)$ (Arfken 1985, p. 584), we find that $J_0(1) \approx \frac{3}{4}$. Consequently,

$$\sum_{n=1}^{\infty} J_n^2(1) \approx \frac{7}{32}, \tag{38}$$

and T_c^p becomes

$$T_c^p = \frac{7\pi}{8} \frac{1}{\Omega^2} \left(\frac{\delta U_\perp}{l_{c\perp}}\right) \frac{\langle \delta B_\perp^2 \rangle}{B_0^2}. \tag{39}$$

Substitution of equation (39) into equation (2) and assuming $v \gg V_A$ and $\sigma_c \neq 0$ result in the following set of diffusion coefficients:

$$\begin{aligned} D_{\mu\mu} &\approx \frac{7\pi}{8} (1 - \mu^2) \left(\frac{\delta U_\perp}{l_{c\perp}}\right) \frac{\langle \delta B_\perp^2 \rangle}{B_0^2}, \\ D_{pp} &= D_{\mu\mu} \left(\frac{p V_A}{v}\right)^2 r_A, \\ D_{p\mu} &= D_{\mu p} = -D_{\mu\mu} \left(\frac{p V_A}{p}\right) \left[\frac{1}{2} \sigma_c (r_A + 1)\right]. \end{aligned} \tag{40}$$

5.4. *Comparison of the Diffusion Coefficients for Different Scale Lengths*

To compare the contribution to the diffusion coefficients for energy-containing–scale, small-scale, and gyroscale two-dimensional turbulence, it is sufficient to consider the ratio of the pitch-angle diffusion coefficients $D_{\mu\mu}$ because the ratio of the other diffusion coefficients D_{pp}, $D_{\mu p}$, and $D_{p\mu}$ only depends on the $D_{\mu\mu}$ ratio. The $D_{\mu\mu}$ ratio of small-scale over energy-containing–scale two-dimensional turbulence is

$$\frac{D_{\mu\mu}^S}{D_{\mu\mu}^{EC}} = \frac{a^2}{\ln(a l_{c\perp}/r_g)} \approx \frac{0.01}{\ln 200 - \ln(v/U_0)}, \tag{41}$$

where we specified standard parameters at Earth ($l_{c\perp} = 0.01$ AU and $U_0 = 400$ km s^{-1}) and chose the constant a to be an upper limit of $a = 0.1$. This ratio is smaller than 1 for proton kinetic energies $E < 40$ MeV approximately, which indicates that for low-energy particles with super-Alfvénic speeds small-scale two-dimensional turbulence is negligible relative to energy-containing–scale two-dimensional turbulence in affecting particle pitch-angle scattering and momentum diffusion.

The ratio of $D_{\mu\mu}$ for gyroscale over energy-containing–scale turbulence is given by

$$\frac{D_{\mu\mu}^G}{D_{\mu\mu}^{EC}} = \frac{1.5}{\ln 200 - \ln(v/U_0)}. \tag{42}$$

In this case the ratio is smaller than one for proton kinetic energies $E < 2$ MeV approximately. Clearly gyroscale two-dimensional turbulence is more important than small-scale two-dimensional turbulence in causing pitch-angle scattering and by implication momentum diffusion. In fact, the pitch-angle scattering rates for energy-containing–scale and gyroscale two-dimensional turbulence are basically of the same order of magnitude at Earth for a significant range of super-Alfvénic particle energies. Thus, a good estimate of particle pitch-angle scattering and momentum diffusion by dynamic two-dimensional turbulence can be achieved by considering either energy-containing scales or gyroscales.

6. PITCH-ANGLE SCATTERING AND MOMENTUM DIFFUSION BY TWO-DIMENSIONAL TURBULENCE IN THE SOLAR WIND

In the static limit, QLT predicts in general that transverse two-dimensional electric and magnetic field fluctuations have no significant effect on particle pitch-angle scattering and momentum diffusion irrespective of the scale length of the two-dimensional fluctuations as discussed above. Thus, only static slab fluctuations will affect particle pitch-angle scattering and momentum diffusion within the framework of two-component (two-dimensional plus slab) turbulence. The static assumption is good for high-energy particles such as Galactic or anomalous cosmic rays in the solar wind where $v \gg \delta U_\perp$ for the two-dimensional component and $v \gg V_{\rm A}$ in the case of Alfvén waves. It remains to be determined whether particles with super-Alfvénic but sub–cosmic-ray speeds will respond significantly to the time dependence of the two-dimensional turbulence component caused by nonlinear interactions. More specifically, we want to determine whether pitch-angle scattering and momentum diffusion caused by dynamic two-dimensional turbulence can in principle be comparable to the contribution from parallel-propagating Alfvén waves in the quiet low-latitude solar wind. We focus our comparison on the super-Alfvénic solar wind between the Sun and 5 AU in the inner heliosphere.

The particle pitch-angle and momentum diffusion coefficients due to interaction with parallel-propagating Alfvén waves (Schlickeiser 1989) are

$$\begin{aligned} D^{\rm A}_{\mu\mu} &= D_0(1-\mu^2)\left(\frac{1+\sigma_c}{2}\right)\left(1-\frac{\mu V_{\rm A}}{v}\right)^2\left(\frac{|v\mu - V_{\rm A}|}{\Omega l_{c\parallel}}\right)^{s-1} \\ &\quad + D_0(1-\mu^2)\left(\frac{1-\sigma_c}{2}\right)\left(1+\frac{\mu V_{\rm A}}{v}\right)^2\left(\frac{|v\mu + V_{\rm A}|}{\Omega l_{c\parallel}}\right)^{s-1}, \\ D^{\rm A}_{pp} &= D^{\rm A}_{\mu\mu}\left(\frac{pV_{\rm A}}{v}\right)^2 r_{\rm A}, \end{aligned} \tag{43}$$

where $D_0 = 2\pi^2(s-1)(\langle\delta B_\perp^2\rangle_{\rm A}/B_0^2)\Omega$. In equation (43) Ω is the particle gyrofrequency, $\langle\delta B_\perp^2\rangle_{\rm A}$ is the energy density in one-dimensional transverse magnetic field fluctuations associated with parallel-propagating Alfvén waves, and $l_{c\parallel}$ is the correlation length associated with the inertial range of the power spectrum of Alfvén waves with spectral index s.

Now we discuss the parameters specified for calculating the diffusion coefficients. It has been assumed in equations (20), (39), and (40) that the two-dimensional spectral density $E^{\delta B}(k_\perp) \propto (k_\perp l_{c\perp})^{-5/3}$, which is consistent with solar wind observations (Goldstein et al. 1995) and theory (Goldreich & Sridhar 1995; Matthaeus et al. 1998). For the Alfvén wave spectrum we consider two cases, namely, a Kolmogorov spectrum $E^{\delta B}(k_\parallel) \propto (k_\parallel l_{c\parallel})^{-5/3}$ so that $s = 5/3$ in equation (43), and a steeper spectrum $E^{\delta B}(k_\parallel) \propto (k_\parallel l_{c\parallel})^{-2}$. The latter assumption of a steeper spectrum is qualitatively consistent with some theory and simulations (Shebalin et al. 1983; Goldreich & Sridhar 1995; Matthaeus et al. 1998; Cho & Lazarian 2002). However, there is no observational evidence for such significant deviations from a Kolmogorov power law in the solar wind.

It is assumed that 85% of the energy density in MHD fluctuations is located in the two-dimensional component and 15% in the parallel component following the example of Bieber et al. (1996). The dependence of the energy density on heliocentric distance r is assumed to be r^{-3}. This radial dependence agrees well with observations (Zank et al. 1996).

For the correlation length in the power spectra we assumed $l_{c\parallel} = l_{c\perp} = 0.01$ AU at Earth (Goldstein et al. 1995). There is no observational evidence that these correlation scales differ at low latitudes in the solar wind. The correlation length is kept unchanged when the Alfvén wave power-law spectrum is steepened from a power law with index from $s = \frac{5}{3}$ to 2. Observations also suggest that the turbulence correlation length increases with heliocentric distance r. We used $l_c \propto r^{1/2}$, which reproduces the correlation length observations sufficiently well in the inner heliosphere (Smith et al. 2001).

The dependence of the magnitude of the large-scale magnetic field on heliocentric distance is specified according to the Parker model whereby $B_0 \propto 1/r$ for $r \gg 1$ and $B_0 \propto 1/r^2$ for $r \ll 1$. We choose $\delta B_\perp/B_0 = 0.2$ at Earth orbit. The rms of the turbulent two-dimensional velocity fluctuations $\delta U_\perp = (\delta B_\perp/B_0)V_{\rm A}$ assuming equipartition in the energy of two-dimensional velocity and magnetic field fluctuations, where at Earth the Alfvén speed $V_{\rm A} \approx 40$ km s^{-1} so that $\delta U_\perp \approx 2$ km s^{-1}. Close to the Sun $\delta U_\perp \propto r^{-1/2}$, while at large distances $\delta U_\perp \propto r^{-1}$ approximately. For the Alfvén ratio $r_{\rm A}$ we specify $r_{\rm A} = 1$ closer to the Sun and let it decrease linearly to a value of 0.5 at 2 AU from the Sun, beyond which it is kept constant (Goldstein et al. 1995).

In Figure 1 we show the pitch-angle diffusion coefficient $D_{\mu\mu}$ in inverse time units of S^{-1} (1 S $= 3.74 \times 10^5$ s ≈ 4.3 days) as a function of the cosine of the particle pitch angle μ at Earth for 1 keV protons. The top curve labeled "A" denotes $D_{\mu\mu}$ for parallel-propagating Alfvén waves with a Kolmogorov magnetic field fluctuation power spectrum assumed to contain all the power in MHD fluctuations, the curve labeled "$\rm A_{Q2DK}$" represents Alfvén waves with a Kolmogorov power spectrum having 15% of the fluctuation power, while the curve with the label "$\rm A_{Q2D2}$" was calculated for Alfvén waves with the same fraction of total fluctuation power but with a steeper $k_\parallel^{-2}$ spectrum. The bottom two curves, representing dynamic two-dimensional fluctuations in different wavenumber regimes, carry most of the remaining 85% allocated to the total two-dimensional turbulence component. The curve with the label "$\rm 2D_I$" represents energy-containing–scale two-dimensional fluctuations and carries most of the total two-dimensional fluctuation power, while the bottom curve with the label "$\rm 2D_S$" denotes small-scale two-dimensional turbulence carrying about 1% of the total two-dimensional fluctuation power. For all the two-dimensional turbulence curves Kolmogorov power spectra were assumed.

The main result is that the pitch-angle scattering rate by the minor parallel-propagating Alfvén wave component is on the whole larger than for the two-dimensional turbulence component of dynamic quasi–two-dimensional MHD turbulence in the solar wind for 1 keV protons at Earth for all cases shown. It is only for the steeper Alfvén wave power spectrum [$E^{\delta B}(k_\parallel) \propto k_\parallel^{-2}$] that the pitch-angle scattering rate by the energy-containing–scale two-dimensional component approaches the rate of the Alfvén waves, and that only occurs at large pitch angles (small μ-values). It must be remembered, however, that QLTs for energy-containing–scale two-dimensional fluctuations are possibly inaccurate to some degree at 1 AU because of

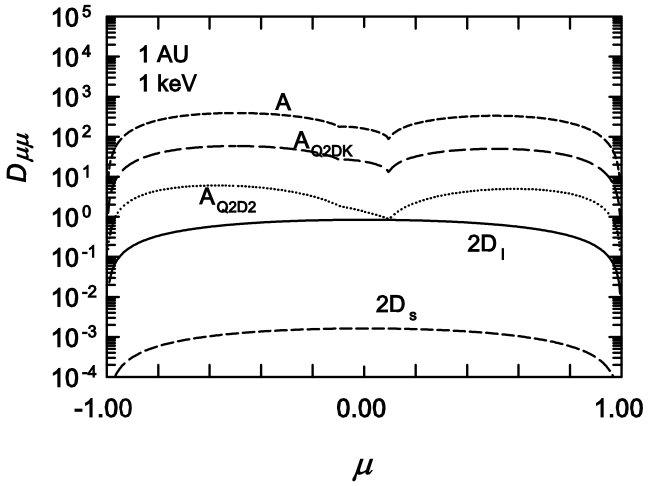

FIG. 1.—Theoretical pitch-angle diffusion coefficient $D_{\mu\mu}$ in units of S^{-1} ($1\ S = 4.33$ days) for 1 keV ions at Earth for quiet solar wind conditions as a function of μ ($\mu = \cos\theta$, where θ is the particle pitch angle). The top dashed curve labeled "A" denotes the $D_{\mu\mu}$ for standard Alfvén waves propagating parallel to the large-scale magnetic field with a Kolmogorov power spectrum in the inertial range. The curve labeled "A_{Q2DK}" represents the same case as the top curve, except for the assumption that only 15% of the total MHD fluctuation energy density is in the Alfvén mode, and the curve labeled "Q_{A2D2}" is valid for the same parameters as the "A_{Q2DK}" case, except that the inertial range power spectrum has a $k_\parallel^{-2}$ dependence on parallel wavenumber as suggested by some MHD turbulence simulations. The solid curve labeled "$2D_I$" denotes $D_{\mu\mu}$ as we predict with our new QLT for dynamic two-dimensional turbulence in the limit of energy-containing–scale fluctuations where $k_\perp \ll 1/r_g$, while the bottom curve labeled "$2D_S$" represents small-scale dynamic two-dimensional turbulence ($k_\perp \gg 1$). In both two-dimensional cases it is assumed that we have a Kolmogorov power spectrum in the inertial range and that 85% of the total MHD magnetic field fluctuation energy density is in the two-dimensional turbulence mode.

a $\sim$20% deviation in the particle magnetic moment that should, strictly speaking, be conserved in slowly varying fluctuations (see § 5.1, eq. [24]).

Originally, people thought that Alfvén waves carried almost all of the power in solar wind MHD fluctuations (see curve labeled "A" in Fig. 1). The consequence of the two-component model is that the Alfvén wave–induced pitch-angle scattering rate of 1 keV protons at Earth is approximately an order of magnitude lower than originally thought, mainly because only 15% of the total magnetic fluctuation energy resides in the wavelike slab modes (Bieber et al. 1994, 1996). If in addition the Alfvén wave power spectrum of magnetic field fluctuations is indeed steeper than a Kolmogorov spectrum as suggested by quasi–two-dimensional MHD theory and simulations, we find that the scattering rate by Alfvén waves for 1 keV protons is about 2 orders of magnitude smaller than people originally thought.

We also confirmed that small-scale dynamic two-dimensional turbulence is much less efficient in causing particle pitch-angle scattering than energy-containing–scale or gyroscale two-dimensional turbulence (see last paragraph of § 5.4), partly because there is much less energy in the small-scale fluctuations. The main cause, however, is that particles see decorrelated fluctuations on a much shorter timescale of less than a gyroperiod in the case of small-scale two-dimensional turbulence, while for intermediate-scale turbulence the decorrelation timescale is determined by the turbulence dynamic timescale, which at 1 AU is about 4 orders of magnitude longer for 1 keV protons (see eqs. [21] and [30]).

In Figure 2 we show $D_{\mu\mu}$ as a function of heliocentric distance between 0.05 and 5 AU in the ecliptic plane for 1 keV protons. The curves are labeled in the same manner as in Figure 1. The $D_{\mu\mu}$ values are averaged over μ for $\mu < 0$ in Figure 2*a* and for $\mu > 0$ in Figure 2*b*. For $\mu < 0$ the results are qualitatively the same as in Figure 1. The Alfvén wave component produces on average more pitch-angle scattering than the two-dimensional turbulence component at all heliocentric distances if $\mu < 0$. However, the scattering rate for energy-containing–scale two-dimensional turbulence is well within an order of magnitude of the lowest scattering rate given by Alfvén waves with a steepened $k_\parallel^{-2}$ magnetic field fluctuation energy density inside 1 AU (the average $D_{\mu\mu}$ value for those Alfvén waves is about a factor of 4 more than for the two-dimensional component at $\sim$0.2 AU). Beyond 1 AU the Alfvén wave pitch-angle scattering rate becomes increasingly dominant.

In the case of $\mu > 0$ (Fig. 2*b*), the domination of Alfvén waves over two-dimensional turbulence in causing pitch-angle scattering is predicted to be less strong inside 1 AU but the same beyond 1 AU compared to Figure 2*a*. The curve representing the steepened Alfvén wave $k_\parallel^{-2}$ magnetic fluctuation power

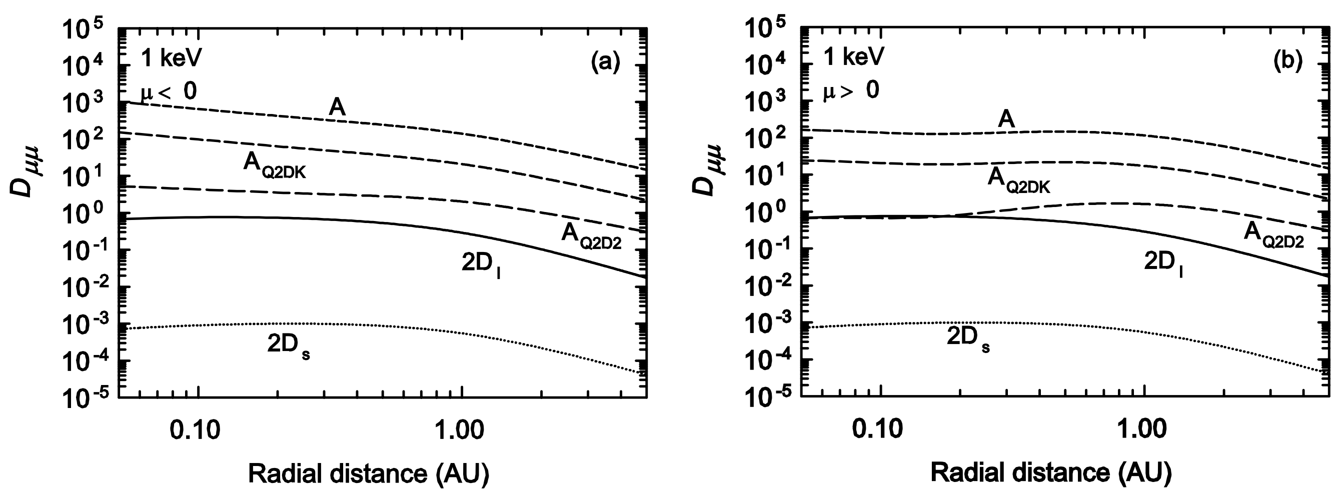

FIG. 2.—(*a*) Theoretical pitch-angle diffusion coefficient $D_{\mu\mu}$ averaged over $\mu < 0$ values in units of S^{-1} ($1\ S = 4.33$ days) for 1 keV ions as a function of heliocentric distance in astronomical units (AU) at low heliolatitudes. The curves are labeled the same way as in Fig. 1. (*b*) Same as (*a*), except that $D_{\mu\mu}$ was averaged over $\mu > 0$ values.

spectrum has a reduced pitch-angle scattering rate that approximately equals the scattering rate of the energy-containing–scale two-dimensional turbulence component for heliocentric distances less than $\sim$0.2 AU from the Sun. The reason is that the cross helicity increases with decreasing heliocentric distance close to the Sun so that Alfvén waves propagating away from the Sun mostly cause pitch-angle scattering (first term in eq. [43]). The first term has a resonance gap for $\mu = +v_{\rm A}/v$, close to $\mu = 1$ at 0.05 AU, for example, so that pitch-angle scattering is predicted to be considerably less for $\mu > 0$ than for $\mu < 0$. In reality, cyclotron wave damping will occur so that particles will see decorrelated fluctuations on the damping timescale and a finite scattering rate will be maintained in the resonance gap (Schlickeiser & Achatz 1993). One might question the accuracy of the diffusion coefficients for energy-containing–scale two-dimensional turbulence with which the Alfvén wave diffusion coefficients are compared as a result of the violation in the magnetic moment in the derivation of the two-dimensional diffusion coefficients. However, the scattering rate due to the energy-containing–scale two-dimensional fluctuations is more accurate closer to the Sun because the violation of the conservation of magnetic moment becomes as small as 6% at 0.05 AU as a result of the decrease in $\delta B_\perp/B_0$ with decreasing heliocentric distance. Thus, we conclude that also for $\mu > 0$ the average $D_{\mu\mu}$ inside 1 AU will be dominated by Alfvén waves in the case of the steepened Alfvén wave power spectrum.

The same conclusions can be drawn from the momentum diffusion coefficients because these coefficients differ from the pitch-angle coefficients by the same factor regardless of whether one considers two-dimensional turbulence or Alfvén waves. In Figure 3 we show the μ-averaged momentum diffusion coefficients in units of GV^2 S^{-1} following the same format as in Figure 2 for $D_{\mu\mu}$. The main differences are that the D_{pp} values are smaller in magnitude and have a stronger dependence on heliocentric distance when compared to the $D_{\mu\mu}$ values in Figure 2 as a result of the presence of the square of the Alfvén speed in D_{pp} expressions (the dependence of the Alfvén speed on heliocentric distance is approximately $V_{\rm A} \propto 1/r$ close to the Sun).

The dominance of the Alfvén wave component over the two-dimensional component in causing quasi-linear particle pitch-angle scattering and momentum diffusion becomes more pronounced with increasing particle energy because $D_{\mu\mu}$ for two-dimensional turbulence is at most weakly dependent on particle speed (see eq. [20]) while $D_{\mu\mu} \propto v^{2/3}$ for Alfvén waves (see eq. [43]) with a Kolmogorov power spectrum. Thus, at high energies in the solar wind, such as cosmic-ray energies, the role of dynamic two-dimensional turbulence in pitch-angle scattering and momentum diffusion will be totally negligible relative to the Alfvén wave component. This suggests that the static two-dimensional turbulence assumption that produces zero pitch-angle scattering and momentum diffusion in QLT is a very good one at cosmic-ray energies. To good approximation it seems to us that one has to consider only the role of the Alfvénic wave component in particle pitch-angle scattering and momentum diffusion at high and low super-Alfvénic particle speeds within the context of transverse MHD fluctuations associated with the two-component model of solar wind MHD turbulence.

7. SUMMARY AND CONCLUSIONS

In the past, quasi-linear theory (QLT) was almost exclusively used to study resonant interaction of energetic charged particles with waves such as Alfvén waves rather than turbulence in the solar wind. The notable exception is the work by Bieber et al. (1994), where a QLT for energetic particle pitch-angle scattering by dynamic two-dimensional MHD magnetic field turbulence is presented but little detail is given (also note recent work on the subject by Shalchi & Schlickeiser 2003). This work was motivated by the emerging viewpoint that the dominating component of MHD turbulence in the solar wind is nonpropagating but highly active two-dimensional turbulence structures convected with the solar wind flow, while Alfvén waves propagating parallel to the background magnetic field (also known as one-dimensional slab turbulence in the static limit) form a minor second component.

In this paper a detailed discussion is presented of QLT from a turbulence perspective. In an extension of the Bieber et al. (1994) work, which treats only dynamic two-dimensional magnetic turbulence fluctuations, turbulent transverse two-dimensional electric fields are also included in this theory. This enables us to study the role of dynamic two-dimensional turbulence in the stochastic acceleration of energetic particles. The level of turbulence, which is treated as a free parameter by

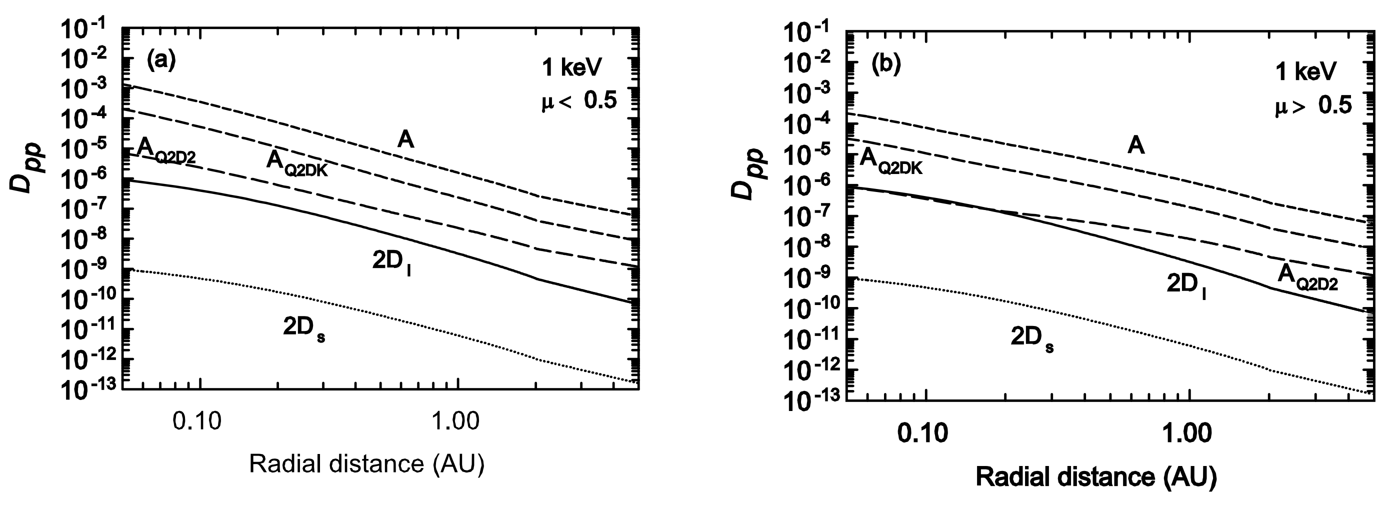

FIG. 3.—(*a*) Momentum diffusion coefficient D_{pp} in units of GV^2 S^{-4} (1 S = 4.33 days) for 1 keV ions averaged over $\mu < 0$ values as a function of heliocentric distance in AU at low heliolatitudes. The curves have the same labels as in Fig. 1. (*b*) Same as (*a*), except that D_{pp} was averaged over $\mu > 0$ values.

Bieber et al. (1994), is estimated in this paper within the framework of Kolmogorov turbulence theory.

The general expressions for the diffusion coefficients are complicated, but we succeeded in deriving tractable expressions when computing the contributions from small-scale ($k_\perp r_g \gg 1$), gyroscale ($k_\perp r_g \sim 1$), and energy-containing–scale two-dimensional fluctuations ($k_\perp r_g \ll 1$). We found that the particle pitch-angle scattering and momentum diffusion rates caused by energy-containing–scale and gyroscale dynamic two-dimensional turbulence are of the same order of magnitude, while the contribution of small-scale two-dimensional turbulence is negligible at low sub–cosmic-ray but super-Alfvénic particle energies.

We also calculated the values for the two-dimensional turbulence Fokker-Planck diffusion coefficients in the quiet solar wind between 0.05 and 5 AU in the ecliptic plane using standard solar wind parameters. These values were compared with the values of the diffusion coefficients due to parallel-propagating Alfvén waves. Our main result is that dynamic two-dimensional turbulence does indeed produce significant pitch-angle scattering of energetic particles, as well as momentum diffusion. However, even at low energies of the order of 1 keV for protons, we find that the Alfvénic component is more important for pitch-angle scattering and momentum diffusion than the two-dimensional component by about 2 orders of magnitude on average. If the Alfvén wave component is weakened from a Kolmogorov power spectrum to a $k_\parallel^{-2}$ dependence for the power spectrum as suggested by some MHD turbulence simulations, we find that the domination of Alfvén waves drops to less than an order of magnitude for distances less than about 1 AU from the Sun at 1 keV proton energies. There is, however, no evidence from solar wind observations that Alfvén wave power spectra deviate significantly from a Kolmogorov power law.

This domination of Alfvén waves in particle diffusive transport grows with increasing energy. Thus, at high energies in the solar wind (cosmic-ray energies), the role of dynamic two-dimensional turbulence in pitch-angle scattering and momentum diffusion will be totally negligible relative to the Alfvén wave component. This suggests that the assumption of static two-dimensional turbulence that produces zero pitch-angle scattering and momentum diffusion in QLT is good at cosmic-ray energies. To good approximation it seems to us that one has to consider only the role of the Alfvénic component in particle pitch-angle scattering and momentum diffusion at high cosmic-ray and low sub–cosmic-ray super-Alfvénic particle speeds. That is true within the context of transverse MHD fluctuations associated with the two-component model of incompressible solar wind MHD turbulence during quiet solar wind conditions.

However, more theoretical work needs to be done in the future before final conclusions can be drawn. The assumption that the particle orbit is undisturbed during the characteristic time of interaction with weak energy-containing–scale two-dimensional turbulence needs to be reinvestigated. Particles that interact with energy-containing–scale turbulence also interact simultaneously with Alfvén waves on a much shorter timescale so that the particle orbit might in principle be significantly disturbed on the characteristic timescale associated with energy-containing–scale two-dimensional turbulence. Our results indicate that small-scale Alfvén wave pitch-angle scattering rates are sufficient to disturb the particle orbit. In addition, several parameters come into play in the determination of the relative importance of resonant and turbulent diffusion; we defer the important further exploration of this issue to future work.

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APPENDIX

QUASI-LINEAR KINETIC THEORY FOR CHARGED PARTICLE TRANSPORT IN TWO-DIMENSIONAL TURBULENCE: THE DETAILS

The Vlasov or collisionless Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + q\left(\boldsymbol{E} + \frac{\boldsymbol{p} \times \boldsymbol{B}}{m}\right) \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \tag{A1}$$

and describes the evolution of a charged particle distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$ as a function of position $\boldsymbol{r}$, momentum $\boldsymbol{p}$, and time t. Particles with mass m and net charge q are influenced by plasma electromagnetic fields where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field.

Following QLT, the electromagnetic fields, plasma flow velocity $\boldsymbol{U}$ (specified in $\boldsymbol{E}$ according to eq. [A3]), and f are separated into slowly evolving large-scale and more rapidly fluctuating smaller scale random parts:

$$\begin{aligned}
\boldsymbol{E} &= \langle \boldsymbol{E} \rangle + \delta \boldsymbol{E}, \qquad \langle \boldsymbol{E} \rangle = \boldsymbol{E}_0, \qquad \langle \delta \boldsymbol{E} \rangle = 0, \\
\boldsymbol{B} &= \langle \boldsymbol{B} \rangle + \delta \boldsymbol{B}, \qquad \langle \boldsymbol{B} \rangle = \boldsymbol{B}_0, \qquad \langle \delta \boldsymbol{B} \rangle = 0, \\
\boldsymbol{U} &= \langle \boldsymbol{U} \rangle + \delta \boldsymbol{U}, \qquad \langle \boldsymbol{U} \rangle = \boldsymbol{U}_0, \qquad \langle \delta \boldsymbol{U} \rangle = 0, \\
f &= \langle f \rangle + \delta f, \qquad \langle f \rangle = f_0, \qquad \langle \delta f \rangle = 0,
\end{aligned} \tag{A2}$$

where $\langle \ldots \rangle$ denote ensemble average quantities. The electromagnetic fields are assumed to vary smoothly on a large scale L, while exhibiting energy-containing–scale random variations on the correlation length scale $l_{c\perp}$, where $l_{c\perp} \ll L$. The power spectrum of fluctuations ranges from scales on the order of the correlation length to smaller than a particle gyroradius r_g. For simplicity, the inertial range extends to infinite wavenumber, thereby ignoring kinetic damping effects. In the large-scale solar wind, on scale lengths $L \geq l_{c\perp}$, away from current sheets or shocks, the most important contribution to the total electric field is from the motional electric field, $\boldsymbol{E} = -\boldsymbol{U} \times \boldsymbol{B}$, where $\boldsymbol{U}$ and $\boldsymbol{B}$ are measured in the observer frame. Using equation (A2) and assuming small-amplitude fluctuations ($\delta\boldsymbol{B} \ll \boldsymbol{B}_0$ and $\delta\boldsymbol{U} \ll \boldsymbol{U}_0$), we find

$$\begin{aligned} \boldsymbol{E}_0 &= -\boldsymbol{U}_0 \times \boldsymbol{B}_0, \\ \delta\boldsymbol{E} &= -\boldsymbol{U}_0 \times \delta\boldsymbol{B} - \delta\boldsymbol{U} \times \boldsymbol{B}_0, \end{aligned} \tag{A3}$$

neglecting terms that are quadratic in the fluctuations. Within the framework of a quasi-linear theory based on linear waves, two-dimensional turbulence has zero frequency (see § 1) whereby the Faraday relationship between electric field and magnetic field fluctuations is lost. However, from a turbulence perspective the relationship between electric and magnetic field fluctuations is established by using equation (A3). The connection between velocity and magnetic fluctuations is described by the Alfvén ratio parameter (see text below eq. [A25]).

We substitute equation (A2) into equation (A1) and subtract the ensemble average of equation (A1) from equation (A1). The resulting equation for δf is linearized by assuming $\delta\boldsymbol{B} \ll \boldsymbol{B}_0$, $\delta\boldsymbol{U} \ll \boldsymbol{U}_0$, $\delta f \ll f_0$ and dropping terms that are quadratic in these small parameters. Further simplification is achieved by assuming the frame comoving with the plasma ($\boldsymbol{U}_0 = 0$). Consequently, equation (A3) in the comoving frame becomes $\boldsymbol{E}_0' = 0$ and $\delta\boldsymbol{E}' = -\delta\boldsymbol{U}' \times \boldsymbol{B}_0'$. For notational convenience, the primes are dropped in the rest of the text below. The equation for $\delta f(\boldsymbol{x}, \boldsymbol{p}, t)$ then becomes

$$\frac{\partial \delta f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial \delta f}{\partial \boldsymbol{x}} + \boldsymbol{p} \times \boldsymbol{\Omega} \cdot \frac{\partial \delta f}{\partial \boldsymbol{p}} = -q\left(\delta\boldsymbol{E} + \frac{\boldsymbol{p} \times \delta\boldsymbol{B}}{m}\right) \cdot \frac{\partial f_0}{\partial \boldsymbol{p}}, \tag{A4}$$

where $\boldsymbol{\Omega} = q\boldsymbol{B}_0/m$ is the particle gyrofrequency. We specify a Cartesian coordinate system (x, y, z) in the comoving frame with its z-axis aligned with the large-scale magnetic field so that $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. For the two-dimensional MHD turbulence component, we define

$$\begin{aligned} \delta\boldsymbol{U}(x, y) &= \delta U_x(x, y)\boldsymbol{e}_x + \delta U_y(x, y)\boldsymbol{e}_y, \\ \delta\boldsymbol{B}(x, y) &= \delta B_x(x, y)\boldsymbol{e}_x + \delta B_y(x, y)\boldsymbol{e}_y, \end{aligned} \tag{A5}$$

so that

$$\delta\boldsymbol{E}(x, y) = -\delta U_y B_0 \boldsymbol{e}_x + \delta U_x B_0 \boldsymbol{e}_y. \tag{A6}$$

Thus, also the induced electric field fluctuations are transverse to $\boldsymbol{B}_0$.

We also introduce Cartesian coordinates (p_x, p_y, p_z) of the particle momentum $\boldsymbol{p}$ in the mean field–aligned comoving frame ($\boldsymbol{U}_0 = 0$) so that p_z is the momentum component along the large-scale magnetic field. Expressed in terms of spherical coordinates, the momentum components become $(p \sin\theta \cos\phi, p \sin\theta \sin\phi, p \cos\theta)$, where p is the magnitude of the particle momentum, θ is the particle pitch angle, and ϕ is the particle phase angle.

The method of characteristic solution to equation (A4) is

$$\begin{aligned} \delta f(\boldsymbol{r}, \boldsymbol{p}, t) = &\int_{t_0}^{t} dt' \left[-q \sin\theta \frac{\partial f_0'}{\partial p}\left(\delta E_x' \cos\phi' + \delta E_y' \sin\phi'\right)\right] + \int_{t_0}^{t} dt' \left[-q \cos\theta \frac{1}{p}\frac{\partial f_0'}{\partial \theta}\left(\delta E_x' \cos\phi' + \delta E_y' \sin\phi'\right)\right] \\ &+ \int_{t_0}^{t} dt' \left[-\frac{\Omega}{B}\frac{\partial f_0'}{\partial \theta}\left(-\delta B_y \cos\phi' + \delta B_x \sin\phi'\right)\right] + \delta f(\boldsymbol{r}_0, \boldsymbol{p}_0, t_0), \end{aligned} \tag{A7}$$

where $\phi' = \phi(t')$, $\delta E_i' = \delta E_i(\boldsymbol{r}(t'), t')$, $\delta B_i(\boldsymbol{r}(t'), t')$, and $f_0 = f_0(\boldsymbol{r}(t'), \boldsymbol{p}(t'), t')$. Furthermore, $\boldsymbol{r}_0 = \boldsymbol{r}(t_0) = [x_0, y_0, z_0]$ is the position of particles with momentum p_0 and phase angle $\phi(t_0) = \phi_0$ at initial time t_0. The characteristic solution of equation (A4) also yields the particle trajectory described by particle phase angle $\phi(t)$ and particle position $\boldsymbol{r}(t)$. Thus,

$$\begin{aligned} \phi(t') &= \phi_0 - \Omega(t' - t_0), \qquad & x(t') &= x_0 - r_g[\sin\phi(t') - \sin\phi_0], \\ y(t') &= y_0 + r_g[\cos\phi(t') - \cos\phi_0], \qquad & z(t') &= z_0 + v\cos\theta(t' - t_0), \end{aligned} \tag{A8}$$

where $r_g = v \sin\theta/\Omega$ is the particle gyroradius. These expressions indicate that the particles execute approximately undisturbed helical trajectories along $\boldsymbol{B}_0$ (no changes in θ and p) on the characteristic timescale of interaction with small-amplitude turbulence fluctuations, namely, the particle correlation time t_c^p. Therefore, the value for t_c^p must be restricted so that $t' - t_0$ is small enough to ensure that $\delta f \ll f_0$ (Jokipii 1972). The particle trajectories are allowed to be distorted significantly only after many interactions with fluctuations on large spatial and long timescales as discussed below.

In equation (A7) we assumed a gyrotropic average particle distribution ($\partial f_0/\partial\phi = 0$), which means that cross field diffusion and drifts are negligible. This implies that the particle gyroperiod $t_g = 1/\Omega \ll t_c^p$ or that the particle gyroradius $r_g \ll l_{c\perp}$, where $l_{c\perp}$ is the correlation length of the turbulence. For typical solar wind parameters at 1 AU ($l_{c\perp} = 0.01$ AU, $B_0 = 5$ nT) the derivation is valid for energetic particles with rigidity $R \ll 2.5$ GV in the solar wind. However, evidence has been presented for large-scale cross field diffusion (mainly due to large-scale turbulent magnetic fields) and for the importance of drifts in the heliosphere (Giacalone & Jokipii 1999) in this rigidity interval, which implies that the theory might be applicable to lower energies than suggested above.

It is useful to substitute $\Delta t = t - t'$ in equations (A7) and (A8), where t is the time of observation and t' denotes the passage of time as the particle follows its helical trajectory. Because $t = [t_0, t]$, where t_0 is the initial time, it implies that $\Delta t = [t - t_0,\ 0]$. With regard to Δt, we are following the particle motion backward in time. The advantage of this substitution is that the final transport equation does not require the specification of the initial values of the particle helical orbit as shown below. The solution for δf then becomes

$$\begin{aligned}\delta f(\boldsymbol{r},\boldsymbol{p},t) = &\int_0^{t-t_0} d(\Delta t)\left[-q\sin\theta\frac{\partial f_0}{\partial p}\left(\delta E_x\cos\phi + \delta E_y\sin\phi\right)\right] + \int_0^{t-t_0} d(\Delta t)\left[-q\cos\theta\frac{1}{p}\frac{\partial f_0}{\partial\theta}\left(\delta E_x\cos\phi + \delta E_y\sin\phi\right)\right]\\ &+ \int_0^{t-t_0} d(\Delta t)\left[-\frac{\Omega}{B}\frac{\partial f_0}{\partial\theta}\left(-\delta B_y\cos\phi + \delta B_x\sin\phi\right)\right] + \delta f(\boldsymbol{r}_0,\boldsymbol{p}_0,t_0),\end{aligned} \tag{A9}$$

where $\phi = \phi(t-\Delta t)$, $\delta B_i = \delta B_i(\boldsymbol{r}(t-\Delta t), t-\Delta t)$, $\delta E_i = \delta E_i(\boldsymbol{r}(t-\Delta t), t-\Delta t)$, and $f_0 = f_0(\boldsymbol{r}(t-\Delta t), \boldsymbol{p}(t-\Delta t), t-\Delta t)$. Expressions for the undisturbed particle helical trajectory $\phi(t-\Delta t)$ and the components of $\boldsymbol{r}(t-\Delta t)$, now independent of initial values, are given by

$$\begin{aligned}\phi(t-\Delta t) &= \phi(t) + \Omega(\Delta t), & x(t-\Delta t) &= x(t) + r_g[\sin\phi(t) - \sin\phi(t-\Delta t)],\\ y(t-\Delta t) &= y(t) - r_g[\cos\phi(t) - \cos\phi(t-\Delta t)], & z(t-\Delta t) &= z(t) - v\cos\theta(\Delta t).\end{aligned} \tag{A10}$$

After substituting equation (A2) into equation (A1), taking the average of equation (A1), and exploiting the comoving frame,

$$\frac{\partial f_0}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\frac{\partial f_0}{\partial \boldsymbol{x}} + \boldsymbol{p}\times\boldsymbol{\Omega}\cdot\frac{\partial f_0}{\partial \boldsymbol{p}} = -q\left\langle\left(\delta\boldsymbol{E} + \frac{\boldsymbol{p}\times\delta\boldsymbol{B}}{m}\right)\cdot\frac{\partial\delta f}{\partial\boldsymbol{p}}\right\rangle, \tag{A11}$$

where the nonlinear term in fluctuating quantities on the right-hand side contains information about the average effect of electromagnetic field fluctuations on the particle distribution. The last term on the right-hand side of equation (A7) can be neglected when equation (A7) is substituted into equation (A11) because, for homogeneous stationary turbulence, $\langle\delta\boldsymbol{F}(\boldsymbol{r},\boldsymbol{p},t)\cdot\nabla_p\,\delta f(\boldsymbol{r}_0,\boldsymbol{p}_0,t_0)\rangle = R(\boldsymbol{r}-\boldsymbol{r}_0, \boldsymbol{p}-\boldsymbol{p}_0, t-t_0) = 0$ on long timescales (Jokipii 1972), where $\delta\boldsymbol{F}$ is the Lorentz force due to electromagnetic fluctuations and t_c^p is the particle correlation time. This means that $t - t_0 \gg t_c^p$ and therefore $|\boldsymbol{r}-\boldsymbol{r}_0| \gg l_{c\perp}$.

In addition, because we are considering weak turbulence (small changes in pitch angle occur during particle interaction with the turbulence), particles will need many pitch-angle scatterings for efficient spatial diffusion so that $l_{c\perp} \ll \lambda_\parallel$, where $\lambda_\parallel$ is the parallel mean free path for spatial diffusion. In addition, we assume that the large-scale magnetic field appears uniform to the diffusing particles so that $\lambda_\parallel \ll L$. Thus, the theory is valid for $r_g \ll l_{c\perp} \ll \lambda_\parallel \ll L$.

Expressed in Cartesian coordinates with the momentum coordinates transformed into spherical coordinates, the right-hand side of equation (A11) becomes

$$\begin{aligned}-q\left\langle\left(\delta\boldsymbol{E} + \frac{\boldsymbol{p}\times\delta\boldsymbol{B}}{m}\right)\cdot\frac{\partial\delta f}{\partial\boldsymbol{p}}\right\rangle = &-\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 q\sin\theta\langle\delta f\,\Psi^{\delta E}\rangle\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(q\sin\theta\cos\theta\frac{1}{p}\langle\delta f\,\Psi^{E}\rangle\right)\\ &-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\Omega}{B_0}\sin\theta\langle\delta f\,\Psi^{\delta B}\rangle\right),\end{aligned} \tag{A12}$$

where

$$\begin{aligned}\Psi^{\delta E}(\boldsymbol{r},t) &= \delta E_x(\boldsymbol{r},t)\cos\phi(t) + \delta E_y(\boldsymbol{r},t)\sin\phi(t),\\ \Psi^{\delta B}(\boldsymbol{r},t) &= -\delta B_y(\boldsymbol{r},t)\cos\phi(t) + \delta B_x(\boldsymbol{r},t)\sin\phi(t).\end{aligned} \tag{A13}$$

Closure of equation (A11) is achieved by the substitution of equation (A9). Consequently, equation (A12) acquires the form of a diffusion equation

$$-q\left\langle\left(\delta\boldsymbol{E} + \frac{\boldsymbol{p}\times\delta\boldsymbol{B}}{m}\right)\cdot\frac{\partial\delta f}{\partial\boldsymbol{p}}\right\rangle = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(D_{\theta\theta}\frac{\partial f_0}{\partial\theta} + D_{\theta p}\frac{\partial f_0}{\partial p}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(D_{p\theta}\frac{\partial f_0}{\partial\theta} + D_{pp}\frac{\partial f_0}{\partial p}\right)\right], \tag{A14}$$

where the diffusion coefficients can be expressed as

$$\begin{aligned}D_{\theta\theta}(\boldsymbol{r},t) = &\,q^2\sin\theta\cos^2\theta\frac{1}{p^2}\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\left(R_{xx}^{\delta E\,\delta E}c^2 + R_{yx}^{\delta E\,\delta E}cs + R_{xy}^{\delta E\,\delta E}cs + R_{yy}^{\delta E\,\delta E}s^2\right)\\ &+ q^2\sin\theta\cos^2\theta\frac{1}{p^2}\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\left(R_{xx}^{\delta E\,\delta E}cs - R_{yx}^{\delta E\,\delta E}c^2 + R_{xy}^{\delta E\,\delta E}s^2 - R_{yy}^{\delta E\,\delta E}cs\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{q\Omega}{B_0}\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(-R_{xy}^{\delta E\,\delta B}c^2-R_{yy}^{\delta E\,\delta B}cs+R_{xx}^{\delta E\,\delta B}cs+R_{yx}^{\delta E\,\delta B}s^2\Big)\\
&+\frac{q\Omega}{B_0}\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(-R_{xy}^{\delta E\,\delta B}cs+R_{yy}^{\delta E}c^2+R_{xx}^{\delta E\,\delta B}s^2-R_{yx}^{\delta E\,\delta B}cs\Big)\\
&+\frac{q\Omega}{B_0}\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(-R_{yx}^{\delta B\,\delta E}c^2+R_{xx}^{\delta B\,\delta E}cs-R_{yy}^{\delta B\,\delta E}cs+R_{xy}^{\delta B\,\delta E}s^2\Big)\\
&-\frac{q\Omega}{B_0}\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{yx}^{\delta B\,\delta E}cs+R_{xx}^{\delta B\,\delta E}c^2+R_{yy}^{\delta B\,\delta E}s^2+R_{xy}^{\delta B\,\delta E}cs\Big)\\
&+\left(\frac{\Omega}{B_0}\right)^2\sin\theta\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(R_{yy}^{\delta B\,\delta B}c^2-R_{xy}^{\delta B\,\delta B}cs+R_{yx}^{\delta B\,\delta B}cs+R_{xx}^{\delta B\,\delta B}s^2\Big)\\
&+\left(\frac{\Omega}{B_0}\right)^2\sin\theta\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{yy}^{\delta B\,\delta B}cs+R_{xy}^{\delta B\,\delta B}c^2-R_{yx}^{\delta B\,\delta B}s^2-R_{xx}^{\delta B\,\delta B}cs\Big),
\end{aligned}
\tag{A15}
$$

$$
\begin{aligned}
D_{pp}(\boldsymbol{r},\boldsymbol{t})=&\,q^2\sin^2\theta\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}c^2+R_{yx}^{\delta E\,\delta E}cs+R_{xy}^{\delta E\,\delta E}cs+R_{yy}^{\delta E\,\delta E}s^2\Big)\\
&+q^2\sin^2\theta\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}cs-R_{yx}^{\delta E\,\delta E}c^2+R_{xy}^{\delta E\,\delta E}s^2-R_{yy}^{\delta E\,\delta E}cs\Big),
\end{aligned}
\tag{A16}
$$

$$
\begin{aligned}
D_{p\theta}(\boldsymbol{r},t)=&\,q^2\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}c^2+R_{yx}^{\delta E\,\delta E}cs+R_{xy}^{\delta E\,\delta E}cs+R_{yy}^{\delta E\,\delta E}s^2\Big)\\
&+q^2\sin\theta\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}cs-R_{yx}^{\delta E\,\delta E}c^2+R_{xy}^{\delta E\,\delta E}s^2-R_{yy}^{\delta E\,\delta E}cs\Big)\\
&+q\frac{\Omega}{B_0}\sin\theta\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(-R_{xy}^{\delta E\,\delta B}c^2-R_{yy}^{\delta E\,\delta B}cs+R_{xx}^{\delta E\,\delta B}cs+R_{yx}^{\delta E\,\delta B}s^2\Big)\\
&+q\frac{\Omega}{B_0}\sin\theta\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(-R_{xy}^{\delta E\,\delta B}cs+R_{yy}^{\delta E\,\delta B}c^2+R_{xx}^{\delta E\,\delta B}s^2-R_{yx}^{\delta E\,\delta B}cs\Big),
\end{aligned}
\tag{A17}
$$

$$
\begin{aligned}
D_{\theta p}(\boldsymbol{r},t)=&\,q^2\sin^2\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}c^2+R_{yx}^{\delta E\,\delta E}cs+R_{xy}^{\delta E\,\delta E}cs+R_{yy}^{\delta E\,\delta E}s^2\Big)\\
&+q^2\sin^2\cos\theta\frac{1}{p}\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{xx}^{\delta E\,\delta E}cs-R_{yx}^{\delta E\,\delta E}c^2+R_{xy}^{\delta E\,\delta E}s^2-R_{yy}^{\delta E\,\delta E}cs\Big)\\
&+q\frac{\Omega}{B_0}\sin^2\theta\int_0^\infty d(\Delta t)\cos(\Omega\Delta t)\Big(-R_{yx}^{\delta B\,\delta E}c^2-R_{yy}^{\delta B\,\delta E}cs+R_{xx}^{\delta B\,\delta E}cs+R_{xy}^{\delta B\,\delta E}s^2\Big)\\
&-q\frac{\Omega}{B_0}\sin^2\theta\int_0^\infty d(\Delta t)\sin(\Omega\Delta t)\Big(R_{yx}^{\delta B\,\delta E}cs+R_{yy}^{\delta B\,\delta E}s^2+R_{xx}^{\delta B\,\delta E}c^2+R_{xy}^{\delta B\,\delta E}cs\Big).
\end{aligned}
\tag{A18}
$$

Here $c=\cos\phi(t-\Delta t)=\cos(\phi(t)+\Omega\Delta t)$ and $s=\sin\phi(t-\Delta t)=\sin(\phi(t)+\Omega\Delta t)$ (see eq. [A10]), and R_{ij} represents two-point, two–time correlation functions for the two-dimensional fluctuations along the undisturbed particle orbit. To derive equations (A14)–(A18), $\phi(t)$ was transformed to $\phi(t-\Delta t)$ according to equation (A10) so that

$$
\begin{aligned}
\cos\phi(t)&=\cos\phi(t-\Delta t)\cos(\Omega\Delta t)+\sin\phi(t-\Delta t)\sin(\Omega\Delta t),\\
\sin\phi(t)&=\sin\phi(t-\Delta t)\cos(\Omega\Delta t)-\cos\phi(t-\Delta t)\sin(\Omega\Delta t).
\end{aligned}
\tag{A19}
$$

The various two-point, two–time correlation functions imply that

$$
\begin{aligned}
R_{ij}^{\delta B\,\delta B}(\boldsymbol{r},\boldsymbol{r}(t-\Delta t),t,t-\Delta t)&=\langle\delta B_i(\boldsymbol{r},t)\delta B_j(\boldsymbol{r}(t-\Delta t),t-\Delta t)\rangle,\\
R_{ij}^{\delta E\,\delta E}(\boldsymbol{r},\boldsymbol{r}(t-\Delta t),t,t-\Delta t)&=\langle\delta E_i(\boldsymbol{r},t)\delta E_j(\boldsymbol{r}(t-\Delta t),t-\Delta t)\rangle,\\
R_{ij}^{\delta E\,\delta B}(\boldsymbol{r},\boldsymbol{r}(t-\Delta t),t,t-\Delta t)&=\langle\delta E_i(\boldsymbol{r},t)\delta B_j(\boldsymbol{r}(t-\Delta t),t-\Delta t)\rangle,\\
R_{ij}^{\delta B\,\delta E}(\boldsymbol{r},\boldsymbol{r}(t-\Delta t),t,t-\Delta t)&=\langle\delta B_i(\boldsymbol{r},t)\delta E_j(\boldsymbol{r}(t-\Delta t),t-\Delta t)\rangle,
\end{aligned}
\tag{A20}
$$

where the components of $\boldsymbol{r}(t-\Delta t)$ are determined by equation (A10).

In addition, it was assumed that the particle correlation time $t_c^p \ll t_\mu$, where t_μ is the timescale for particle pitch-angle scattering. This implies that $R_{ij}\rightarrow 0$ on a much shorter timescale than the timescale over which the particle orbit deviates from an undisturbed helical trajectory. It is only then that the assumption of an undisturbed helical trajectory can be defended. Because the integrand of the time integrals contributes only on a relatively short time t_c^p rather than the timescale t_μ to the time integration, it allows us to do the following in equations (A14)–(A18): (1) extract the derivatives of f_0 in front of the time integrals because they are changing on the longer timescale t_μ as a result of the assumption of gyrotropy (f_0 is not a function of the particle phase angle, which varies on a timescale much shorter than t_μ), and (2) extend the upper time integration boundary to ∞ ($t_0 \rightarrow -\infty$ in eq. [A9]).

Considerable simplification of the expressions in equations (A15)–(A18) can be achieved if one restricts oneself to turbulence that is axisymmetric about B_0. According to Matthaeus & Smith (1981), the axisymmetry condition for the correlation matrix $\boldsymbol{R}(\delta\boldsymbol{r})$ under an arbitrary rotation ϕ' about the direction of the large-scale magnetic field $\boldsymbol{B} = B_0\boldsymbol{e}_z$ is expressed by $\boldsymbol{R}(\delta\boldsymbol{r}) = \boldsymbol{OR}(\boldsymbol{O}^T\,\delta\boldsymbol{r})\boldsymbol{O}^T$, where both the left- and right-hand sides are independent of ϕ'. Thus,

$$\boldsymbol{R}(\delta\boldsymbol{r}) = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} = \boldsymbol{OR}\left(\boldsymbol{O}^T\,\delta\boldsymbol{r}\right)\boldsymbol{O}^T$$
$$= \begin{bmatrix} R_{xx}c^2 + R_{xy}sc + R_{yx}sc + R_{yy}s^2 & -R_{xx}sc + R_{xy}c^2 - R_{yx}s^2 + R_{yy}sc & R_{xz}c + R_{yz}s \\ -R_{xx}sc - R_{xy}s^2 + R_{yx}c^2 + R_{yy}sc & R_{xx}s^2 - R_{xy}sc - R_{yx}sc + R_{yy}c^2 & -R_{xz}s + R_{yz}c \\ R_{zx}c + R_{zy}s & -R_{zx}s + R_{zy}c & R_{zz} \end{bmatrix}, \tag{A21}$$

where $c = \cos\phi'$, $s = \sin\phi'$, $\boldsymbol{O}$ is the rotation matrix given by

$$\boldsymbol{O} = \begin{bmatrix} \cos\phi' & \sin\phi' & 0 \\ -\sin\phi' & \cos\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A22}$$

and $\boldsymbol{O}^T$ is its transpose so that all the matrix elements on the left- and right-hand sides of equation (A21) are independent of ϕ'. Inspection of equations (A15)–(A18) reveals that the R_{ij} terms in each bracket correspond exactly with the matric element expressions on the right-hand side of equation (A21). By assuming that $\phi' = \phi(t - \Delta t)$ and substituting equation (A19) into equation (A10), one finds, for the case $\phi(t - \Delta t) = 0$,

$$\delta\boldsymbol{r} = \left[-r_g \sin(\Omega\Delta t), r_g \sin\theta(1 - \cos(\Omega\Delta t), -v\cos\theta\Delta t\right]. \tag{A23}$$

Based on equation (A23), one finds that $\boldsymbol{O}^T\,\delta\boldsymbol{r}$ becomes

$$\begin{aligned} \left(\boldsymbol{O}^T\,\delta\boldsymbol{r}\right)_x &= -r_g[\cos\phi(t-\Delta t)\sin\Omega t' + \sin\phi(t-\Delta t)(1 - \cos\Omega\Delta t)], \\ \left(\boldsymbol{O}^T\,\delta\boldsymbol{r}\right)_y &= r_g[-\sin\phi(t-\Delta t)\sin\Omega\Delta t + \cos\phi(t-\Delta t)(1 - \cos\Omega\Delta t)], \\ \left(\boldsymbol{O}^T\,\delta\boldsymbol{r}\right)_z &= -v\cos\theta\Delta t, \end{aligned} \tag{A24}$$

which agrees with equation (A10) after substitution of equation (A19) into equation (A10). Thus, for the case of axisymmetric turbulence the R_{ij} terms in each bracket in equations (A15)–(A18) are independent of $\phi(t - \Delta t)$ so that without loss of generality we can set $\phi(t - \Delta t) = \pi/2$ in these terms. This useful simplification was to our knowledge first pointed out by Bieber (J. W. Bieber 2001, private communication) and is discussed in detail by Hattingh (1998, p. 41) after communication with Bieber.

Further simplification of equations (A15)–(A18) is possible if the electric field fluctuations are expressed in terms of velocity fluctuations using equation (A6). We now assume, in accord with nearly incompressible MHD theory (Zank & Matthaeus 1992), that the leading-order turbulence fields are incompressible so that both velocity and magnetic field fluctuations are subject to the solenoidal condition. In addition, we assume that all the relevant correlation functions are similar to one another, apart from possibly differences in the overall energy budgets. In accord with this structural similarity hypothesis (Townsend 1976, p. 105), we assume that

$$R_{ij}^{fg}(\boldsymbol{r}, \boldsymbol{r}(t-\Delta t), t, t-\Delta t) = \lambda^{fg} R_{ij}(\boldsymbol{r}, \boldsymbol{r}(t-\Delta t), t, t-\Delta t), \tag{A25}$$

where both f and g could be either δU or δB and R_{ij} is a universal correlation function for each pair i and j independent of the combination of δU and δB one considers. We assume that the trace of $R_{ij}(0) = 1$ whereby λ^{fg} can be interpreted as the energy in two-dimensional fluctuations. By taking the trace of equation (A25), one finds that

$$\lambda^{\delta B\,\delta B} = \langle\delta\boldsymbol{V}_\mathrm{A}\cdot\delta\boldsymbol{V}_\mathrm{A}\rangle 4\pi\rho, \quad \lambda^{\delta U\,\delta U} = \langle\delta\boldsymbol{U}\cdot\delta\boldsymbol{U}\rangle, \quad \lambda^{\delta U\,\delta B} = \langle\delta\boldsymbol{U}\cdot\delta\boldsymbol{V}_\mathrm{A}\rangle\sqrt{4\pi\rho}. \tag{A26}$$

Upon introducing definitions for the Alfvén ratio r_A and the cross helicity σ_c given by

$$r_\mathrm{A} = \frac{\langle\delta\boldsymbol{U}\cdot\delta\boldsymbol{U}\rangle}{\langle\delta\boldsymbol{V}_\mathrm{A}\cdot\delta\boldsymbol{V}_\mathrm{A}\rangle}, \qquad \sigma_c = \frac{2\langle\delta\boldsymbol{U}\cdot\delta\boldsymbol{V}_\mathrm{A}\rangle}{\langle\delta\boldsymbol{U}\cdot\delta\boldsymbol{U}\rangle + \langle\delta\boldsymbol{V}_\mathrm{A}\cdot\delta\boldsymbol{V}_\mathrm{A}\rangle}, \tag{A27}$$

where ρ is the mass density of the solar wind plasma, it follows that

$$R_{ij}^{\delta U\,\delta U} = r_{\rm A} V_{\rm A}^2 \frac{R_{ij}^{\delta B\,\delta B}}{B_0^2}, \qquad R_{ij}^{\delta U\,\delta B} = \frac{1}{2}\sigma_c (r_{\rm A}+1) B_0 V_{\rm A} \frac{R_{ij}^{\delta B\,\delta B}}{B_0^2}. \tag{A28}$$

Substitution of the expressions in equation (A28) into the diffusion coefficient expressions given by equations (A15)–(A18) and setting $\phi(t-\Delta t) = \pi/2$ in the terms in square brackets of the diffusion coefficient expressions result in compact expressions for the diffusion coefficients that only depend on $R_{ij}^{\delta B\,\delta B}$. After simplification, the expressions for the diffusion coefficients in equations (A15)–(A18) become

$$\begin{aligned}
D_{\theta\theta} &= \sin\theta \Omega^2 T_c^p \left[\cos^2\theta r_{\rm A}\left(\frac{v_{\rm A}}{v}\right)^2 + \cos\theta\left(\frac{V_{\rm A}}{v}\right)\sigma_c(r_{\rm A}+1) + 1\right],\\
D_{pp} &= \sin^2\theta r_{\rm A}\Omega^2\left(\frac{pV_{\rm A}}{v}\right)^2 T_c^p,\\
D_{p\theta} &= \sin\theta\Omega^2\left(\frac{pV_{\rm A}}{v}\right)T_c^p\left[\cos\theta r_{\rm A}\left(\frac{v_{\rm A}}{v}\right) + \frac{1}{2}\sigma_c(r_{\rm A}+1)\right] = D_{p\theta},
\end{aligned} \tag{A29}$$

where time T_c^p, related to the particle decorrelation time, is given by

$$T_c^p = \frac{\int_0^\infty d\Delta t\left[\cos(\Omega\Delta t) R_{xx}^{\delta B\,\delta B} - \sin(\Omega\Delta t) R_{yx}^{\delta B\,\delta B}\right]}{B_0^2}. \tag{A30}$$

Based on the condition for axisymmetric turbulence discussed above, we also set $\phi(t-\Delta t) = \pi/2$ in $\boldsymbol{r}(t-\Delta t)$ given by equation (A10) after substituting the expressions of equation (A19) into it so that we have the following simplified components for the arguments of the two-point, two–time correlation functions:

$$x(t-\Delta t) = x(t) + r_g[\cos(\Omega\Delta t) - 1], \quad y(t-\Delta t) = y(t) - r_g\sin(\Omega\Delta t), \quad z(t-\Delta t) = z(t) - v\cos(\theta\Delta t). \tag{A31}$$

What remains to be done is the evaluation of the time integral T_c^p. This is accomplished by applying the standard method of Fourier transforms. For this purpose we assume homogeneous stationary turbulence, which means that

$$R_{ij}^{\delta B\,\delta B}(\boldsymbol{r}, \boldsymbol{r}(-\Delta t), t, t-\Delta t) = R_{ij}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(-\Delta t), -\Delta t) = R_{ji}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t), \tag{A32}$$

where the components of $\Delta\boldsymbol{r}$ are

$$\Delta x(\Delta t) = -r_g[\cos(\Omega\Delta t) - 1], \quad \Delta y(\Delta t) = +r_g\sin(\Omega\Delta t), \quad \Delta z(\Delta t) = +v\cos\theta(\Delta t). \tag{A33}$$

The correlation functions can now conveniently be expressed in terms of a spatial Fourier transform so that

$$R_{ji}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\Delta\boldsymbol{r}(\Delta t)} P_{ji}(\boldsymbol{k}, \Delta t) = \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\Delta\boldsymbol{r}(\Delta t)} P_{ji}(\boldsymbol{k})\Gamma(\boldsymbol{k}, \Delta t), \tag{A34}$$

where $\boldsymbol{k}$ is the turbulence wavenumber vector, $P_{ji}(\boldsymbol{k})$ is the energy spectrum tensor, and the time dependence is parameterized following Bieber et al. (1994) by the function Γ. Note that $\mathrm{Tr}\left(R_{ji}^{\delta B\,\delta B}\right)(0) = \int d\boldsymbol{k}\, P_{ii}(\boldsymbol{k}) = \langle\delta\boldsymbol{B}\cdot\delta\boldsymbol{B}\rangle$. The particle correlation function $R_{ji}^{\delta B\,\delta B}$ is expected to decay in time, and we adopt the exponential damping model (Bieber et al. 1994) $\Gamma(\boldsymbol{k}, \Delta t) = e^{-\gamma(\boldsymbol{k})\Delta t}$. Thus, we have

$$R_{ji}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\Delta\boldsymbol{r}(\Delta t)} e^{-\gamma(\boldsymbol{k})\Delta t} P_{ji}(\boldsymbol{k}). \tag{A35}$$

It is convenient for axisymmetric turbulence to transform the turbulence wavevector $\boldsymbol{k}$ into cylindrical coordinates $(k_\perp\cos\Psi, k_\perp\sin\Psi, k_\parallel)$, where $k_\perp$ ($k_\parallel$) is the wavevector component perpendicular (parallel) to $\boldsymbol{B}_0$ and Ψ is the phase angle of the wavevector. Consequently, equation (A35) becomes

$$R_{ji}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \int_0^{2\pi} d\Psi \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty dk_\perp\, k_\perp e^{ik_\perp r_g[\cos\Psi - \cos(\Psi+\Omega\Delta t)]} e^{ik_\parallel v\cos\theta\Delta t} e^{-\gamma(k_\parallel, k_\perp)\Delta t} P_{ji}(\boldsymbol{k}). \tag{A36}$$

To simplify the integrals in equation (A34), it is useful to introduce the Jacobi-Anger expansion in terms of Bessel Functions $J_n(x)$, $e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} i^n e^{in\theta} J_n(x)$, and to make use of the relationships $J_n(x) = (-1)^n J_n(-x)$ and $J_{-n}(x) = (-1)^n J_n(x)$. Thus,

$$R_{ji}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t)y, \Delta t) = \int_0^{2\pi} d\Psi \int_{-\infty}^{\infty} dk_\parallel \int_0^{\infty} dk_\perp\, k_\perp \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} i^{m-n} e^{i(m-n)\Psi} J_m(x) J_n(x)$$
$$\times\, e^{-in\Omega\Delta t} e^{ik_\parallel v\cos\theta\Delta t} e^{-\gamma(k_\parallel, k_\perp)\Delta t} P_{ji}(\boldsymbol{k}), \tag{A37}$$

where $x = k_\perp r_g$.

In order to do the time integrations for the two-dimensional turbulence component, we have to find expressions for $P_{ji}(k_\parallel, k_\perp)$.

The general expression (Oughton, Rädler, & Matthaeus 1997) for P_{ij} simplifies considerably in the case of axisymmetric two-dimensional turbulence because it depends only on a single scalar function. In particular, one finds that

$$P_{ij}(\boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k_\perp^2}\right) E^{\delta B}(\boldsymbol{k}), \tag{A38}$$

where $k_\perp$ is the wavevector perpendicular to the large-scale magnetic field (Oughton et al. 1997). Again using cylindrical coordinates,

$$P_{xx}(\boldsymbol{k}) = E^{\delta B}(\boldsymbol{k}) \sin^2\psi, \qquad P_{yx}(\boldsymbol{k}) = P_{xy}(\boldsymbol{k}) = -E^{\delta B}(\boldsymbol{k}) \cos\psi \sin\psi, \tag{A39}$$

where ψ is the wavevector angle.

By substituting equation (A39) into equation (A37), expressing the spectral energy density in the fluctuations in equation (A40) as $E^{\delta B}(\boldsymbol{k}) = E^{\delta B}(k_\perp)\delta(k_\parallel)$, and assuming $\gamma(k_\perp)$ as appropriate for the two-dimensional turbulence component, we find that

$$R_{xx}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = \frac{\pi}{2} \int_0^{\infty} dk_\perp\, k_\perp \sum_{n=-\infty}^{\infty} \left[J_{n-2}(x)J_n(x) + J_{n+2}(x)J_n(x) - 2J_n^2(x)\right] e^{-in\Omega\Delta t} e^{-\gamma(k_\perp)\Delta t} E^{\delta B}(k_\perp),$$
$$R_{xy}^{\delta B\,\delta B}(\Delta\boldsymbol{r}(\Delta t), \Delta t) = R_{yx}(\Delta\boldsymbol{r}(\Delta t), \Delta t)$$
$$= -\frac{\pi}{2} i \int_0^{\infty} dk_\perp\, k_\perp \sum_{n=-\infty}^{\infty} [J_{n-2}(x)J_n(x) - J_{n+2}(x)J_n(x)] e^{-in\Omega\Delta t} e^{-\gamma(k_\perp)\Delta t} E^{\delta B}(k_\perp). \tag{A40}$$

After substitution of equation (A40) into equation (A30) and resumming the series of Bessel functions, we find that

$$T_c^p = \frac{\pi}{2} \int_0^{\infty} d(\Delta t) \int_0^{\infty} dk_\perp\, k_\perp \sum_{n=1}^{\infty} \left[J_{1-n}^2 + 2J_{1-n}J_{-1-n} + J_{-1-n}^2\right] e^{+in\Omega(\Delta t)} e^{-\gamma(k_\perp)\Delta t} \frac{E^{\delta B}(k_\perp)}{B_0^2}. \tag{A41}$$

Using the identities $J_{-n}(x) = (-1)^n J_n(x)$ and $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$, this becomes

$$T_c^p = 2\pi \int_0^{\infty} d(\Delta t) \int_0^{\infty} dk_\perp\, k_\perp e^{-\gamma(k_\perp)\Delta t} \frac{E^{\delta B}(k_\perp)}{B_0^2} \sum_{n=-\infty}^{\infty} \frac{n^2}{x^2} J_n^2(x) e^{+in\Omega(\Delta t)}. \tag{A42}$$

Completion of the time integration leads to the result that

$$T_c^p = 4\pi \int_0^{\infty} dk_\perp\, k_\perp \gamma(k_\perp) \frac{E^{\delta B}(k_\perp)}{B_0^2} \sum_{n=1}^{\infty} \frac{n^2}{x^2} \frac{J_n^2(x)}{[\gamma(k_\perp)]^2 + [n\Omega]^2}. \tag{A43}$$

It is reasonable to assume that the exponential decay of the particle correlation function due to the inherent time dependence of the turbulence, described by $\gamma(k_\perp)$ in equation (A43), can be characterized by the timescale for nonlinear interactions between two-dimensional MHD turbulence eddies. This timescale is plausibly given by the nonlinear time $\tau_{nl}(k_\perp) = 1/k_\perp \delta U(k_\perp)$, where $\delta U(k_\perp)$ is the convective speed of the turbulence at a perpendicular wavenumber $k_\perp$ in the inertial range. In the Kolmogorov turbulence model, the transfer rate of energy in the inertial range is independent of the scale size of the two-dimensional turbulence so that we can write $\delta U(k_\perp) = \delta U_\perp (k_\perp l_{c\perp})^{-1/3}$, where $\delta U_\perp$ is the rms of the velocity fluctuations associated with the two-dimensional turbulence correlation length $l_{c\perp}$. Invoking approximate equipartition between the energy density of the velocity and magnetic field fluctuations (Oughton et al. 1998), $\delta U_\perp \approx (\delta B_\perp/B_0)V_{\rm A}$, where $V_{\rm A}$ is the Alfvén speed. It then follows that

$\tau_{nl}(k_\perp) \approx (k_\perp l_{c\perp})^{1/3}/[k_\perp(\delta B/B_0)V_A]$. This indicates that temporal decay of the correlation function due to dynamic turbulence is faster for smaller scale eddies. Consequently,

$$\gamma(k_\perp) = k_\perp\,\delta U(k_\perp) = \frac{k_\perp(\delta B/B_0)V_A}{(k_\perp l_{c\perp})^{1/3}} \tag{A44}$$

in equation (A43). This model corresponds to the exponential damping model suggested by Bieber et al. (1994) if $(k_\perp l_{c\perp}) = 1$.

When the two-dimensional magnetic field fluctuation spectral energy $E^{\delta B}(k_\perp)$ in the inertial range is specified as a Kolmogorov power law according to

$$E^{\delta B}(k_\perp) = \frac{2}{3} l_{c\perp}\langle \delta B_\perp^2\rangle \frac{1}{k_\perp}\frac{1}{(k_\perp l_{c\perp})^{5/3}}, \tag{A45}$$

the expression for T_c^p (eq. [A44]) becomes

$$T_c^p = \frac{8\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle \delta B_\perp^2\rangle}{B_0^2}\frac{1}{\Omega^2}\int_{r_g/l_{c\perp}}^{\infty} dx\,\frac{1}{x^3}\sum_{n=1}^{\infty}\frac{n^2 J_n^2(x)}{y^2+n^2}, \tag{A46}$$

where

$$y^2 = \frac{\langle \delta B_\perp^2\rangle}{B_0^2}\left(\frac{V_A}{v_\perp}\right)^2\left(\frac{r_g}{l_{c\perp}}\right)^{2/3} x^{4/3}, \tag{A47}$$

$v_\perp = v\sqrt{1-\mu^2}$ with μ the cosine of the particle pitch angle, and $U_\perp = \delta U(l_{c\perp}) = (\delta B_\perp/B_0)V_A$.

Then assuming that the ratios $\langle \delta B_\perp^2\rangle/B_0^2 \ll 1$, $(V_A/v_\perp)^2 \ll 1$, and $(r_g/l_{c\perp})^{2/3} \ll 1$, which is appropriate for particles with super-Alfvénic but sub–cosmic-ray speeds interacting with weak two-dimensional turbulence in the solar wind, it follows that $y \ll 1$ provided that x is not too large or μ is not too close to 1. However, there is probably no need to consider the $y > 1$ case because for large x-values or μ-values close to 1, the coefficients are negligibly small. This follows because the diffusion coefficients are proportional to $1-\mu^2$ (see eq. [A50]) and $T_c^p \propto 1/x^3$. Thus, for $y \ll 1$, T_c^p simplifies to

$$T_c^p \approx \frac{8\pi}{3}\frac{\delta U_\perp}{l_{c\perp}}\frac{\langle \delta B_\perp^2\rangle}{B_0^2}\frac{1}{\Omega^2}\int_{r_g/l_{c\perp}}^{\infty} dx\,\frac{1}{x^3}\sum_{n=1}^{\infty} J_n^2(x). \tag{A48}$$

For more information on the $y > 1$ case, see Shalchi & Schlickeiser (2003).

The final Fokker-Planck equation is found by introducing a uniform large-scale magnetic $\boldsymbol{B} = B_0\boldsymbol{e}_z$ into the left-hand side of equation (A11) and by averaging the equation over particle phase angle ϕ. This is consistent with the assumption mentioned above that particles see a uniform large-scale magnetic field on the timescale they experience diffusion in momentum space. The final transport equation is

$$\frac{\partial f_0}{\partial t} + v\mu\frac{\partial f_0}{\partial z} = \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial f_0}{\partial\mu} + D_{\mu p}\frac{\partial f_0}{\partial p}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D_{p\mu}\frac{\partial f_0}{\partial\mu} + D_{pp}\frac{\partial f_0}{\partial p}\right), \tag{A49}$$

where the expressions for the Fokker-Planck diffusion coefficients in momentum space are

$$\begin{aligned} D_{\mu\mu} &= \Omega^2(1-\mu^2)T_c^p\left[\mu^2 r_A\left(\frac{V_A}{v}\right)^2 + \mu\sigma_c(r_A+1)\left(\frac{V_A}{v}\right) + 1\right], \\ D_{pp} &= \Omega^2(1-\mu^2)\left(\frac{pV_A}{v}\right)^2 T_c^p r_A, \\ D_{\mu p} &= D_{p\mu} = -\Omega^2(1-\mu^2)\left(\frac{pV_A}{v}\right)T_c^p\left[\mu\left(\frac{V_A}{v}\right)r_A + \frac{1}{2}\sigma_c(r_A+1)\right], \end{aligned} \tag{A50}$$

where $\mu = \cos\theta$ is the cosine of the particle pitch angle and T_c^p is given by equation (A43), equation (A46), or equation (A48).

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