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# COSMOLOGICAL PERTURBATION THEORY USING THE SCHRÖDINGER EQUATION

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## ABSTRACT

We introduce the theory of nonlinear cosmological perturbations using the correspondence limit of the Schrödinger equation. The resulting formalism is equivalent to using the collisionless Boltzmann (or Vlasov) equations, which remain valid during the whole evolution, even after shell crossing. Other formulations of perturbation theory explicitly break down at shell crossing, e.g., Eulerean perturbation theory, which describes gravitational collapse in the fluid limit. This Letter lays the groundwork by introducing the new formalism, calculating the perturbation theory kernels that form the basis of all subsequent calculations. We also establish the connection with conventional perturbation theories, by showing that third-order tree-level results, such as bispectrum, skewness, cumulant correlators, and three-point function, are exactly reproduced in the appropriate expansion of our results. We explicitly show that cumulants up to $N = 5$ predicted by Eulerian perturbation theory for the dark matter field $\delta$ are exactly recovered in the corresponding limit. A logarithmic mapping of the field naturally arises in the Schrödinger context, which means that tree-level perturbation theory translates into (possibly incomplete) loop corrections for the conventional perturbation theory. We show that the first loop correction for the variance is $\sigma^2 = \sigma_L^2 + (-1.14 - n)\sigma_L^4$ for a field with spectral index $n$. This yields 1.86 and 0.86 for $n = -3$ and $-2$, respectively, to be compared with the exact loop order corrections 1.82 and 0.88. Thus, our tree-level theory recovers the dominant part of first-order loop corrections of the conventional theory, while including (partial) loop corrections to infinite order in terms of $\delta$.

*Subject headings:* cosmic microwave background — cosmology: theory — methods: statistical

## 1. INTRODUCTION

The most successful theories of structure formation assume that small initial fluctuations grew by gravitational amplification in an expanding cosmological background. This simple assumption has been remarkably successful in explaining nonlinear structures in simulations and observations, especially on larger scales. The basic equation governing the statistics of the dark matter field is the collisionless Boltzmann (or Vlasov) equation. These nonlinear equations are considered to be intractable; therefore, the most usual approach is to take their moments.

Taking only the first few moments of the equation results in Euler's ideal (pressureless) fluid equations coupled with the Poisson equation for gravity. The fluid approximation, however, breaks down at shell crossing. Arbitrarily high moments yield an infinite hierarchy of equations (e.g., Peebles 1980), the BBGKY equations. These relate time evolution of $N$th-order moments to $(N + 1)$th-order moments. The equations can be "closed" only under certain assumptions (e.g., Davis & Peebles 1977; Fry 1984a; Hamilton 1988). Equations of motion in Lagrangian space (e.g., Bouchet et al. 1995) are as successful as the Eulerian method. The above analytical tools are checked by a set of numerical and seminumerical methods, such as $N$-body simulations, and approximations thereof such as Zeldovich, truncated Zeldovich, frozen flow, and adhesion approximation (e.g., Gurbatov, Saichev, & Shandarin 1989).

One particularly successful method to solve the above equations before shell crossing is perturbation theory (PT); Eulerian and Lagrangian perturbation theories (EPT and LPT) are used with similar success. Contributions can be ordered systematically, represented and enumerated by the use of Feynman graphs pioneered by Goroff et al. (1986). The first nontrivial order, tree-level PT penetrates the nonlinear regime to surprisingly high degree. Cumulants of the dark matter density field have been calculated for Gaussian initial conditions by several authors[2] (e.g., Peebles 1980; Juszkiewicz, Bouchet, & Colombi 1993; Bernardeau 1992; Bernardeau et al. 2002). For instance, the skewness $S_3$ of a mildly nonlinear field is $34/7 - (n + 3)$, where $n$ is the local power index of the power spectrum. Similar calculations have been performed for the full three-point correlation function and bispectrum (Fry 1984b). The theory has been confirmed by simulations (e.g., Colombi, Bouchet, & Schaeffer 1994; Szapudi et al. 1999, 2000; Columbi et al. 2000), and data appear to be in broad agreement as well (e.g., Szapudi, Meiksin, & Nichol 1996; Szapudi & Gaztanaga 1998; Szapudi & Szalay 1997; Szapudi et al. 2002).

The most straightforward way to improve tree-level PT is to include the next to leading order "loop" corrections. Such calculations (Scoccimarro & Frieman 1996a, 1996b), albeit fairly complicated, do improve the agreement with simulations on small scales. Another extension to PT is the spherical infall model, in which one calculates angle-averaged Feynman vertices (Fosalba & Gaztanaga 1998). It is a fairly simple alternative to the tedious calculation of loop corrections, but it yields only a fraction of the corrections owing to neglecting tidal effects. Besides physical theories, several phenomenological assumptions exist to fit smaller scale behavior (e.g., "extended" and "hyperextended" PT; see Bernardeau et al. 2002 for details).

In this Letter, we propose an entirely different approach based on the correspondence limit of the Schrödinger equations. These are equivalent to the full Boltzmann-Vlasov description (Widrow & Kaiser 1993) but involve a complex scalar field depending on $3 + 1$ variables, similarly to the Euler equations. As shown below, PT of the Schrödinger theory (SPT) is not significantly more complicated than EPT. Section 2 presents the underlying theory, § 3 outlines the connection with conventional PT, § 4 deals with the cumulants in more detail, and

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[2] The last paper contains comprehensive references on the subject.

finally § 5 discusses the results and provides an outlook for research possible using this formalism.

## 2. PERTURBATION THEORY WITH THE SCHRÖDINGER EQUATION

The Schrödinger equation in the correspondence limit is a viable alternative to the collisionless Boltzmann (Vlasov) equations. In the expanding universe (with $\Omega = 1$ for simplicity),

$$i\hbar\dot{\psi} + \frac{3}{2}H\psi + \boldsymbol{H}\psi = 0, \tag{1}$$

where $\psi$ is the complex scalar field representing dark matter density with $\rho = |\psi|$, $H$ is Hubble's constant, and $\boldsymbol{H}$ is the Hamiltonian of the gravitational field. This is a nonlinear equation, since $\boldsymbol{H}$ depends on the density field. If new variables are introduced as

$$\psi(r, t) = \psi_0 \left(\frac{a}{a_0}\right)^{-3/2} e^{A(r, t) + iB(r, t)/\hbar}, \tag{2}$$

the above equation reduces to equations for two real scalar fields,

$$\begin{aligned} \dot{A} &= -\frac{1}{2ma^2}(\nabla^2 B + 2\nabla A \nabla B), \\ \dot{B} &= \frac{\hbar^2}{2ma^2}(\nabla^2 A + |\nabla A|^2) - \frac{1}{2ma^2}|\nabla B|^2 - mV, \\ \nabla^2 V &= 4\pi G\bar{\rho}a^2(a^{2A} - 1), \end{aligned} \tag{3}$$

where the last equation is the Poisson equation coupled to the Schrödinger equation. The structure of these equations is similar to Euler's fluid equations in terms of the density contrast and velocity potential, despite the fact that we have not taken moments of the underlying full equations. The main difference and extra complication arise from the exponential in the Poisson equation. The PT of these equations will be entirely analogous to that of the Euler equations, and it can be presented in Fourier space in the simplest manner.

In what follows, we work in the correspondence limit, i.e., $\hbar \to 0$; we neglect "wavy" features of the equations. Fourier transforming the equations yields

$$\begin{aligned} \dot{A}_k &= -\frac{1}{2a^2}[k^2 B_k + 2(kA_k)(kB_k)], \\ \dot{B}_k &= -\frac{1}{2a^2}(kB_k)(kB_k) - \frac{3H^2a^2}{2k^2}\sum_{N\geq 1}\frac{2^N}{N!}A_k^N, \end{aligned} \tag{4}$$

where the multiple and power of transforms are understood as convolutions, $m = 1$ is assumed for simplicity, the Poisson equation was substituted for $-V$, and the exponential was expanded.

These equations can be rendered homogeneous in $a$ and $H$ using the following *Ansätze*:

$$\begin{aligned} A_k &= \sum A_k^{(N)} a^N, \\ B_k &= -H\sum B_k^{(N)} a^{N+2}. \end{aligned} \tag{5}$$

Perturbations can be ordered according to powers of the growth factor. We can introduce the usual kernels with the following definition:

$$\begin{aligned} A_k^{(N)} &= \int d^3k_1 \ldots d^3k_N \delta(k = k_1 + \ldots + k_N) \\ &\quad \times F^{(N)}(k_1, \ldots, k_n) A_{k_1}^{(1)} \ldots A_{k_N}^{(1)}, \\ B_k^{(N)} &= \frac{2}{k^2}\int d^3k_1 \ldots d^3k_N \delta(k = k_1 + \ldots + k_N) \\ &\quad \times G^{(N)}(k_1, \ldots, k_n) A_{k_1}^{(1)} \ldots A_{k_N}^{(1)}. \end{aligned} \tag{6}$$

Substituting to the equations leads to the following recursions:

$$\begin{aligned} NF^{(N)} &= G^{(N)} + 2\sum_S \alpha(q_1, q_2) F^{(S)} G^{(N-S)}, \\ \left(N + \frac{1}{2}\right) G^{(N)} &= \sum_S \beta(q_1, q_2) G^{(S)} G^{(N-S)} \\ &\quad + 3\sum_{M, s_1, s_2, \ldots} \frac{2^{(M-2)}}{M!}\delta\left(N = \sum s_i\right) F^{(s_1)} F^{(s_2)} \ldots, \end{aligned} \tag{7}$$

where mode coupling functions are similar to the Eulerian case,

$$\begin{aligned} \alpha(q_1, q_2) &= \frac{(q_1 q_2)}{k_2^2}, \\ \beta(q_1, q_2) &= \frac{k^2(q_1 q_2)}{q_1^2 q_2^2}, \end{aligned} \tag{8}$$

and the exponential is expanded; $q_1$, $q_2$, and $k$ correspond to the sum of $S$, $N - S$, and all the wavevectors. To solve the recursion at any order, one has to separate $F^N$ ($M = 1$) in the expansion of the exponential and subtract the two equations from each other. This procedure leads to terms up to $N - 1$ on the right-hand side of the equations. Here we give explicitly the $N = 2$ case

$$F^{(2)}(k_1, k_2) = \frac{3}{7} + \frac{10}{7}\alpha(k_1, k_2) + \frac{2}{7}\beta(k_1, k_2). \tag{9}$$

Higher order kernel functions can be obtained trivially from the recursion relation. These kernels can be used to calculate quantities to the accuracy of tree-level PT. In what follows, we will show the connection with EPT via explicitly calculating tree-level quantities at third order.

## 3. CONNECTION WITH EULERIAN PERTURBATION THEORY

Since $\delta = e^{2A} - 1$, $\delta_1$ and $\delta_2$, the perturbative corrections to $\delta$ growing with the first and second power of the growth factor, are

$$\begin{aligned} \delta^{(1)} &= 2A^{(1)}, \\ \delta^{(2)} &= 2(A^{(2)} + A^{(1)^2}). \end{aligned} \tag{10}$$

Thus, $\delta^3$ to tree-level PT is

$$\begin{aligned} 3\langle \delta^{(1)2}\delta^{(2)}\rangle &= 3a^4 8A^{(1)2}(A^{(2)} + A^{(1)2}) \\ &= 3a^4 8\int d^3k_1 \ldots d^3k_4[1 + F^2(k_2,\ k_3)]A^{(1)}_{k_1} \ldots A^{(1)}_{k_4} e^{ix\sum k_i} \\ &= \frac{3}{2}a^4\int d^3k_1 \ldots d^3k_4[1 + F^2(k_2,\ k_3)]\delta^{(1)}_{k_1} \ldots \delta^{(1)}_{k_4} e^{ix\sum k_i}. \end{aligned} \tag{11}$$

When taking the ensemble average (assuming Gaussian fields), $k_2$ and $k_3$ cannot be paired, yielding a combinatorial factor of 2. From this, $S_3 = 3\langle 1 + F^2\rangle$, where the angle brackets here mean angle averaging. Since $\langle\alpha\rangle = 0$ and $\langle\beta\rangle = \frac{2}{3}$, $S_3 = 34/7$, as expected from PT.

In fact, it is easy to show $1 + F^{(2)} = 2\tilde{F}^{(2)}$ (after the necessary symmetrization of the kernels), where $\tilde{F}^{(2)}$ is the second-order kernel (naked vertex) in EPT. Thus, calculations entirely analogous to the above show that tree-level bispectrum, three-point function, and the skewness after smoothing, cumulant correlators, etc., are all exactly matching the tree-level EPT results.

## 4. CUMULANTS

To derive cumulants, one has to consider the angle-averaged recursion relations. If the angle-averaged kernels are defined as $\nu_N = N!\langle F_N\rangle$ and $\mu_N = N!\langle G_N\rangle$, the first equation gives a very simple relation:

$$N\nu_N = \mu_N. \tag{12}$$

From this the second equation gives a simple recursion for $\nu_N$, which reads

$$\nu_N = \frac{2}{(2N+1)N-3}\left[\sum_{s=1}^{N-1}\frac{2}{3}\binom{N}{s}s(N-s)\nu_s\nu_{N-s} + 3N!\sum_{M=2}\frac{2^{M-2}}{M!}\sum_{s_1,\ldots,s_M}\delta\left(N = \sum s_i\right)\frac{\nu_{s_1}}{s_1!}\ldots\frac{\nu_{s_M}}{s_M!}\right]. \tag{13}$$

With the initial condition $\nu_1 = \mu_1 = 1$, one can solve to arbitrary order in a trivial fashion. The first two naked vertices are

$$\begin{aligned} \nu_2 &= \frac{26}{21}, \\ \nu_3 &= \frac{568}{189}, \\ \nu_4 &= \frac{473{,}744}{43{,}659}. \end{aligned} \tag{14}$$

From these the tree-level cumulants of the $A$-field can be calculated as

$$\begin{aligned} S_3^A &= 3\nu_2 = \frac{26}{7}, \\ S_4^A &= 4\nu_3 + 12\nu_2^2 = \frac{40{,}240}{1323}, \\ S_5^A &= 5\nu_4 + 60\nu_3\nu_2 + 60\nu_2^3 = \frac{119{,}609{,}680}{305{,}613}. \end{aligned} \tag{15}$$

Projecting the cumulants of the $A$-field using the formalism of biasing recovers cumulants of $\delta$. In this context, the primary field is $A$ and $\delta = e^{2A} - 1 = \sum b_k A^k/k!$ is a nonlinearly biased field, where the usual bias coefficients read $b = 2$ and $c_N = b_N/b = 2^{N-1}$. According to the formula of Fry & Gaztanaga (1993),

$$\begin{aligned} S_3 &= b^{-1}(S_3 + 3c_2) = \frac{34}{7}, \\ S_4 &= b^{-2}(S_4^A + 12c_2S_3^A + 4c_3 + 12c_2^2) = \frac{60{,}712}{1323}, \\ S_5 &= b^{-3}[S_5^A + 20c_2S_4^A + 15c_2S_3^A + (30c_3 + 120c_2)S_3^A \\ &\quad + 5c_4 + 60c_3c_2] = \frac{200{,}575{,}880}{305{,}613}; \end{aligned} \tag{16}$$

i.e., we recover the results of tree-level PT exactly.

Because of the nonlinear transformation, tree-level calculations of the $A$-field translate into (possibly incomplete) loop corrections for the $\delta$-field. To demonstrate this, we calculate the loop corrections arising from the tree-level SPT for the variance $\sigma$. According to Fry & Gaztanaga (1993), the variance $\sigma^2 = 4\langle A^2\rangle + 4\langle A^2\rangle^2(2S_3^A + 6) + \ldots$ . Initially, there is no skewness; thus, for the initial conditions, the linear variance $\sigma_L$ follows the same equation with $S_3 = 0$. Expanding and collecting terms to second order in $\langle A^2\rangle$ yields

$$\sigma^2 = \sigma_L^2 + \frac{S_3^A}{2}\sigma_L^4. \tag{17}$$

Numerically the coefficient $S_3^A/2 = 13/7 \simeq 1.857$ is within 2% of the exact loop correction 1.82. This means that the dominant part of the loop correction arises from tree-level perturbations of the logarithm of the field. The same expansion can be generalized to any order; although tedious, it is fairly simple. It will be presented elsewhere. Here we have used the general theory of Fry & Gaztanaga (1993). However, exponential bias was explicitly treated by Grinstein & Wise (1986); their results will probably be useful for future calculations of this sort.

Note that the above calculations are for the unsmoothed field, or $n = -3$. Detailed calculation for other spectral indices will be shown elsewhere; as a preview, simple considerations suggest $S_3^A = 26/7 - 2(n+3)$, and consequently $\sigma^2 = \sigma_L^2 + (-1.14 - n)\sigma_L^4$, for $n < -1.14$, in excellent agreement with PT.

## 5. SUMMARY AND OUTLOOK

We have introduced SPT based on the Schrödinger equations. In the correspondence limit, our description is equivalent with the full Boltzmann (Vlasov) equations. Other versions of PT, in particular EPT, explicitly break down at shell crossing; therefore, our ultimate aim is to penetrate the nonlinear regime deeper.

A unique feature of SPT is that it naturally uses the logarithm of the dark matter field, which remains close to Gaussian throughout the nonlinear evolution. This suggests that the SPT expansion should converge faster. Related techniques such as Edgeworth expansion, etc., are expected to be more accurate. Indeed, we have shown that, at least for the variance, this is the case. This feature of our theory sheds new light on the validity of the lognormal prescription for $\delta$ (e.g., Coles & Jones 1991) and suggests using $A$, i.e., the logarithm of the underlying dark matter field, to construct statistics, such as correlation function, skewness, kurtosis, and bispectrum (in $A$-space). The first encouraging steps toward analyzing data in log-space have been done by Colombi (1994).

Our principal aim here is to present the basic theory and the calculation of the recursions for the tree-level PT kernels. These constitute the groundwork for a spectrum of future research. We have also given the recursion for the angle-averaged kernels and elaborated the results up to $N = 5$ explicitly. Higher orders are calculated trivially from the results shown.

We have explored the connection of our theory with EPT. We have recovered the first nontrivial tree-level PT kernel in the appropriate limit. All third-order tree-level PT results (e.g., bispectrum, three-point function) are exactly reproduced by the SPT expansion. In addition, we have applied the exponential bias formalism to recover exactly the tree-level PT cumulants up to $N = 5$. The theoretical exercise of proving the equivalence to arbitrary order is left for subsequent research.

Our tree-level results correspond to (possibly incomplete) infinite-order loop corrections. To demonstrate that, we have obtained the first nontrivial loop correction for the variance by simply expanding our tree-level results to second order. The agreement of this simple expansion with the complex EPT loop calculations is remarkable. A large, perhaps dominant, fraction of the loop corrections arises from the nonlinear projection of tree-level results from $A$-space to $\delta$-space.

The following extensions and generalizations will be presented in subsequent publications: detailed exploration of the higher order cumulants, cumulant correlators, $N$-point correlation functions, and $N - 1$ spectra, and the probability distribution function; other statistics, such as genus and void probability; smoothing; non-Gaussian initial conditions; application to angular correlations and lensing; general cosmological background; approximate loop and nonperturbative corrections for higher order quantities using our nonlinear projection; exact loop corrections in $A$-space; and comparison of tree-level statistics of the $A$-field, such as bispectrum, three-point correlation function, and cumulants (variance, skewness, and kurtosis), with simulations. Other interesting applications of the theory are possible, such as studying the wavy term, which was neglected so far (and thus modeling a dark matter with large Compton wavelength). A fast approximation scheme similar to second-order LPT will be obtained and implemented from our formalism, possibly including wavy terms.

Note that we have not mentioned the interpretation of the $B$-field ($G^N$ and $\mu_N$). Initially, this corresponds to the usual velocity potential; the interpretation after shell crossing is unclear. Nevertheless, our calculations yield $G^N$ and $\mu_N$ as well; the connection with measurements is left for future work.

The method we have put forward follows almost exactly the prescription of EPT for a slightly different set of equations. This is not the only way to deal with these equations; other possibilities exist, such as using the interaction picture analogously to quantum mechanics. As a first step toward this direction, the authors have shown that the Zeldovich approximation is recovered in the zeroth-order approximation. This in turn (along with the logarithmic mapping) hints at a connection with LPT, which will be explored later.

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