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STABILITY AND ECCENTRICITY FOR TWO PLANETS IN A 1:1 RESONANCE, AND THEIR POSSIBLE OCCURRENCE IN EXTRASOLAR PLANETARY SYSTEMS

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ABSTRACT

The nonlinear stability domain of Lagrange's celebrated 1772 solution of the three-body problem is obtained numerically as a function of the masses of the bodies and the common eccentricity of their Keplerian orbits. This domain shows that this solution can be realized in extrasolar planetary systems similar to those that have been discovered recently with two Jupiter-size planets orbiting a solar-size star. For an exact 1 : 1 resonance, the Doppler shift variation in the emitted light would be the same as for stars that have only a single planetary companion. But it is more likely that in actual extrasolar planetary systems there are deviations from such a resonance, raising the interesting prospect that Lagrange's solution can be identified by an analysis of the observations. The existence of another stable 1 : 1 resonance solution that would have a more unambiguous Doppler shift signature is also discussed.

Key words: celestial mechanics — planetary systems

1. INTRODUCTION

In many extrasolar planetary systems discovered recently, the observed Doppler shift of the emitted light is well described by assuming that the central star is moving in a Keplerian elliptic orbit due to its gravitational interaction with a single Jupiter-size planet. If residuals are present in a least-squares fit to the data after possible chromospheric fluctuations in the star have been taken into account, these residuals signal the presence of an additional planet or, possibly, several planets (Fischer et al. 2001; Marcy et al. 2002). There is, however, an important exception when such a star also travels on a Keplerian orbit even though there are *two* planets orbiting it. We have in mind Lagrange's celebrated solution of the three-body problem (Lagrange 1873), for which he won the prize of the Royal Academy of Science of Paris. In this solution, each body moves on a Keplerian orbit with a common plane, period, eccentricity, and focus that is located at their center of mass, in such a manner that at all times the relative positions of these bodies form the vertices of an equilateral triangle of variable size (see Fig. 1). This solution will be discussed in § 2. It is therefore interesting to consider the possible occurrence of such a 1 : 1 resonance in extrasolar planetary system with two Jupiter-size planets, particularly in view of the recent discovery of a remarkable 2 : 1 resonance of two such large planets in GJ 876 (Marcy et al. 2001). To be relevant to current astronomical discoveries, however, it is necessary that Lagrange's solution be stable in the range of observed masses and eccentricities. In the past, linear stability analyses have been carried out that were primarily focused on the restricted three-body problem (where one of the three masses vanishes), which is applicable to the study of the motion of asteroids, but these results can also be extended to the general Lagrange solution, as will be discussed in § 3. In this paper, we present the results of a numerical analysis of the nonlinear stability domain of Lagrange's solution as a function of the masses of the three bodies and the eccentricity of the common elliptical orbits (see § 4). In practice, there will be deviations from Lagrange's solution, and the interesting question arises as to whether there are distinct characteristics that would distinguish these deviations from other types of perturbations due to additional planets. While variations in the eccentricity and major axis of the approximate elliptical orbit are common to all perturbations caused by the presence of a second planet, one of the most distinguishing feature of a slightly off-resonance but stable Lagrange solution is that the rotation rate of the axis of the elliptic orbit of the star is much smaller than for other types of perturbations (see Fig. 2). For example, for a 1% deviation in the position of the lighter planet (Fig. 2, *red*), assumed to be about a Jupiter mass and half as large as the heavier planet (*green*), we find that for a solar-mass central star it takes about 800 periods to complete a revolution of the major axis. In contrast, a fit to the recently discovered 2 : 1 resonance in GJ 876 indicates that the major axis of the planets should complete a revolution in about 53 periods of the heavier planet (Nauenberg 2002).

Lagrange's solution is not the only 1 : 1 resonance for the three-body problem. We also consider another solution that we found to be stable in the domain of masses and eccentricities relevant to extrasolar planetary systems (G. Laughlin 2002, private communication). In this case, the two lighter bodies (planets) and the heaviest body (star) have different orbits, and therefore the Doppler shift data should be readily distinguishable from the case of a single planet. The characteristic feature of this solution is that at each half-period the three bodies are aligned, and if the heavier planet is in a nearly circular orbit, the lighter planet moves in a highly eccentric elliptical orbit (see Fig. 3). A periodic alignment of the two planets and the star is also a characteristic of other resonance solutions, as in the case of the 2 : 1 resonance in GJ 876 (Laughlin & Chambers 2001; Rivera & Lissauer 2001; Lee & Peale 2002; Nauenberg 2002). The evolution of this configuration when the planets are slightly off resonance is shown in Figure 4, which illustrates the clockwise rotation of the major axis of the eccentric orbit

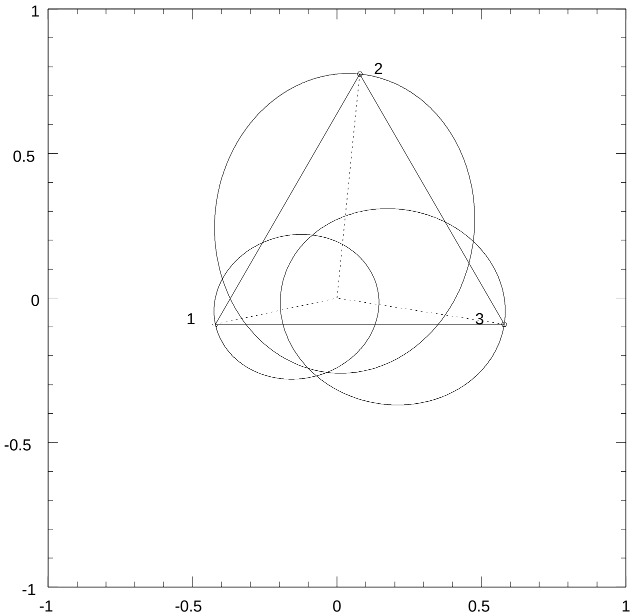

FIG. 1.—Lagrange's periodic 1 : 1 resonance solution for the three-body problem, showing the locations of the bodies at apogee on the vertices of an equilateral triangle (*solid lines*). The dashed lines show the direction of the major axis for each ellipse, and their intersection at the center of mass, which is their common focus.

of the lighter planet for the case in which $m_3/m_1 = 0.005$ and $m_2/m_1 = 0.001$.

2. LAGRANGE'S SOLUTION OF A THREE-BODY PROBLEM

In Lagrange's solution (Lagrange 1873) of a three-body problem, each body travels on a separate elliptic orbit with a common period, eccentricity, and focus that is located at their center of mass, in such a way that these bodies are always at the vertices of an equilateral triangle of variable size. An example is illustrated in Figure 1, where $m_2/m_1 = 0.2$ and $m_3/m_1 = 0.7$ and the common eccentricity is $\epsilon = 0.5$, showing the equilateral triangle for the relative positions of the three bodies at apogee. Apart from the over-

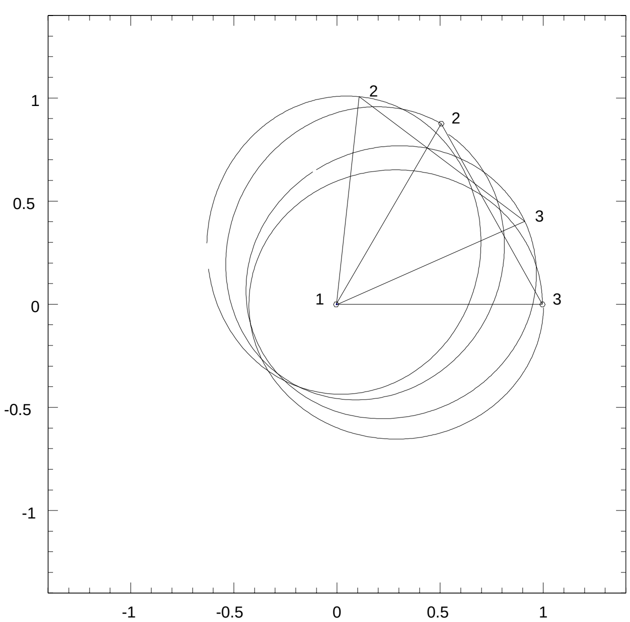

FIG. 2.—Rotation of the axes of the ellipses in Lagrange's solution after 800 periods, for a 1% deviation in the initial position of planet 2 from exact resonance. The two pairs of orbital curves for the lighter (*red*) and heavier (*green*) planet cover the first and last period.

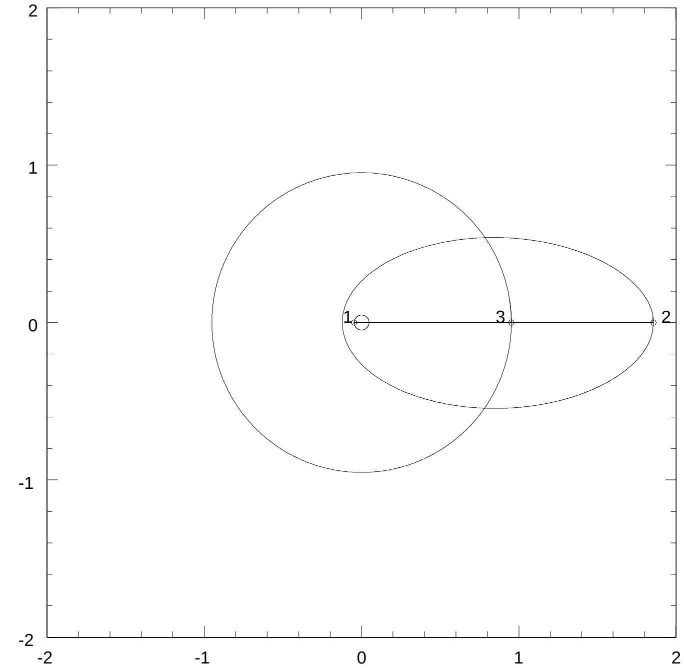

FIG. 3.—Another 1 : 1 resonance solution for the three-body problem, showing the inner heavier planet (*green*) on a nearly circular orbit, the lighter planet (*red*) on an eccentric elliptic orbit with eccentricity $\epsilon = 0.8$, and the central star (*blue*), as they appear aligned at maximum elongation of the ellipse.

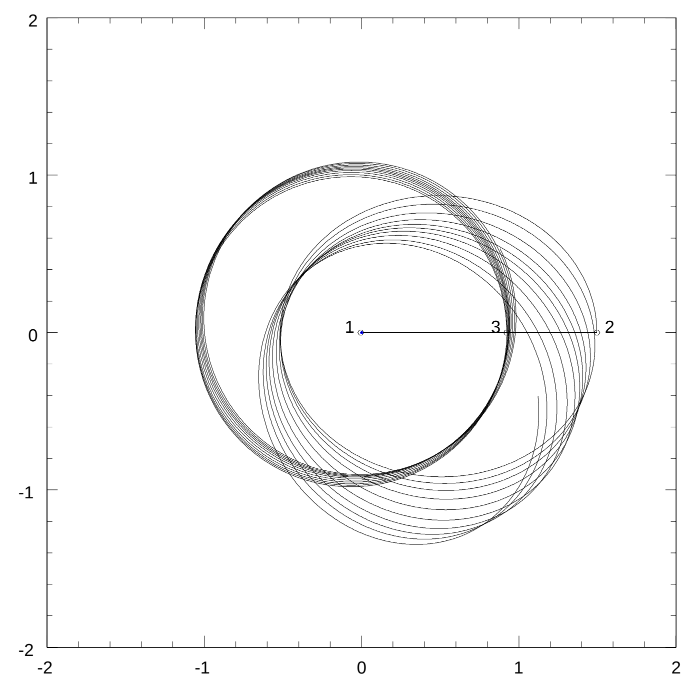

FIG. 4.—Evolution of the orbits shown in Fig. 3 for a small deviation from 1 : 1 resonance, showing the rotation of the major axes of the planets during 10 periods.

all orientation of the system, these parameters uniquely describe the solution. In this solution, the relation between variables is somewhat different from the corresponding ones in the two-body problem. For example, the common frequency ω or period $P = 2\pi/\omega$ of the motion is given by the relation

$$\omega = \sqrt{Gm(1+\epsilon)^3/R^3} \,, \tag{1}$$

where G is Newton's constant, $m = m_1 + m_2 + m_3$ is the total mass, m_1 is the mass of the heaviest body (star), m_2 and m_3 are the masses of the lighter bodies (planets), ϵ is the common eccentricity, and R is the maximum size of the equilateral triangle on which the three bodies are located. Then the major axis a_i of each of the elliptic orbits is

$$a_i = \frac{\sqrt{m_j^2 + m_k^2 + m_j m_k}}{m(1+\epsilon)} R \,, \tag{2}$$

where the subscripts i, j, and k are permutations of the integers 1, 2, and 3, while the mean of the velocity of the star at the maximum and minimum distance from the foci of the ellipse is

$$K = \left(\frac{2\pi G}{P}\right)^{1/3} \frac{\sqrt{m_2^2 + m_3^2 + m_2 m_3}}{m^{2/3}\sqrt{1-\epsilon^2}} \,. \tag{3}$$

If the planetary masses m_2 and m_3 are small compared with the mass of the star, m_1, then $a_2 \approx a_3 \approx a$, where $a = R/(1+\epsilon)$ and $\omega \approx (Gm/a^3)^{1/2}$ as in the corresponding two-body problem. In principle, any fit to the data with a single planet of mass m_p can also be attributed to two planets that, according to equation (3) for K, have masses satisfying the relation $m_p = (m_2^2 + m_3^2 + m_2 m_3)^{1/2}$, which is somewhat less than the sum of the masses of the two planets. In practice, however, it is unlikely that extrasolar planetary systems would occur in an exact 1 : 1 resonance, and therefore the presence of a second planet manifests itself in the occurrence of residuals in a single Keplerian orbit fit to the data.

3. LINEAR STABILITY ANALYSIS

The first linear stability analysis of Lagrange's solution was carried out by Routh for the special case of circular orbits (Routh 1875).[1] Assuming that the attractive forces between the bodies depends on the relative distance r as $1/r^\kappa$, Routh demonstrated that Lagrange's solution was stable provided that the masses satisfied the inequality

$$\gamma < \frac{1}{3}\left(\frac{3-\kappa}{1+\kappa}\right)^2 \,, \tag{4}$$

where

$$\gamma = \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} \,. \tag{5}$$

For gravitational interactions where $\kappa = 2$, the constant on the right-hand side[2] of equation (4) is $1/27$, which implies that the masses of the two lighter bodies (planets), m_2 and m_3, must be much smaller than the mass of the heaviest body (star), m_1. For example, setting $m_2 = 0$, which corresponds to the restricted three-body problem and applies to the motion of asteroids such as the Trojans, this inequality implies that $m_3/(m_1 + m_3) < 0.03852$, which has become known as Routh's critical point. Neglecting quadratic terms in the mass ratios m_2/m_1 and m_3/m_1, Routh's inequality becomes approximately $(m_2 + m_3)/m_1 < 1/27$, indicating that the stability depends to a very good approximation only on the sum of the masses of the lighter bodies relative to the mass of the heaviest one.

For the stable configurations, Routh obtained the frequencies ω_1, ω_2, and ω_3 of the normal modes in the plane of the orbits, which for $\kappa = 2$ are given by

$$\omega_1 = \omega\sqrt{\tfrac{1}{2}\left(1 + \sqrt{1-27\gamma}\right)} \,, \tag{6}$$

$$\omega_2 = \omega\sqrt{\tfrac{1}{2}\left(1 - \sqrt{1-27\gamma}\right)} \,, \tag{7}$$

$$\omega_3 = \omega \,, \tag{8}$$

where ω is the fundamental Kepler frequency (eq. [1]), and he determined the corresponding amplitudes for these modes. While Routh did not discuss the stability with respect to perturbations perpendicular to the plane of the orbit, it is straightforward to show that for this case the Lagrange orbits are linearly stable, and that these perturbations oscillate at the fundamental Kepler period for all mass ratios. Remarkably, he also briefly considered second-order deviations, which he remarked could "ultimately disturb the stability," but this part of his analysis was incomplete, calling attention only to two of the possible commensurability or resonance relations, $\omega_3 = 2\omega_2$ and $\omega_1 = 2\omega_2$, for which $\gamma = 1/36$ and $\gamma = 16/675$, respectively.

For elliptic orbits, a linear stability analysis of Lagrange's solution was not carried out until some 90 years later, when Danby (1964) numerically integrated the Floquet equations for the first-order deviations from the Lagrange solution of the restricted three-body problem (see also Bennett 1965). Subsequently, a majority of stability studies have been confined to this special case, but as we shall see, the results can also be applied to the general solution (Marchal 1990). For a modern discussion of the stability of the Lagrange solution, see Siegel & Moser (1971).

4. NONLINEAR STABILITY DOMAIN OF LAPLACE'S SOLUTION

The nonlinear stability domain of Lagrange's solution for the three-body problem that is presented here was obtained by integrating the equations of motion numerically and determining whether, for a small initial deviation from the solution, the orbits were either confined or unconfined after a large number, n, of periods. The criterion for an unconfined orbit was that the linear size of one of the planetary orbits become 3 times the initial size or larger. Changing this factor to 5 did not alter the results, and numerous spot checks of such orbits indicated that if the integration time were continued the orbital size would diverge further. We found that when n was increased from 400 to 800, there were no significant changes in the results except near the critical points discussed below, where we increased n until no further changes occurred. For the deviations, we consider small

[1] In this paper, Routh incorrectly attributed Lagrange's solution to Laplace.

[2] Routh found that he had been anticipated in this important result by someone called M. Gascheau in a thesis on mechanics.

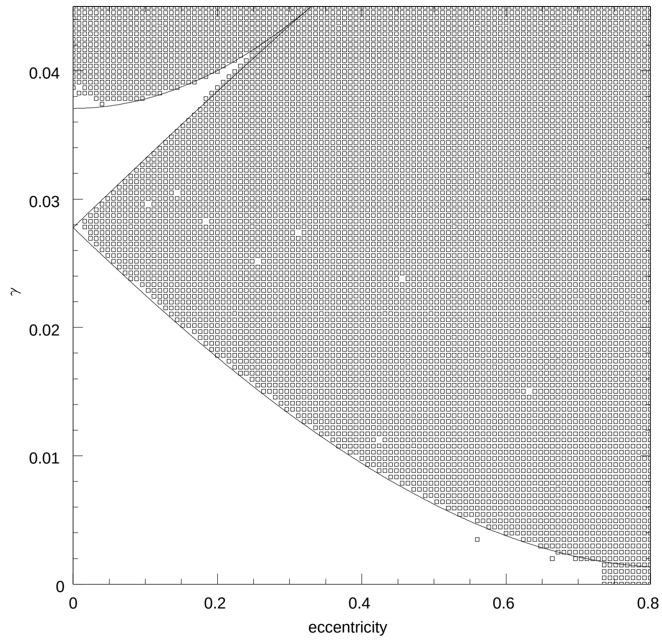

FIG. 5.—Stability domain for Lagrange's solution as a function of the eccentricity ϵ and the Routh parameter γ (eq. [5]), obtained for initial deviations perpendicular to the plane of the orbit.

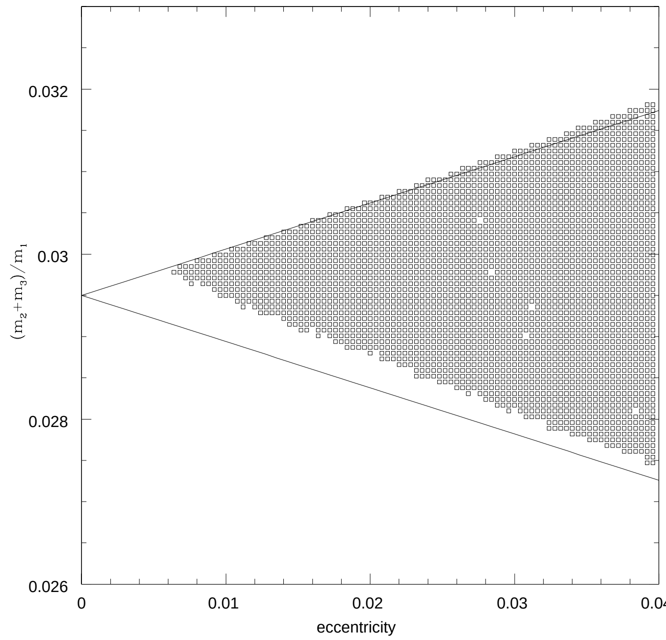

FIG. 6.—Enlargement of the stability domain shown in Fig. 5 near the critical point for the parametric resonance at $\omega_3 = 2\omega_2$.

displacements of the position of one of the planets in the plane of the orbit and also perpendicular to this plane. As we shall see, except in the case when $\omega_3 = 2\omega_2$, only initial deviations in the plane of the orbit gave evidence for some of the expected nonlinear instabilities near the commensurability relations for the three fundamental frequencies, equations (6)–(8).

Starting with a small deviation in the velocity of one of the lighter bodies of the order of 1% in the direction perpendicular to the plane of the orbits, our numerical result for the nonlinear stability domain is shown in Figure 5, where the unstable regions are indicated by small squares. In agreement with the linear stability analysis (see § 3), we found that the nonlinear stability domain depends only on the common eccentricity ϵ of the orbits (*horizontal axis*), and the Routh parameter γ, equation (5) (*vertical axis*). This parameter is determined by the ratios m_2/m_1 and m_3/m_1 of the masses of the lighter bodies (planets) to the mass m_1 of the heaviest body (star). We have verified this result by computing this domain for different fixed values of the ratio of the masses of the planets, without observing any changes when plotting the results with Routh's parameter γ. For example, the computation shown in Figure 5 was carried out for $m_2/m_3 = 1$, while the boundary curves shown are the linear stability computations of Danby and Bennett (Danby 1964; Bennett 1965), which were originally obtained for the restricted three-body problem where either m_2 or m_3 is set equal to zero.

This nonlinear stability domain looks surprisingly similar to the stability domain of the Mathieu equation (Grimshaw 1990, p. 62) with an additional nonlinear restoring or damping term. Indeed, for the restricted three-body problem, a linear stability analysis of the equations of motion, to first order in the common eccentricity ϵ, in a frame of reference rotating with the frequency of the orbits, yields a bifurcation at $\omega_3 = 2\omega_2$ which can be viewed as a parametric resonance between the fundamental orbital frequency for elliptic motion and the oscillation frequency of the first-order deviations of the massless body. In the general case, we have seen that this bifurcation occurs for $\epsilon = 0$ at $\gamma = 1/36 = 0.02777$. For the restricted problem, $m_2 = 0$, this corresponds to $\Delta_0 = 2\sqrt{2}/3 \approx 0.9428$, where $\Delta = (m_1 - m_3)/(m_1 + m_3)$ and $m_3/m_1 \approx 0.02944$. We evaluated the instability domain to first order in the eccentricity ϵ and found that it lies inside the wedge

$$\Delta_0 - \delta\epsilon \le \Delta \le \Delta_0 + \delta\epsilon\ , \tag{9}$$

where $\delta = 3/20\sqrt{2} \approx 0.10606$. This wedge is shown in Figure 6, together with the corresponding nonlinear domain where the orbit actually becomes unstable. These linear boundaries mark the onset of a bifurcation in which the orbits first become aperiodic, filling out a confined region of space. The vertical axis in this figure is shown as the variable $\delta = (m_2 + m_3)/m_1$, although this calculation was carried out for the case $m_2 = 0$. Inside the stable domain δ is approximately the same as the Routh parameter γ, equation (5) [the exact value of γ can be obtained from this figure by setting $\gamma = \delta/(1 + \delta^2)$].

An enlargement of the stability domain near the critical point at $\epsilon = 0$ and $\gamma = 1/27 \approx 0.037037$ is shown in Figure 7, which displays our results for the case that $m_2 = m_3$. This critical point is often discussed in connection with the stability of the Lagrangian points for the restricted three-body problem in a frame of reference rotating with the frequency of the circular orbits, and has been applied to the study of the Trojan and other asteroids (Murray & Dermott 1999, p. 77). As in the previous case, however, this critical point marks only a bifurcation to an aperiodic but confined motion, while numerically we find that the nonlinear instability begins at a somewhat higher value of $\gamma \approx 0.0391$, corresponding for $m_2 = 0$ to $m_3/m_1 \approx 0.0425$. A curve quadratic in the eccentricity fits well the boundary of the upper part of the nonlinear instability domain, as

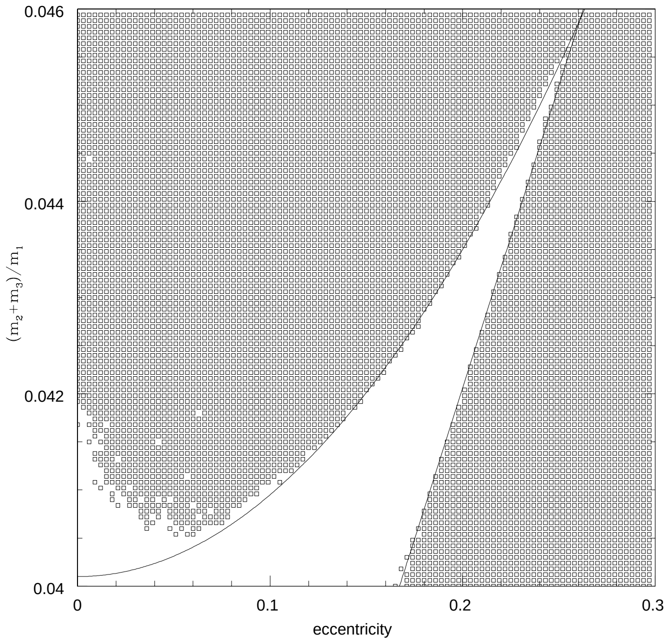

FIG. 7.—Enlargement of the stability domain shown in Fig. 5 near the critical point at $\omega_1 = \omega_2$.

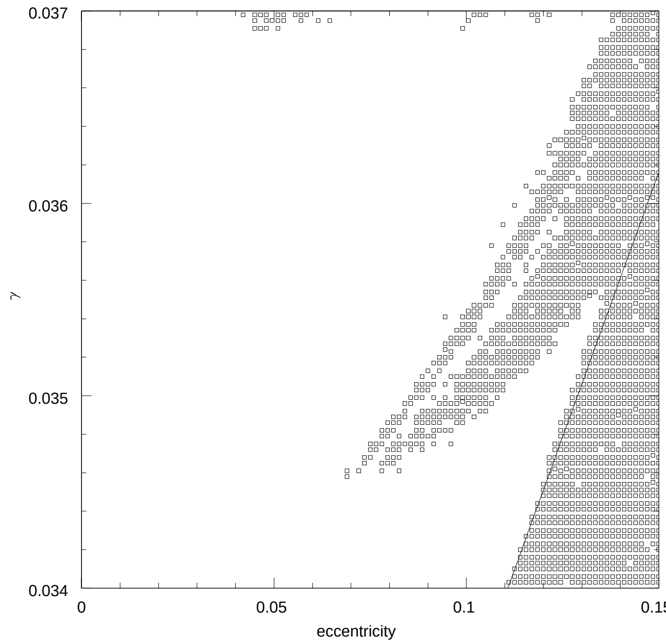

FIG. 9.—Enlargement of the stability domain for Lagrange's solution in the upper wedge shown in Fig. 8 as a function of the eccentricity ϵ and the Routh parameter γ (eq. [5]), obtained for initial deviations in the plane of the orbit.

shown in Figure 7, while a linear curve fits the lower part of the domain with a slope that is somewhat higher than the one that we calculated analytically in the linear approximation (eq. [9]), but in good agreement with the numerical linear stability results of Danby (1964) and Bennett (1965).

Up to now, we have considered only initial deviations in the direction perpendicular to the plane of Lagrange's orbits. In Figure 8, we show the nonlinear stability domain obtained by taking an initial displacement $dx = dy = 0.001$ in this plane for one of the two lighter bodies. This result was obtained by fixing the ratio $m_2/m_3 = 1$, but the same results are obtained for other values of the ratios of these masses when the results are plotted as a function of the Routh parameter γ (eq. [5]). We see evidence in this figure for the instabilities at the nonlinear resonances $\omega_1 = 2\omega_2$ and $\omega_3 = 3\omega_2$, corresponding to $\gamma = 16/675 \approx 0.02370$ and $\gamma = 32/2187 \approx 0.01463$. Also, the domain of stability shrinks in the upper wedge with indications of additional resonances in this region. This is confirmed by an enlargement of this region, which shows the tail of the instability due to the resonance at $3\omega_1 = 4\omega_2$ corresponding to $\gamma = 576/16875 \approx 0.03413$, as well as other resonances (see Fig. 9).

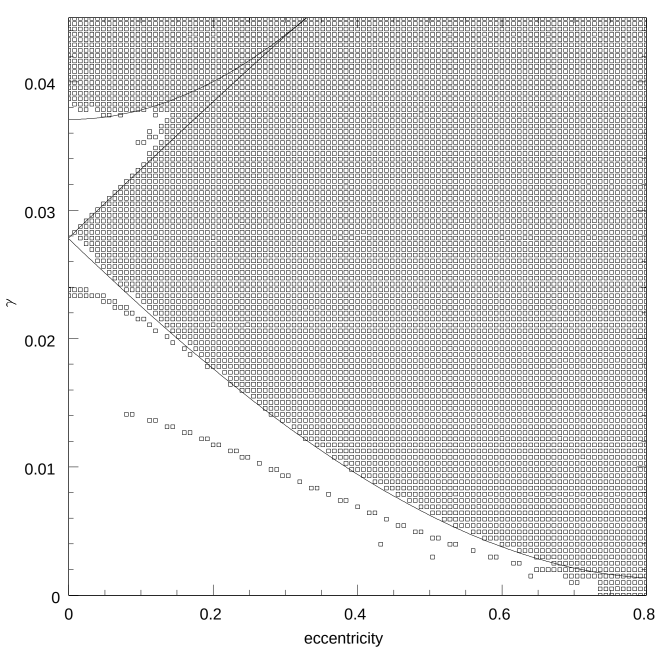

FIG. 8.—Stability domain for Lagrange's solution as a function of the eccentricity ϵ and the Routh parameter γ (eq. [5]), obtained for initial deviations in the plane of the orbit.

5. ANOTHER 1 : 1 RESONANCE SOLUTION

Lagrange's solution is not the only stable 1 : 1 resonance system for the three-body problem. Another solution is illustrated in Figure 3, in which the two lighter bodies or planets are traveling on two different orbits that are approximately elliptic in such a way that the central star and the planets are aligned when located at the maximum or minimum distance from the center of mass (G. Laughlin 2002, private communication). We found these orbits by a new method based on an expansion of the coordinates in a Fourier series with a common period, where the Fourier coefficients of this expansion were determined by finding the minima of the action integral with respect to these coefficients by an efficient iterative process (Nauenberg 2001). A characteristic feature of these orbits is that when the heavier of the two planets is in a nearly circular orbit, the lighter planet is in an elliptical orbit with a very large eccentricity, which rises slowly with increasing mass of the heavier planet. This is shown in Figure 10 for the case in which $m_2/m_1 = 0.001$.

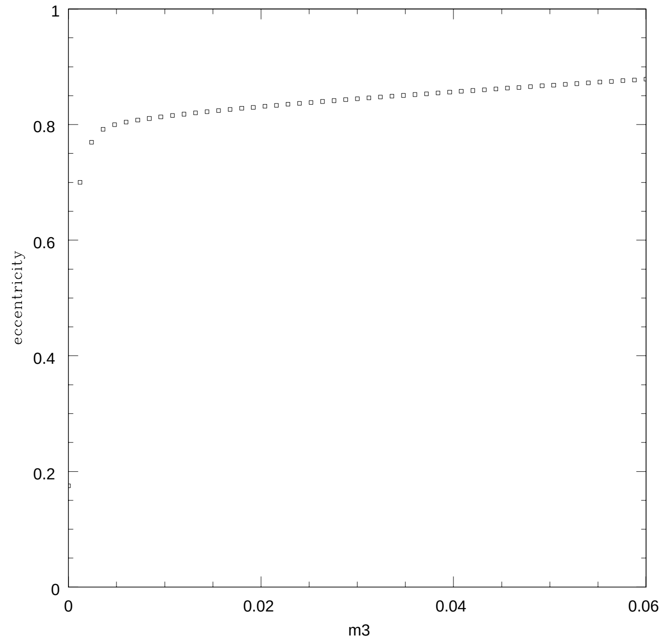

FIG. 10.—Dependence of the eccentricity of the lighter planet on the mass of the heavier planet for the 1 : 1 resonance solution shown in Fig. 3.

In contrast to Lagrange's solution discussed previously, a small deviation from exact 1 : 1 resonance in this case leads to a relatively rapid rotation of the major axis of the elliptical orbit, as illustrated in Figure 3. This leads to characteristic modulations in the Doppler shift oscillations of the light emitted by the star. An example is shown in Figure 11, where we fixed the velocity scale by assuming that the mass of the central star is one-third of a solar mass, and the period of the planetary orbit is 60 days. As in the case of Lagrange's solution discussed in the previous section, we found that these orbits are stable over a range of masses relevant to

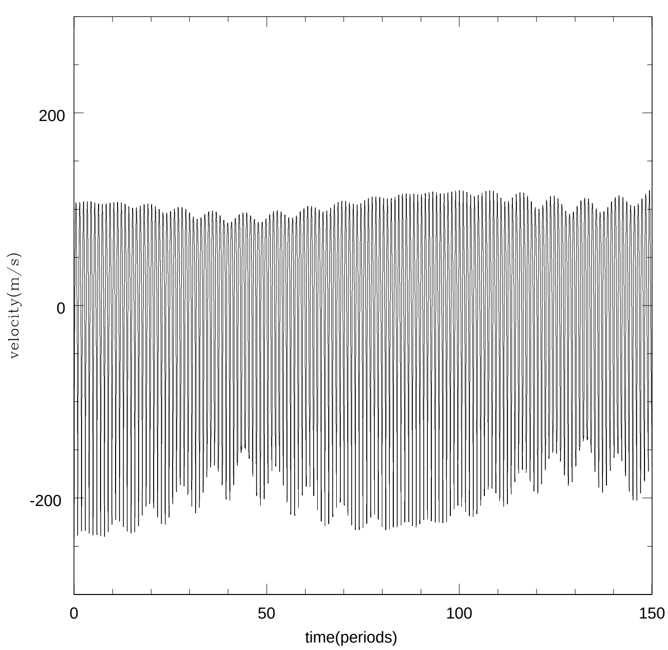

FIG. 11.—Doppler shift as a function of time for the 1 : 1 resonance configurations discussed in § 5.

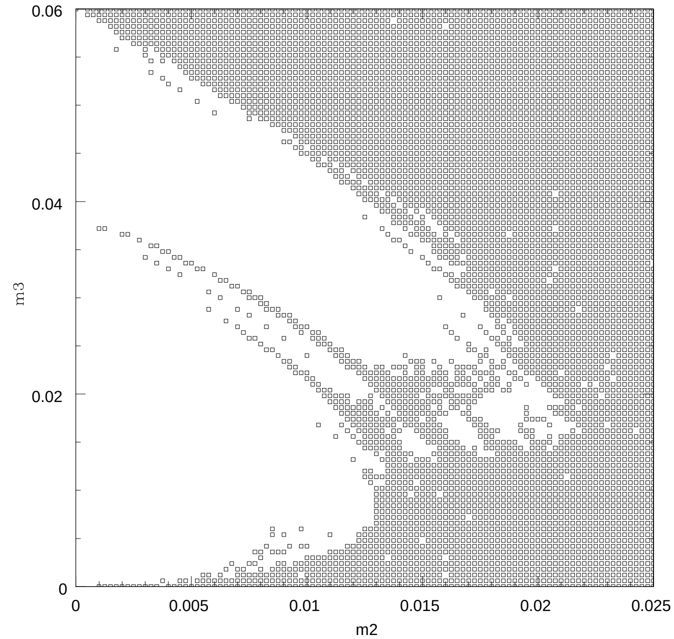

FIG. 12.—Stability domain for the 1 : 1 resonance configurations discussed in § 5.

extrasolar planetary systems (see Fig. 12). This stability domain was generated with initial 1 : 1 resonance configurations obtained by a general method to evaluate periodic orbits discussed in Nauenberg (2001). For the case in which the masses of the two planets are approximately equal and there is a small deviation from exact resonance, the planets exchange their orbital eccentricity each time the major axis rotates through 180°.

6. CONCLUSIONS

The nonlinear stability domain for Lagrange's solution of the three-body problem shown in Figure 8 indicates that there is a wide range of Jupiter-size planetary masses (including brown dwarfs) and eccentricities for which such solutions can exist in extrasolar planetary systems. For example, for an eccentricity of $\epsilon \approx 0.6$, the ratio of the total mass of the two planets to the mass of the star for which the solutions are stable is 0.004, except for a small region where nonlinear resonances occur. This mass correspond to 4.2 Jupiter masses for a 1 $M_\odot$ star, while for smaller eccentricities, $\epsilon \leq 0.2$, there is a wedge of stable solutions for higher mass ratios up to approximately 0.04. In principle, any Doppler shift data that can be fitted under the assumption that only a single planet is orbiting the central star can equally well be attributed to two planets orbiting the star according to Lagrange's solution. In practice, however, it is very unlikely that two planets are in an exact 1 : 1 resonance, and therefore one expects to find residuals in the data that signal the present of a second planet. One of the main effects due to the perturbations caused by a second planet is the secular rotation of the major axes of the approximate Keplerian ellipses that characterize the orbits of the planets and the star. This effect, however, would not occur in the case of an exact 1 : 1 Lagrangian resonance, and it is strongly suppressed in the case in which the resonance is approximate. It may be thought that in view of the greater number of degrees of freedom present in Lagrange's solution, a better least-squares fit to the data should be readily available. This is not the case, because Lagrange's equations also allow for unphysical solutions in which the mass of one or even both of the planets can have negative values, provided the total mass is positive. Indeed, in a preliminary attempt to obtain a least-squares fit to data which show residuals, the optimal mass of the smaller of two Lagrange planets turned out to be negative, which ruled out this solution.

The stability domain for the type of 1 : 1 resonance solution discussed in § 5 is shown in Figure 12, demonstrating that this solution also encompasses the possibility of two Jupiter-size planets orbiting a solar-size star. In this analysis, we restricted the heavier planet to be in a nearly circular orbit and found that the lighter planet is in a highly eccentric orbit with $\epsilon \approx 0.8$ (see Fig. 4). If this restriction is relaxed, we also find similar stable solutions, and for equal masses the two planets exchange eccentricity when the major axis rotates through 180°.

In summary, it is likely that extrasolar planetary systems that have several Jupiter-size planets that are close enough to give rise to significant gravitational perturbations will be in resonance, because numerical investigations have shown that such systems can be stable. In such cases, the planets and the central star are periodically aligned. An interesting exception is the 1 : 1 resonance solution of Lagrange, where the planets and the star are located at all times on the vertices of an equilateral triangle of varying size. It would be very exciting if this solution, discovered by Lagrange 230 years ago, and realized thus far only in the motion of the Trojan and other asteroids in our solar system (Murray & Dermott 1999, p. 77), were also present in the orbits of planets in extrasolar systems. Likewise, a search should also be undertaken to find two planets in extrasolar systems that are in a 1 : 1 resonance of the type discussed in § 5, which does not occur in our solar system.

Note added in manuscript.—After the completion of this paper, I received a preprint from G. Laughlin on his recent work with J. E. Chambers on the 1 : 1 resonances (Laughlin & Chambers 2002). I also learned that a search is underway at the Appalachian State University (Caton 2002) to find a Trojan planet in binary stars by observing the photometric signature of an eclipse near the Lagrangian equilateral points. The nonlinear stability domain evaluated here provides limits for the ratio of the two masses in a binary star that are suitable for such a search.

I would like to thank Richard Montgomery and Carles Simo for useful references to the vast literature on the Lagrange solution of the three-body problem, and Greg Laughlin for discussions.

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