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A COSMIC BATTERY RECONSIDERED

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ABSTRACT

We revisit the problem of magnetic field generation in accretion flows onto black holes owing to the excess radiation force on electrons. This excess force may arise from the Poynting-Robertson effect. Instead of a recent claim of the generation of dynamically important magnetic fields, we establish the validity of earlier results from 1977 that show that only small magnetic fields are generated. The radiative force causes the magnetic field to initially grow linearly with time. However, this linear growth holds for only a *restricted* time interval that is of the order of the accretion time of the matter. The large magnetic fields recently found result from the fact that the linear growth is unrestricted. A model of the Poynting-Robertson magnetic field generation close to the horizon of a Schwarzschild black hole is solved exactly using general relativity, and the field is also found to be dynamically insignificant. These weak magnetic fields may however be important as seed fields for dynamos.

Subject headings: accretion, accretion disks — galaxies: active — magnetic fields — plasmas — stars: magnetic fields — X-rays: stars

1. INTRODUCTION

The classical battery mechanism of magnetic field generation is connected with a noncoincidence of surfaces of constant pressure and constant density, where forces connected with pressure gradients become nonpotential or rotational. In this situation no static equilibrium in the gravitational field is possible. When considering separately the motion of electrons and ions, there is always a difference in the velocities of electrons and ions that creates electric currents and an associated magnetic field. Self-induction is very important in the battery mechanism, determining the rate of increase of the magnetic field.

Along with ion and electron pressure gradients, a nonpotential force field may arise as a result of the radiation force that acts predominantly on the electrons. In a spherically symmetric star, the radiation force has a potential so that no magnetic field is generated: equilibrium in the two-fluid plasma results from a distribution of an electric charge and a static radial electric field. For *geometrically thin, optically thick* accretion disks, Bisnovatyi-Kogan & Blinnikov (1977, hereafter BKB) showed that the radiation force above the disk has a nonpotential or rotational component. Under this condition, no electric charge distribution can give a static equilibrium. Instead, electric currents and a corresponding magnetic field are generated. The radiation forces above a thin disk give rise to poloidal electrical current flow and a toroidal magnetic field.

In accretion flows at very low mass accretion rates, an *optically thin*, *geometrically thick* accretion flow is possible (Shapiro, Lightman, & Eardley 1976) where the ion temperature is close to the virial temperature. In the absence of a magnetic field, and neglecting relaxation processes between electrons and ions except for binary collisions, these flows are referred to as advection-dominated accretion flows (ADAFs; Ichimaru 1977; Narayan & Yi 1995). In the ADAF regime the radiative efficiency of accretion may be very low, $\sim 10^3$ times less than the standard value for a geometrically thin, optically thick accretion disk. Account of processes connected with the presence of a magnetic field increases the efficiency up to at least $\frac{1}{3}$ of the standard value (Bisnovatyi-Kogan & Lovelace 1997, 2000, 2001). Nevertheless, the disk remains geometrically thick in the optically thin regime as a result of high ion temperature.

Contopoulos & Kazanas (1998, hereafter CK) proposed that a cosmic battery may operate in ADAFs owing to the Poynting-Robertson (PR) effect. The PR effect acts to generate a toroidal electrical current and poloidal magnetic field. The authors found that the magnetic field may be amplified up to $\sim 10^7$ G in the vicinity of a black hole of stellar mass. Note that the PR mechanism of magnetic field generation is similar to the mechanism of BKB based on the nonpotential radiative force, where the magnetic field reached values of $\lesssim 10$ G for a stellar mass black hole. In an optically thin disk both mechanisms act together, leading to the generation of toroidal and poloidal components of the magnetic field. The influence of the PR effect on the dynamics of the surface layer of an accretion disk was treated by Mott & Lovelace (1999).

Here we analyze the difference in conclusions between CK and BKB. The radiative force initially causes the magnetic field to grow linearly with time. However, this linear growth holds for only a *restricted* time interval that is of the order of the accretion time of the matter. In CK the interval of linear growth is unrestricted. Even though we conclude that the magnetic field due to the radiation force is weak, it may have a role as a seed field for an α-ω dynamo (see, e.g., Brandenburg et al. 1995; Colgate, Li, & Pariev 2001).

Section 2 treats the generation of toroidal field for the case of a thin disk, while § 3 treats the generation of poloidal field in an ADAF. Section 4 gives a general relativistic treatment of a simplified model of PR magnetic field generation in an accretion flow close to a Schwarzschild black hole. Appendix A gives an explicit solution to the nonrelativistic induction equation for the magnetic field generated by the PR effect in an ADAF. Appendix B gives an alternative form of the relative MHD equations.

2. RADIATIVELY INDUCED CURRENT AND TOROIDAL MAGNETIC FIELD PRODUCTION IN ACCRETION DISKS

Above a geometrically thin accretion disk around a black hole, the electrons are acted on by a nonpotential radiation force $\boldsymbol{F}_L$ due to Thomson scattering. This was calculated by BKB,

$$\boldsymbol{F}_L = -R\cos\theta\nabla\phi_L = (F_{LR}, F_{L\theta}, 0) , \tag{1}$$

where a spherical coordinate system (R, θ, ϕ) is used, and ϕ_L is the " pseudopotential " of the radiation force, which may be expressed as

$$\phi_L = \frac{\sigma_T}{c}\int_{r_{\rm in}}^{\infty}\frac{H(r)r\,dr}{(R^4 + r^4 + 2R^2r^2\cos 2\theta)^{1/2}} . \tag{2}$$

Here the disk thickness is neglected and the cylindrical radius is $r = R\cos\theta$. The function $H(r)$ is the radiative flux emitted per unit area from one side of the disk. In the standard local accretion model,

$$H(r) = \frac{3}{8\pi}\frac{GM\dot{M}}{r^3}\mathscr{J} , \tag{3}$$

where $\mathscr{J} \equiv 1 - (r_{\rm in}/r)^{1/2}$, and $r_{\rm in} = 3r_{\rm S} = 6GM/c^2$ is the inner radius of the disk for a nonrotating black hole of Schwarzschild radius $r_{\rm S}$. In the disk plane, $\theta = \pi/2$, the radiative force is perpendicular to the disk,

$$F_{L\theta} = -\frac{\sigma_{\rm T}}{c}H(r) = \frac{3GMm_p}{r^2}\frac{r_{\rm in}}{r}\frac{L}{L_{\rm Edd}}\mathscr{J} , \tag{4}$$

where

$$L = \frac{GM\dot{M}}{2r_{\rm in}}, \quad L_{\rm Edd} = \frac{4\pi cGMm_p}{\sigma_{\rm T}} , \tag{5}$$

and $\sigma_{\rm T}$ is the Thomson cross section. Because of the interaction of the radiation flux mainly with the electrons, the accretion disk becomes positively charged up to a value where the electrostatic attraction of the electrons balances the radiation force. The vertical component of the electrical field strength E_θ in the disk plane is written as

$$E_\theta(r) = -\frac{\sigma_{\rm T}}{c|e|}H(r) = -E_0\frac{L}{L_{\rm Edd}}\left(\frac{r_{\rm in}}{r}\right)^3\mathscr{J} , \tag{6}$$

where

$$E_0(M) \equiv \frac{m_pc^4}{12|e|GM} \approx 1.76\frac{M_\odot}{M}\ {\rm cgs} \approx 528\frac{M_\odot}{M}\frac{\rm V}{\rm cm} . \tag{7}$$

Thus, the surface charge density of the disk is

$$\Sigma_e(r) = \frac{E_\theta(r)}{2\pi} .$$

The influence of this charge on the structure and stability of the accretion disk is negligible.

Both the gravitational and electrical forces have a potential, so that they cannot balance the nonpotential radiation force. Because of the radiation and electric forces, electrons move with respect to protons, which to a first approximation do not acquire the poloidal motion. Thus, a poloidal electrical current is generated with an associated toroidal magnetic field. The finite disk thickness may create poloidal motion of all of the matter of the accretion disk, similar to meridional circulation in rotating stars (Kippenhahn & Thomas 1982). The absence of this circulation occurs for a unique dependence of the rotational velocity over the disk thickness, $\Omega(z)$.

To estimate the magnetic field strength, we write the electromotive force (EMF) as

$$\mathscr{E} = \frac{1}{e}\oint \boldsymbol{F}_L\times d\boldsymbol{l} = \frac{1}{e}\iint d\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{F}_L \sim E_\theta h , \tag{8}$$

where h is the half-thickness of the disk. Thus, the stationary current density is

$$J_{\rm st} \sim \frac{\sigma_e\mathscr{E}}{r} \sim \frac{\sigma_eE_\theta h}{r} , \tag{9}$$

where σ_e is the conductivity of the disk plasma. The stationary state results from a balance of the driving radiation force against the ohmic diffusion of the magnetic field.

The stationary toroidal magnetic field (BKB) is

$$B_{\phi0} \sim \frac{4\pi}{c}J_{\rm st}h \sim \frac{4\pi\sigma_e}{c}\frac{E_\theta h^2}{r} . \tag{10}$$

In the radiation-dominated inner region of the standard α-disk model, h can be written as

$$h = 3\frac{L}{L_{\rm Edd}}\mathscr{J}r_{\rm in} \tag{11}$$

(Shakura 1972; Shakura & Sunyaev 1973). Finally, we obtain the stationary value of toroidal magnetic field in the disk,

$$\begin{aligned} B_{\phi0} &\sim \frac{36\pi\sigma_e}{c}E_0\left(\frac{L}{L_{\rm Edd}}\right)^3\left(\frac{r_{\rm in}}{r}\right)^4 r_{\rm in}\mathscr{J}^3 \\ &= \frac{12\pi\sigma_e h}{c}E_0\left(\frac{L}{L_{\rm Edd}}\right)^2\left(\frac{r_{\rm in}}{r}\right)^4\mathscr{J}^2 . \end{aligned} \tag{12}$$

We next discuss the value of conductivity σ_e.

BKB considered different values of the plasma conductivity σ_e, namely, the conductivity owing to binary collisions $\sigma_{\rm Coul}$ and the effective conductivity $\sigma_{\rm eff}$ derived by Vainshtein (1971),

$$\sigma_{\rm Coul} \approx 3\times 10^6 T^{3/2}\ {\rm s}^{-1}, \quad \sigma_{\rm eff} = \frac{\sigma_{\rm Coul}}{\sqrt{{\rm Re}_{m0}}} . \tag{13}$$

The magnetic Reynolds number in a turbulent plasma is defined as

$${\rm Re}_{m0} \equiv \frac{4\pi\sigma_{\rm Coul}v_t h}{c^2} , \tag{14}$$

where the turbulent velocity v_t in an α-disk is $v_t = \alpha c_s$, where $c_s = (p/\rho)^{1/2}$.

In addition to the values of equation (13), we consider the conductivity in the presence of well-developed turbulence (Bisnovatyi-Kogan & Ruzmaikin 1976),

$$\sigma_{\rm turb} = \frac{c^2}{4\pi\alpha h c_s} = \frac{\sigma_{\rm Coul}}{{\rm Re}_{m0}} \ . \tag{15}$$

Estimations of the stationary field $B_{\phi 0}$ of equation (12) in the radiation-dominated inner region of the disk around a stellar mass black hole give values (BKB) for two cases in equation (13) of $B_{\phi 0} \sim 10^{13}$ G for $\sigma_e = \sigma_{\rm Coul}$ and $B_{\phi 0} \sim 10^8$ G for $\sigma_e = \sigma_{\rm eff}$, with ${\rm Re}_{m0} \sim 3 \times 10^{10}$. For a turbulent conductivity $\sigma_e = \sigma_{\rm turb}$, we obtain $B_{\phi 0} \sim 10$ G for $\alpha = 0.1$.

The timescale τ_m for reaching the stationary field given by equation (12) is determined by the self-induction of the disk. This is equivalent to the "$\mathscr{L}$ over $\mathscr{R}$ time" of circuit with inductance $\mathscr{L}$ and resistance $\mathscr{R}$. This timescale is equal to

$$\tau_m \simeq \frac{4\pi\sigma_e h r}{c^2} \ . \tag{16}$$

The crossing time of matter passing through the radiation-dominated region of the disk is

$$t_c \approx \frac{r}{v_r} \ . \tag{17}$$

During the time t_c there is linear growth of the magnetic field after which the matter falls into the black hole. Thus, the stationary value of the large-scale toroidal magnetic field is

$$B_\phi \approx B_{\phi 0}\frac{t_c}{\tau_m} = 3\frac{c}{v_r}E_0\left(\frac{L}{L_{\rm Edd}}\right)^2\left(\frac{r_{\rm in}}{r}\right)^4\mathscr{J}^2 \ . \tag{18}$$

For the case of a turbulent conductivity $\sigma_e = \sigma_{\rm turb}$, the growth timescale of the magnetic field, using equations (15) and (16), is equal to

$$\tau_m^{\rm turb} \approx \frac{r}{\alpha c_s} \ . \tag{19}$$

Taking into account that $v_r = \alpha c_s (L/L_{\rm Edd})(r_{\rm in}/r) < \alpha c_s$ and $t_m^{\rm turb} < t_c$, we find

$$B_\phi = B_{\phi 0}^{\rm turb} \approx 3\frac{c}{\alpha c_s}E_0\left(\frac{L}{L_{\rm Edd}}\right)\left(\frac{r_{\rm in}}{r}\right)^3\mathscr{J}^2 \ , \tag{20}$$

where

$$c_s = \left(7 \times 10^9 \ {\rm cm \ s^{-1}}\right)\left(\frac{L}{L_{\rm Edd}}\right)\left(\frac{r_{\rm in}}{r}\right)^{3/2}\mathscr{J} \ .$$

The strength of the stationary toroidal magnetic field produced by the battery effect in the radiation-dominated region of an accretion disk with the turbulent or higher conductivity from equation (13) is equal to

$$B_\phi \approx \frac{22}{\alpha}\left(\frac{M_\odot}{M}\right)\left(\frac{r_{\rm in}}{r}\right)^{3/2}\mathscr{J} \ . \tag{21}$$

At a distance $r = 3r_{\rm in}$, we have

$$B_\phi \approx \frac{2}{\alpha}\left(\frac{M_\odot}{M}\right) \ {\rm G} \ . \tag{22}$$

This agrees with the findings of BKB. The corresponding magnetic energy density is very much less than the energy density associated with the turbulent motion in the disk $\rho v_t^2/2$.

3. PRODUCTION OF A POLOIDAL MAGNETIC FIELD IN OPTICALLY THIN ACCRETION FLOWS BY THE POYNTING-ROBERTSON EFFECT

In optically thin accretion flows (Shapiro et al. 1976; Ichimaru 1977; Narayan & Yi 1995), the radiation flux interacts with the in-spiraling matter by the PR effect (Robertson 1937). Analysis by Shakura (1972) showed that the PR effect was negligible for optically thick accretion disks. CK studied the PR effect as a mechanism for generating poloidal magnetic field in an optically thin accretion flow. They concluded that dynamically important magnetic field strengths could result from this effect. Here we reconsider the PR effect for quasi-spherical ADAFs (Narayan & Yi 1995).

The linear growth of the magnetic field due to the radiative force on the electrons found by CK is similar to that analyzed by BKB, but the PR effect implies an additional (small) numerator, (v_ϕ/c). In addition, for a quasi-spherical accretion flow the characteristic scale is r instead of h, and the quasi-spherical luminosity is $L/(4\pi r^2)$ instead of H in equation (3). Then, using equations (10), (16), and (18), we obtain the rate of growth of the poloidal magnetic field due to the PR effect, which is equivalent to the expression obtained by CK,

$$B_z \approx B_{z0}\frac{t}{\tau_m} = \frac{E_0}{3\alpha}\left(\frac{L}{L_{\rm Edd}}\right)\left(\frac{r_{\rm in}}{r}\right)^2\left(\frac{t v_r}{r}\right) \ . \tag{23}$$

Here E_0 is defined in equation (6) and $r_{\rm in}$ in equation (3).

Now it is essential to take into account that an element of matter with the induced magnetic field reaches the black hole in time $t_c \approx r/v_r$. (The magnetic field behavior near the black hole horizon is discussed in § 4.) This means that the magnetic field grows *only* during the time t_c. Consequently, the magnetic field reaches a maximum value

$$B_z \approx \frac{E_0}{3\alpha}\frac{L}{L_{\rm Edd}}\left(\frac{r_{\rm in}}{r}\right)^2 \approx \frac{0.7}{\alpha}\frac{L}{L_{\rm Edd}}\frac{M_\odot}{M}\left(\frac{r_{\rm in}}{r}\right)^2 \ . \tag{24}$$

Taking into account that for optically thin accretion the luminosity is $\lesssim 10^{-3}L_{\rm Edd}$, and taking $\alpha = 0.1$, we get a maximum value of the magnetic field created by the PR effect in an ADAF to a black hole,

$$B_z \approx 7 \times 10^{-3}\left(\frac{M_\odot}{M}\right) \ {\rm G} \ . \tag{25}$$

This estimate of the field is about 10 orders of magnitude less than the value obtained by CK. The difference in the estimates results from the fact that CK assume that magnetic flux accumulates continuously near the black hole during a long time, reaching the equipartition with the kinetic energy. The accumulation actually occurs only during the time the plasma (which carries the field or current loops) takes to move inward to the black hole horizon (Bisnovatyi-Kogan & Ruzmaikin 1976). The current loops created by the PR effect disappear as the matter approaches the horizon (see § 4). In the case of accretion onto a neutron star or a white dwarf, matter containing the current loops merges with the stellar matter, which is typically much more strongly magnetized. After merging, the matter becomes optically thick, the action of PR effect stops, penetration of matter into the magnetosphere of the star occurs, and interaction of the accretion flux with the stellar surface takes place.

4. MAGNETIC FIELD GENERATION IN THE VICINITY OF THE SCHWARZSCHILD RADIUS

Here we consider the PR magnetic field generation on the accretion flow near a Schwarzschild black hole. The accreting matter is assumed to move radially toward the black hole with velocity v_r. As before we consider the case of high-conductivity matter where $t_m \gg t_c$. In a nonaccreting flow, the magnetic field can grow linearly with time as accepted by CK. As mentioned above, there is an important relativistic effect close to the black hole: current loops in the accreting matter that approach the horizon of a black hole cannot produce a magnetic field visible by an external observer. This effect is related to the damping of the magnetic field in a collapsing star (Ginzburg & Ozernoi 1964).

The azimuthal force and the corresponding azimuthal electric field due to the PR effect are

$$F_{\rm PR} = \frac{L\sigma_{\rm T}}{4\pi c r^2}\frac{v_{\phi 0}}{c}\sin\theta, \quad E_\phi^{(\rm ph)} = \frac{1}{|e|}F_\phi^{\rm PR}, \tag{26}$$

where

$$v_{\phi 0} = A\sqrt{\frac{GM}{r}}, \quad A \sim 1 , \tag{27}$$

and where the (ph) superscript indicates the physical value of the field component. Note that in a strictly spherical accretion flow there is no azimuthal EMF. However, in the approximate model considered here the infalling matter rotates relatively slowly so that the PR force affects the radial inflow only slightly.

We assume a Schwarzschild metric,

$$ds^2 = g_{00}c^2\,dt^2 + g_{11}\,dr^2 + g_{22}\,d\theta^2 + g_{33}\,d\phi^2 , \tag{28}$$

where

$$g_{00} = \left(1 - \frac{r_g}{r}\right), \quad g_{11} = -\left(1 - \frac{r_g}{r}\right)^{-1} ,$$

$$g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta ,$$

and $r_g \equiv 2GM/c^2$. The matter is free-falling in the radial direction with the nonzero components of a four-velocity

$$u^0 = \left(1 - \frac{r_g}{r}\right)^{-1}, \quad u_0 = 1 ,$$

$$u^r = -\sqrt{\frac{r_g}{r}}, \quad u_r = \sqrt{\frac{r_g}{r}}\left(1 - \frac{r_g}{r}\right)^{-1} . \tag{29}$$

In any four-dimensional spacetime with metric g_{ik} (Latin indices takes values 0, 1, 2, 3), the electric E_α and magnetic B^α fields (Greek indices run over the values 1, 2, 3) in three-space are defined through the antisymmetric electromagnetic field tensor $F_{ik} = -F_{ki}$, with zero diagonal components (Lichnerowicz 1967; Landau & Lifshitz 1988),

$$B^\alpha = -\frac{1}{2\sqrt{\gamma}}\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma}, \quad E_\alpha = F_{0\alpha} ,$$

$$F_{\alpha\beta} = -\sqrt{\gamma}\varepsilon_{\alpha\beta\gamma}B^\gamma , \tag{30}$$

where $\varepsilon_{\alpha\beta\gamma} \equiv \varepsilon^{\alpha\beta\gamma}$ is the three-dimensional Levi-Civitta tensor ($\varepsilon_{123} = 1$) and γ is the determinant of the three-dimensional metric tensor, obtained by splitting the metric tensor g_{ik} into space ($\gamma_{\alpha\beta}$) and time (h) parts as

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}, \quad h = g_{00} . \tag{31}$$

For the Schwarzschild metric (eq. [28]), $\gamma_{\alpha\beta} = -g_{\alpha\beta}$. The first pair of Maxwell's equations can be written as

$$F_{ik;l} + F_{li;k} + F_{kl;i} = 0$$

or

$$\frac{1}{c}\frac{\partial(\sqrt{\gamma}B^\alpha)}{\partial t} + \varepsilon^{\alpha\beta\gamma}\frac{\partial E_\gamma}{\partial x^\beta} = 0, \quad \frac{\partial(\sqrt{\gamma}B^\alpha)}{\partial x^\alpha} = 0 . \tag{32}$$

Note that the physical r and θ components of the magnetic field in this reference frame are $(-g_{11})^{1/2}B^r$ and $(-g_{22})^{1/2}B^\theta$, respectively. Thus, the dimensions of B^θ are length times B^r.

In a perfectly conducting medium moving with four-velocity u^i, we have $F_{ik}u^k = 0$, which corresponds to a vanishing electric field in the comoving frame. This gives

$$E_\alpha = -\sqrt{-g}\varepsilon_{\alpha\beta\gamma}\left(\frac{v^\beta}{c}\right)B^\gamma . \tag{33}$$

In the presence of an externally imposed electric field $E_\alpha^{\rm ext}$, the electrical field E_α is

$$E_\alpha = -\sqrt{-g}\varepsilon_{\alpha\beta\gamma}\left(\frac{v^\beta}{c}\right)B^\gamma - E_\alpha^{\rm ext} . \tag{34}$$

The three-velocities v^α are given by

$$v^\alpha = \frac{c\,dx^\alpha}{\sqrt{h}\,dx^0}, \quad dx^0 = c\,dt , \tag{35}$$

$$u^\alpha = \frac{v^\alpha}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} , \tag{36}$$

where

$$u^0 = \frac{1}{\sqrt{h}}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad v^2 = \gamma_{\alpha\beta}v^\alpha v^\alpha = v_\alpha v^\alpha .$$

Substituting equation (34) into equation (32) gives the following equation for the magnetic field B^α:

$$\frac{\partial(\sqrt{\gamma}B^\alpha)}{\partial t} = \frac{\partial}{\partial x^\beta}\left[\sqrt{-g}\left(B^\beta v^\alpha - B^\alpha v^\beta\right)\right] + \varepsilon^{\alpha\beta\gamma}\frac{\partial E_\gamma^{\rm ext}}{\partial x^\beta}c , \tag{37}$$

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}B^\alpha\right) = 0 . \tag{38}$$

For the Schwarzschild metric (eq. [28]) $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$, it follows from equations (29), (35), and (36) that $v^\alpha = (v^r,\ 0,\ 0)$ and

$$v^r = c\sqrt{1 - \frac{r_g}{r}}u^r = -c\sqrt{1 - \frac{r_g}{r}}\sqrt{\frac{r_g}{r}} , \tag{39}$$

where $v^2/c^2 = r_g/r$. The value of h, the determinant g of the four-metric tensor g_{ik}, and the determinant γ of the metric tensor $\gamma_{\alpha\beta}$ in the Schwarzschild metric are

$$h = 1 - \frac{r_g}{r}, \quad \sqrt{-g} = r^2\sin\theta,$$

$$\sqrt{\gamma} = \left(1 - \frac{r_g}{r}\right)^{-1/2} r^2\sin\theta . \tag{40}$$

With $B^\alpha = (B^r, B^\theta, 0)$ and $E^{\rm ext}_\alpha = (0, 0, E^{\rm ext}_\phi)$, and with all quantities independent of the azimuthal angle ϕ, equations (37) and (38) give

$$\frac{\partial(\sqrt{\gamma}B^r)}{\partial t} + \sqrt{h}v^r \frac{\partial(\sqrt{\gamma}B^r)}{\partial r} = c\frac{\partial E^{\rm ext}_\phi}{\partial\theta} , \quad (41)$$

$$\frac{\partial\left(\sqrt{-g}v^r B^\theta\right)}{\partial t} + \sqrt{h}v^r \frac{\partial\left(\sqrt{-g}v^r B^\theta\right)}{\partial r} = -cv^r\sqrt{h}\frac{\partial E^{\rm ext}_\phi}{\partial r} , \quad (42)$$

$$\frac{\partial}{\partial r}\left(\sqrt{\gamma}B^r\right) + \frac{\partial}{\partial\theta}\left(\sqrt{\gamma}B^\theta\right) = 0 . \quad (43)$$

Equations (41) and (42) with known right-hand sides are solved using the method of characteristics.

The integrals of the equations for the characteristics can be written as

$$t - \int_{r_0}^r \frac{dr}{\sqrt{h}v^r} = C_1 , \quad (44)$$

$$\sqrt{\gamma}B^r - c\int_{r_0}^r \frac{\partial E^{\rm ext}_\phi}{\partial\theta}\frac{dr}{v^r\sqrt{h}} = C_2 , \quad (45)$$

$$\sqrt{-g}v^r B^\theta + c\int_{r_0}^r \frac{\partial E^{\rm ext}_\phi}{\partial r}dr = C_3 . \quad (46)$$

The constants C_i are determined by the initial conditions. For the present problem these are

$$B^r = B^\theta = 0, \quad r = r_0 \quad \text{at } t = 0 ,$$

which implies $C_i = 0$. In the general case the constants C_2, C_3 are determined by the initial values of B^r, B^θ, which should satisfy the zero divergence condition given by equation (43). We may in general take $C_1 = 0$, fixing the reference frame $r = r_0$ at $t = 0$. With account of equations (39) and (40), equations (45) and (46) can be written as

$$B^r = -\frac{\sqrt{r-r_g}}{r^{5/2}\sqrt{r_g}\sin\theta}\frac{\partial}{\partial\theta}\left(\int_{r_0}^r E^{\rm ext}_\phi \frac{r^{3/2}\,dr}{r-r_g}\right) , \quad (47)$$

$$B^\theta = \frac{1}{r\sqrt{r-r_g}\sqrt{r_g}\sin\theta}\left[E^{\rm ext}_\phi(r,\theta) - E^{\rm ext}_\phi(r_0,\theta)\right] . \quad (48)$$

Carrying out the integration in equation (44) with $C_1 = 0$ gives a relation between t, r, and r_0 in the form

$$\frac{ct}{r_g} + \frac{2}{3}x^{3/2} + 2x^{1/2} + \ln\frac{\sqrt{x}-1}{\sqrt{x}+1} = \frac{2}{3}x_0^{3/2} + 2x_0^{1/2} + \ln\frac{\sqrt{x_0}-1}{\sqrt{x_0}+1} , \quad (49)$$

where $x = r/r_g$ and $x_0 = r_0/r_g$ (Bisnovatyi-Kogan & Ruzmaikin 1974).

Consider now the PR EMF, $E^{\rm ext}_\phi(r,\theta)$. First we show that $E^{\rm ext}_\phi$ must tend to zero as $r \to r_g$ at least as fast as $(r - r_g)$ in order to avoid singularity at r_g. This means that in the comoving reference system with metric

$$ds^2 = c^2\,d\tau^2 - \frac{r_g}{r}d\rho^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) , \quad (50)$$

there is no singularity at the black hole horizon. The connection between Schwarzschild and comoving coordinates (τ, ρ) (the angle coordinates θ and ϕ are the same) is

$$c\tau = ct + r_g\left(2\sqrt{x} + \ln\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) ,$$

$$\rho = ct + r_g\left(\frac{2}{3}x^{3/2} + 2x^{1/2} + \ln\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) . \quad (51)$$

We can now connect the magnetic field in a comoving system $\acute{B}^\alpha = (\acute{B}^\rho, \acute{B}^\theta, 0)$ with the field in the Schwarzschild system in terms of Schwarzschild variables (r, t) as

$$\acute{B}^\rho = \frac{r}{\sqrt{r-r_g}\sqrt{r_g}}B^r ,$$

$$\acute{B}^\theta = \frac{\sqrt{r-r_g}}{\sqrt{r}}B^\theta + \sqrt{\frac{r_g}{r}}\frac{E^{\rm ext}_\phi}{r(r-r_g)\sin\theta} . \quad (52)$$

It follows from equations (47), (48), and (52) that there is no singularity in the comoving frame if $E^{\rm ext}_\phi$ tends to zero as $(r - r_g)$ or faster as $r \to r_g$. The metric tensor in this system (eq. [50]) is regular on the horizon, so with finite $\acute{B}^\alpha$ all four-invariants (e.g., $F_{ik}F^{ik}$) are also regular there. In fact, we can obtain the dependence of $E^{\rm ext}_\phi$ from equation (26), taking into account that the covariant component $E^{\rm ext}_\phi$ in equations (41) and (42) is connected with the physical component $E^{\rm (ph)}_\phi$ from equation (26) as

$$E^{\rm ext}_\phi = \sqrt{\gamma_{\phi\phi}}E^{\rm (ph)}_\phi = r\sin\theta E^{\rm (ph)}_\phi . \quad (53)$$

The luminosity L seen by a distant observer viewing collapsing matter with constant comoving luminosity L_0 is

$$L = L_0\left(1 - \frac{r_g}{r}\right)^4 \quad (54)$$

(Zeldovich & Novikov 1971). Thus, we have from equations (26), (27), (53), and (54)

$$E^{\rm ext}_\phi = Dr^{-3/2}\left(1 - \frac{r_g}{r}\right)^4 \sin^2\theta , \quad (55)$$

where

$$D = \frac{L_0\sigma_{\rm T}A\sqrt{GM}}{4\pi c^2|e|} . \quad (56)$$

Equation (55) is of course simplified, but it allows an estimate to be made of the magnetic field generation by the PR effect close to a black hole. It is necessary that $E^{\rm (ph)}_\phi$ vanishes sufficiently rapidly at the horizon in order to avoid a physical singularity, but the exact dependence is not important.

Substituting equation (55) into equations (41) and (42) gives

$$B^r = \frac{2D}{\sqrt{r_g}}\frac{\sqrt{r-r_g}}{r^{5/2}}\left[\ln\frac{r_0}{r} - \frac{1}{3}\left(\frac{r_g}{r}\right)^3 + \frac{3}{2}\left(\frac{r_g}{r}\right)^2 - 3\frac{r_g}{r} + \frac{1}{3}\left(\frac{r_g}{r_0}\right)^3 - \frac{3}{2}\left(\frac{r_g}{r_0}\right)^2 + 3\frac{r_g}{r_0}\right]\cos\theta , \quad (57)$$

$$B^\theta = \frac{D}{\sqrt{r_g}r\sqrt{r-r_g}}\left[\frac{1}{r^{3/2}}\left(1 - \frac{r_g}{r}\right)^4 - \frac{1}{r_0^{3/2}}\left(1 - \frac{r_g}{r_0}\right)^4\right]\sin\theta . \quad (58)$$

There are several limiting cases in which expressions for B^r and B^θ can be written in a simpler form.

Nonrelativistic, Newtonian regime.—Here $r, r_0 \gg r_g$, and from equation (49) we have

$$x_0 = x\left(1 + \frac{3}{2x^{3/2}}\frac{ct}{r_g}\right)^{2/3} , \tag{59}$$

and from equations (57) and (58),

$$B^r = \frac{4D\cos\theta}{3r^2\sqrt{r_g}}\ln\left(1 + \frac{3}{2x^{3/2}}\frac{ct}{r_g}\right) , \tag{60}$$

$$B^\theta = \frac{D\sin\theta}{r^3\sqrt{r_g}}\left[1 - \left(1 + \frac{3}{2x^{3/2}}\frac{ct}{r_g}\right)^{-1}\right] . \tag{61}$$

We see here that for large t the physical value rB^θ tends to a finite limit of the order of equation (24), while B^r grows but only logarithmically. During the accretion time to a massive black hole this logarithm does not exceed $\sim$25. Thus, the magnetic field is larger by a factor of $\sim$25 than the estimations of equations (24) and (25). Still, the magnetic field is enormously less than the value found by CK.

Vicinity of the gravitational radius.—Here $(x - 1) \ll 1$, and we have from equation (49)

$$x - 1 = 4\frac{x_0 - 1}{\left(\sqrt{x_0} + 1\right)^2}\exp\left(-\frac{ct}{r_g} - \frac{8}{3} + \frac{2}{3}x_0^{3/2} + 2x_0^{1/2}\right) . \tag{62}$$

For matter initially situated in the vicinity of the horizon the Lagrangian x_0 coordinate satisfies $x_0 - 1 \ll 1$. In this case we have

$$\ln\frac{x_0 - 1}{x - 1} = \frac{ct}{r_g} . \tag{63}$$

From equation (57),

$$B^r = \frac{D\cos\theta}{2r_g^{5/2}}(x - 1)^{9/2}\left[\exp\left(\frac{4ct}{r_g}\right) - 1\right] . \tag{64}$$

From equation (58),

$$B^\theta = -\frac{D\sin\theta}{r_g^{7/2}}(x - 1)^{7/2}\left[\exp\left(\frac{4ct}{r_g}\right) - 1\right] . \tag{65}$$

These relations are valid also at large t provided that $(x - 1)e^{ct/r_g} \ll 1$.

For matter with an intermediate Lagrangian coordinate, $(x_0 - 1) \sim 1$, the vicinity of the horizon r_g is reached only at very large t, so that

$$B^r = 2D\cos\theta\frac{\sqrt{x - 1}}{r_g^{5/2}}\left(\ln x_0 - \frac{11}{6} + \frac{1}{3x_0^3} - \frac{3}{2x_0^2} + \frac{3}{x_0}\right) , \tag{66}$$

$$B^\theta = -\frac{D\sin\theta}{r_g^{7/2}\sqrt{x - 1}}\frac{(x_0 - 1)^4}{x_0^{11/2}} . \tag{67}$$

We see that in the vicinity of the horizon $(x - 1) \ll 1$ for matter with an intermediate x_0, $(x_0 - 1) \sim 1$, the component B^θ in the Schwarzschild coordinates diverges. However, this is a natural coordinate singularity, which means only that a Schwarzschild observer cannot exist physically in this region. Consequently, the magnetic field in Schwarzschild coordinates has observable consequences only in the region $x \gg 1$, $r \gg r_g$.

Matter at a very large Lagrangian radius reaches the vicinity of the gravitational radius at very late times. Consider a case with

$$x_0 - 1 \gg x, \quad x - 1 \ll 1 . \tag{68}$$

Here, for $ct/r_g \gg |\ln(x - 1)|$, we have $x_0 = (3ct/2r_g)^{2/3}$ and

$$B^r = \frac{4D\cos\theta}{3r_g^{5/2}}(x - 1)^{1/2}\ln\left(\frac{3ct}{2r_g}\right) . \tag{69}$$

If we assume, in addition, that $x_0 \gg (x - 1)^{-8/3}$, we obtain

$$B^\theta = \frac{D\sin\theta}{r_g^{7/2}}(x - 1)^{7/2} . \tag{70}$$

From a comparison of the asymptotic relations given by equations (64)–(69), we see that for r approaching r_g, the Schwarzschild component B^r grows exponentially with time very close to the horizon while remaining zero at the horizon. For larger r it grows logarithmically, which is the dependence in the Newtonian domain. The Schwarzschild component B^θ has the mentioned singularity on the horizon, which is unobservable because a physically realizable observer cannot measure it. However, at any fixed value of r close enough to r_g, the temporal behavior of the B^θ can be obtained from equations (65), (67), and (70). In this region B^θ starts to grow exponentially with time, but at long times it tends to a constant value. This component rapidly decreases with increasing r.

It is of interest to determine the magnetic field "seen" by an observer located far from the black hole who measures the field near the gravitational radius by means of the cyclotron radiation coming from this region. The observer determines the field strength by measuring the cyclotron frequency of the radiation. The frequency of the emitted radiation is determined by the comoving field strength, that is, by $\acute{B}^\rho$ and $\acute{B}^\theta$ from equation (52) evaluated in the vicinity of r_g. We now obtain estimates of these fields. It follows from equation (62) that matter with Lagrangian coordinate x_0 approaches the horizon as $t \to \infty$. The coordinate x_0 parameterizes the horizon points, and the radial coordinate x parameterizes points of the initial hypersphere $t = 0$. From equations (52), (57), (58), (66), and (67) we obtain the comoving fields in this region,

$$\acute{B}^\rho = \frac{2D\cos\theta}{r_g^{5/2}}\left(\ln x_0 - \frac{11}{6} + \frac{1}{3x_0^3} - \frac{3}{2x_0^2} + \frac{3}{x_0}\right) , \tag{71}$$

$$\acute{B}^\theta = -\frac{D\sin\theta}{r_g^{7/2}}\frac{(x_0 - 1)^4}{x_0^{11/2}} . \tag{72}$$

It is easy to verify that for any value of r_0, $r_g < r_0 < \infty$, we have

$$\acute{B}^\rho < \frac{2D|\cos\theta|}{r_g^{5/2}}\ln x_0 . \tag{73}$$

For accretion to a stellar mass or massive black hole the logarithmic factor does not exceed $\sim$25, and the value of $\acute{B}^\rho$ remains of the same order of magnitude as the Schwarzschild component B^r in the Newtonian region, equation (60) at large times, formally extrapolated to $r \sim r_g$. The poloidal component $\acute{B}^\theta$, given by equation (72), is equal to zero at $r_0 = r_g$ and $r_0 = \infty$ and has a maximum at $r = 11r_g/3$. Thus,

$$\acute{B}^\theta < \left|\acute{B}^\theta\right|_{\max} = \frac{\lambda D \sin\theta}{r_g^{7/2}} , \quad (74)$$

where $\lambda = (3^3 8^8 11^{-11})^{1/2} \approx 0.04$. Therefore, the possible values of $\acute{B}^\theta$ are less than the Newtonian value of the Schwarzschild component B^θ (eq. [61]) at large times near r_g.

The proper cyclotron frequency ω_0 of radiation emitted at (r, t), but measured with respect to the comoving time, is

$$\omega_0 = \frac{|e|}{mc}\left[\sqrt{\frac{r_g}{r}}\left(\acute{B}^\rho\right)^2 + r^2\left(\acute{B}^\theta\right)^2\right]^{1/2} , \quad (75)$$

where m is the electron rest mass. Using equations (73) and (74), we obtain an estimate for the upper limit on this frequency,

$$(\omega_0)_{\max} \sim \frac{D|e|\sqrt{2}}{mcr_g^{5/2}}\sqrt{\ln^2 x_0 + \frac{\lambda^2}{4}} . \quad (76)$$

The frequency measured by a distant observer ω is related to ω_0 as

$$\omega = \omega_0\sqrt{h}\,\frac{\sqrt{1 - v^2/c^2}}{1 - v\cos\psi/c} , \quad (77)$$

where $v/c = \sqrt{r_g/r}$ and ψ is the angle between v^α (which is in the negative r-direction) and the direction of the photon trajectory in Schwarzschild coordinates. For a radially emitted photon $(\psi = \pi)$ we have from equation (77) $\omega = \omega_0(1 - \sqrt{r_g/r})$ near the horizon, and for the tangential direction $(\psi = \pi/2)$ we have $\omega = \omega_0(1 - r_g/r)$. As a result of the upper limit on ω_0, a distant observer does not see the light emitted very close to the horizon because of the very large redshift. Thus, for a distant observer the relativistic region close to the horizon is unobservable.

Therefore, we conclude that the magnetic field produced by the PR effect close to the horizon of a black hole can be safely estimated using the Newtonian approximation given by equations (60) and (61). The estimated magnetic fields are dynamically insignificant.

5. CONCLUSION

We have reconsidered the battery effect in accretion flows due to the nonpotential nature of the radiation force on the electrons. We considered cases of a geometrically thin, optically thick disk where a toroidal magnetic field is generated and a geometrically thick, optically thin ADAF where a poloidal magnetic field is generated as a result of the PR effect. For a stellar mass black hole the generated toroidal field is estimated to be $\lesssim 10$ G, while the poloidal field in an ADAF is $\lesssim 0.01$ G. The fields vary inversely with the black hole mass. In both cases the fields are dynamically insignificant. The very large fields obtained by CK resulted from assuming unrestricted linear growth of the magnetic field. The field grows only during the accretion time. A general relativistic treatment of the PR-generated magnetic field close to the horizon of a black hole shows that the field magnitude may be larger by a factor of $\lesssim 25$ than the values obtained with a nonrelativistic treatment. Even though the magnetic field due to the radiation force is weak, it may have a role as a seed field for an α-ω dynamo (see, e.g., Brandenburg et al. 1995; Colgate et al. 2001). The importance of the seed field may depend on its symmetry; for example, the dipole symmetry of the poloidal seed field in the ADAF may not couple to the typically most unstable quadrupolar mode of the α-ω dynamo.

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APPENDIX A

EQUATION FOR POLOIDAL MAGNETIC FIELD DUE TO PR EFFECT

Here we treat in more detail the influence of the azimuthal radiation force $F_\phi^{\rm PR}$, which is *rotational* and *cannot* be balanced by any axisymmetric electrostatic field. The radiation is mainly from the central region of the flow so that the radiation flux density is $S \approx L/(4\pi R^2)$, where L is the accretion luminosity and R is the distance from the origin. The PR radiation force on an electron is $F_\phi^{\rm PR} = -S\sigma_{\rm T} v_\phi/c^2$, where $\sigma_{\rm T}$ is the Thomson cross section and v_ϕ is the azimuthal velocity of the accreting matter. Including the radiation force, Ohm's law for the plasma is

$$\boldsymbol{J} = \sigma_e\left(\boldsymbol{E}^{\rm PR} + \boldsymbol{E} + \boldsymbol{v}\times\frac{\boldsymbol{B}}{c}\right) , \quad (\rm A1)$$

where σ_e is the electrical conductivity and $\boldsymbol{E}^{\rm PR} = \hat{\phi}S\sigma_{\rm T}v_\phi/(|e|c^2)$ is the PR electric field.

Combining Faraday's and Ampere's laws and equation (A1) gives

$$\frac{d\Psi}{dt} \equiv \frac{\partial\Psi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\Psi = crE_\phi^{\rm PR} + \eta_e\Delta^*\Psi , \quad (\rm A2)$$

where $\eta_e = c^2/(4\pi\sigma_e)$ is the magnetic diffusivity and $\Psi = rA_\phi$ is the flux function, with A_ϕ being the toroidal component of the vector potential. In addition, $\Delta^* = \partial^2/\partial r^2 - (1/r)\partial/\partial r + \partial^2/\partial z^2$ in cylindrical coordinates and $\Delta^* = \partial^2/\partial R^2 + [(1-\mu^2)/R^2]\partial^2/\partial\mu^2$ in spherical coordinates where $\mu = \cos\theta$. Note that $B_r = -(1/r)\partial\Psi/\partial z$ and $B_z = (1/r)\partial\Psi/\partial r$ in cylindrical coordinates, while $B_R = (R^2 \sin\theta)^{-1}\partial\Psi/\partial\theta$ and $B_\theta = -(R\sin\theta)^{-1}\partial\Psi/\partial R$ in spherical coordinates. Taking $v_\phi = (GM/R)^{1/2}g(\theta)$, with $g(\pi/2) = 1$ and $g(\theta \to 0, \pi) = 0$, we find

$$E_\phi^{\rm PR} = \frac{m_p c^2 g}{6^{3/2}|e|}\frac{L}{L_{\rm Edd}}\frac{r_{\rm in}^{3/2}}{R^{5/2}} \approx \frac{2gE_0}{6^{3/2}}\frac{L}{L_{\rm Edd}}\left(\frac{r_{\rm in}}{R}\right)^{5/2}, \tag{A3}$$

where $E_0(M)$ is given by equation (7). Equation (A2) is equivalent to equation (8) of CK.

For an ADAF, the poloidal velocity is $v_R = -\alpha\xi(GM/R)^{1/2}$, where α is the Shakura-Sunyaev parameter and $\xi \lesssim 1$ is a constant (Narayan & Yi 1995). Following CK we write $\eta_e = \mathscr{P}R|v_R|$, where $\mathscr{P}$, the magnetic Prandtl number, is the ratio of magnetic diffusivity to viscosity. Measuring R in units of $r_{\rm in}$ and t in units of $t_0 = r_{\rm in}^{3/2}/[\alpha\xi(GM)^{1/2}] = \sqrt{6}(r_{\rm in}/c)/(\alpha\xi)$, equation (A2) becomes

$$\frac{\partial\Psi}{\partial t} = \frac{Kg(\theta)\sin\theta}{R^{3/2}} + \frac{1}{\sqrt{R}}\frac{\partial\Psi}{\partial R} + \mathscr{P}\sqrt{R}\left(\frac{\partial^2\Psi}{\partial R^2} + \frac{1-\mu^2}{R^2}\frac{\partial^2\Psi}{\partial\mu^2}\right), \tag{A4}$$

where $K \equiv r_{\rm in}^2 E_0(L/L_{\rm Edd})/(3\alpha\xi)$.

The timescale of the linear growth of Ψ is $\tau_m = t_0/\mathscr{P}$. For a turbulent magnetic diffusivity where $\mathscr{P} = \mathscr{O}(1)$, this timescale is quite short, $\sim t_0 \approx (7.3 \times 10^{-4}\ {\rm s})(M/M_\odot)(0.1/\alpha)(1/\xi)$. Therefore, the physically relevant solution to equation (A4) is the stationary one where the PR term $\propto K$ is balanced by diffusion. This gives $Kg(\theta)\sin\theta = -\mathscr{P}(1-\mu^2)\partial^2\Psi/\partial\mu^2$ so that Ψ is independent of R. For example, for $g(\theta) = \sin(\theta)$, $B_R = (K/\mathscr{P})\cos\theta/R^2$ or

$$B_R^{\rm PR} \approx \frac{0.6\cos\theta}{\alpha}\frac{L}{\xi\mathscr{P}}\frac{M}{L_{\rm Edd}}\frac{M}{M_\odot}\left(\frac{r_{\rm in}}{R}\right)^2 \ {\rm G}, \tag{A5}$$

and $B_\theta = 0$. This estimate agrees with equation (24). Equation (A5) corresponds to a radially outward field in the northern hemisphere and a radially inward field in the southern hemisphere. The polarity of the field agrees with the PR drag on the electrons in the $-\hat{\phi}$ direction giving a ring current in the $+\hat{\phi}$ direction, while the simple nature of the field results from the approximation that $S \approx L/(4\pi R^2)$

APPENDIX B

FOUR-VECTOR E AND B FIELDS

The field evolution during free-fall accretion without an external EMF was solved by Bisnovatyi-Kogan & Ruzmaikin (1974), using only four-vector magnetic field B^i,

$$\mathrm{B}^i = \frac{1}{2\sqrt{-g}}\varepsilon^{iklm}u_k F_{lm}, \qquad \mathrm{B}^k u_k = 0. \tag{B1}$$

For conditions of high (infinite) conductivity, the four-vector electric field is

$$\mathrm{E}^i = F^{ik}u_k = 0; \qquad \mathrm{E}_i = F_{ik}u^k = 0. \tag{B2}$$

The equation for the magnetic field can be written in the form

$$\frac{\partial}{\partial x^k}\left[\sqrt{-g}\left(\mathrm{B}^i u^k - \mathrm{B}^k u^i\right)\right] = 0 \tag{B3}$$

(Lichnerowicz 1967). For radial free fall in the absence of a toroidal field ($B_\phi = 0$), the equations following from equation (B3) are

$$\frac{d}{dt}\left(\sqrt{-g}u_0^{-1}\mathrm{B}^r\right) = 0, \qquad \frac{d}{dt}\left(\sqrt{-g}u^r\mathrm{B}^\theta\right) = 0, \tag{B4}$$

with $d/dt = \partial/\partial t + c(u^r/u^0)\partial/\partial r$. The connection between the three-vectors B^α, E_α and the four-vector B^i is

$$\mathrm{B}^r = B^r\left(1 - \frac{r_g}{r}\right)^{-1/2}, \qquad \mathrm{B}^\theta = B^\theta\left(1 - \frac{r_g}{r}\right)^{-1/2} - \sqrt{\frac{r_g}{r}}\frac{E_\phi}{r^2\sin\theta}\left(1 - \frac{r_g}{r}\right)^{-1}, \tag{B5}$$

$$\mathrm{B}^0 = -B^r\sqrt{\frac{r_g}{r}}\left(1 - \frac{r_g}{r}\right)^{-3/2}, \qquad \mathrm{B}^\phi = 0. \tag{B6}$$

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