

DARK ENERGY AND THE OBSERVABLE UNIVERSE

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ABSTRACT

We consider ever-expanding big bang models with a cosmological constant, Λ , and investigate in detail the evolution of the observable part of the universe. We also discuss quintessence models from the same point of view. A new concept, the Λ -sphere (or Q -sphere, in the case of quintessence) is introduced. This is the surface in our visible universe that bounds the region where dark energy dominates the expansion, and within which the universe is accelerating. We follow the evolution of this surface as the universe expands, and we also investigate the evolution of the particle and event horizons as well as the Hubble surface. We calculate the extent of the observable universe and the portion of it that can be seen at different epochs. Furthermore, we trace the changes in redshift, apparent magnitude and apparent size of distant sources through cosmic history. Our approach is different from, but complementary to, most other contemporary investigations, which concentrate on the past light cone at the present epoch. When presenting numerical results we use the FRW world model with $\Omega_{m0} = 0.30$ and $\Omega_{\Lambda0} = 0.70$ as our standard cosmological model. In this model the Λ -sphere is at a redshift of 0.67, and within a few Hubble times the event horizon will be stationary at a fixed proper distance of 5.1 Gpc (assuming $h_0 = 0.7$). All cosmological sources with present redshift larger than 1.7 have by now crossed the event horizon and are therefore completely out of causal contact.

Subject headings: cosmology: observations — cosmology: theory — relativity

1. INTRODUCTION

At present there is growing evidence that the expansion of our visible universe is accelerating. Two independent research groups using Type Ia supernovae as standard candles have discovered signs of a small positive cosmological constant in the Hubble diagram (Perlmutter et al. 1999; Garnavich et al. 1998; Riess et al. 1999, 2001). Further evidence comes from investigations of anisotropies in the cosmic microwave background as well as large-scale structure and the age of the universe (for recent reviews and references see Carroll 2001, Sahni & Starobinsky 2000, and Bahcall et al. 1999).

Assuming standard cosmological theory (Peacock 1999; Peebles 1993) these observations indicate that matter, including dark matter, contributes about 30% of the critical density and an effective cosmological constant or dark energy about 70%. These results indicate that our universe is close to being flat and has a big bang origin. Furthermore, we are living at a time in cosmic history when the cosmological constant, or something that mimics its effects, is already dominating the expansion.

In this paper we shall investigate the evolution of ever-expanding big bang world models with a cosmological constant, or a quintessence field, with particular emphasis on the evolution of the observable universe, i.e., our past light cone and relevant observables such as redshift of cosmic sources, their apparent magnitude, and their apparent size. The properties of the particle horizon, the Hubble surface and the event horizon will also be discussed.

We introduce a new concept, the Λ -sphere (or Q -sphere, in the case of quintessence), which is the surface in our visible universe that bounds the region where dark energy dominates the expansion, and within which the universe is accelerating. We track the evolution of this surface through cosmic history.

Our methods are in many ways similar to the ones used in our work on the evolution of closed-world models

without a cosmological constant (Björnsson & Gudmundsson 1995), in which an extensive list of references to earlier work on cosmic evolution can be found (see also Adams & Laughlin 1997 for a different perspective and further references). We emphasize that our approach is different from, but complementary to, most other recent investigations, which concentrate on applying new and old cosmological tests, such as the m - z relation, to the present light cone.

The paper is organized as follows. We begin in § 2 by reviewing the basic definitions and results of standard cosmology that are relevant to our discussion. In § 3 we discuss the light cone, the particle and event horizons as well as the Hubble surface for a fundamental observer. In §§ 4, 5, and 6 we present our results for the evolution of observable quantities such as redshift, apparent magnitude, and apparent angular size and discuss the properties of the Λ -sphere as well as the question of causal connections. Similar methods are then used in § 7 to investigate the effects of quintessence, and in § 8 we conclude the paper.

2. THE WORLD MODELS

The spacetime metric of the standard spatially homogeneous and isotropic Friedmann-Robertson-Walker (FRW) world models can be written in the form (see, e.g., Weinberg 1972; Peebles 1993; Peacock 1999)

$$ds^2 = -c^2 dt^2 + R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, (r, θ, ϕ) are comoving spherical coordinates, t is the cosmic proper time, and c is the velocity of light. $R = R(t)$ is the universal scale factor and k the curvature scalar, which takes one of three possible values according to whether the universe is open ($k = -1$), flat ($k = 0$), or closed ($k = +1$).

The time evolution of the models is determined by the Einstein field equations for a universe composed of a perfect

fluid and by the equation of energy-momentum conservation. These equations can be reduced to a system of two differential equations in the form

$$\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 \quad (2)$$

and

$$\frac{d\rho}{dR} = -\frac{3}{R} \left(\rho + \frac{P}{c^2} \right). \quad (3)$$

Here the dot means differentiation with respect to t , G is Newton's constant of gravitation and the total mass-energy density, ρ , and the total pressure, P , are given by

$$\rho = \rho_m + \rho_r + \rho_\Lambda + \rho_Q \quad (4)$$

and

$$P = P_m + P_r + P_\Lambda + P_Q, \quad (5)$$

respectively. Above, ρ_m is the mass-energy density of non-relativistic matter, including nonrelativistic dark matter, and ρ_r that of radiation, including all relativistic particles. Similarly, P_m and P_r are the pressure of matter and radiation, respectively. The term ρ_Λ is the mass-energy density associated with Einstein's cosmological constant Λ ,

$$\rho_\Lambda = \frac{c^2 \Lambda}{8\pi G}, \quad (6)$$

and the corresponding effective pressure is given by

$$P_\Lambda = -\rho_\Lambda c^2. \quad (7)$$

The mass-energy density and pressure of quintessence are denoted by ρ_Q and P_Q , respectively.

Note that in order to solve equations (2) and (3) one also needs equations of state for matter, radiation, and quintessence. In general one can write for each component

$$P_i = w_i \rho_i c^2, \quad (8)$$

where i stands for each of m , r , Λ , or Q with $w_m = 0$, $w_r = \frac{1}{3}$, $w_\Lambda = -1$, and $-1 < w_Q < 0$ (for a further discussion on the cosmic equation of state see, e.g., Gudmundsson & Rögnvaldsson 1990 and references therein. That paper uses α_i instead of w_i as a pressure parameter, with $\alpha_i = 3w_i$).

In this paper we shall primarily be concerned with the effects of a cosmological constant on the evolution of the observable universe with emphasis on cosmic epochs in which matter dominates over radiation. The effects of quintessence, which could mimic the effects of a true cosmological constant at the present epoch, will be considered in § 7. In later sections we shall assume that $\rho_r = 0$ and $P_r = 0$.

The total mass-energy density parameter is defined by

$$\Omega = \sum \Omega_i, \quad (9)$$

where $\Omega_i = \rho_i/\rho_c$ is the contribution of each component, m , r , Λ , and Q , and $\rho_c = 3H^2/8\pi G = 1.9 \times 10^{-29} h^2 \text{ g cm}^{-3}$ is the critical density. Here $h = H/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$, where $H = \dot{R}/R$ is the Hubble constant. Note that since H is a function of time, so is ρ_c and hence Ω (for $k = \pm 1$). The relation between Ω and other cosmological parameters can be found by rewriting equation (2) in the form

$$\Omega = 1 + \frac{kc^2}{R^2 H^2} = 1 + \frac{kc^2}{\dot{R}^2}. \quad (10)$$

Using equation (10) and the fact that after decoupling each of the cosmic components m , r , Λ and Q separately satisfy equation (3), the dynamical equation (2) can be written in a convenient form as

$$\frac{da}{d\tau} = \left[\Omega_{m0} \left(\frac{1}{a} - 1 \right) + \Omega_{r0} \left(\frac{1}{a^2} - 1 \right) + \Omega_{\Lambda 0} (a^2 - 1) + \Omega_{Q0} \left(\frac{1}{a^{1+3w_Q}} - 1 \right) + 1 \right]^{1/2} \quad (11)$$

for ever-expanding models. Here w_Q is assumed to be constant, $\tau = t/t_{H0}$ is a new dimensionless cosmic time variable defined in terms of the Hubble time at the present epoch, $t_{H0} = 1/H_0 = 9.8 h_0^{-1} \text{ Gyr}$, and $a = R/R_0$ is the scale factor normalized to its present value. In equation (11) and in what follows we denote the values of quantities at the present time by the subscript 0. Note that $a(\tau_0) = (da/d\tau)_{\tau=\tau_0} = 1$ and $H = \dot{R}/R = H_0(1/a)(da/d\tau)$. For future reference we also remind the reader of the definition of the Hubble radius, $R_H = c/H$. Its present day value is $R_{H0} = c/H_0 = 3.0 h_0^{-1} \text{ Gpc}$.

In order to determine a as a function of τ one simply integrates equation (11) for given values of Ω_{m0} , Ω_{r0} , $\Omega_{\Lambda 0}$ and Ω_{Q0} , as well as w_Q . Note that the age of the universe at the present epoch is given by

$$\tau_0 = \int_0^1 \left(\frac{da}{d\tau} \right)^{-1} da. \quad (12)$$

In the following sections we shall be concerned with the evolution of the subclass of FRW models that have a big bang origin and continue to expand forever, since recent observations indicate that we live in such a universe. For the relevant region in the $(\Omega_m, \Omega_\Lambda)$ plane we refer the reader to equations (11) and (12) and Figure 1 in Carroll, Press, & Turner (1992) and Figure 7 in Perlmutter et al. (1999).

3. THE OBSERVABLE UNIVERSE

In this section we shall discuss various concepts which are necessary for an understanding of our observable universe. For this purpose it is convenient to start by reminding the reader of the definition of the conformal time η :

$$d\eta = \frac{cdt}{R(t)}. \quad (13)$$

In a big bang universe the relation between η and the time variable τ is therefore given by

$$\eta = \frac{R_{H0}}{R_0} \int_0^\tau \frac{d\tau}{a(\tau)}. \quad (14)$$

The comoving conformal radial distance, χ , is defined by

$$d\chi^2 = \frac{dr^2}{1 - kr^2}. \quad (15)$$

Assuming that we are in the position of a fundamental observer at the origin, $\chi = 0$ (corresponding to $r = 0$), we therefore have

$$\chi = \begin{cases} \sinh^{-1}(r) & k = -1, \\ r & k = 0, \\ \sin^{-1}(r) & k = +1. \end{cases} \quad (16)$$

For our distance calculations we only need the radial coordinate. Setting $d\theta = d\phi = 0$ in equation (1) and using

conformal representation, the spacetime metric becomes very simple:

$$ds^2 = R^2(\eta)[d\chi^2 - d\eta^2]. \quad (17)$$

Figure 1 shows the scale factor, a , both as a function of η in units of R_{H0}/R_0 , and τ for selected values of Ω_{m0} and $\Omega_{\Lambda0}$ with $\Omega_{r0} = \Omega_{Q0} = 0$. Note in particular that while τ covers the whole range from 0 to ∞ , η has a maximum value, which we denote by $\eta_{\max}$. A detailed discussion of $\eta_{\max}$ will be given in § 3.4. In what follows, in the main text as well as in figure captions, both η and χ will be presented in units of R_{H0}/R_0 .

We now discuss in turn the light cone of a fundamental observer, his Hubble sphere and his event horizon.

3.1. The Light Cone

The light cone of a fundamental observer is determined by $ds = 0$, i.e., by

$$d\eta = \pm d\chi, \quad (18)$$

where we have used equation (17). Here the plus sign corresponds to the future light cone and the minus sign to the past light cone.

Assuming that at time η_0 we receive a signal emitted at time η from a source at χ , integration of equation (18) gives the equation of our past light cone at η_0 as

$$\eta = \eta_0 - \chi. \quad (19)$$

Similarly our future light cone at η_0 is given by

$$\eta = \eta_0 + \chi, \quad (20)$$

where η is now the time in the future at which an observer at χ receives a signal emitted by us at η_0 . The future light cone at the big bang is sometimes called “the creation light cone” (Rindler 1956).

Figure 2 shows our past and future light cones in a model with $\Omega_{m0} = 0.30$, $\Omega_{\Lambda0} = 0.70$, and $\Omega_{Q0} = \Omega_{r0} = 0$. In what follows, this will be our standard cosmological model when presenting numerical results, and we shall refer to it as our standard Λ -model. In this model, $\tau_0 = 0.96$, which corresponds to $\eta_0 = 3.3$, whereas the end of time $\tau = \infty$ corresponds to $\eta = \eta_{\max} = 4.5$ (our numerical results will generally be given with two significant figures). The distance d_p is the normalized proper distance $d_p = d/R_{H0}$, with the proper distance given by $d = d(\tau) = R(\tau)\chi$. Figure 2 also shows the creation light cone as well as the Hubble surface,

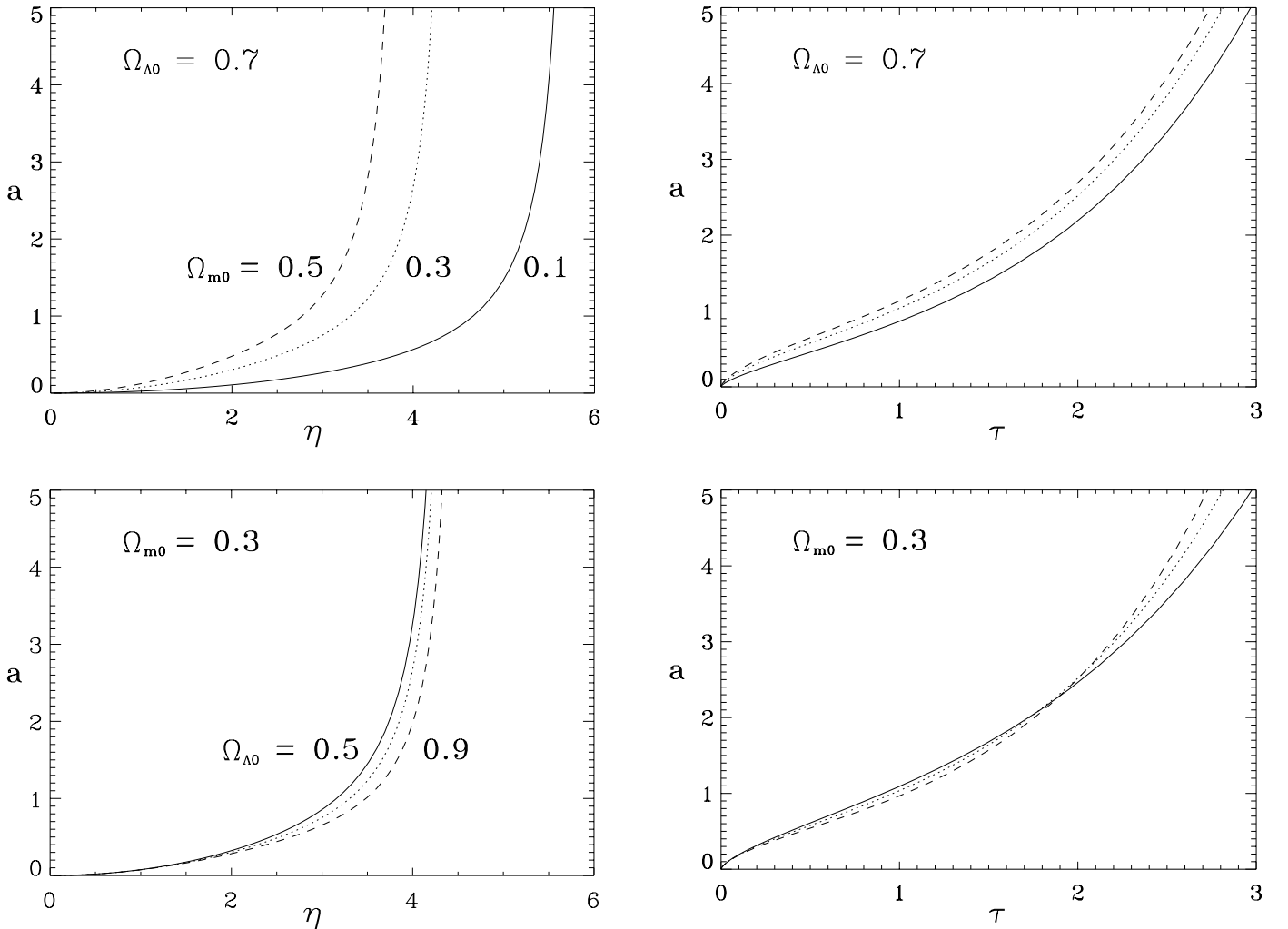


FIG. 1.—*Top left*: Scale factor a as a function of η for $\Omega_{\Lambda0} = 0.70$ and $\Omega_{m0} = 0.10$ (solid curve), 0.30 (dotted curve) and 0.50 (dashed curve). Here and in all figures that follow, η is in units of R_{H0}/R_0 . *Top right*: Scale factor a as a function of τ for the same values of $\Omega_{\Lambda0}$ and Ω_{m0} as in the top left-hand panel. *Bottom left*: The scale factor as a function of η for $\Omega_{m0} = 0.30$ and $\Omega_{\Lambda0} = 0.50$ (solid curve), 0.70 (dotted curve), and 0.90 (dashed curve). *Bottom right*: a as a function of τ for the same values of $\Omega_{\Lambda0}$ and Ω_{m0} as in the bottom left-hand panel.

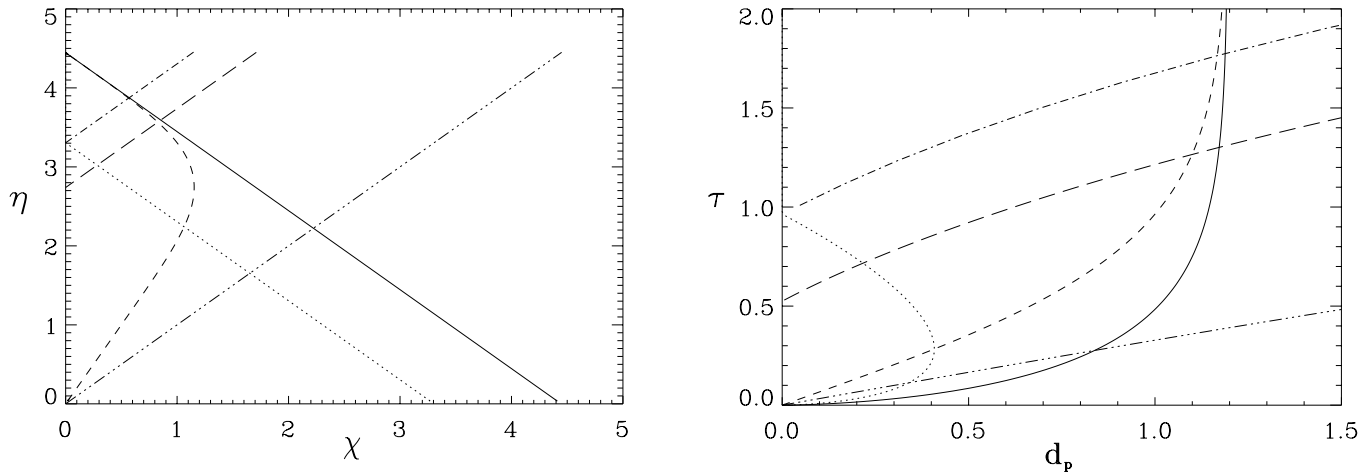


FIG. 2.—*Left*: Our past light cone (dotted line) and the future light cone (dot-dashed line) at the present epoch in a η - χ diagram for our standard Λ -model with $\Omega_{\Lambda 0} = 0.70$, $\Omega_{m0} = 0.30$, and $\Omega_{\Lambda 0} = \Omega_{r0} = 0$. Also shown are the Hubble surface (dashed curve), the particle or visual horizon (triple-dot-dashed line), the event horizon (solid line), and the Λ -sphere (long-dashed line). Here $\eta_0 = 3.3$ and $\eta_{\max} = 4.5$. Note that the creation light cone lies right on top of the particle horizon (visual horizon). *Right*: Same curves in a τ - d_p diagram, where $d_p = d/R_{H0}$ and d is the proper distance. Note that $\tau_0 = 0.96$, and therefore $t_0 = 13.5$ Gyr if $h_0 = 0.70$.

the particle horizon, the event horizon and the Λ -sphere which will all be discussed in detail in sections 3.2–3.4 and 4.1 below.

In terms of the time variable τ and the proper distance, $d = R\chi$, the past light cone (lc) at τ_0 is given by

$$d_{lc}(\tau) = R_{H0} a(\tau) \left[\int_0^{\tau_0} \frac{d\tau}{a(\tau)} - \int_0^{\tau} \frac{d\tau}{a(\tau)} \right], \quad (21)$$

with $0 \leq \tau \leq \tau_0$.

3.2. The Particle Horizon

At time η our particle horizon (ph) is situated at $\chi = \eta$ (Rindler 1956). The proper distance to this horizon is therefore

$$d_{ph}(\eta) = R\eta, \quad (22)$$

and as a function of τ it is given by

$$d_{ph}(\tau) = R\eta = R_{H0} a(\tau) \int_0^{\tau} \frac{d\tau}{a(\tau)}. \quad (23)$$

The horizon is moving away from us at speed

$$v_{ph} = \frac{d}{dt} (d_{ph}) = c + H d_{ph}(\eta) = c \left(1 + \frac{\eta}{R_H/R} \right). \quad (24)$$

Note that comoving sources momentarily at the particle horizon are moving away from us at speed $H d_{ph}$. In a universe with a big bang beginning we “see” these sources as they were at $\tau = 0$ and with infinite redshift.

In our numerical calculations we assume a universe with ordinary matter and a cosmological constant (or quintessence), i.e., a universe with $\Omega_{r0} = 0$. This basically means that we ignore the expansion dynamics of the early universe. For our purposes this is a good approximation. However it should be kept in mind that the early universe probably went through an inflationary period. With inflation the real particle horizon is much further away than the particle horizon obtained by assuming a dust universe with a cosmological constant (see, e.g., Harrison 1991) and references therein). The particle horizon presented in our calcu-

lations is therefore approximately equal to the particle horizon looking back to the cosmic microwave background. This horizon is sometimes called the “visual horizon” (Ellis & Rothman 1993). For our standard Λ -model we find that $d_{ph}(\tau_0) = 3.3R_{H0}$ and $v_{ph}(\tau_0) = 4.3c$.

3.3. The Hubble Surface

The “Hubble surface” (hs; Harrison 1991) is the instantaneous set of points which at time η are moving away from us at the speed of light. Their proper distance is given by the velocity-distance law as

$$d_{hs}(\eta) = \frac{c}{H} = R_H, \quad (25)$$

and hence the conformal distance is

$$\chi_{hs}(\eta) = \frac{d_{hs}(\eta)}{R}. \quad (26)$$

In terms of τ , the proper distance to the Hubble surface is given by

$$d_{hs}(\tau) = R\chi_{hs} = R_{H0} a(\tau) \left(\frac{da}{d\tau} \right)^{-1} \quad (27)$$

and it is moving away from us at speed

$$v_{hs} = \frac{d}{dt} (d_{hs}) = c(1 + q), \quad (28)$$

where $q = -R\ddot{R}/\dot{R}^2 = -a(d^2a/d\tau^2)/(da/d\tau)^2$ is the deceleration parameter. Note that if the cosmological constant dominates the expansion then $q = -1$ and $v_{hs} = 0$. For our standard Λ -model we find that $q_0 = -0.85$ and hence the present speed of the Hubble surface is $v_{hs}(\tau_0) = 0.15c$.

In Figure 2 we show the evolution of the Hubble surface by the dashed curve. Note that at a given time sources beyond the Hubble surface are moving away from us with speed greater than c , whereas sources inside the surface are receding with speed less than c . From the figure one can also see that on a cosmic timescale the Hubble surface rapidly approaches the event horizon and within only a few

Hubble times after the big bang the two have practically merged, never to part again.

3.4. The Event Horizon

If the world line of a source at conformal distance χ intersects our past light cone at time η_0 , then we see it as it was at $\eta = \eta(\tau)$, where

$$\chi = \eta_0 - \eta = \int_{\eta}^{\eta_0} d\eta = \frac{R_{H0}}{R_0} \int_{\tau}^{\tau_0} \frac{d\tau}{a(\tau)}. \quad (29)$$

Now suppose that for an ever-expanding world model the integral $\int_{\tau}^{\infty} a^{-1} d\tau$ is finite. Then the conformal time

$$\eta_{\max} = \frac{R_{H0}}{R_0} \int_0^{\infty} \frac{d\tau}{a(\tau)} \quad (30)$$

corresponding to $\tau_0 = \infty$ is finite and the conformal distance χ_{eh} given by

$$\begin{aligned} \chi_{\text{eh}} &= \eta_{\max} - \eta = \int_{\eta}^{\eta_{\max}} d\eta = \frac{R_{H0}}{R_0} \int_{\tau}^{\infty} \frac{d\tau}{a(\tau)} \\ &= \frac{R_{H0}}{R_0} \left[\int_0^{\infty} \frac{d\tau}{a(\tau)} - \int_0^{\tau} \frac{d\tau}{a(\tau)} \right], \end{aligned} \quad (31)$$

is the finite distance to our event horizon (eh) at time η (or τ). This is because the event horizon is our final or ultimate light cone (Rindler 1956). It is defined by equation (31) and shown for our standard model by the solid curve in Figure 2. All events on the event horizon will first be “seen” by us at the end of time ($\eta = \eta_{\max}$ corresponding to $\tau = \infty$) and with infinite redshift (see § 4). Events beyond this horizon will never be seen by us.

The proper distance to the event horizon at time τ is given by

$$d_{\text{eh}}(\tau) = R\chi_{\text{eh}} = R_{H0} a(\tau) \left[\int_0^{\infty} \frac{d\tau}{a(\tau)} - \int_0^{\tau} \frac{d\tau}{a(\tau)} \right], \quad (32)$$

and it is moving away from us with speed

$$v_{\text{eh}} = \frac{d}{dt} (d_{\text{eh}}) = c \left(\frac{\eta_{\max}}{R_{H0}/R_0} \right) \left(\frac{da}{d\tau} \right)_{\tau} - v_{\text{ph}}, \quad (33)$$

where v_{ph} is given by equation (24). For our standard Λ -model we have that $v_{\text{eh}}(\tau_0) = 0.14c$.

From Figure 2 we see that for the ever-expanding big bang models with a cosmological constant, the event horizon is stationary at a particular proper distance after a certain time. Furthermore the Hubble surface approaches the event horizon quite rapidly. For the standard Λ -model in Figure 2, we have that the proper distance to the Hubble surface and the event horizon is fixed at $\approx 1.2R_{H0}$ at late cosmic epochs. For comparison with this ultimate value, we remind the reader that the present day value of d_{hs} is $1.0R_{H0}$, and from equation (32) we see that $d_{\text{eh}}(\tau_0) = 1.1R_{H0}$.

The reason for this limiting behavior can be understood in the following way. For an ever-expanding model the scale factor increases without limit. After a certain time, say $\tau_{\bullet}$, we see from equation (11) that the Λ term dominates completely and hence it is a good approximation to write

$$a(\tau) = a(\tau_{\bullet}) e^{\Omega_{\Lambda 0}^{1/2}(\tau - \tau_{\bullet})}, \quad \tau > \tau_{\bullet}. \quad (34)$$

From this it follows that

$$d_{\text{eh}}(\tau) = d_{\text{hs}}(\tau) = R_{H0} a(\tau) \left(\frac{da}{d\tau} \right)^{-1} = \frac{R_{H0}}{\Omega_{\Lambda 0}^{1/2}}, \quad \tau > \tau_{\bullet}. \quad (35)$$

Note that although the proper distance to the event horizon is finite, its luminosity distance, d_L , is infinite. However, as will be discussed in more detail in §4, the angular diameter distance, d_A , is finite.

For later purposes we also express equation (34) in terms of conformal time. For η in the range $\eta_{\bullet} < \eta \leq \eta_{\max}$, where $\eta_{\bullet}$ corresponds to $\tau_{\bullet}$, we find from equations (14) and (34) that

$$\eta \approx \eta_{\max} - \frac{R_{H0}/R_0}{\Omega_{\Lambda 0}^{1/2} a(\eta)}, \quad \eta_{\bullet} < \eta \leq \eta_{\max}, \quad (36)$$

and hence

$$a(\eta) \approx \frac{R_{H0}/R_0}{\Omega_{\Lambda 0}^{1/2} (\eta_{\max} - \eta)}, \quad \eta_{\bullet} < \eta \leq \eta_{\max}. \quad (37)$$

Finally, we remind the reader that only universes with finite $\eta_{\max}$ have event horizons. For example if the scale factor grows as a power of time, i.e., $a \propto \tau^n$, then $\eta_{\max}$ is finite only if $n > 1$. Models with scale factors growing more slowly than this have no event horizons, e.g., the open or flat FRW-universes with $\Lambda = 0$, and quintessence models with $-\frac{1}{3} < w_Q < 0$ (see also § 7).

4. EVOLUTION OF OBSERVABLE QUANTITIES

In this section we shall discuss the time evolution of various observational quantities in ever-expanding big bang models with a cosmological constant.

Consider first our past light cone. In Figure 3 we show how it evolves with cosmic time in our standard Λ -model. As time advances for the observer, his light cone gets closer and closer to the event horizon, demonstrating that the event horizon corresponds to his final light cone. One can clearly see how his observable part of the universe is enclosed for all time within a finite proper volume with proper radius $R_{H0}/\Omega_{\Lambda 0}^{1/2} \approx 1.2R_{H0}$. As expected, the right-hand panel of Figure 3 also shows that the Hubble surface (short-dashed curve) crosses the observer's light cones at their maximum proper distance from the time axis.

The world lines of sources at several different conformal distances, $\chi = 0.10, 0.60, 1.0$, and 2.0 , are also shown in the right-hand panel of Figure 3 (dotted curves). All sources taking part in the cosmic expansion leave our observable part of the universe in a finite proper time. However, just as an observer at rest far away from a black hole never sees infalling objects pass the event horizon of the black hole, we shall never see the sources pass through our cosmic event horizon, although they will rapidly fade away once the cosmological constant dominates the expansion.

4.1. The Redshift and the Λ -Sphere

Assume that a given cosmic source is located at comoving conformal distance χ . Its redshift, z , as observed at time $\eta_{\text{obs}} = \eta(\tau_{\text{obs}})$ is given by

$$1 + z = \frac{R(\eta_{\text{obs}})}{R(\eta_{\text{em}})} = \frac{a(\tau_{\text{obs}})}{a(\tau_{\text{em}})}, \quad (38)$$

where $\eta_{\text{em}} = \eta(\tau_{\text{em}})$ is the time at emission, i.e., the time at which the world line of the source crosses the observer's past light cone as it is at the time of observation.

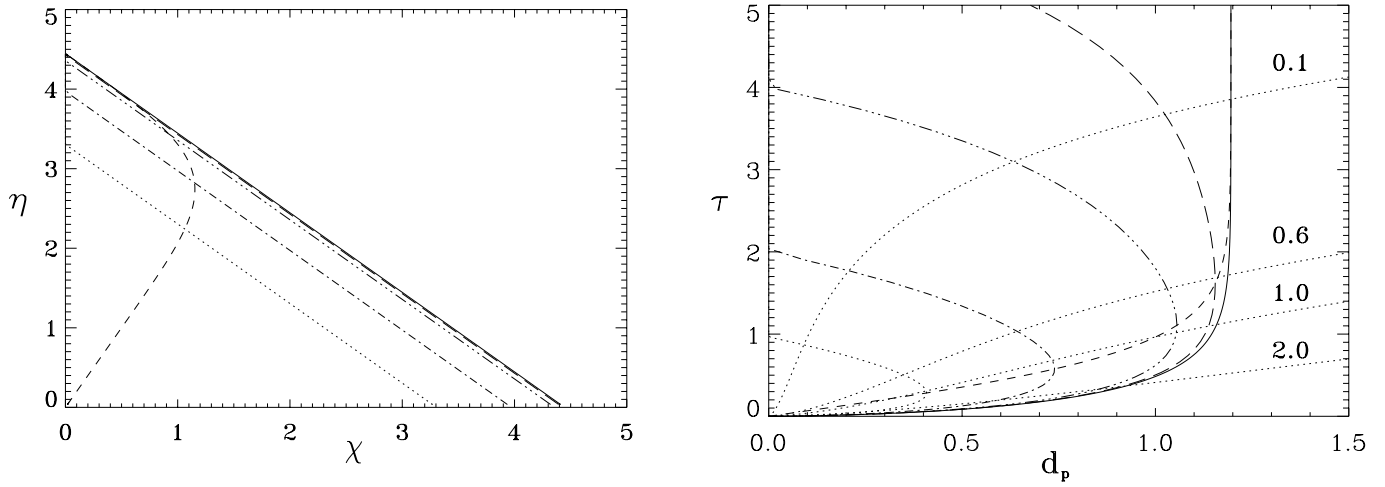


FIG. 3.—*Left*: The past light cone at various cosmic epochs in a η - χ diagram. The world model is the same as in Fig. 2. Also shown are the event horizon (solid line) and the Hubble surface (short-dashed curve). *Right*: Same as a, but in a τ - d_p diagram. In addition, world lines of several sources are also shown as dotted curves, marked by the positions of the sources at $\chi = 0.10, 0.60, 1.0$, and 2.0 .

In Figure 4 we plot z as a function of the time of observation for sources at various distances, χ . For simplification we have dropped the subscript “obs” on both η and τ . We first “see” each source when it comes within our particle horizon, i.e., as it was at the big bang ($\eta_{\text{em}} = 0$). It therefore enters our observable universe with infinite redshift at cosmic time $\eta = \chi$. Before the cosmological constant becomes dynamically important the redshift decreases more or less as in a universe without Λ because the expansion is slowing down. As the effects of the cosmological constant begin to manifest themselves, the source’s redshift reaches a minimum, and once Λ completely dominates the expansion (i.e., for $\eta_{\bullet} < \eta \leq \eta_{\text{max}}$ or equivalently for $\tau_{\bullet} \ll \tau$; see the discussion after eq. [32]) the behavior of the source’s redshift with time is given by

$$1 + z \approx 1 + \frac{\chi}{\eta_{\text{max}} - \eta} \approx (1 + z_{\bullet}) e^{\Omega_{\Lambda 0}^{1/2}(\tau - \tau_{\bullet})}, \quad (39)$$

where $z_{\bullet} = z(\tau_{\bullet})$ and $\tau_{\text{em}} \leq \tau_{\bullet}$. For $\tau_{\bullet} < \tau_{\text{em}} < \tau$ we have that $1 + z = e^{\Omega_{\Lambda 0}^{1/2}(\tau - \tau_{\text{em}})}$. Hence all sources will redshift away on a timescale $\Delta t \approx \Omega_{\Lambda 0}^{-1/2} t_{H0}$, the redshift going to

infinity at $\eta = \eta_{\text{max}}$ corresponding to $\tau = \infty$ when the scale factor becomes infinite.

In order to investigate this in more detail we introduce a redshift evolutionary timescale, $T_z(\tau) = \{[d(1+z)/d\tau]/(1+z)\}^{-1}$, which can be compared to the expansion timescale (normalized Hubble time), $T_a(\tau) = (H_0/H) = [(da/d\tau)/a]^{-1}$. Note that T_z can take both positive and negative values, depending on whether the redshift is increasing or decreasing. By use of equation (38) we find for the FRW models in general that

$$\frac{1}{T_z} = \frac{1}{(1+z)} \frac{d(1+z)}{d\tau} = \frac{1}{a} \left[\left(\frac{da}{d\tau} \right)_{\tau} - \left(\frac{da}{d\tau} \right)_{\tau_{\text{em}}} \right], \quad (40)$$

where the time derivative of a is given by equation (11). Also note that in deriving this result we have used cosmic time dilation: $d\tau = (1+z)d\tau_{\text{em}}$. In terms of the timescales, equation (40) can be rewritten as

$$\frac{1}{T_z(\tau)} = \frac{1}{T_a(\tau)} - \frac{1}{(1+z)T_a(\tau_{\text{em}})}. \quad (41)$$

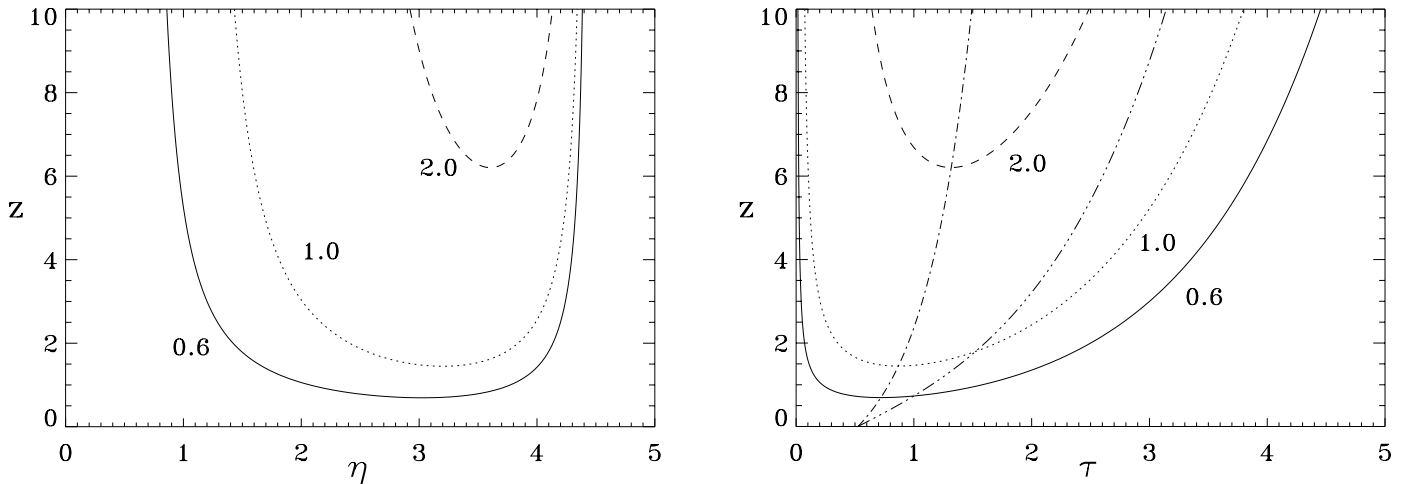


FIG. 4.—Evolution of the redshift of three sources in the Λ -model. The sources are at $\chi = 0.60$ (solid curve), 1.0 (dotted curve), and 2.0 (dashed curve) and their worldlines are shown in the right-hand panel of Fig. 3. *Left*: Redshift as a function of η ; *right*: redshift as a function of τ . In the right-hand panel the redshifts z_{Λ} (triple-dot-dashed curve) and z_{eq} (dot-dashed curve) are also shown as functions of τ . See text for further explanations.

Next consider the right hand side of equation (41) as a function of z . It has a maximum at the redshift corresponding to a minimum value of T_z . For big bang models with $\Omega_{\Lambda 0} = \Omega_{Q 0} = 0$ we have that $da/d\tau$ is a decreasing function of τ , i.e., $d^2a/d\tau^2 < 0$, and thus $d(1+z)/d\tau$ is negative during expansion. In such models the redshift of a given source always decreases with time (in recollapsing models the redshift eventually changes into blueshift, see, e.g., Björnsson & Gudmundsson 1995). However, for big bang models with a positive cosmological constant the situation is different. Owing to the dynamical effects of Λ , accelerated expansion starts at cosmic time τ_Λ given by

$$\left(\frac{d^2a}{d\tau^2}\right)_{\tau_\Lambda} = 0 \quad (42)$$

and continues forever. It is clear from the discussion above that at a given cosmic time, $\tau > \tau_\Lambda$, the redshift corresponding to the epoch τ_Λ is also the redshift that minimizes T_z and maximizes the change in redshift. This particular redshift, which we shall denote by z_Λ , thus locates the surface on the past light cone which bounds the region where Λ dominates the expansion and within which the universe is accelerating. We shall refer to this surface as the Λ -sphere. Beyond the Λ -sphere the universe is still decelerating.

We next determine the redshift z_Λ . By use of equation (11) it is easy to show that equation (42) is equivalent to the following algebraic equation for a :

$$\frac{2\Omega_{\Lambda 0}}{\Omega_{m 0}} a^3 - \frac{(1 + 3w_Q)\Omega_{Q 0}}{\Omega_{m 0}} a^{-3w_Q} = 1, \quad (43)$$

where we have assumed that $\Omega_{r 0} = 0$. In the case $\Omega_{Q 0} = 0$ the solution is $a = a_\Lambda = a(\tau_\Lambda)$, where

$$a_\Lambda = \left(\frac{\Omega_{m 0}}{2\Omega_{\Lambda 0}}\right)^{1/3}. \quad (44)$$

At any time $\tau > \tau_\Lambda$ the observed redshift, z_Λ , of a source which emitted its light at τ_Λ is therefore given by

$$1 + z_\Lambda = \frac{a(\tau)}{a_\Lambda} = \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m 0}}\right)^{1/3} a(\tau). \quad (45)$$

In the right-hand panel of Figure 4 the relation $z_\Lambda = z_\Lambda(\tau)$ is shown by the triple-dot-dashed curve. Note that equation (45) has a physical solution only if $a > a_\Lambda$. For our standard Λ -model, we find that $a_\Lambda = 0.60$, corresponding to time $\tau_\Lambda = 0.52$, and $z_\Lambda = 0.67$. Hence, in this model, cosmic acceleration started $\Delta t = (\tau_0 - \tau_\Lambda)t_{H 0} = 4.3h_0^{-1}$ Gyr ago. For $h_0 \approx 0.70$ this is 6.1 Gyr, and hence the acceleration started well before the formation of the solar system.

As time advances the Λ -sphere moves away from the observer and its conformal distance at time η is given by

$$\chi = \eta - \eta_\Lambda, \quad (46)$$

where $\eta_\Lambda = \eta(\tau_\Lambda)$. Hence the proper distance to the Λ -sphere is

$$d_\Lambda(\eta) = R\chi = R_{H 0} a(\eta) \frac{(\eta - \eta_\Lambda)}{R_{H 0}/R_0}, \quad (47)$$

and it is moving away from the observer with speed

$$v_\Lambda = \frac{d}{dt}(d_\Lambda) = c \left[1 + \frac{(\eta - \eta_\Lambda)}{R_{H 0}/R_0} \left(\frac{da}{d\tau} \right) \right]. \quad (48)$$

For our standard Λ -model we have that $\eta_\Lambda = 2.7$ and therefore $v_\Lambda(\tau_0) = 1.6c$. Furthermore $d_\Lambda(\tau_0) = 0.56R_{H 0}$, which is

about 47% of the proper distance to the ultimate event horizon. The evolution of the Λ -sphere is shown by the long-dashed curve in Figure 2. Note that this curve is the same as the observer's future light cone at time η_Λ .

From equations (40) and (41) we see that in a Λ dominated universe $d(1+z)/d\tau = dz/d\tau$ is zero, and $T_a(\tau) = (1+z)T_a(\tau_{em})$, when $(da/d\tau)_\tau = (da/d\tau)_{\tau_{em}}$. This corresponds to redshift z_{eq} given by

$$\begin{aligned} 1 + z_{eq} &= \left(\frac{\Omega_{\Lambda 0}}{2\Omega_{m 0}}\right) a^3(\tau) \left[1 + \sqrt{1 + \frac{4\Omega_{m 0}}{\Omega_{\Lambda 0} a^3(\tau)}} \right] \\ &= \frac{1}{4} \left[\frac{a(\tau)}{a_\Lambda} \right]^3 \left[1 + \sqrt{1 + 8 \left[\frac{a_\Lambda}{a(\tau)} \right]^3} \right], \end{aligned} \quad (49)$$

where we have used equations (40) and (11). Note that $d(1+z)/d\tau$ (and hence T_z) is positive for $z < z_{eq}$ and negative for $z > z_{eq}$. The relation $z_{eq} = z_{eq}(\tau)$ is shown by the dot-dashed curve in the right-hand panel of Figure 4 for our standard Λ -model. Note that equation (49) has a physical solution only if $a > a_\Lambda$.

From the discussion above we see that although the Λ -sphere bounds the region in our visible universe where the cosmological constant dominates the expansion, the influence of Λ extends beyond z_Λ and is in principle observable approximately out to redshift z_{eq} . We shall return to this point in § 5.

4.2. Brightness and Angular Size

At time τ the luminosity distance, d_L , of a source at conformal distance χ is given by

$$d_L(\tau) = \left(\frac{L}{4\pi F} \right)^{1/2} = R_0 r(\chi)(1+z)a(\tau), \quad (50)$$

where L is the luminosity of the source, F its apparent flux and z its redshift. The relation $r = r(\chi)$ is given by equation (16).

The distance modulus of the source is

$$\begin{aligned} m - M &= 25 + 5 \log \left(\frac{d_L}{\text{Mpc}} \right) \\ &= (5 \log e) \ln [r(\chi)(1+z)a(\tau)] + \text{constant}, \end{aligned} \quad (51)$$

where $\log$ stands for the logarithm with base 10, d_L is measured in Mpc, and m and M are the apparent and the absolute magnitude of the source, respectively.

The angular diameter distance of the source is given by

$$d_A(\tau) = \frac{d_L(\tau)}{(1+z)^2} = \frac{R_0 r(\chi)a(\tau)}{(1+z)} = R_0 r(\chi)a(\tau_{em}), \quad (52)$$

and if the source has proper diameter D , its apparent angular size ϕ is

$$\phi(\tau) = \frac{D}{d_A} = \left(\frac{D}{R_0} \right) \frac{1}{r(\chi)a(\tau_{em})}. \quad (53)$$

In order to understand the evolution of m and ϕ it is convenient to have an expression for their derivatives. By use of equations (51), (53), (40), and (41) we find that

$$\begin{aligned} \frac{dm}{d\tau} &= (5 \log e) \left[\frac{1}{(1+z)} \frac{d(1+z)}{d\tau} + \frac{1}{a} \frac{da}{d\tau} \right] \\ &= (5 \log e) \left[\frac{1}{T_z(\tau)} + \frac{1}{T_a(\tau)} \right] \end{aligned} \quad (54)$$

and

$$\begin{aligned} \frac{1}{\phi} \frac{d\phi}{d\tau} &= \left[\frac{1}{(1+z)} \frac{dz}{d\tau} - \frac{1}{a} \frac{da}{d\tau} \right] = \left[\frac{1}{T_z(\tau)} - \frac{1}{T_a(\tau)} \right] \\ &= - \frac{1}{(1+z)T_a(\tau_{\text{em}})}. \end{aligned} \quad (55)$$

In Figure 5 we show the luminosity distance, the distance modulus and the apparent angular size of sources at various distances χ as functions of cosmic time in our standard Λ -model. The sources are assumed to have the same intrinsic luminosity and the same proper diameters at all times (i.e., to be standard candles and standard rods). A given source comes within our observable universe after a time determined by its comoving conformal distance, as long as it is less than η_{max} , since sources further away are always beyond our event horizon and we never see them. Both d_L and $m - M$ for the source are infinite when the source appears with infinite redshift at $\eta = \chi$. These quantities decrease to a minimum and then increase to infinity at $\eta = \eta_{\text{max}}$. Each source appears with infinite angular size at $\eta = \chi$, then decreases monotonically with time to a

minimum at $\eta = \eta_{\text{max}}$, the minimum size being given by

$$\phi(\eta_{\text{max}}) = \left(\frac{D}{R_0} \right) \frac{1}{r(\chi)a(\eta_{\text{max}} - \chi)} = \left(\frac{D}{R_{H0}} \right) \Omega_{\Lambda 0}^{1/2} \frac{\chi}{r(\chi)}, \quad (56)$$

corresponding to the final and finite angular diameter distance

$$d_A(\eta_{\text{max}}) = \frac{R_{H0}}{\Omega_{\Lambda 0}^{1/2}} \frac{r(\chi)}{\chi}. \quad (57)$$

In deriving these expressions we have used equation (37).

Once the cosmological constant completely dominates the expansion (i.e., for $\tau \gg \tau_\bullet$ corresponding to $\eta_{\text{max}} > \eta > \eta_\bullet$) the source's redshift increases according to equation (39) and its luminosity distance is given by

$$d_L(\tau) \sim e^{2\Omega_{\Lambda 0}^{1/2} \tau}. \quad (58)$$

Hence its apparent magnitude grows linearly with time:

$$m(\tau) \approx \text{constant} + 10(\log e) \Omega_{\Lambda 0}^{1/2} \tau. \quad (59)$$

It is therefore clear that once a source has entered our observable universe we never see it leave, although once

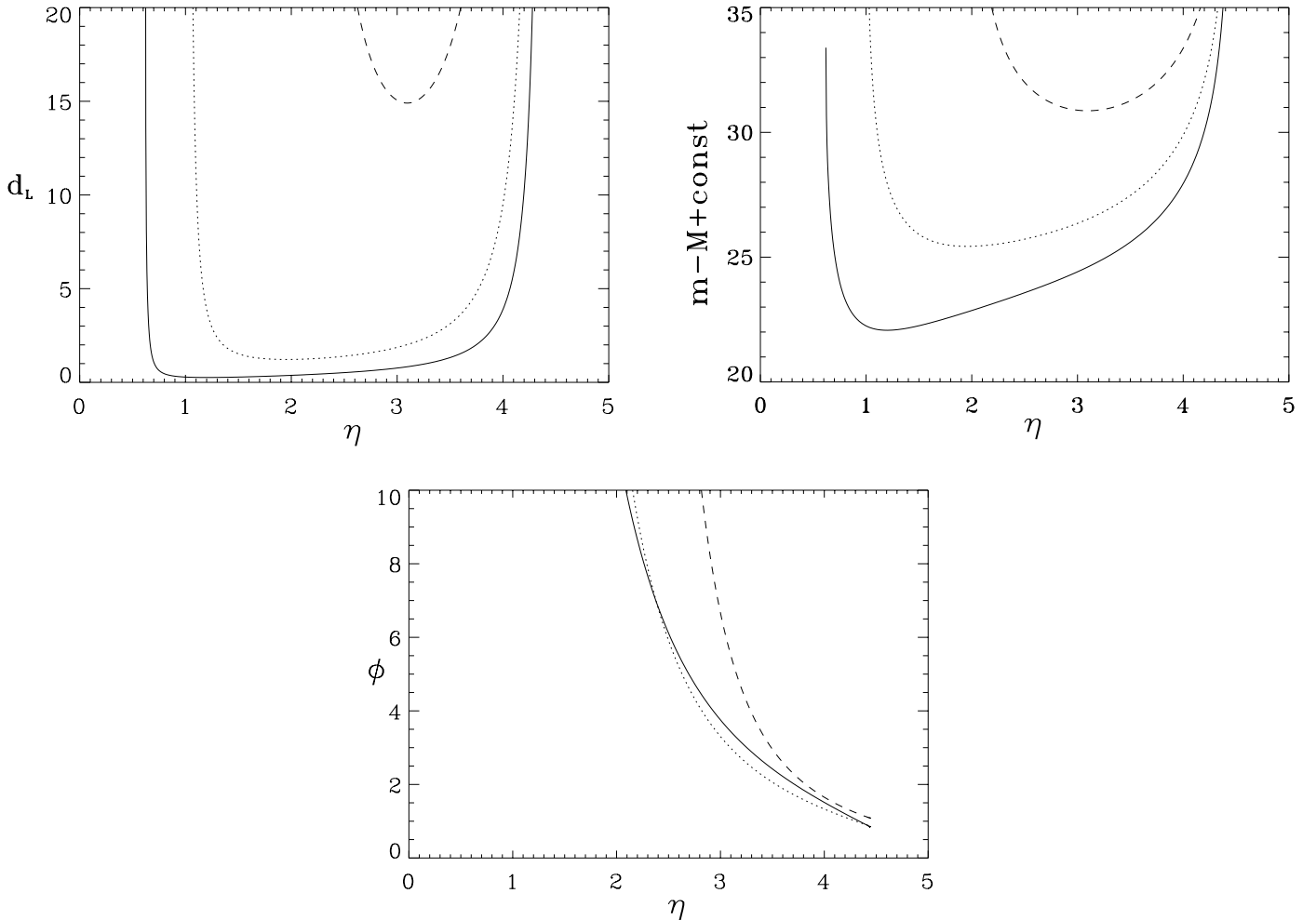


FIG. 5.—*Top left:* The luminosity distances (in units of R_{H0}) of the same sources as in Fig. 4 as functions of η . *Top right:* The distance moduli of the sources as functions of η . *Bottom:* The apparent angular sizes (in units of D/R_{H0}) as functions of η .

cosmic acceleration has started, its brightness gets below any finite detection limit in a few Hubble times.

5. CHANGES IN OBSERVABLE PROPERTIES AT THE PRESENT EPOCH

The main astronomical evidence for cosmic acceleration comes from investigations based on classical cosmological tests, in particular the m - z relation with Type Ia supernovae as standard candles. This has been treated in great detail by the research groups who discovered the acceleration (Perlmutter et al. 1999; Garnavich et al. 1998; Riess et al. 1999, 2001), and we shall therefore not discuss these tests here.

A related but more difficult approach, at least observationally, is to consider changes in cosmological observables over extended periods of observing time and see how they are affected by a cosmological constant or a quintessence field. We have laid the foundation for such a discussion in the previous section.

Let us consider a source with redshift z on our present past light cone and determine its change in redshift, Δz , during a time interval $\Delta\tau_0 \ll \tau_0$. Using equation (40) we find

$$\frac{1}{(1+z)} \frac{\Delta z}{\Delta\tau_0} = \frac{1}{(1+z)} \frac{\Delta(1+z)}{\Delta\tau_0} \approx \left[1 - \left(\frac{da}{d\tau} \right)_{\tau_{em}} \right], \quad (60)$$

and by equation (11) this can be written in terms of observables as

$$\begin{aligned} \frac{1}{(1+z)} \frac{\Delta(1+z)}{\Delta\tau_0} \approx & \left\{ 1 - \left[\Omega_{m0}(1+z) \right. \right. \\ & + \Omega_{r0}(1+z)^2 + \frac{\Omega_{\Lambda0}}{(1+z)^2} \\ & \left. \left. + \frac{\Omega_{Q0}}{(1+z)^{-(1+3w_Q)}} + (1-\Omega_0) \right]^{1/2} \right\}, \end{aligned} \quad (61)$$

where $\Omega_0 = \Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda0} + \Omega_{Q0}$.

In Figure 6 we plot $[\Delta(1+z)/\Delta\tau_0]/(1+z)$ as a function of z for selected values of Ω_{m0} and $\Omega_{\Lambda0}$ with $\Omega_{Q0} = \Omega_{r0} = 0$.

For this choice of parameters the zeros are at $z = 0$ and $z = z_{eq}$, where by equation (49)

$$1 + z_{eq} = \left(\frac{\Omega_{\Lambda0}}{2\Omega_{m0}} \right) \left(1 + \sqrt{1 + \frac{4\Omega_{m0}}{\Omega_{\Lambda0}}} \right). \quad (62)$$

The presence of the cosmological constant makes $[\Delta(1+z)/\Delta\tau_0]/(1+z)$ positive for $0 < z < z_{eq}$ and its maximum value at the Λ -sphere (with $1 + z_{\Lambda} = (2\Omega_{\Lambda0}/\Omega_{m0})^{1/3}$) is given by

$$\begin{aligned} \left[\frac{1}{(1+z)} \frac{\Delta(1+z)}{\Delta\tau_0} \right]_{z=z_{\Lambda}} = & 1 - \left\{ 3 \left[\left(\frac{\Omega_{m0}}{2} \right)^2 \Omega_{\Lambda0} \right]^{1/3} \right. \\ & \left. + (1 - \Omega_{m0} - \Omega_{\Lambda0}) \right\}^{1/2}. \end{aligned} \quad (63)$$

For our standard Λ -model we find that $z_{eq} = 2.1$ and that the maximum change given by equation (63) is 0.13 (corresponding to $T_z = 7.7$) at $z_{\Lambda} = 0.67$. Hence the maximum effects of the cosmological constant on the change in redshift is given by $\Delta z = 1.67 \times 0.13 \Delta\tau_0 = 0.22 \Delta t_0 / t_{H0}$. For $\Delta t_0 = 100$ years, say, Δz at maximum is therefore only of the order of 10^{-9} . This is a very small number and such minute changes in z will probably not be observable in the near future (see, however, Loeb 1998 for a detailed discussion of the observational situation).

In a similar way one can investigate changes in the apparent magnitude and the apparent angular size of a given source with present redshift z . Using equations (54) and (55) we find to first order in $\Delta\tau_0$:

$$\frac{1}{(5 \log e)} \frac{\Delta m}{\Delta\tau_0} \approx \left[1 + \frac{1}{(1+z)} \frac{\Delta(1+z)}{\Delta\tau_0} \right] \quad (64)$$

and

$$\frac{1}{\phi} \frac{\Delta\phi}{\Delta\tau_0} \approx \left[\frac{1}{(1+z)} \frac{\Delta(1+z)}{\Delta\tau_0} - 1 \right], \quad (65)$$

with $(1+z)^{-1}[\Delta(1+z)/\Delta\tau_0]$ given by equation (61). It is clear that Figure 6 can be used to investigate the behavior of both $\Delta m/\Delta\tau_0$ and $\Delta\phi/\phi\Delta\tau_0$ as functions of z for the selected values of Ω_{m0} and $\Omega_{\Lambda0}$. From the results already

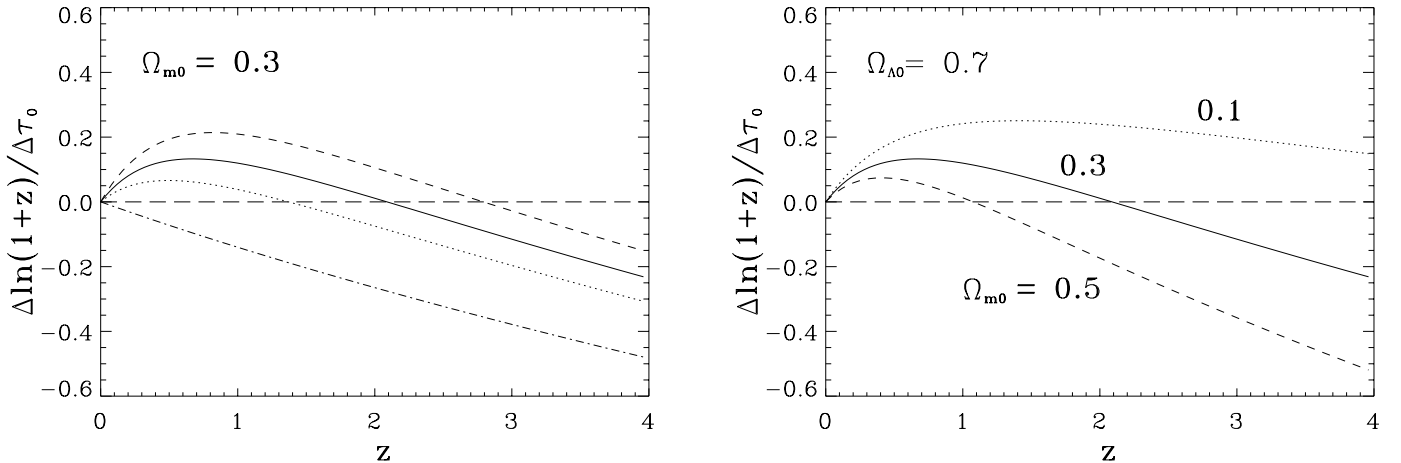


FIG. 6.—Left: $(\Delta(1+z)/\Delta\tau_0)/(1+z)$ as a function of z for $\Omega_{m0} = 0.30$ and $\Omega_{\Lambda0} = 0$ (dash-dotted curve), $\Omega_{\Lambda0} = 0.50$ (dotted curve), 0.70 (solid curve), and 0.90 (dashed curve). Right: The same quantity as a function of z for $\Omega_{\Lambda0} = 0.70$ and $\Omega_{m0} = 0.10$ (dotted curve), 0.30 (solid curve), and 0.50 (dashed curve). All models in this figure have $\Omega_{Q0} = \Omega_{r0} = 0$.

obtained we find for our standard Λ -model that at z_Λ the change in magnitude is $\Delta m = 2.5\Delta\tau_0 \approx 10^{-8}$ over a period of 100 years. In a similar way the maximum relative changes in ϕ due to the accelerated expansion is $\Delta\phi/\phi = -0.87\Delta\tau_0 \approx -10^{-8}$ over the same period of time. Such small changes will presumably not be observable in the near future.

We conclude this section by emphasizing that although the prospects for actually measuring the changes in redshift, apparent brightness and apparent size of cosmological sources do not seem promising, it is of interest to investigate Figure 6 with respect to the general effects of a cosmological constant at the present epoch. We see e.g., that for distant sources with high z (i.e., $z > z_{\text{eq}} > z_\Lambda$), corresponding to emission at cosmic times before Λ becomes dynamically dominant, their redshift is decreasing and their apparent brightness is also decreasing, but relatively slowly. However, for cosmological sources with $z < z_{\text{eq}}$ the redshift is increasing due to the repulsive effects of Λ , and the apparent brightness is decreasing relatively fast. Thus the influence of Λ is considerable outside the Λ -sphere, out to $z \approx z_{\text{eq}}$. This is in agreement with the time evolution of the observables discussed in § 4.1 and 4.2 and demonstrated in Figures 4 and 5.

6. CAUSAL CONNECTIONS AND THE EXTENT OF OUR OBSERVABLE UNIVERSE

Consider Figure 7. The future light cone (*dot-dashed curve*) at observing time η crosses the event horizon (*solid curve*) at the event (χ_*, η_*) . The world line of a source that passes through that event is also shown (*long-dashed curve*). We will not be able to receive any signal from that source sent later than η_* . Because of symmetry, the source can not receive any signal from us sent after η_* . Hence the source passes out of our sphere of influence at time η_* . However, our evolving past light cone (shown as it is at the present epoch by the dotted curve) crosses the source's world line right till the end at η_{max} , at which time the source is seen as it was at η_* but with infinite redshift. At time η we see the source with redshift z_* as it was at time $\eta_{\text{em}*}$. It is easy to see

that

$$\eta_* = \frac{\eta_{\text{max}} + \eta}{2} \quad (66)$$

and

$$\chi_* = \frac{\eta_{\text{max}} - \eta}{2}. \quad (67)$$

Also

$$\eta_{\text{em}*} = \frac{3\eta - \eta_{\text{max}}}{2}, \quad (68)$$

and for this last equation to be valid we must have $\eta \geq \eta_{\text{max}}/3$, since at earlier times the source is outside the particle horizon. Its redshift is given by

$$1 + z_* = \frac{a(\eta)}{a(\eta_{\text{em}*})}. \quad (69)$$

Next consider the source at χ_c in Figure 7 (*triple-dot-dashed curve*) that is crossing the event horizon at time η . Clearly

$$\chi_c = \eta_{\text{max}} - \eta = 2\chi_*, \quad (70)$$

and we see this source with redshift z_c given by

$$1 + z_c = \frac{a(\eta)}{a(\eta_{\text{emc}})}. \quad (71)$$

the light we see being emitted at time

$$\eta_{\text{emc}} = 2\eta - \eta_{\text{max}}. \quad (72)$$

Note that we must have $\eta > \eta_{\text{max}}/2$ for equation (72) to be valid. At earlier times the source is outside the particle horizon.

For our standard world model shown in Figure 7 we have that the present time of observation, η , is equal to $\eta_0 = 3.3$. Also $\eta_{\text{max}} = 4.5$, $\eta_* = 3.9$, $\chi_* = 0.57$ and $\eta_{\text{em}*} = 2.7$ (we remind the reader that numerical values of all η 's and χ 's are in units of R_{H0}/R_0). We also find that $z_* = 0.68$. This means, that the light being emitted now by sources having

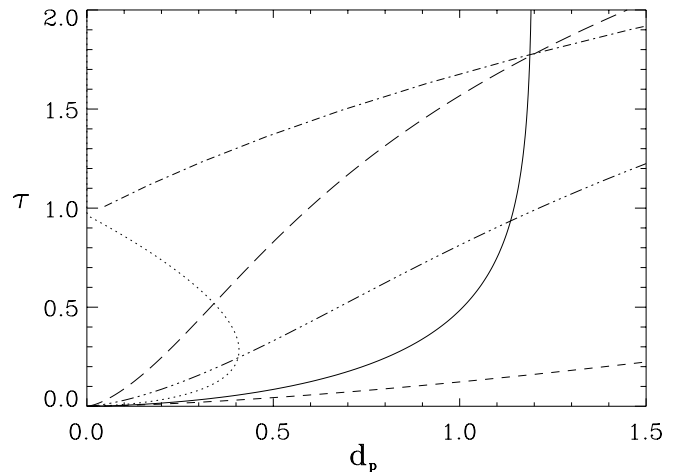
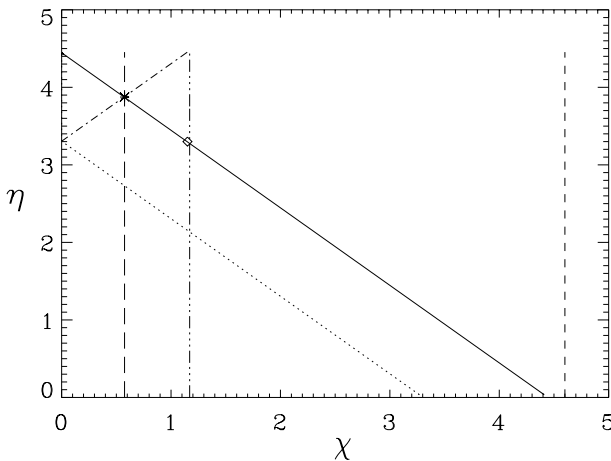


FIG. 7.—Left: Sources at χ_* (*long-dashed line*) and χ_c (*dash-triple dotted line*) crossing the event horizon at times η_* and η , respectively. Also shown is the world line of a source which is always outside the event horizon (*short-dashed line*). Right: The same situation in proper coordinates τ vs. $d_p = d/R_{H0}$. The model is our standard Λ -model with the time of observation equal to $\eta_0 = 3.3$. See text for further explanations.

present redshift greater than 0.68, will not reach us until the sources have crossed our event horizon, at which time they are completely out of causal contact.

In this model we also have $\chi_c = 1.1$, $\eta_{\text{emc}} = 2.2$ and $z_c = 1.7$. Hence all sources with present redshift greater than 1.7 have already crossed our event horizon, and are thus completely out of causal contact with us (see also Starkman, Trodden, & Vachaspati 1999 and Starobinsky 2000, who reach similar conclusions). From this one can easily estimate the number of sources that are still within the event horizon as compared with the initial matter content. The number of sources is proportional to the comoving proper volume and hence the relative number of sources presently within the horizon is equal to $[\chi_c(\eta_0)/\chi_{\text{eh}}(0)]^3 = [\chi_c(\eta_0)/\eta_{\text{max}}]^3 = 0.015$. This means that more than 98% of all sources initially within our observable part of the universe have already crossed the event horizon.

We can also estimate the portion of observable sources that we could have seen by now, at least in principle. It is simply given by $[\chi_{\text{ph}}(\eta_0)/\chi_{\text{eh}}(0)]^3 = (\eta_0/\eta_{\text{max}})^3 \approx 0.40$. Hence, we still have not seen 60% of the observable sources.

To summarize: In our standard Λ -model the event horizon will ultimately be stationary at a proper distance of $1.2R_{H0} = 3.6h_0^{-1}$ Gpc (note that this is also the ultimate angular diameter distance to the event horizon). We have already seen about 40% of the sources that are in principle observable, but about 98% of all cosmic sources originally within our observable part of the universe have already left. This includes all sources that we presently see and have redshift higher than 1.7. Because of the finite speed of light we will eventually be able to see all the sources originally inside our event horizon, but only as they were in the past before crossing the horizon. Once the cosmological constant dominates the expansion their redshift will, however, increase exponentially with time and they will fade away on a timescale measured in a few Hubble times. Of course this applies only to sources that are distant enough to participate in the cosmic expansion, i.e., sources outside our local supercluster but originally within our event horizon.

7. QUINTESSENCE

Although by invoking Einstein's cosmological constant it is easy to explain why the expansion of the universe is presently accelerating, there has recently been considerable discussion of the possibility that something else is causing the acceleration, mimicking the effects of a cosmological constant at the present epoch. The main reason for this is simply that theoretical particle physicists have so far been unable to explain why the cosmological constant has such a small nonzero value ($\Lambda \approx 10^{-56} \text{ cm}^{-2}$ corresponding to $\Omega_{\Lambda 0} \approx 0.7$), their most natural estimates giving values of $\Lambda \sim 10^{64} \text{ cm}^{-2}$ corresponding to $\Omega_{\Lambda 0} \sim 10^{120}$ (see, e.g., Weinberg 1989, 2000; Witten 2000 and references therein).

Several other possible causes for the cosmic acceleration have been discussed in the literature, including quintessence, frustrated network of topological defects, time-varying particle masses and effects from extra dimensions (see, e.g., Huterer & Turner 2000, Binétruy 2000, and Weinberg 2001 for further discussion and references).

In this paper we shall only investigate the effects of one of these alternatives. We choose the case of a slowly evolving scalar field (or quintessence; Peebles & Ratra 1988; Ratra & Peebles 1988; Wetterich 1988; Caldwell, Dave, & Steinhardt 1998; Zlatev, Wang, & Steinhardt 1999) leading to an

equation of state of the form $P_Q = w_Q \rho_Q c^2$, where w_Q may or may not be a function of cosmic time. We assume as before that we are investigating epochs where radiation can be neglected ($\Omega_{r0} = 0$) and that the quintessence field is decoupled from matter.

In order to investigate the effects of quintessence on the evolution of the observable universe we shall furthermore assume that $\Omega_{\Lambda 0} = 0$ and that w_Q is a constant (assuming a constant w_Q is not a serious restriction since our results can easily be extended to the time dependent case). It then follows from equation (3) that the mass-energy density of the quintessence component is given by

$$\rho_Q = \rho_{Q0} a^{-3(1+w_Q)}. \quad (73)$$

Realistic expanding models, with ρ_Q decreasing with time, thus require $w_Q > -1$. The case corresponding to $w_Q = -1$ is of course equivalent to the cosmological constant case.

Equation (11) now reduces to

$$\frac{da}{d\tau} = \left[\Omega_{m0} \left(\frac{1}{a} - 1 \right) + \Omega_{Q0} \left(\frac{1}{a^{1+3w_Q}} - 1 \right) + 1 \right]^{1/2}, \quad (74)$$

from which we see that for ever-expanding big bang models, the quintessence field will ultimately dominate the expansion if $w_Q < 0$, and at late times the scale factor will grow with τ according to

$$a \sim (n\Omega_{Q0}^{1/2} \tau)^n, \quad (75)$$

where

$$n = \frac{2}{3(1+w_Q)}. \quad (76)$$

This should be compared with the corresponding behavior of a at late times in a Λ -dominated universe (eq. [34]). Examples demonstrating the difference are shown in Figure 8.

From the results above and the discussion at the end of § 3.4 we see that only models with $w_Q < -1/3$ (corresponding to $n > 1$) have event horizons. The theoretically interesting range for w_Q is therefore $-1 < w_Q < -\frac{1}{3}$ (corresponding to $\infty > n > 1$). This should be compared to observations, which seem to indicate that quintessence can only be viable at present if $-1 < w_Q < w_{\text{max}}$ with the upper limit, w_{max} , not greater than -0.4 and probably lower (see e.g., Huterer & Turner 2000 and Wang et al. 2000 for a discussion of the observational situation). Thus, if the cosmic acceleration is due to quintessential dark energy, the universe has event horizons just as in the case of a true cosmological constant (see in this context Hellerman, Kaloper, & Susskind 2001 and Fischler et al. 2001, who discuss the problems this poses for string theory). It should be pointed out, however, that if for some reason w_Q changes in the future, so that it ultimately becomes larger than $-\frac{1}{3}$ (corresponding to $n < 1$), then the event horizons disappear. Also note that for a given Ω_{Q0} the maximum conformal time, η_{max} , tends to infinity as w_Q approaches $-\frac{1}{3}$ from below.

In what follows we shall investigate in detail the evolution of the observable universe when quintessence is the single cause of cosmic acceleration. We use the same methods as above and compare the results to those of models with a positive cosmological constant.

In Figure 8 we show the normalized scale factor a as a function of time for $\Omega_{m0} = 0.30$, $\Omega_{Q0} = 0.70$ and two values

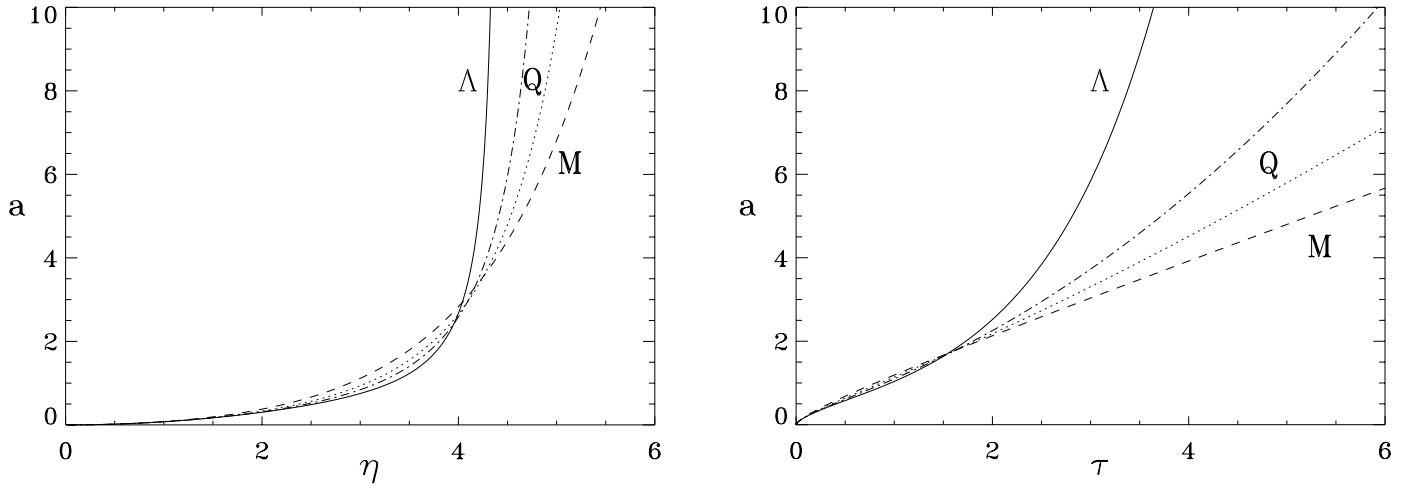


FIG. 8.—Comparison of the time evolution of the scale factor, a , in different cosmological models. *Left*: Scale factor a as a function of η for the standard Λ -model (solid curve, marked Λ) and two quintessence models with $\Omega_{m0} = 0.30$ and $\Omega_{Q0} = 0.70$. One has $w_Q = -0.50$ (dotted curve, marked Q), the other $w_Q = -2/3$ (dash-dotted curve). Also shown is the M -model with $\Omega_{m0} = 0.30$ and $\Omega_{\Lambda0} = \Omega_{Q0} = 0$ (dashed curve, marked M). *Right*: Scale factor a as a function of τ for the same models. Compare with Fig. 1.

of w_Q : -0.50 (dotted curve, marked Q) and $-\frac{2}{3}$ (dash-dotted curve). In the following we shall refer to the model with $w_Q = -0.50$ as the Q -model. For this model $\eta_{\max} = 7.7$ and $\eta_0 = 3.1$. Also shown for comparison are two other models: our standard Λ -model (solid curve, marked Λ), and a model, which we shall call the M -model, with $\Omega_{m0} = 0.30$ and both Ω_Q and $\Omega_{\Lambda0}$ equal to zero (dashed curve, marked M). Note that since $w_Q < -\frac{1}{3}$ for the quintessence models, they both have event horizons.

From Figure 8 one sees how the expansion of quintessence models, with Ω_{Q0} and Ω_{m0} both fixed, depends on w_Q . As w_Q takes decreasing values in the range from $-\frac{1}{3}$ to -1 , the corresponding curves change continuously from the M -curve to the Λ -curve.

Figure 9 shows the past and future light cones, the Hubble surface, the particle horizon and the event horizon for our Q -model, and should be compared to Figure 2 for the standard Λ -model. Applying the methods introduced in

§ 3 to quintessence models in general it is easy to see that once quintessence completely dominates the expansion, the proper distance to the Hubble surface is given by

$$d_{\text{hs}}(\tau) \approx \frac{R_{H0}}{n} \tau = \frac{3(1+w_Q)R_{H0}}{2} \tau \quad (77)$$

and that the proper distance to the event horizon is

$$d_{\text{eh}}(\tau) \approx \frac{2R_{H0}}{n|1+3w_Q|} \tau = \frac{3(1+w_Q)R_{H0}}{|1+3w_Q|} \tau. \quad (78)$$

We emphasize that if $w_Q \geq -\frac{1}{3}$ there is no event horizon, and hence the last expression (78) is only valid for models with $w_Q < -\frac{1}{3}$. Both d_{hs} and d_{eh} increase linearly with time, but since $d_{\text{hs}}(\tau) = |1+3w_Q|d_{\text{eh}}(\tau)/2 < d_{\text{eh}}(\tau)$ for all $-1 < w_Q < -\frac{1}{3}$, the Hubble surface is always considerably closer than the event horizon. This is in contrast to models

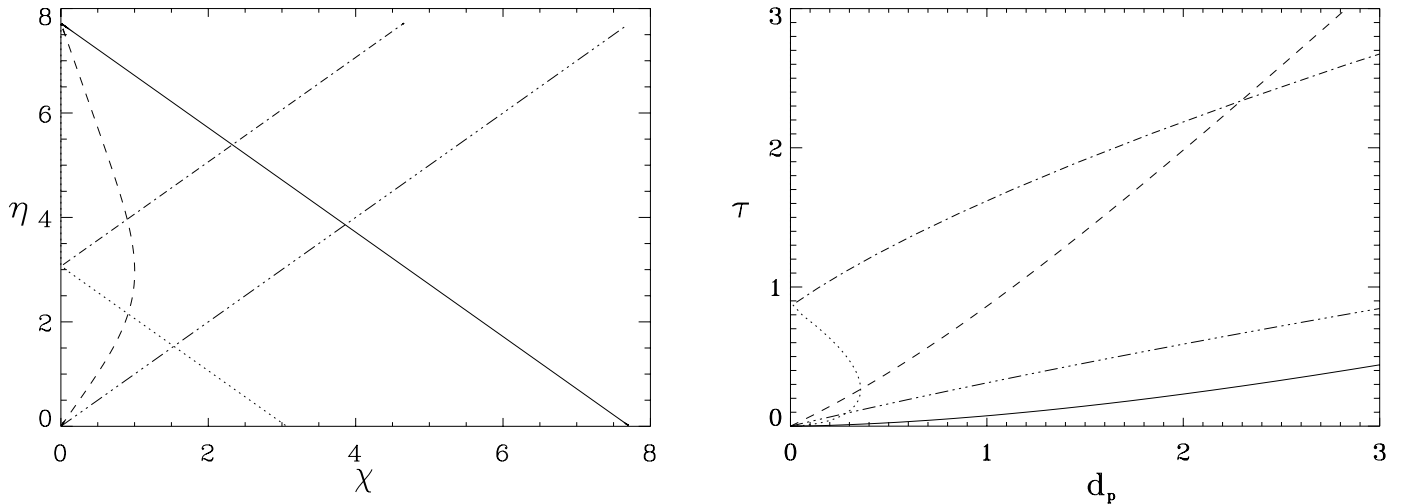


FIG. 9.—Quintessence universe with model parameters $\Omega_{m0} = 0.30$, $\Omega_{Q0} = 0.70$, and $w_Q = -0.50$. *Left*: The past light cone (dotted line) and the future light cone (dot-dashed line) at the present epoch in a $\eta - \chi$ diagram. Also shown are the Hubble surface (dashed curve), the particle or visual horizon (dash-triple dotted line), and the event horizon (solid line). Here $\eta_0 = 3.1$ and $\eta_{\max} = 7.7$. *Right*: Same as a in a $\tau - d_p$ diagram.

with a cosmological constant where both distances quickly approach the same finite limit given by equation (35).

In order to investigate the evolution of the redshift, the apparent magnitude and the angular size of distant sources in a quintessential universe we pick a source at a typical conformal distance $\chi = 0.60$. The results are shown in Figure 10 for the models of Figure 8. Note that at late times

$$z \sim (n\Omega_{Q0}^{1/2} \tau)^n \quad (79)$$

and

$$d_L \sim (n\Omega_{Q0}^{1/2} \tau)^{2n}. \quad (80)$$

Hence m is a linear function of $\log(\tau)$. Furthermore the minimum size of ϕ is given by equation (56) as before. These results should be compared to the late time behavior of z , d_L and m in models with a cosmological constant, shown by equations (39), (58), and (59) and in Figures 4 and 5.

7.1. The Q -Sphere

In § 4.1 we defined the Λ -sphere as the surface bounding the region in our visible universe where the expansion is

accelerating due to the presence of the cosmological constant. For the quintessence models we similarly define a Q -sphere as the surface in the visible universe bounding the accelerating region driven by the quintessence component.

We denote the time when quintessence acceleration becomes dominant by τ_Q and use equation (43) with $\Omega_{\Lambda 0} = 0$ to find the value $a_Q = a(\tau_Q)$. It is given by

$$a_Q = \left(\frac{|1 + 3w_Q| \Omega_{Q0}}{\Omega_{m0}} \right)^{1/3w_Q}. \quad (81)$$

As a result, at any time $\tau > \tau_Q$ the observed redshift, z_Q , of a source which emitted its light at τ_Q is given by

$$1 + z_Q = \frac{a(\tau)}{a_Q} = \left(\frac{|1 + 3w_Q| \Omega_{Q0}}{\Omega_{m0}} \right)^{-1/3w_Q} a(\tau). \quad (82)$$

In the same way as in the previous section, we find from equations (40) and (41) that in a universe dominated by quintessence, $d(1+z)/d\tau = dz/d\tau$ is zero, and $T_a(\tau) = (1+z)T_a(\tau_{em})$, when $(da/d\tau)_\tau = (da/d\tau)_{\tau_{em}}$. This particular epoch corresponds to a redshift $z = z_{eq,Q}$ given by the solu-

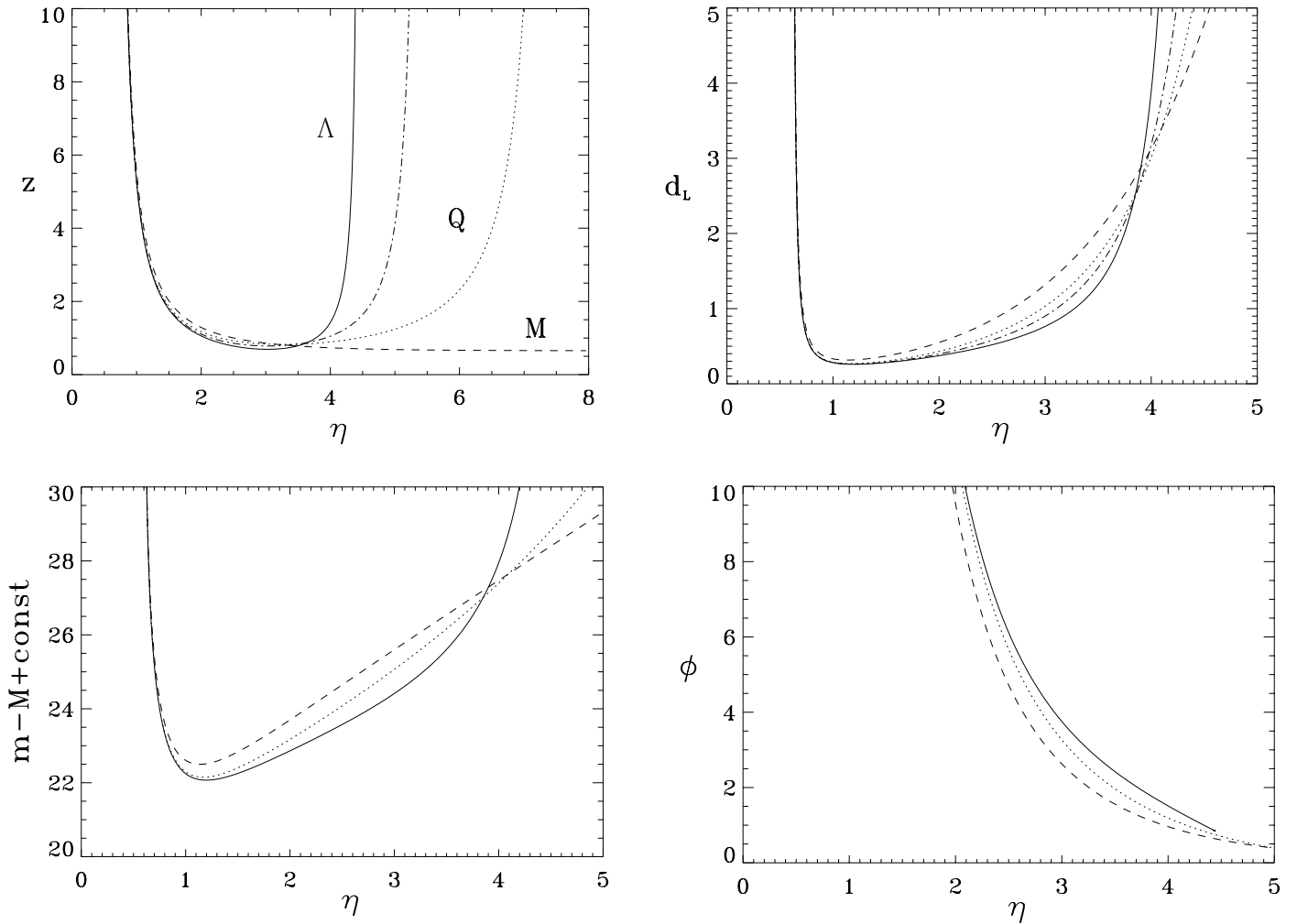


FIG. 10.—Top left: Evolution of the redshift of a source at $\chi = 0.60$ for the model universes of Fig. 8: The standard Λ -model (solid curve), the Q -model (dotted curve), and the M -model (dashed curve). Also shown is the evolution of the quintessence model with $w_Q = -2/3$ (dash-dotted curve). Top right: The luminosity distances (in units of R_{H0}) for the same source and models as in the top left-hand panel. Bottom left: The distance moduli for the Λ , Q , and M -models. Bottom right: The apparent angular sizes (in units of D/R_{H0}) for the Λ , Q , and M -models.

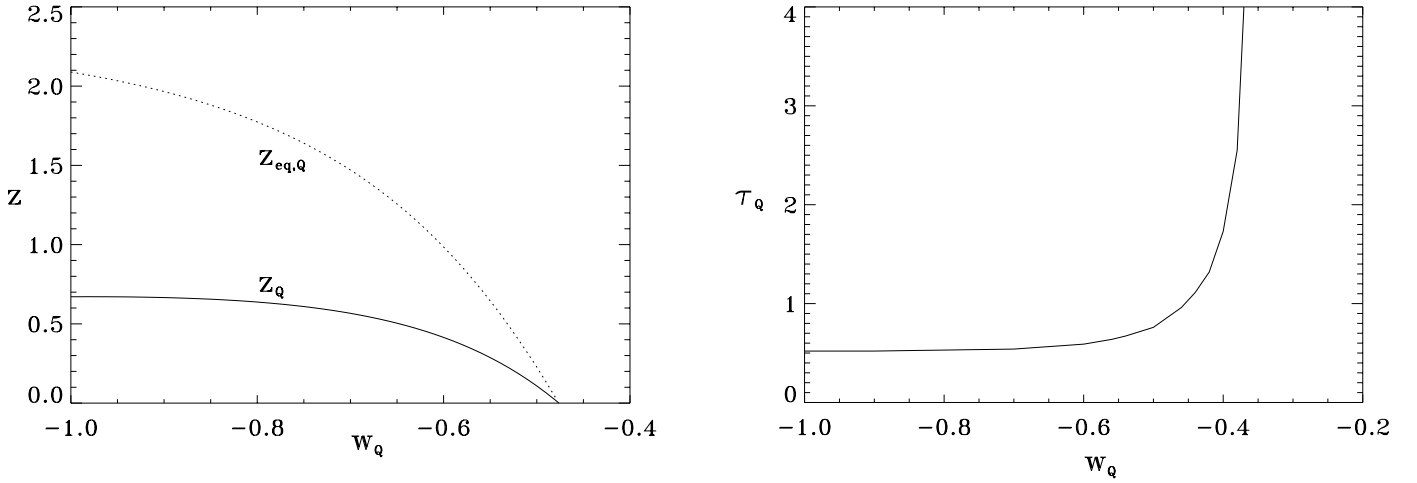


FIG. 11.— Q -sphere at the present epoch for models with $\Omega_{m0} = 0.30$ and $\Omega_{Q0} = 0.70$. *Left*: The redshift z_Q as a function of w_Q . Also shown is the redshift $z_{eq,Q}$. *Right*: The time τ_Q as a function of w_Q .

tion of the algebraic equation

$$\frac{\Omega_{m0}}{\Omega_{Q0}} a^{3w_Q} [(1+z) - 1] = 1 - (1+z)^{1+3w_Q}. \quad (83)$$

where we have used equation (74). This equation can easily be solved numerically, and Figure 11 shows $z_{eq,Q}$ at the present epoch, together with z_Q and τ_Q , as functions of w_Q for models with $\Omega_{m0} = 0.30$ and $\Omega_{Q0} = 0.70$. In general, as w_Q approaches -1 , the redshift $z_{eq,Q}$ approaches the value z_{eq} for the Λ -model (eq. [49] with Ω_{Q0} instead of $\Omega_{\Lambda0}$). In the same way z_Q and τ_Q approach z_Λ and τ_Λ , respectively. For the special case $w_Q = -\frac{2}{3}$ equation (83) has a simple solution: $1 + z_{eq,Q} = (\Omega_{Q0}/\Omega_{m0})a^2(\tau) = [a(\tau)/a_Q]^2$. Note that since $1 + z \geq 1$, equations (82) and (83) have physical solutions only if $a \geq a_Q$.

Figure 11 also shows that if w_Q is less than $-\frac{2}{3}$, then z_Q and τ_Q vary slowly with changing w_Q , indicating that at the present epoch numerical values of observables are not very sensitive to w_Q in the range $-1 < w_Q < -\frac{2}{3}$. The same thing can be inferred from Figures 8 and 10. This is similar to the corresponding results of Gudmundsson & Rögnvaldsson (1990) for FRW-models without a cosmological constant, which show that for models with the same value of q_0 the classical cosmological tests are degenerate at low redshifts with respect to different values of the pressure parameter w_i (see eq. [8]). It is therefore necessary to go to high redshifts in order to distinguish between the various models (see also Maor, Brustein, & Steinhardt 2001).

Continuing with the general approach already introduced for models with a cosmological constant, we can similarly determine changes in observable quantities over extended periods of observing time. In Figure 12 we show the relative change $(\Delta(1+z)/\Delta\tau_0)/(1+z) = 1/T_z$ at the present epoch as a function of z for our Q -model. We find that $a_Q = 0.90$, $\tau_Q = 0.77$, $z_Q = 0.11$ and $z_{eq,Q} = 0.23$. In this case $\tau_0 = 0.87$, so cosmic acceleration has started relatively recently, i.e., about 1.4 Gyr ago if $h_0 \approx 0.70$.

Finally, let us investigate the extent of the observable universe in the Q -model as well as the question of causal connections. We use the same terminology and notation as in § 6 so the results can easily be compared to the corresponding results for the Λ -model. In the Q -model with

$\Omega_{m0} = 0.30$, $\Omega_{Q0} = 0.70$, and $w_Q = -0.50$, we find that $\eta_0 = 3.1$, $\eta_{\max} = 7.7$, $\eta_{\text{em*}} = 0.72$, and $z_* = 24$. This means, that the light being emitted now by sources having redshift greater than 24, will not reach us until the sources have crossed our event horizon. This particular value is much larger than the highest measured redshift at the present time.

In this model we also have that $\chi_c = 4.7$, which is further away than the particle horizon at $\chi_{\text{ph}} = 3.1$. Hence, none of the sources that we can observe at the present epoch have yet crossed our event horizon. As a result we presently have causal contact with all of them. The relative number of sources presently within the event horizon is equal to $[\chi_c(\eta_0)/\chi_{\text{eh}}(0)]^3 = 0.23$. This means that in this model 77% of all sources initially within our observable part of the universe have already crossed the event horizon. So far we have not seen any of these departed sources but they will appear in our sky in the future and show themselves as they were in a younger universe.

The portion of the observable sources that we could already have seen in principle is given by

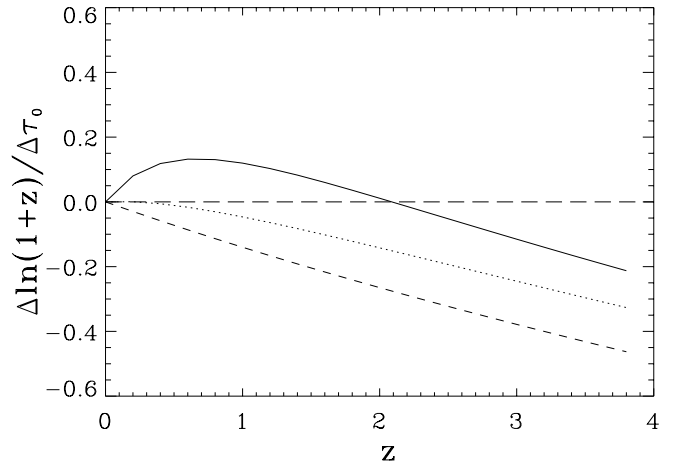


FIG. 12.— $(\Delta(1+z)/\Delta\tau_0)/(1+z)$ at the present epoch as a function of z for different model universes of Fig. 8: The standard Λ -model (solid curve), the Q -model (dotted curve), and the M -model (dashed curve).

$[\chi_{\text{ph}}(\eta_0)/\chi_{\text{ch}}(0)]^3 = 0.065$. Hence, in this model, we have yet to see about 93% of the observable sources.

It should be emphasized that we use this Q -model only as a pedagogical example in order to show the effects of w_Q on the various numerical results. The astronomical observations indicate that the high value of w_Q corresponding to this model may not be realistic, and that a value closer to the Λ -value of -1 is more likely to be correct. One should also keep in mind that observational results are not very sensitive to the value of w_Q , if $-1 < w_Q < -\frac{2}{3}$.

8. DISCUSSION AND CONCLUSIONS

In this paper we have discussed the evolution of ever-expanding big bang universes which undergo acceleration, due either to a cosmological constant, Λ , or to a quintessence field, Q . In particular we have investigated the evolution of our observable part of the universe with emphasis on the evolution of our past light cone, the Hubble sphere and the particle and event horizons. The Λ -sphere (or Q -sphere, in the case of quintessence), which is the surface bounding the region in our visible universe where cosmic acceleration dominates, has also been investigated in detail. We have traced observables such as redshift, apparent magnitude and apparent angular size of distant sources through cosmic history, and shown in considerable detail how their images change and fade once the cosmic expansion is accelerating.

Taking at face value recent observations which indicate that $\Omega_{\Lambda 0} = 0.70$, $\Omega_{m0} = 0.30$ and $h_0 = 0.70$ we find that the universe is presently about 13.5 Gyr old, and that cosmic acceleration started 6.1 Gyr ago, well before the formation of the solar system. The Λ -sphere is presently at a redshift of 0.67, and the redshifts of sources out to a redshift of 2.1 are increasing with time owing to the influence of Λ . Further out on the light cone the redshift is decreasing as in a universe without a cosmological constant. Within a few Hubble times the event horizon will be stationary at a fixed proper distance of 5.1 Gpc. This distance limits the extent of our observable universe for all time.

Cosmic sources with redshifts in the range 0.68–1.7 are now emitting light that will not reach us until these sources have crossed our event horizon. At that time they will be completely out of causal contact with us. All sources with redshift larger than 1.7 have already crossed the event horizon and are thus out of causal contact.

About 98% of all sources originally within our observable part of the universe have by now crossed the event horizon. Because of the finite speed of light we still “see” a large portion of these sources (about 40%, the ones inside our particle horizon) and will eventually be able to see them all. They will appear as they were in the distant past before crossing the event horizon. Because of redshift effects, all these sources will, however, fade away and disappear from view on a timescale measured in a few Hubble times.

The quintessence models become equivalent to a model with a cosmological constant in the limit when w_Q tends to -1 , and all quintessence models with w_Q in the range $-1 < w_Q < -\frac{1}{3}$ have event horizons. For w_Q less than about $-\frac{2}{3}$ the numerical values of various observables at the present epoch do not depend critically on w_Q , and are very similar to the values for $w_Q = -1$. Hence there is an observational degeneracy with respect to w_Q in that range.

Returning to the standard Λ -model it is clear that the future evolution of the observable universe is rather bleak from the human point of view. The receding galaxies will approach the cosmic event horizon on a timescale measured in a few Hubble times. Observers will not see the event horizon as such, but as galaxies approach it, their apparent motion slows down because of time dilation, and finally they will appear to be hovering at the horizon. This is why the apparent angular size of galaxies tends to a finite value at infinite time.

Because of the exponentially increasing redshift, the apparent luminosity of the galaxies decreases on a timescale of a few Hubble times, making them disappear from view. If observers had instruments sensitive enough so that they could follow the galaxies for all time, they would eventually see all the matter originally within the observable universe forming a membrane at the event horizon. This is analogous to what an observer, stationed far away from a black hole, would see if he was watching luminous matter falling into the black hole.

From this discussion it is clear that in Λ -models (and also in quintessence models, as long as $-1 < w_Q < -\frac{1}{3}$) any fundamental observer (more precisely his local supercluster) will be left alone in his observable universe within a few Hubble times after the big bang. This rather dismal prospect raises interesting questions about the future evolution of life in the universe and cosmic communication. We shall not tackle such questions here but instead refer the reader to the papers by Gott (1996) and Krauss & Starkman (2000), which discuss the fate of life in an accelerating universe.

Finally we mention that if the expansion is dominated by a cosmological constant one expects that eventually there will be upward quantum fluctuations making bubbles of high density vacuum which consequently undergo inflation (Garriga & Vilenkin 1998; Linde 1986). Also, if for some reason the cosmological constant (or the quintessence field) were to decay, the future evolution of the observable universe would be different from the scenario presented here (see e.g., Starobinsky 2000 and Barrow, Bean, & Magueijo 2000 for a discussion of various possibilities).

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